

Séparation de sources. Principes et algorithmes

Pierre Comon (1), Christian Jutten (2)

(1) : I3S, CNRS, Univ. of Nice, Sophia-Antipolis

(2) : GIPSA-lab, CNRS, Univ. de Grenoble

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I. Le problème et les principes de résolution



1 Le problème de séparation de sources

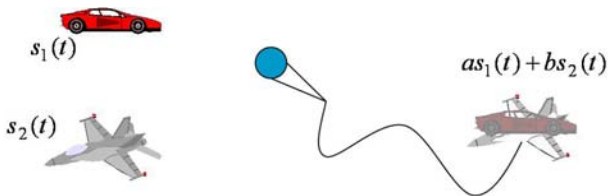
2 Principes de l'ACI

3 Mesure de dépendance

Séparation de sources : un problème courant ?

Un exemple

Le signal délivré par un capteur (électrode, antenne, microphone, etc.) est un mélange de signaux

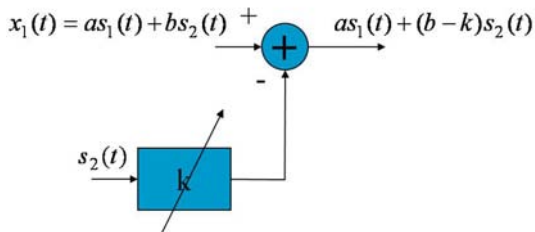


La question

Est-il possible de retrouver les différents signaux (les sources) à partir du mélange observé en sortie du capteur ?

Si oui, à quelle condition ? Et comment ?

Séparation de sources avec des a priori (1/3)

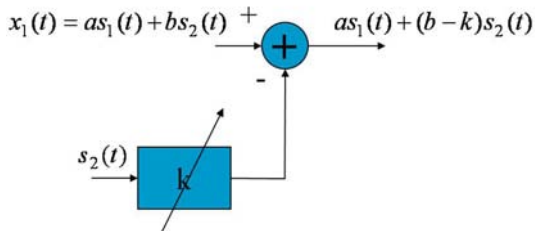


L'idée de Widrow et Hoff (1960)

On dispose alors d'une référence du signal de bruit :

- la référence est le bruit à un gain près, $s_2(t)$, ou une version filtrée du bruit, $G(s_2)$;
- on estime un filtre qui annule la contribution de s_2 dans l'observation, c'est-à-dire tel que la différence n'est plus corrélée avec la référence de bruit.

Séparation de sources avec des a priori (2/3)



$$E[(as_1(t) + (b - k)s_2(t))s_2(t)] = 0 \quad (1)$$

$$E[x_1(t)s_2(t)] = kE[s_2^2(t)], \quad (2)$$

d'où :

$$k = E[x_1(t)s_2(t)]/E[s_2^2(t)] \quad (3)$$

Séparation de sources avec des a priori (3/3)

Approche classique

Si le signal utile et le bruit sont dans des bandes de fréquences différentes

- on peut faire la séparation par filtrage.

En absence d'a priori

Que faire...

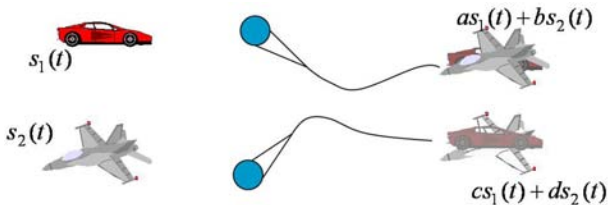
- si le signal utile et le bruit sont dans la même gamme de fréquences ?
- si on ne dispose pas d'une référence de bruit ?

Séparation de sources : l'idée fondamentale

Plusieurs observations

La séparation devient possible si l'on a plusieurs observations :

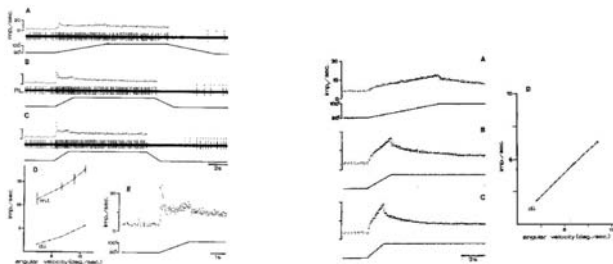
- plus de capteurs que de sources,
- mélanges, différents, non proportionnels $ad \neq bc$



C'est la **diversité spatiale**

Origine du problème

Décodage du mouvement chez les vertébrés [HJA85]



$$f_I(t) = a_{11}p(t) + a_{12}v(t)$$

$$f_{II}(t) = a_{21}p(t) + a_{22}v(t).$$

Questions ?

- Peut-on retrouver $p(t)$ et $v(t)$ à partir des mélanges ?
- Si oui, à quelles conditions ? Et comment ?

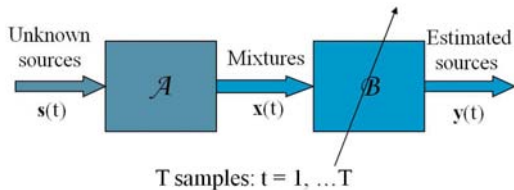
Formalisation mathématique

Les hypothèses sur le mélange (inversible)

- Mélange linéaire instantané
- Mélange linéaire convolutif
- Mélange non linéaire

Principes de solution

- Bloc de séparation, \mathcal{B} , adapté au mélange \mathcal{A} pour l'inverser
- Comment estimer \mathcal{B} ?



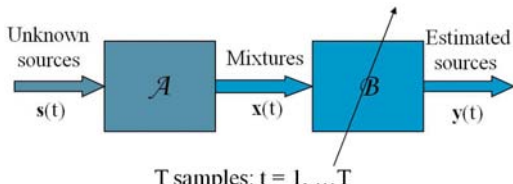
Un problème mal posé ?

Impossible sans hypothèses supplémentaires

- Hypothèse : on connaît \mathcal{A} , \mathcal{B} adapté à \mathcal{A}
- Le problème est mal posé, quel que soit le type de mélange
- Il faut ajouter des hypothèses sur les sources $s_1(t), \dots, s_P(t)$

Quels a priori ?

- Les sources sont non corrélées : insuffisant
- Les sources sont statistiquement indépendantes \rightarrow ICA
- Autres a priori : sources discrètes, sources à support borné, sources parcimonieuses, sources positives, etc.



Résultat de Darmois (1953) [Dar53]

Analyse factorielle linéaire

- Mélange linéaire : $\mathbf{x} = \mathbf{A}\mathbf{s}$
- Hypothèse : les composantes du vecteur aléatoire \mathbf{s} sont mutuellement indépendantes

Résultat théorique

- Séparation impossible si les sources sont indépendantes et identiquement distribuées (iid) ET gaussiennes

Pistes pour la séparation

- Séparation possible si les sources sont iid ET NON gaussiennes : ICA, et statistiques d'ordre supérieur à 2
- Séparation possible si les sources sont gaussiennes (statistiques d'ordre 2) ET NON iid :
 - sources temporellement corrélées
 - sources non stationnaires

Commentaires sur les hypothèses

iid non gaussiennes

- non gaussiennes : on peut (doit) utiliser des statistiques d'ordre supérieur à deux
- iid : on n'exploite pas les relations temporelles entre échantillons...
- par conséquent, l'ACI peut séparer des sources iid ou non

Gaussiennes non iid

- non iid : on exploite les relations temporelles entre échantillons...
- gaussiennes : on se restreint aux statistiques jusqu'à l'ordre 2
- par conséquent, ces approches à l'ordre 2 sont optimales pour des sources gaussiennes, mais peuvent séparer des sources non gaussiennes

Analyse en composantes indépendantes (ICA)

Principe de la méthode

- Mélange général : $\mathbf{x} = \mathcal{A}(\mathbf{s})$, avec $P = K$ et \mathcal{A} inversible
- Hypothèse : les composantes de la source \mathbf{s} sont mutuellement indépendantes
- Idée : estimer \mathcal{B} de sorte que $\mathbf{y} = \mathcal{B}(\mathbf{x})$ ait des composantes mutuellement indépendantes

Questions

- Si \mathbf{y} est indépendante, est-ce que $\mathcal{B} = \mathcal{A}^{-1}$?
- Critère pour estimer \mathcal{B} = indépendance de \mathbf{s} . Comment le mesurer ?

ICA : mélanges linéaires instantanés

Modèle de mélange

- $\mathbf{x} = \mathbf{A}\mathbf{s}$

Résultat théorique (Comon, HOS 1991 et SP 1994) [Com94a]

- Soit $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t)$, où \mathbf{A} est une matrice régulière et $\mathbf{s}(t)$ est un vecteur source à composantes statistiquement indépendantes, dont au plus une est gaussienne, alors $\mathbf{y}(t) = \mathbf{B}\mathbf{x}(t)$ est un vecteur à composantes mutuellement indépendantes si et seulement si $\mathbf{B}\mathbf{A} = \mathbf{D}\mathbf{P}$, où \mathbf{D} est une matrice diagonale et \mathbf{P} est une matrice de permutation.

Commentaires

- Indépendance = séparation, avec des hypothèses faibles : \mathbf{A} régulière, sources non gaussiennes indépendantes
- Indétermination d'échelle et de permutation
- Si plusieurs sources sont gaussiennes...?

ICA : mélanges linéaires convolutifs

Modèle de mélange

- $\mathbf{x}(t) = \mathbf{A}(t) * \mathbf{s}(t)$

ou en temps discret :

- $x_i(n) = \sum a_{ij}(k) s_j(n - k), \forall i = 1, \dots, K$

Premiers résultats théoriques [YW94, TJ95]

- Soient $\mathbf{x}(t) = [\mathbf{A}(z)]\mathbf{s}(t)$, où $\mathbf{A}(z)$ est une matrice de filtres inversible et $\mathbf{s}(t)$ est un vecteur source à composantes statistiquement indépendantes, dont au plus une est gaussienne, alors $\mathbf{y}(t) = [\mathbf{B}(z)]\mathbf{x}(t)$ est un vecteur à composantes mutuellement indépendantes si et seulement si $\mathbf{B}(z)\mathbf{A}(z) = \mathbf{D}(z)\mathbf{P}$, où $\mathbf{D}(z)$ est une matrice de filtres diagonale et \mathbf{P} est une matrice de permutation.

Commentaires

- Sources retrouvées à un filtre et une permutation près



ICA : mélanges non linéaires (NL)

Modèle de mélange

- $\mathbf{x}(t) = \mathcal{A}(s(t))$

Impossible en général [Dar53, HP99]

- Dans la cas général, l'indépendance de \mathbf{y} n'implique pas la séparation de sources : $\mathbf{y}(t) = \mathcal{B}(\mathbf{x}(t))$ peut avoir des composantes indépendantes avec $\mathcal{B} \circ \mathcal{A}$ non diagonale !
- Si l'ACI permet la séparation, la solution est indéterminée à une permutation et une fonction non linéaire près : $f(s_i)$ avec $f(\cdot)$ fonction NL inconnue.

Possible pour certains mélanges

- Mélanges post-non-linéaires [TJ99a, BZJN02, JBZH04, AJ05]
- Mélanges NL linéarisables [KLR73, EK02]
- Mélanges bilinéaires et linéaires-quadratiques (sans preuve d'identifiabilité : [HD03, DH09])

Aveugle, vous avez dit aveugle ?

La séparation de sources est un problème...

- non aveugle : nature du mélange
- non aveugle : hypothèses supplémentaires sur les sources
- indépendance des sources : hypothèse très large (même si mise en oeuvre complexe), d'où le terme (abusif) *aveugle associé à l'ACI*

Autres informations a priori

- Autres a priori : autres critères et autres méthodes
- Sources avec dépendances temporelles entre échantillons : méthodes à l'ordre 2
- Sources discrètes, à support borné : méthodes algébriques ou géométriques
- Parcimonie : masquage dans le plan t-f, analyse en composantes parcimonieuses (SCA)

Mélanges déterminés et sur-déterminés

Mélanges déterminés

- Autant de mélanges que de sources, $K = P$
- Identifier \mathbf{A} ou son inverse fournit directement les sources.

Mélanges sur-déterminés

- Plus de mélanges que de sources, $K > P$
- Dans le cas linéaire : solution si la matrice de mélange est de rang plein
- Pré-traitement par PCA pour projeter dans l'espace signal avant séparation

Dans les deux cas, on peut utiliser

- la méthode directe : identifier \mathbf{A} puis l'inverser,
- la méthode indirecte : identifier \mathbf{B} , une inverse de \mathbf{A} .

Mélanges sous-déterminés

Plus de sources que de mélanges, $P > K$

- Si on connaît \mathbf{A} (pas d'inverse !), on ne déduit pas simplement \mathbf{s}
- Identification de \mathbf{A} et restauration des sources sont deux problèmes distincts et difficiles
- Sans a priori supplémentaire, pas de solution
- Dans la cas linéaire, si \mathbf{A} est connu, on peut retrouver les sources à un vecteur aléatoire près [TJ99a]
- Solution possible si les sources sont discrètes ou parcimonieuses

La méthode indirecte est donc impossible

Indéterminations des solutions revisitées

Propriétés de l'indépendance

- Si deux sources s_i et s_j sont indépendantes, alors
 - s_j et s_i sont aussi indépendantes,
 - $k_j s_j$ et $k_i s_i$, où k_i, k_j sont deux scalaires quelconques, sont aussi indépendantes
 - $f_j(s_j)$ et $f_i(s_i)$, où f_i, f_j sont deux fonctions non linéaires quelconques, sont aussi indépendantes
 - $H_j(s_j)$ et $H_i(s_i)$, où H_i, H_j sont deux filtres quelconques, sont aussi indépendantes

Propriétés des observations : cas linéaire instantané

- Les observations $x_i(t) = \sum_j a_{ij} s_j(t)$ ne sont pas discernables de $x_i(t) = \sum_j \frac{a_{ij}}{\alpha_j} \alpha_j s_j(t)$.

Conséquences des indéterminations

Directes, sur la matrice de mélange

- On ne peut pas estimer la puissance des sources
- P paramètres de \mathbf{A} ne sont pas identifiables ; on ne peut estimer que les vecteurs colonne de \mathbf{A} , c'est-à-dire les "directions" des sources
- on peut imposer à \mathbf{A} la contrainte : $a_{ij} = 1$

Indirectes, sur la matrice de séparation

- On peut fixer arbitrairement la puissance des sources estimées : normalisation ou pré-blanchiment
- P paramètres de \mathbf{B} ne sont pas identifiables : on peut fixer $b_{ij} = 1$, ou normaliser les lignes de \mathbf{B}

Indépendance de variables et vecteurs aléatoires

Factorisation des lois ou densités de probabilité (ddp)

Deux variables aléatoires

- 2 variables aléatoires S_1 et S_2 sont indépendantes ssi

$$p_{S_1 S_2}(u_1, u_2) = p_{S_1}(u_1)p_{S_2}(u_2)$$

Vecteurs aléatoires

- Soit \mathbf{S} un vecteur aléatoire à P composantes S_1, \dots, S_P
- Les composantes sont mutuellement indépendantes ssi

$$p_{S_1 S_2 \dots S_P}(u_1, u_2, \dots, u_P) = p_{S_1}(u_1)p_{S_2}(u_2) \dots p_{S_P}(u_P)$$

Commentaires

- Critère délicat à mettre en oeuvre
- Egalité entre deux fonctions multivariées

Fonctions caractéristiques

Première fonction caractéristique

- C'est la T. Fourier inverse de la ddp de \mathbf{S}
- Conséquence : même information que dans la ddp
- $\Phi(\boldsymbol{\nu}) = E[\exp(j\boldsymbol{\nu}^T \mathbf{S})] = \int \dots \int p_{\mathbf{S}}(\mathbf{u}) \exp(j\boldsymbol{\nu}^T \mathbf{S}) d\mathbf{u}$

Seconde fonction caractéristique (SFC)

- C'est le log de la première fonction caractéristique
- Conséquence : même information que dans la ddp
- $\Psi(\boldsymbol{\nu}) = \log \Phi(\boldsymbol{\nu})$

Propriétés de la SFC

- $\Psi(\boldsymbol{\nu}) = 0$ pour $\boldsymbol{\nu} = 0$
- Elle est continue et dérivable en $\boldsymbol{\nu} = 0$

Développement de la SFC et cumulants

Développement de la SFC

$$\blacksquare \Psi(\nu) = \sum_{p=1}^{\infty} \kappa_p \frac{(j\nu)^p}{p!}$$

Cumulant d'ordre p

$$\blacksquare \kappa_p = (-j)^p \left. \frac{d^p \Psi(\nu)}{d\nu^p} \right|_{\nu=0}$$

Cumulants d'une variable centrée

$$\kappa_1 = \mu_1 = 0 \quad (4)$$

$$\kappa_2 = \mu_2 \quad (5)$$

$$\kappa_3 = \mu_3 \quad (6)$$

$$\kappa_4 = \mu_4 - 3\mu_2^2 \quad (7)$$

Petit exercice

Calcul des cumulants d'ordre 2, 3 et 4 d'une variable centrée

Définition

- $\Psi(\nu) = \sum_{p=1}^{\infty} \kappa_p \frac{(j\nu)^p}{p!}$

Développement en $\nu = 0$

- $\Psi(\nu) = \ln \Phi(\nu) = \ln \Phi(0) + \left. \frac{d \ln \Phi(\nu)}{d\nu} \right|_{\nu=0} \nu + \left. \frac{d^2 \ln \Phi(\nu)}{d\nu^2} \right|_{\nu=0} \frac{\nu^2}{2} + \dots$

Détermination des cumulants...

- par simple identification

Indépendance, fonction caractéristique et cumulants

Indépendance et fonction caractéristique

$$p_{S_1 S_2 \dots S_P}(u_1, u_2, \dots, u_P) = p_{S_1}(u_1) p_{S_2}(u_2) \dots p_{S_P}(u_P) \quad (8)$$

$$\Phi(\nu) = \Phi(\nu_1) \Phi(\nu_2) \dots \Phi(\nu_P) \quad (9)$$

$$\ln \Phi(\nu) = \ln \Phi(\nu_1) + \dots + \ln \Phi(\nu_P) \quad (10)$$

$$\Psi(\nu) = \Psi(\nu_1) + \Psi(\nu_2) + \dots + \Psi(\nu_P) \quad (11)$$

Développement de la SFC

- le terme de gauche contient des termes croisés
- les termes de droites ne contiennent que des dérivées par rapport à une seule variable
- indépendance ssi tous les cumulants croisés sont nuls

Problème

- Il y a une infinité de cumulants
- Peut-on s'arrêter à un ordre donné ? Comment le définir ?

Divergence de Kullback-Leibler

Définition

- $KL(f\|g) = \int \dots \int f(\mathbf{u}) \log \frac{f(\mathbf{u})}{g(\mathbf{u})} d\mathbf{u}$
- $KL(f\|g)$ mesure l'écart entre les deux distributions f et g
- c'est un scalaire positif qui s'annule ssi $f = g$ (la démonstration est aisée en utilisant l'inégalité de Jensen)

Divergence KL pour mesure l'indépendance

- $KL(p_{\mathbf{Y}}\|\prod p_{Y_i}) = \int \dots \int p_{\mathbf{Y}}(\mathbf{u}) \log \frac{p_{\mathbf{Y}}(\mathbf{u})}{\prod p_{Y_i}(u_i)} d\mathbf{u}$
- $KL(p_{\mathbf{Y}}\|\prod p_{Y_i})$ est un scalaire positif qui s'annule ssi les composantes du vecteur aléatoire \mathbf{Y} sont mutuellement indépendantes
- On peut identifier cette mesure, $KL(p_{\mathbf{Y}}\|\prod p_{Y_i})$, à l'information mutuelle (IM) $I(\mathbf{Y})$ définie en théorie de l'information

Décomposition de l'IM à l'aide d'entropie

Définition

$$I(\mathbf{Y}) = \int \dots \int p_{\mathbf{Y}}(\mathbf{u}) \log \frac{p_{\mathbf{Y}}(\mathbf{u})}{\prod p_{Y_i}(u_i)} d\mathbf{u} \quad (12)$$

$$= \sum_i H(Y_i) - H(\mathbf{Y}) \quad (13)$$

avec les définitions suivantes des entropies marginales et jointes

$$H(Y_i) = \int p_{Y_i}(u_i) \log p_{Y_i}(u_i) du_i \quad (14)$$

$$H(\mathbf{Y}) = \int \dots \int p_{\mathbf{Y}}(\mathbf{u}) \log p_{\mathbf{Y}}(\mathbf{u}) d\mathbf{u} \quad (15)$$

Problème

L'estimation des entropies, jointe et marginales, nécessite l'estimation des densités, jointe et marginales, resp.

Une astuce dans le cas linéaire

Dans le cas déterminé, avec une matrice de mélange \mathbf{A} inversible, on a :

L'astuce

$$I(\mathbf{Y}) = \sum_i H(Y_i) - H(\mathbf{Y}) \quad (16)$$

$$= \sum_i H(Y_i) - H(\mathbf{X}) - E[\log |\det \mathbf{B}|] \quad (17)$$

Conséquence

- $\min_{\mathbf{B}} \sum_i H(Y_i) - H(\mathbf{Y}) \Leftrightarrow \min_{\mathbf{B}} \sum_i H(Y_i) - \log |\det \mathbf{B}|$
- L'astuce évite l'estimation de l'entropie jointe !

Minimisation de l'IM et fonction score

Dans le cas linéaire, on estime une matrice de séparation \mathbf{B} qui minimise $l(\mathbf{y})$.

Dérivée de l'IM par rapport à \mathbf{B}

$$\begin{aligned} \frac{dl(\mathbf{Y})}{d\mathbf{B}} &= \sum_i \frac{dH(Y_i)}{d\mathbf{B}} + \frac{d \log |\det \mathbf{B}|}{d\mathbf{B}} \\ &= \sum_i E \left[\underbrace{\frac{-d \log p_{Y_i}(y_i)}{dy_i}}_{\text{fonction score de } y_i} \frac{dy_i}{d\mathbf{B}} \right] + \mathbf{B}^{-T} \end{aligned}$$

Minimisation de l'IM et annulation de HOS

On écrit $\frac{dl(\mathbf{Y})}{d\mathbf{B}} = 0$: $\frac{dl(\mathbf{Y})}{d\mathbf{B}} = E[\varphi_{\mathbf{Y}}(\mathbf{Y})\mathbf{X}^T] + \mathbf{B}^{-T} = 0$
 soit, en multipliant par \mathbf{B}^T :

$$\frac{dl(\mathbf{Y})}{d\mathbf{B}} = 0 \Leftrightarrow E[\varphi_{\mathbf{Y}}(\mathbf{Y})\mathbf{Y}^T] + \mathbf{I} = 0$$

Exploiter des a priori (1/4)

Pourquoi ?

- Critères mieux adaptés
- Algorithmes plus simples
- Relâchement d'hypothèses
- Sélection des sources séparées

Sources colorées et non stationnaires

- Exploite les dépendances temporelles (sources non iid)
- Voir chapitre sur les méthodes au second ordre

Sources positives

- Exploite la positivité des sources et/ou des mélanges
- Voir chapitre consacré aux mélanges non négatifs

Exploiter des a priori (2/4)

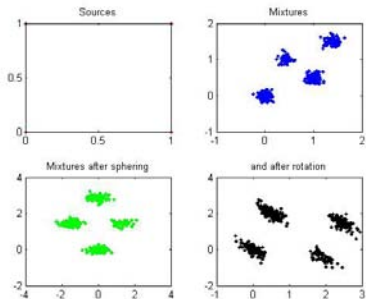
Sources discrètes

- Si D est le nombre d'états,
- Les mélanges (bruités) sont des ensembles de points (nuages)
- Au total, D^P points (nuages)
- Mélange linéaire de sources binaires : points sur les sommets d'un parallépipède

$$s_1 = 0 \Rightarrow \mathbf{x} = \mathbf{a}_2 s_2$$

$$s_2 = 0 \Rightarrow \mathbf{x} = \mathbf{a}_1 s_1$$

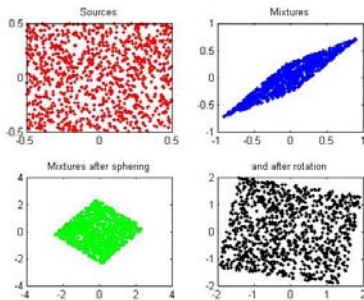
où $\mathbf{a}_i =$ colonne i de \mathbf{A}



Exploiter des a priori (3/4)

Sources à support borné

- Sources dont la ddp est à support borné, avec valeurs non nulles sur les bornes
- Dans le cas de mélange linéaire : les observations sont inscrites dans un parallépipède



$$s_1 = 0 \Rightarrow \mathbf{x} = \mathbf{a}_2 s_2$$

$$s_2 = 0 \Rightarrow \mathbf{x} = \mathbf{a}_1 s_1$$

où \mathbf{a}_i = colonne i de \mathbf{A}

Exploiter des a priori (4/4)

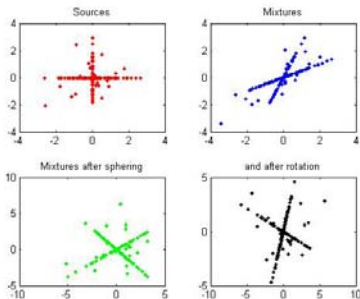
Sources parcimonieuses

- Sources dont les échantillons sont presque toujours nuls
- Parcimonie dans un espace de représentation adapté
- Cas de mélange linéaire : $\mathbf{x} = \mathbf{A}\mathbf{s} \Rightarrow \mathcal{T}(\mathbf{x}) = \mathcal{T}(\mathbf{A}\mathbf{s}) = \mathbf{A}\mathcal{T}(\mathbf{s})$
- Voir les cours de Stark et Daudet

$$s_1 = 0 \Rightarrow \mathbf{x} = \mathbf{a}_2 s_2$$

$$s_2 = 0 \Rightarrow \mathbf{x} = \mathbf{a}_1 s_1$$

où $\mathbf{a}_i =$ colonne i de \mathbf{A}



Séparation de mélanges bruités

Cas de mélanges linéaires

- Les observations sont : $\mathbf{x}(t) = \mathbf{A}s(t) + \mathbf{n}(t)$, où \mathbf{n} est un bruit additif, indépendant de s .
- La présence de bruit entraîne généralement une erreur sur l'estimation de \mathbf{A} ou de son inverse.
- Si on suppose que \mathbf{B} est parfaitement estimée, soit $\mathbf{B} = \mathbf{A}^{-1}$, alors:

$$\mathbf{y}(t) = \mathbf{B}\mathbf{x}(t) = s(t) + \mathbf{B}\mathbf{n}(t)$$

- La séparation idéale conduit évidemment au meilleur SIR, mais pas forcément au meilleur SNR
- Pour des détails voir Chapitres 1 (1.4) et 4 (4.7) dans [CJ10]

ACI : supervisée ou non ?

ACI : une méthode d'apprentissage

- Estimation à partir de données empiriques
- Apprentissage au sens “machine learning”

Supervisé ou non ?

- Critère d'indépendance
- Mesure d'indépendance repose sur les données
- Non supervisé... aveugle par opposition à “supervisé”, qui demande des informations extérieures (superviseur, valeurs désirées)

II. Tools and optimization criteria



Contents of course II

4 Algebraic tools

- Spatial whitening
- Tensors

5 Statistical tools

- Independence
- Cumulants
- Mutual Information

6 Optimization Criteria

- Cumulant matching
- Contrasts

Spatial whitening (1)

Standardization via Cholesky or QR Let \mathbf{x} be a zero-mean r.v. with covariance matrix:

$$\mathbf{\Gamma}_x \stackrel{\text{def}}{=} \mathbb{E}\{\mathbf{x} \mathbf{x}^H\}$$

Then Cholesky yields:

$$\exists \mathbf{L} / \quad \mathbf{L} \mathbf{L}^H = \mathbf{\Gamma}_x$$

Consequence: $\mathbf{L}^{-1}\mathbf{x}$ has a unit variance.

Variable $\tilde{\mathbf{x}} \stackrel{\text{def}}{=} \mathbf{L}^{-1}\mathbf{x}$ is a *standardized random variable*.

- QR factorization of data matrix as $\mathbf{X} = \mathbf{L} \tilde{\mathbf{X}}$ yields same \mathbf{L} as Cholesky factorization of sample covariance, but more accurate.
- Limitation: \mathbf{L} may not be invertible if the covariance $\mathbf{\Gamma}_x$ is not full rank.

Spatial whitening (2)

Standardization via PCA

Definition

PCA is based on second order statistics

- Observed random variable \mathbf{x} of dimension K . Then $\exists(\mathbf{U}, \mathbf{z})$:

$$\mathbf{x} = \mathbf{U}\mathbf{z}, \mathbf{U} \text{ unitary}$$

where *Principal Components* z_i are uncorrelated
 i th column \mathbf{u}_i of \mathbf{U} is called *i th PC Loading vector*

- Two possible calculations:
 - EVD of Covariance \mathbf{R}_x : $\mathbf{R}_x = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^H$
 - Sample estimate by SVD: $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$

Spatial whitening (3)

Summary

Find a linear transform \mathbf{L} such that vector $\tilde{\mathbf{x}} \stackrel{\text{def}}{=} \mathbf{L}\mathbf{x}$ has unit covariance. Many possibilities, including:

- PCA yields $\tilde{\mathbf{x}} = \mathbf{\Sigma}^{-1} \mathbf{U}^H \mathbf{x}$
- Cholesky $\mathbf{R}_x = \mathbf{L} \mathbf{L}^H$ yields $\tilde{\mathbf{x}} = \mathbf{L}^{-1} \mathbf{x}$

Remarks

- Infinitely many possibilities: \mathbf{L} is as good as $\mathbf{L} \mathbf{Q}$, for any unitary \mathbf{Q} .
- If \mathbf{R}_x not invertible, then \mathbf{L} not invertible (ill-posed). One may use pseudo-inverse of $\mathbf{\Sigma}$ in PCA to compute \mathbf{L} , or regularize \mathbf{R}_x .

Bibliography

- Antenna Array Processing: [BK83] [Sch86]
- Implementations: [CJ10], LaPack, LinPack
- Adaptive versions: via QR [Com89a], or via SVD [CG90], or stochastic relative gradient [CL96]

Arrays and Multi-linearity

- A tensor of *order* d is a *multi-linear* map:

$$\mathcal{S}_1^* \otimes \dots \otimes \mathcal{S}_m^* \rightarrow \mathcal{S}_{m+1} \otimes \dots \otimes \mathcal{S}_d$$

- Once bases of spaces \mathcal{S}_ℓ are fixed, they can be represented by *d-way arrays* of coordinates

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- bilinear form, or linear operator: represented by a *matrix*
- trilinear form, or bilinear operator: by a *3rd order tensor*.

Arrays and Tensors

Definitions Table $\mathbf{T} = \{T_{ij..k}\}$

- *Order* of $\mathbf{T} \stackrel{\text{def}}{=} \#$ of its ways = $\#$ of its indices
- *Dimension* $K_\ell \stackrel{\text{def}}{=} \text{range of the } \ell\text{th index}$
- \mathbf{T} is *Cubic* when all dimensions $K_\ell = K$ are equal
- \mathbf{T} is *Symmetric* when it is square and when its entries do not change by *any* permutation of indices

Multi-linearity

Compact notation

Linear change in contravariant spaces:

$$T'_{ijk} = \sum_{npq} A_{in} B_{jp} C_{kq} T_{npq}$$

Denoted compactly

$$\mathcal{T}' = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{T} \quad (18)$$

Example: covariance matrix

$$\mathbf{z} = \mathbf{A}\mathbf{x} \Rightarrow \mathbf{R}_z = (\mathbf{A}, \mathbf{A}) \cdot \mathbf{R}_x = \mathbf{A} \mathbf{R}_x \mathbf{A}^T$$

Decomposable tensor

- A d th order “*decomposable*” tensor is the tensor product of d vectors:

$$\mathcal{T} = \mathbf{u} \otimes \mathbf{v} \otimes \dots \otimes \mathbf{w}$$

and has coordinates $T_{ij\dots k} = u_i v_j \dots w_k$.

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- Nothing else but *rank-1* tensors, with forthcoming definition

Example

- Take

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example

■ Take

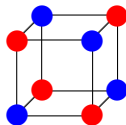
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■ Then

$$\mathbf{v}^{\otimes 3} \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} = \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array} \right]$$

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- Then
$$\mathbf{v}^{\otimes 3} \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} = \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array} \right]$$
- This is a “decomposable” symmetric tensor \rightarrow *rank-1*



blue bullets = 1, red bullets = -1.

From SVD to tensor decompositions

Matrix SVD, $\mathbf{M} = (\mathbf{U}, \mathbf{V}) \cdot \mathbf{\Sigma}$, may be extended in at least two ways to tensors

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$$\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L} \quad (19)$$

\mathcal{L} is $R \times R \times R$ diagonal, $\lambda_i \neq 0$: *rank* = R .

Canonical Polyadic (CP) decomposition

- Any $I \times J \times \dots \times K$ tensor \mathcal{T} can be decomposed as

$$\mathcal{T} = \sum_q \lambda_q \mathbf{u}^{(q)} \otimes \mathbf{v}^{(q)} \otimes \dots \otimes \mathbf{w}^{(q)} \quad (20)$$

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- The *tensor rank* of \mathcal{T} is the minimal number $R(\mathcal{T})$ of “decomposable” terms such that equality holds.
- May impose unit norm vectors $\mathbf{u}^{(q)}, \mathbf{v}^{(q)}, \dots, \mathbf{w}^{(q)}$

Hitchcock



Frank Lauren Hitchcock
(1875-1957)
[Courtesy of L-H.Lim]

Hitchcock



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Claude Elwood Shannon
(1916-2001)

Towards a unique terminology

- Minimal *Polyadic Form* [Hitchcock'27]
- *Canonical decomposition* [Weinstein'84, Carroll'70, Chiantini-Ciliberto'06, Comon'00, Khoromskij, Tyrtysnikov]
- *Parafac* [Harshman'70, Sidiropoulos'00]
- *Optimal computation* [Strassen'83]
- Minimum-length *additive decomposition* (AD) [Iarrobino'96]

Suggestion:

- *Canonical Polyadic decomposition (CP)* [Comon'08, Grasedyk, Espig...]
- *CP* does also already stand for Candecomp/Parafac [Bro'97, Kiers'98, tenBerge'04...]

Psychometrics



Richard A. Harshman
(1970)



J. Douglas Carroll
(1970)

Uniqueness: Kruskal 1/2



The *Kruskal rank* of a matrix \mathbf{A} is the maximum number k_A , such that *any* subset of k_A columns are linearly independent.

Uniqueness: Kruskal 2/2

Sufficient condition for uniqueness of CP

[Kruskal'77, Sidiropoulos-Bro'00, Landsberg'09]:

Essential uniqueness is ensured if tensor rank R is below the so-called *Kruskal's bound*:

$$2R + 2 \leq k_A + k_B + k_C \quad (21)$$

Uniqueness: Kruskal 2/2

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or *generically*, for a tensor of order d and dimensions N_ℓ :

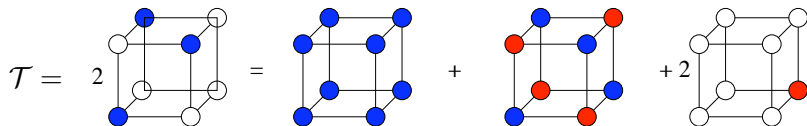
$$2R \leq \sum_{\ell=1}^d \min(N_\ell, R) - d + 1$$

➡ Bound much smaller than expected rank:

∃ *a much better bound*, in almost sure sense

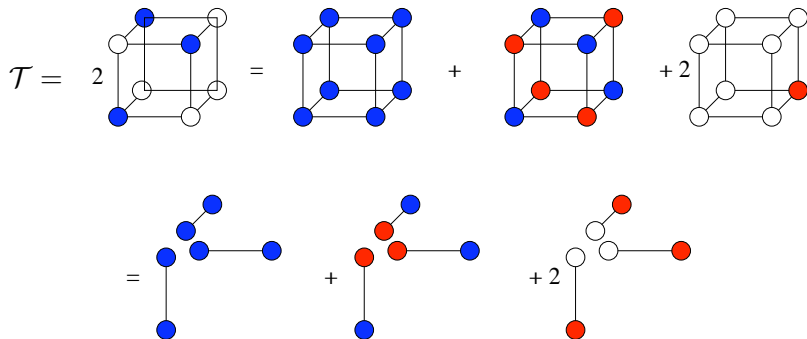
Rank-3 example 1/2

Example



Rank-3 example 1/2

Example



blue bullets = 1, red bullets = -1.

Rank-3 example 2/2

Conclusion: the $2 \times 2 \times 2$ tensor

$$\mathcal{T} = 2 \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

admits the CP

$$\mathcal{T} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}^{\otimes 3} + 2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}^{\otimes 3}$$

and has rank 3, hence *larger* than dimension

Why need for approximation?

- Additive noise in measurements
- Noise has a continuous probability distribution
- Then tensor rank is *generic*
- Hence there are often *infinitely many* CP decompositions

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- Additive noise in measurements
 - Noise has a continuous probability distribution
 - Then tensor rank is *generic*
 - Hence there are often *infinitely many* CP decompositions
- ➔ Approximations aim at getting rid of noise, *and* at restoring uniqueness:

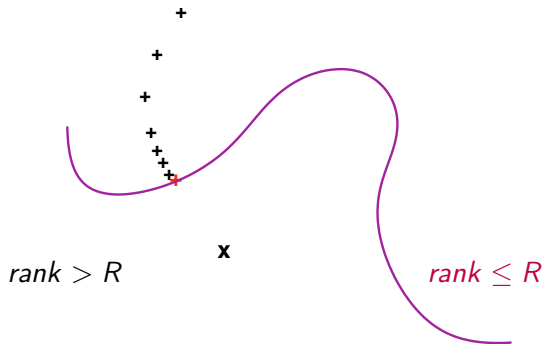
$$\mathop{\text{Arg}}_{\mathbf{a}(p), \mathbf{b}(p), \mathbf{c}(p)} \inf \left\| \mathcal{T} - \sum_{p=1}^r \mathbf{a}(p) \otimes \mathbf{b}(p) \dots \otimes \mathbf{c}(p) \right\|^2 \quad (22)$$

But infimum may be reached for tensors of rank $> r$!

Border rank

\mathcal{T} has *border rank* R iff it is the limit of tensors of rank R , and not the limit of tensors of lower rank.

[Bini'79, Schönhage'81, Strassen'83, Likteig'85]



Example

Let \mathbf{u} and \mathbf{v} be non collinear vectors. Define \mathcal{T}_0 [CGLM08]:

$$\mathcal{T}_0 = \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$$

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And define sequence $\mathcal{T}_\varepsilon = \frac{1}{\varepsilon} [(\mathbf{u} + \varepsilon \mathbf{v})^{\otimes 4} - \mathbf{u}^{\otimes 4}]$.

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Then $\mathcal{T}_\varepsilon \rightarrow \mathcal{T}_0$ as $\varepsilon \rightarrow 0$

➡ Hence $\text{rank}\{\mathcal{T}_0\} = 4$, but *border rank* is 2

Ill-posedness

Tensors for which the border rank is $<$ than rank are such that the approximating sequence contains several decomposable tensors which

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Ideas towards a well-posed problem:

- Prevent collinearity *or* bound columns.

Remedies

- 1 Impose *orthogonality* of columns within factor matrices [Com92a]
- 2 Impose *orthogonality* between decomposable tensors [Kol01]
- 3 Prevent tendency to infinity by *norm constraint* on factor matrices [Paa00]

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- 1 Impose *orthogonality* of columns within factor matrices [Com92a]
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- 3 Prevent tendency to infinity by *norm constraint* on factor matrices [Paa00]
- 4 Nonnegative tensors: impose decomposable tensors to be *nonnegative* [LC09] → “nonnegative rank”
- 5 Impose *minimal angle* between columns of factor matrices [CL11]

Tensors as linear operators

- **Warning:** Considering a tensor as a linear operator is restrictive (eg. rank is bounded by smallest dimension)

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NB: do not confuse tensor \otimes and Kronecker \otimes products...

Matrix products (1/2)

Kronecker product between two matrices **A** and **B** of dimensions $m \times n$ and $p \times q$ respectively, is a matrix of size $mp \times nq$ defined as:

$$\mathbf{A} \otimes \mathbf{B} \stackrel{\text{def}}{=} \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} & \cdots \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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Khatri-Rao product between two matrices with the same number of columns:

$$\mathbf{A} \odot \mathbf{B} \stackrel{\text{def}}{=} \left(\mathbf{a}_1 \otimes \mathbf{b}_1 \quad \mathbf{a}_2 \otimes \mathbf{b}_2 \quad \cdots \right).$$

 This is a *column-wise Kronecker* product.

Matrix products (2/2)

Example: Let

$\mathbf{a} \in \mathcal{S}$, space of basis $\{\mathbf{e}(i)\}$ of dim. I ,

$\mathbf{b} \in \mathcal{S}'$, space of basis $\{\mathbf{e}'(j)\}$ of dim. J ,

$\mathbf{c} \in \mathcal{S}''$, space of basis $\{\mathbf{e}''(k)\}$ of dim. K

Then tensor $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$ has coordinates given by vector $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$ in basis $\{\mathbf{e}(i) \otimes \mathbf{e}'(j) \otimes \mathbf{e}''(k)\}$

Decorrelation vs Independence

Example 1: Mixture of 2 iid sources

Consider the mixture of two independent sources

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

where $E\{s_i^2\} = 1$ and $E\{s_i\} = 0$. Then x_i are *uncorrelated*:

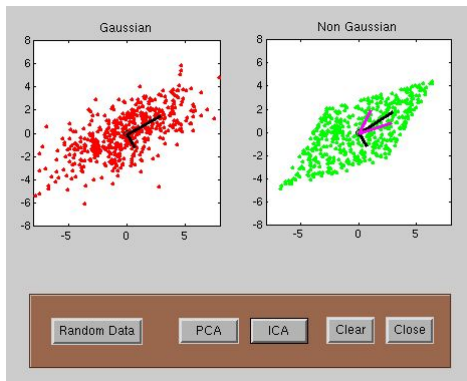
$$E\{x_1 x_2\} = E\{s_1^2\} - E\{s_2^2\} = 0$$

But x_i are *not independent* since, for instance:

$$E\{x_1^2 x_2^2\} - E\{x_1^2\}E\{x_2^2\} = E\{s_1^4\} + E\{s_2^4\} - 6 \neq 0$$

PCA vs ICA

Example 2: 2 sources and 2 sensors



Statistical independence

Definition

Components s_k of a K -dimensional r.v. s are *mutually independent*



The *joint* pdf equals the *product of marginal* pdf's:

$$p_s(\mathbf{u}) = \prod_k p_{s_k}(u_k) \quad (23)$$

Definition

Components s_k of s are *pairwise independent* \Leftrightarrow Any pair of components (s_k, s_ℓ) are mutually independent.

Mutual and Pairwise independence

Darmois's Theorem (1953)

Let two random variables be defined as linear combinations of independent random variables x_j :

$$X_1 = \sum_{i=1}^N a_i x_i, \quad X_2 = \sum_{i=1}^N b_i x_i$$

Then, if X_1 and X_2 are independent, those x_j for which $a_j b_j \neq 0$ are Gaussian.

[Proof](#)

Mutual and Pairwise independence (cont'd)

Corollary [Com92a]

If $\mathbf{z} = \mathbf{C} \mathbf{s}$, where s_i are independent r.v., with at most one of them being Gaussian, then the following properties are equivalent:

- 1 Components z_i are pairwise independent
- 2 Components z_i are mutually independent
- 3 $\mathbf{C} = \mathbf{\Lambda} \mathbf{P}$, with $\mathbf{\Lambda}$ diagonal and \mathbf{P} permutation

Characteristic functions

First c.f.

- Real Scalar: $\Phi_x(t) \stackrel{\text{def}}{=} E\{e^{j t x}\} = \int_{\mathbf{u}} e^{j t u} dF_x(u)$
- Real Multivariate: $\Phi_x(\mathbf{t}) \stackrel{\text{def}}{=} E\{e^{j \mathbf{t}^T \mathbf{x}}\} = \int_{\mathbf{u}} e^{j \mathbf{t}^T \mathbf{x}} dF_x(\mathbf{u})$

Second c.f.

- $\Psi(\mathbf{t}) \stackrel{\text{def}}{=} \log \Phi(\mathbf{t})$
- Properties:
 - Always exists in the neighborhood of 0
 - Uniquely defined as long as $\Phi(\mathbf{t}) \neq 0$

Definition of Multivariate cumulants

- Moments:

$$\mu'_{ij..k} \stackrel{\text{def}}{=} \mathbb{E}\{\underbrace{x_i x_j \dots x_k}_{r \text{ times}}\} = (-j)^r \left. \frac{\partial^r \Phi(t)}{\partial t_i \partial t_i \dots \partial t_k} \right|_{t=0} \quad (24)$$

- Cumulants:

$$C_{ij..k} \stackrel{\text{def}}{=} \text{Cum}\{\underbrace{x_i, x_j, \dots, x_k}_{r \text{ times}}\} = (-j)^r \left. \frac{\partial^r \Psi(t)}{\partial t_i \partial t_i \dots \partial t_k} \right|_{t=0} \quad (25)$$

NB: *Leonov Shirayev* relationship between Moments and Cumulants obtained by expanding both sides in Taylor series:

$$\text{Log } \Phi_x(t) = \Psi_x(t)$$

Definition of Multivariate cumulants (cont'd)

- First cumulants:

$$\begin{aligned}\mu'_i &= C_i \\ \mu'_{ij} &= C_{ij} + C_i C_j \\ \mu'_{ijk} &= C_{ijk} + [3] C_i C_{jk} + C_i C_j C_k\end{aligned}$$

with $[n]$: McCullagh's *bracket notation*.

- Next, for zero-mean variables:

$$\begin{aligned}\mu_{ijkl} &= C_{ijkl} + [3] C_{ij} C_{kl} \\ \mu_{ijklm} &= C_{ijklm} + [10] C_{ij} C_{klm}\end{aligned}$$

Definition of Complex Cumulants

Definition

Let $\mathbf{z} = \mathbf{x} + j\mathbf{y}$. Then pdf $p_{\mathbf{z}}$ = joint pdf $p_{\mathbf{x},\mathbf{y}}$

Notation

- Characteristic function:

$$\Phi_{\mathbf{z}}(\mathbf{w}) = \mathbb{E}\{\exp[j(\mathbf{x}^T \mathbf{u} + \mathbf{y}^T \mathbf{v})]\} = \mathbb{E}\{\exp[j\Re(\mathbf{z}^H \mathbf{w})]\}$$

where $\mathbf{w} \stackrel{\text{def}}{=} \mathbf{u} + j\mathbf{v}$.

- Generates Moments & Cumulants, e.g.:

$$\text{Variance: } \text{Var}\{z\}_{ij} = C_{z_i}^j$$

$$\text{Higher orders: } \text{Cum}\{z_i, \dots, z_j, z_k^*, \dots, z_\ell^*\} = C_{z_{ij}}^{kl}$$

where *conjugated* r.v. are labeled *in superscript*.

Properties of Cumulants

- **Multi-linearity** (also enjoyed by moments):

$$\text{Cum}\{\alpha X, Y, \dots, Z\} = \alpha \text{Cum}\{X, Y, \dots, Z\} \quad (26)$$

$$\text{Cum}\{X_1 + X_2, Y, \dots, Z\} = \text{Cum}\{X_1, Y, \dots, Z\} + \text{Cum}\{X_2, Y, \dots, Z\}$$

- **Cancellation:** If $\{X_i\}$ can be partitioned into 2 groups of independent r.v., then

$$\text{Cum}\{X_1, X_2, \dots, X_r\} = 0 \quad (27)$$

- **Additivity:** If \mathbf{X} and \mathbf{Y} are independent, then

$$\begin{aligned} \text{Cum}\{X_1 + Y_1, X_2 + Y_2, \dots, X_r + Y_r\} &= \text{Cum}\{X_1, X_2, \dots, X_r\} \\ &+ \text{Cum}\{Y_1, Y_2, \dots, Y_r\} \end{aligned}$$

Mutual Information: definition

- According to the definition of page 95, one should measure a divergence:

$$\delta \left(p_{\mathbf{x}}, \prod_{i=1}^N p_{x_i} \right)$$

- If the *Kullback divergence* is used:

$$K(p_{\mathbf{x}}, p_{\mathbf{y}}) \stackrel{\text{def}}{=} \int p_{\mathbf{x}}(\mathbf{u}) \log \frac{p_{\mathbf{x}}(\mathbf{u})}{p_{\mathbf{y}}(\mathbf{u})} d\mathbf{u},$$

then we get the *Mutual Information* as an independence measure:

$$I(p_{\mathbf{x}}) = \int p_{\mathbf{x}}(\mathbf{u}) \log \frac{p_{\mathbf{x}}(\mathbf{u})}{\prod_{i=1}^N p_{x_i}(u_i)} d\mathbf{u}. \quad (28)$$

Mutual Information: properties

- MI always positive
- Cancels if r.v. are mutually independent
- MI is invariant by scale change

Proof...

- **Example 3: Gaussian case**

$$I(\mathbf{g}_x) = \frac{1}{2} \log \frac{\prod V_{ii}}{\det \mathbf{V}}$$

Mutual Information: decomposition

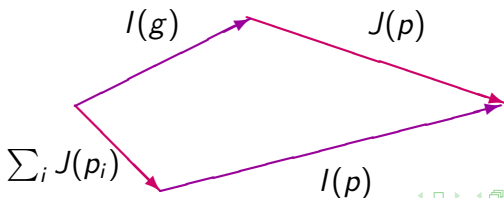
- Define the Negentropy as the divergence:

$$J(p_x) = K(p_x, g_x) = \int p_x(\mathbf{u}) \log \frac{p_x(\mathbf{u})}{g_x(\mathbf{u})} d\mathbf{u}. \quad (29)$$

Negentropy is invariant by invertible transforms

- Then MI can be decomposed into:

$$I(p_x) = I(g_x) + J(p_x) - \sum_i J(p_{x_i}). \quad (30)$$



Edgeworth expansion of a pdf (2)



Francis Edgeworth (1845-1926).

$$\frac{p_x(u)}{g_x(u)} = 1 + \frac{1}{3!} \kappa_3 h_3(v) + \frac{1}{4!} \kappa_4 h_4(v) + \frac{10}{6!} \kappa_3^2 h_6(v) \\ + \frac{1}{5!} \kappa_5 h_5(v) + \frac{35}{7!} \kappa_3 \kappa_4 h_7(v) + \frac{280}{9!} \kappa_3^3 h_9(v) + \dots$$

Edgeworth expansion of the MI

For standardized random variables \mathbf{x} , the approximation [Com92a]:

$$I(p_{\mathbf{x}}) = J(p_{\mathbf{x}}) - \frac{1}{48} \sum_i 4C_{iii}^2 + C_{iiii}^2 + 7C_{iii}^4 - 6C_{iii}^2 C_{iiii} + o(m^{-2}). \quad (31)$$

- If 3rd order $\neq 0$, then $I(p_{\mathbf{x}}) \approx J(p_{\mathbf{x}}) - \frac{1}{12} \sum_i C_{iii}^2$
- If 3rd order ≈ 0 , then $I(p_{\mathbf{x}}) \approx J(p_{\mathbf{x}}) - \frac{1}{48} \sum_i C_{iiii}^2$

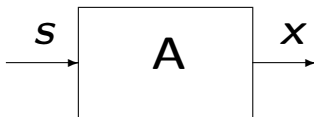
Optimization Criteria

- Cumulant matching
- Contrast criteria
 - Mutual Information
 - Likelihood
 - CoM family (after prewhitening)
 - STO, JAD (after prewhitening)
 - Entropy (after NL transform)

Identification by Cumulant matching

Principle

- Estimate the mixture by solving the I/O Multi-linear equations
- Apply a separating filter based on the latter estimate



Noiseless mixture of 2 sources

Example 4: 2×2 by Cumulant matching (cf. [demo PCA-ICA](#))

- After standardization, the mixture takes the form

$$\mathbf{x} = \begin{pmatrix} \cos \alpha & -\sin \alpha e^{j\varphi} \\ \sin \alpha e^{-j\varphi} & \cos \alpha \end{pmatrix} \mathbf{s} \quad (32)$$

Noiseless mixture of 2 sources

Example 5: 2×2 by Cumulant matching (cf. [demo PCA-ICA](#))

- After standardization, the mixture takes the form

$$\mathbf{x} = \begin{pmatrix} \cos \alpha & -\sin \alpha e^{j\varphi} \\ \sin \alpha e^{-j\varphi} & \cos \alpha \end{pmatrix} \mathbf{s} \quad (32)$$

- Denote $\gamma_{ij}^{k\ell} = \text{Cum}\{x_i, x_j, x_k^*, x_\ell^*\}$ and $\kappa_i = \text{Cum}\{s_i, s_i, s_i^*, s_i^*\}$.
Then by *Multi-linearity*:

$$\gamma_{12}^{12} = \cos^2 \alpha \sin^2 \alpha (\kappa_1 + \kappa_2)$$

$$\gamma_{11}^{12} = \cos^3 \alpha \sin \alpha e^{j\varphi} \kappa_1 - \cos \alpha \sin^3 \alpha e^{j\varphi} \kappa_2$$

$$\gamma_{12}^{22} = \cos \alpha \sin^3 \alpha e^{j\varphi} \kappa_1 - \cos^3 \alpha \sin \alpha e^{j\varphi} \kappa_2$$

Noiseless mixture of 2 sources

Example 6: 2×2 by Cumulant matching (cf. [demo PCA-ICA](#))

- After standardization, the mixture takes the form

$$\mathbf{x} = \begin{pmatrix} \cos \alpha & -\sin \alpha e^{j\varphi} \\ \sin \alpha e^{-j\varphi} & \cos \alpha \end{pmatrix} \mathbf{s} \quad (32)$$

- Denote $\gamma_{ij}^{k\ell} = \text{Cum}\{x_i, x_j, x_k^*, x_\ell^*\}$ and $\kappa_i = \text{Cum}\{s_i, s_i, s_i^*, s_i^*\}$.
Then by *Multi-linearity*:

$$\gamma_{12}^{12} = \cos^2 \alpha \sin^2 \alpha (\kappa_1 + \kappa_2)$$

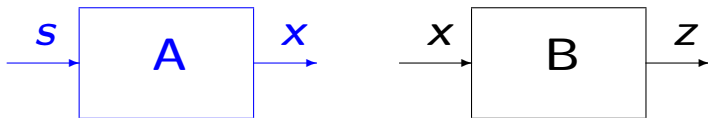
$$\gamma_{11}^{12} = \cos^3 \alpha \sin \alpha e^{j\varphi} \kappa_1 - \cos \alpha \sin^3 \alpha e^{j\varphi} \kappa_2$$

$$\gamma_{12}^{22} = \cos \alpha \sin^3 \alpha e^{j\varphi} \kappa_1 - \cos^3 \alpha \sin \alpha e^{j\varphi} \kappa_2$$

- Compact solution: $\frac{\gamma_{12}^{22} - \gamma_{11}^{12}}{\gamma_{12}^{12}} = -2 \cot 2\alpha e^{j\varphi}$

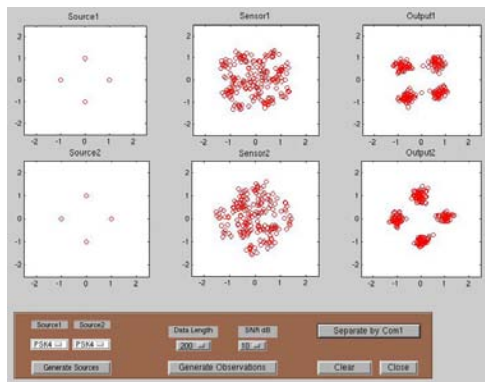
Now the inverse approach

- Cumulant matching (direct approach: identification)
- Contrast Criteria (inverse approach: equalization):



Noisy Mixtures of 2 sources

Example 7: Separation of 2 non Gaussian sources by contrast maximization



Source additional hypotheses

- **H1.** Each sources $s_j[k]$ is an i.i.d. sequence, for any fixed j
- H2.** Sources s_j are mutually statistically independent
- H3.** At most one source is Gaussian
- H4.** At most one source has a null marginal cumulant
- **H5.** Sources are Discrete, and belong to some known alphabet (but may be stat. dependent)
- H6.** Sources $s_j[k]$ are sufficiently exciting
- **H7.** Sources are colored, and the set of source spectra forms a family of linearly independent functions
- **H8.** Sources are non stationary, and have different time profiles

Trivial Filters

- They account for Inherent indeterminacies, remaining after assuming Source additional hypotheses

For instance:

- For dynamic (convolutive) mixtures, under **H1, H2, H3**, $\check{\mathbf{T}}[z] = \mathbf{P} \check{\mathbf{D}}[z]$, where \mathbf{P} is a permutation, and $\check{\mathbf{D}}[z]$ a diagonal filter, with entries of the form $\check{D}_{pp}[z] = \lambda_p z^{\delta_p}$, where δ_p is an integer.
 - For static mixtures, under **H2, H3**, $\mathbf{T} = \mathbf{P}\mathbf{D}$, where \mathbf{P} permutation and \mathbf{D} diagonal invertible.
- In other words, if s satisfies **Hi**, then so does $\mathbf{T}s$

Contrast criteria: definition

Axiomatic definition

A *Contrast* optimization criterion Υ should enjoy 3 properties:

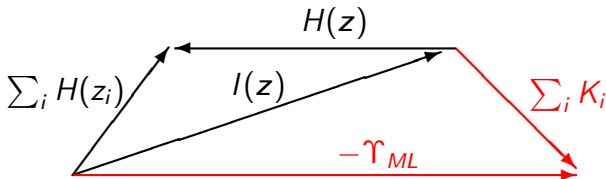
- *Invariance*: Υ should not change under the action of trivial filters (cf. [definition](#) slide 116)
- *Domination*: If sources are already separated, any filter should decrease (or leave unchanged) Υ
- *Discrimination*: The maximum achievable value should be reached only when sources are separated (i.e. all absolute maxima are related to each other by trivial filters)

To summarize:

$$\forall \mathbf{Q}, \Upsilon(\mathbf{Q}) \leq \Upsilon(\mathbf{I}), \text{ with equality iff } \mathbf{Q} \text{ trivial} \quad (33)$$

Examples of contrasts (1/2)

- *Minimize* Mutual Information $I(\mathbf{z})$
- *Maximize* Noiseless Likelihood $\Upsilon_{ML} = -I(\mathbf{z}) - \sum_i K(z_i, s_i)$, if p_{s_i} are known
- *Maximize* Entropy $H(\mathbf{z})$, if $H(z_i)$ are kept constant



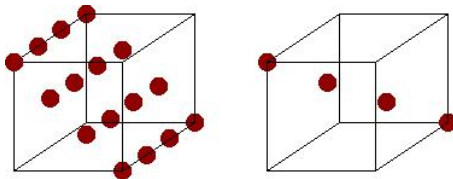
Examples of contrasts (2/2)

Contrasts based on tensor slices

$$\Upsilon_{CoM}^{(r)} \stackrel{\text{def}}{=} \sum_i |C_{iiii}|^r, \quad \Upsilon_{STO} \stackrel{\text{def}}{=} \sum_{ij} |C_{iiij}|^2, \quad \Upsilon_{JAD} \stackrel{\text{def}}{=} \sum_{ijk} |C_{ijik}|^2$$

➔ $\Upsilon_{CoM}^{(2)} \leq \Upsilon_{STO} \leq \Upsilon_{JAD}$ shows that Υ_{CoM} is more sensitive

Example for $4 \times 4 \times 4$ tensors



Matrix slices diagonalization \neq Tensor diagonalization

Bibliographical notes

Optimization criteria

- Expansion of the MI in terms of cumulants [Com92a] [Com94a]
- Contrasts: CoM2 [Com92a] [Com94a], JADE [CS93], STO [LMV01]
- Link between Likelihood and Infomax [Car99] [Car97]
- Likelihood, Estimating equations: [CJ10, ch.4]
- Pairwise sweeping in tensors: [Com89b] [Com92a] [CS93] [Com94a] [Com94b] [MvL08]

III. Second Order Methods



Plan du cours III

- 7** Introduction
- 8** Colored sources
- 9** Nonstationary sources
- 10** Related approaches
- 11** Discussion

Independence or decorrelation?

Decorrelation ?

- Independence \Rightarrow decorrelation, but reverse is wrong
- However, decorrelation is a first step to independence, hence 2-step approaches with whitening or sphering

Algebraic point of view

- For 2 linear mixtures of 2 sources (assumed zero mean, iid)
 - \mathbf{B} has 4 unknown parameters,
 - using SOS, one has 3 equations related to $E(Y_1^2)$, $E(Y_2^2)$ and $E(Y_1 Y_2)$
- For P linear mixtures of P sources (assumed zero mean, iid)
 - \mathbf{B} has P^2 unknown parameters,
 - using SOS, one has $P + P(P - 1)/2 = P(P + 1)/2$ equations related to $E(Y_1^2), \dots, E(Y_P^2)$ and $E(Y_1 Y_2), E(Y_1 Y_3), \dots, E(Y_P Y_{P-1})$

Zeroing the decorrelation

Assumptions

For 2 linear mixtures of two sources

- $a_{jj} = 1$ and enforcing $b_{jj} = 1$
- zero-mean sources with variance σ_i^2

Decorrelation equation

- In the plane (b_{12}, b_{21}) , solution to $E[Y_1 Y_2] = 0$ is a set of hyperboles

$$b_{21} = \frac{b_{12} \left(a_{21}^2 + \frac{\sigma_2^2}{\sigma_1^2} \right) + a_{21} + a_{12} \frac{\sigma_2^2}{\sigma_1^2}}{b_{12} \left(a_{21} + a_{12} \frac{\sigma_2^2}{\sigma_1^2} \right) + 1 + a_{21}^2 \frac{\sigma_2^2}{\sigma_1^2}}$$

- Each hyperbole depends on the mixture coefficients and on the variance ratio.

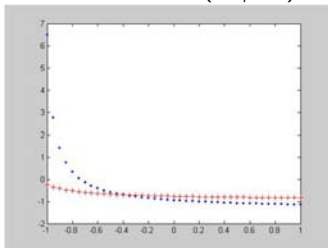
Zeroing the correlation

Decorrelation hyperboles

- Hyperboles are intersected in points related to mixing matrix coefficients

$$\mathbf{A} = \begin{pmatrix} 1 & 0.4 \\ 0.7 & 1 \end{pmatrix}$$

$(\sigma_2/\sigma_1)^2 = 1$ in blue dots, $(\sigma_2/\sigma_1)^2 = 2$ in red +.



Second order separation

Gaussian non iid sources

- Colored sources: AMUSE, Tong et al. [TSLH90]; SOBI, Belouchrani et al. [BAM93]
- Nonstationary sources: Matsuoka et al. [MOK95]; Pham and Cardoso [PC01]

Related approaches

- GEVD problem
- Common Spatial Pattern
- Periodic Component Analysis [TSS⁺09, SJS10]

Discussion

- Joint diagonalization
- Criteria and practical issues
- Conclusions

A 2-step approach

For real linear mixtures

- **B** can be factorized in 2 matrices:



- a whitening (or sphering) matrix **W**
 - an orthogonal matrix **U**
- **W** is computed so that $E[\mathbf{Z}\mathbf{Z}^T] = \mathbf{I}$, i.e.

$$E[\mathbf{W}\mathbf{A}\mathbf{S}(\mathbf{W}\mathbf{A}\mathbf{S})^T] = \mathbf{W}\mathbf{A}E[\mathbf{S}\mathbf{S}^T](\mathbf{W}\mathbf{A})^T = \mathbf{W}\mathbf{A}(\mathbf{W}\mathbf{A})^T = \mathbf{I}$$
- It means that **WA** is an orthogonal matrix, consequently **U** must be an orthogonal matrix too.
- **W** is a symmetric matrix, and its estimation requires $P(P + 1)/2$ parameters
- **U** is associated to $P(P - 1)/2$ plane (Givens) rotations

Decorrelation not enough for iid Gaussian

After whitening

- samples $\mathbf{z}(t)$ are spatially non correlated: $E[\mathbf{Z}\mathbf{Z}^T] = \mathbf{I}$
- for Gaussian iid sources, after whitening, sources are statistically independent: there is no way for estimating \mathbf{U} !

Using mutual information

- Remember the estimating equations: $E[\varphi_{y_i}(y_i)y_j] = 0, \forall i \neq j$, where φ_{y_i} is the score function
- for Gaussian (iid or not) sources with unit variance, $\varphi_{y_i} = +y_i$
- Consequently, for Gaussian iid variables, MI minimisation is equivalent to decorrelation !

Short visual demo

Basic idea and early works

Idea

- Computing a transform \mathbf{B} which decorrelates simultaneously $y_i(t)$ and $y_j(t - \tau)$, for various values of τ ,
- Equivalent to cancel simultaneously $E[y_i(t)y_j(t - \tau)]$, $\forall i \neq j$, or to diagonalize simultaneously $E[\mathbf{y}(t)\mathbf{y}^T(t - \tau)]$, for at least two values of τ .

Early works

- Féty, in his PhD (1988) [F88]
- AMUSE, proposed by Tong et al. in 1990 [TSLH90]
- SOBI, proposed by Belouchrani et al. in 1993 [BAM93]
- Molgedey and Schuster in 1994 [MS94]

Notations and assumptions

Model

- Linear instantaneous mixture: $\mathbf{x}(n) = \mathbf{A}\mathbf{s}(n)$, $n = 1, \dots, N$

Assumptions

- Sources $s_p(\cdot)$, $p = 1, \dots, P$, are zero-mean, real, second order stationay processes; they are spatially decorrelated
- \mathbf{A} is a real valued, square and regular matrix

Notations

- Covariance matrices of sources \mathbf{s} : $\mathbf{R}_{\mathbf{S}}(\tau) = E[\mathbf{s}(t)\mathbf{s}^T(t - \tau)]$
- Due to source spatial decorrelation, $\mathbf{R}_{\mathbf{S}}(\tau)$ are real diagonal matrices. We assume $\mathbf{R}_{\mathbf{S}}(0) = \mathbf{I}$.
- Covariance matrices of observations \mathbf{x} :

$$\mathbf{R}_{\mathbf{X}}(\tau) = E[\mathbf{x}(t)\mathbf{x}^T(t - \tau)]$$
- Since $\mathbf{R}_{\mathbf{X}}(\tau) = \mathbf{A}\mathbf{R}_{\mathbf{S}}(\tau)\mathbf{A}^T$, they can be simultaneously diagonalized

Identifiability: first step

First step: diagonalization of $\mathbf{R}_x(0)$

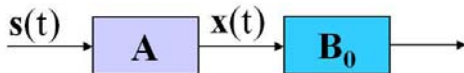
- Diagonalization of $\mathbf{R}_x(0)$ = Whitening of \mathbf{x}

$$\mathbf{B}_0 \mathbf{R}_x(0) \mathbf{B}_0^T = \mathbf{I}$$

$$\mathbf{B}_0 (\mathbf{A} \mathbf{R}_s(0) \mathbf{A}^T) \mathbf{B}_0^T = \mathbf{I}$$

$$(\mathbf{B}_0 \mathbf{A}) \mathbf{R}_s(0) (\mathbf{B}_0 \mathbf{A})^T = \mathbf{I}$$

- Since $\mathbf{R}_s(0) = \mathbf{I}$, we deduce $\mathbf{B}_0 \mathbf{A} = \mathbf{U}_0^T$ is orthogonal
- \mathbf{B}_0 provides a source estimation $s_0(\cdot) = \mathbf{B}_0 \mathbf{x}(\cdot) = \mathbf{U}_0^T s(\cdot)$



Identifiability: second step

Second step: diagonalization of $\mathbf{R}_{S_0}(1)$

- One computes the orthogonal matrix \mathbf{U}_1 , which diagonalizes $\mathbf{R}_{S_0}(1)$.

$$\begin{aligned}\mathbf{R}_{S_0}(1) &= \mathbf{U}_1^T \mathbf{D}_1 \mathbf{U}_1 \\ &= \mathbf{B}_0 \mathbf{R}_X(1) \mathbf{B}_0^T = \mathbf{U}_0^T \mathbf{R}_S(1) \mathbf{U}_0\end{aligned}$$

- It is clearly equivalent to diagonalize $\mathbf{R}_X(1)$
- $\mathbf{B}_1 = \mathbf{U}_1 \mathbf{B}_0$ then jointly diagonalizes $\mathbf{R}_X(0)$ and $\mathbf{R}_X(1)$:

$$\mathbf{B}_1 \mathbf{R}_X(0) \mathbf{B}_1^T = \mathbf{I} \text{ and } \mathbf{B}_1 \mathbf{R}_X(1) \mathbf{B}_1^T = \mathbf{D}_1$$
- The new source estimation $s_1(\cdot) = \mathbf{B}_1 \mathbf{x}(\cdot) = \mathbf{U}_1 \mathbf{U}_0^T s(\cdot)$ is a source separation solution if $\mathbf{R}_S(1)$ has distinct diagonal entries



If $\mathbf{R}_S(1)$ has equal diagonal entries

Case of 2 mixtures of 2 sources

- Since $\mathbf{R}_S(\tau) = \rho(\tau)\mathbf{I}$, it is invariant by any orthogonal transform \mathbf{U} . In fact:

$$\mathbf{U}\mathbf{R}_S(\tau)\mathbf{U}^T = \rho(\tau)\mathbf{U}\mathbf{U}^T = \rho(\tau)\mathbf{I}$$

- Hence:

$$\begin{aligned} \mathbf{x}(\cdot) &= \mathbf{A}\mathbf{s}(\cdot) = [\mathbf{A}\mathbf{U}^T][\mathbf{U}\mathbf{s}(\cdot)] \\ \mathbf{R}_X(\cdot) &= \mathbf{A}\mathbf{R}_S(\cdot)\mathbf{A}^T = [\mathbf{A}\mathbf{U}^T]\mathbf{R}_S(\cdot)[\mathbf{A}\mathbf{U}^T]^T \end{aligned}$$

- It is thus impossible to distinguish $\mathbf{A}\mathbf{s}(\cdot)$ and $[\mathbf{A}\mathbf{U}^T][\mathbf{U}\mathbf{s}(\cdot)]$
- In other words, \mathbf{A} can be estimated, only up to any orthogonal matrix !

If $\mathbf{R}_S(1)$ has equal diagonal entries

General case

- If two entries of $\mathbf{R}_S(1)$ are equal, say $\rho_i(\tau) = \rho_j(\tau)$, it is impossible to separate the two sources s_i and s_j , due to the orthogonal indeterminacy, i.e. \mathbf{A} is only identifiable up to a plane (Givens) rotation in the (s_i, s_j) plane
- All the other sources, with different entries, can be separated after the second step

Solution

- For separating the sources with same entries in $\mathbf{R}_S(1)$, we continue with a third step, similar to the second one, i.e. one computes an orthogonal matrix \mathbf{U}_2 which diagonalizes $\mathbf{R}_X(2)$,
- $\mathbf{B}_2 = \mathbf{U}_2\mathbf{U}_1\mathbf{B}_0$ then jointly diagonalizes $\mathbf{R}_X(0)$, $\mathbf{R}_X(1)$ and $\mathbf{R}_X(2)$:

$$\mathbf{B}_2\mathbf{R}_X(0)\mathbf{B}_2^T = \mathbf{I}, \text{ and } \mathbf{B}_2\mathbf{R}_X(1)\mathbf{B}_2^T = \mathbf{D}_1 \text{ and } \mathbf{B}_2\mathbf{R}_X(2)\mathbf{B}_2^T = \mathbf{D}_2$$

Identifiability theorem

Theorem

- The mixing matrix \mathbf{A} is identifiable from second order statistics (up to scale and permutation indeterminacies) iff the correlation sequences of all sources are pairwise linearly independent, i.e. if $(\rho_i(1), \dots, \rho_i(K)) \neq (\rho_j(1), \dots, \rho_j(K))$, $\forall i \neq j$

In frequency domain

- Due to uniqueness of Fourier transform, the identifiability condition in the time domain can be transposed in the frequency domain:
The mixing matrix \mathbf{A} is identifiable from second order statistics (up to scale and permutation indeterminacies) iff the sources have distinct spectra i.e. pairwise linearly independent spectra.

Consequences

- Since the separation is achieved using second order statistics (SOS), Gaussian sources can be separated
- Since we just use SOS, maximum likelihood approaches can be developed assuming Gaussian densities.

Noisy mixtures

Notations and assumptions

- $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{v}(t)$
- Sources zero-mean, real second order stationary processes; spatially decorrelated
- \mathbf{A} is $K \times P$ full rank matrix, with rank equal to $K \geq P$
- \mathbf{v} real, zero-mean, second order stationary process, assumed uncorrelated of the sources, and spatially as well as temporally white

Equations

- $\mathbf{R}_X(0) = \mathbf{A}\mathbf{A}^T + \sigma_{\mathbf{v}}^2\mathbf{I}$ and $\mathbf{R}_X(\tau) = \mathbf{A}\mathbf{R}_S(\tau)\mathbf{A}^T, \forall \tau \geq 1$
- The condition of different $\rho_i(\tau)$ is no longer sufficient for identifying \mathbf{A}
- In fact, the extra parameter $\sigma_{\mathbf{v}}^2$ must be known or identified
- It can be done iff $\mathbf{A}\mathbf{A}^T$ is identifiable

Example of non identifiable noisy mixtures

Notations and assumptions

- $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{v}(t)$
- In addition to previous assumptions, sources are first order MA, so that $\mathbf{R}_S(\tau) = 0$ for $\tau \geq 2$.
- Consequently, one only has 2 equations:

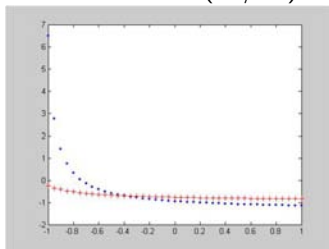
$$\mathbf{R}_X(0) = \mathbf{A}\mathbf{A}^T + \sigma_V^2\mathbf{I} \text{ and } \mathbf{R}_X(1) = \mathbf{A}\mathbf{R}_S(1)\mathbf{A}^T$$
- Since $\mathbf{R}_X(0)$ and $\mathbf{R}_X(1)$ are symmetric $P \times P$, there are $P + P(P - 1)/2 = P(P + 1)/2$ equations for each matrix equation, i.e. a total number of $P(P + 1)$ equations
- But the number of unknown is $P(P + 1) + 1 : P^2$ (matrix \mathbf{A}), P (entries of $\mathbf{R}_S(1)$), 1 for σ_V^2 !

Non stationary sources

Covariance matrices on two time windows

- Assume $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t)$, with the same assumptions as above
- In addition, we assume that sources are nonstationary, with pairwise variance ratio $(\sigma_i/\sigma_j)^2(t) \neq 1$
- On two time windows with different variance ratios, one get two covariance matrices, $\mathbf{R}_1(0)$ and $\mathbf{R}_2(0)$, which are jointly diagonalizable

$(\sigma_2/\sigma_1)^2 = 1$ in blue dots, $(\sigma_2/\sigma_1)^2 = 2$ in red +.



Non stationary sources (con't)

Pham and Cardoso approach [PC01]

- Time is divided into p time blocks, W_i with length T_i ,
- Covariance matrix on each window is estimated using

$$\hat{\mathbf{R}}_{\mathbf{x},i} = \frac{1}{T_i} \sum_{t \in W_i} \mathbf{x}(t)\mathbf{x}^T(t), \quad \forall i = 1 \dots, p$$

- Assuming Gaussian sources (SOS method), Pham and Cardoso [PC01] prove that ML estimate of the separating matrix \mathbf{B} , for unknown source variances, is provided by joint diagonalization of the estimated matrices $\hat{\mathbf{R}}_{\mathbf{x},i}$.

General Eigen Value Decomposition (GEVD) problem

GEVD problem

- Observation: $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) = \mathbf{x}_s(t) + \mathbf{x}_n(t)$, where \mathbf{x}_s and \mathbf{x}_n are desired and undesired signals, resp.
- Objective: estimate \mathbf{b} such that $y = \mathbf{b}^T \mathbf{x}$ has desired property
- Maximizing the power according to 2 conditions can be associated to the joint diagonalization of two matrices \mathbf{M}_1 and \mathbf{M}_2 , called GEVD problem
- Using Rayleigh-Ritz theorem, maximizing the above ratio is equivalent to solve the GEVD of $(\mathbf{M}_1, \mathbf{M}_2)$, i.e.

$$\max_{\mathbf{B}} \frac{\mathbf{B}^T \mathbf{M}_1 \mathbf{B}}{\mathbf{B}^T \mathbf{M}_2 \mathbf{B}} \Leftrightarrow \mathbf{B} \text{ such that } \mathbf{M}_1 \mathbf{B} = \mathbf{M}_2 \mathbf{B} \mathbf{D}$$

where \mathbf{D} is the eigenvalue diagonal matrix

- The method is used for extracting sources with given properties

General Eigen Value Decomposition (GEVD) problem (con't)

SNR maximizer

$$\max_{\mathbf{b}} \text{SNR}(\mathbf{b}) = \max_{\mathbf{b}} \frac{E[y_s^2]}{E[y_n^2]} = \max_{\mathbf{b}} \frac{\mathbf{b}^T \mathbf{R}_{X_s} \mathbf{b}}{\mathbf{b}^T \mathbf{R}_{X_n} \mathbf{b}} \Leftrightarrow \mathbf{b} = \text{GEVD}(\mathbf{R}_{X_s}, \mathbf{R}_{X_n})$$

General Eigen Value Decomposition (GEVD) problem (con't)

Periodicity maximizer [TSS⁺09, SJS10]



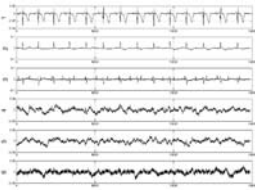
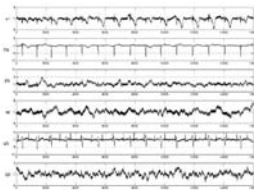
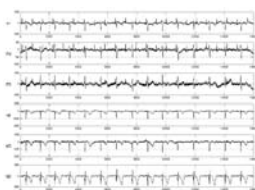
$$\min_{\mathbf{b}} \frac{E_t[(y(t+\tau) - y(t))^2]}{E[y(t)^2]} \Leftrightarrow \mathbf{b} = \text{GEVD}(\mathbf{R}_x(\tau), \mathbf{R}_x(0))$$

- Application for ECG extraction: component are sorted by similarity to ECG (R peaks)

Mixtures

Est. sources ICA (JADE)

Est. sources PiCA



General Eigen Value Decomposition (GEVD) problem (con't)

Spectral-contrast maximizer



$$\max_{\mathbf{b}} \frac{E_{f \in BP}[|\mathbf{Y}(f)|^2]}{E_f[|\mathbf{Y}(f)|^2]} = \max_{\mathbf{b}} \frac{\mathbf{b}^T \mathbf{S}_{\mathbf{X}, BP} \mathbf{b}}{\mathbf{b}^T \mathbf{S}_{\mathbf{X}} \mathbf{b}} \Leftrightarrow \mathbf{b} = \text{GEVD}(\mathbf{S}_{\mathbf{X}, BP}, \mathbf{S}_{\mathbf{X}})$$

- Application for extraction of signal with prior that power in important in a given frequency band, BP. Similar idea used for atrial fibrillation extraction [PZL10, LISC11]

General Eigen Value Decomposition (GEVD) problem (con't)

Nonstationary maximizer

$$\max_{\mathbf{b}} \frac{E_{t \in W_1}[\mathbf{y}^2(t)]}{E_{t \in W_2}[\mathbf{y}^2(t)]} = \max_{\mathbf{b}} \frac{\mathbf{b}^T \mathbf{R}_{\mathbf{x}, W_1}(0) \mathbf{b}}{\mathbf{b}^T \mathbf{R}_{\mathbf{x}, W_2}(0) \mathbf{b}} \Leftrightarrow \mathbf{b} = \text{GEVD}(\mathbf{R}_{\mathbf{x}, W_1}(0), \mathbf{R}_{\mathbf{x}, W_2}(0))$$

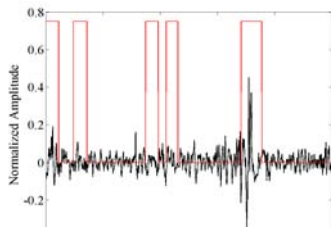
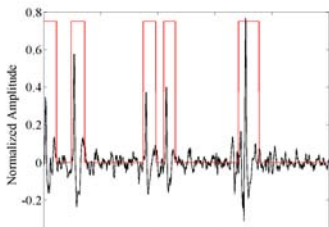
Common spatial pattern

- CSP has been introduced by K. Fukunaga and W. L. G. Koontz (Application of the K-L expansion to feature selection and ordering, IEEE Trans. on Computers, , no. 4, pp. 311-318, 1970),
- CSP aims at maximizing the ratio of variance between two classes, associated to windows W_i , $i = 1, 2$
- CSP is then nothing but a “nonstationarity maximizer”

Common Spatial Pattern: Example

Epileptic focus localization, Samadi et al. [SASZJ11]

- Data : set of intracranial EEG for epileptic patients
- Objectives: estimate the spatial filter \mathbf{B} which provides iEEG sources the most strongly related to interictal discharges (IED)
- The two classes are then samples in either “IED” (class 1) or “NON IED” (Class 2) time intervals
- CSP provides sources, sorted in decreasing order, according to the power in class 1 condition



Joint diagonalization: exact or approximate ?

Exact joint diagonalization (EJD)

- EJD of two symmetric or positive-definite matrices exists
- For more than 2 matrices, with exact matrices, EJD still exists
- However, in practice, due to matrix estimation errors:
 - EJD does not exist for more than 2 matrices
 - EJD of two matrices sensitive to matrix estimation errors

Approximate joint diagonalization (AJD)

- AJD of a set of matrices more robust than EJD of 2 matrices
- How to choose the set of matrices (delays τ_1, \dots, τ_K) for the best estimation ?
- AJD is a usual problem in data analysis \Rightarrow huge number of algorithms, with different structures (with or without whitening), selection of optimal delays τ_i , criteria, etc.

Criteria for approximate joint diagonalization

Criteria

- Algorithms aim at jointly diagonalizing a set of p matrices \mathbf{R}_i , i.e. estimating \mathbf{U} such that

$$\mathbf{U}\mathbf{R}_i\mathbf{U}^T = \mathbf{D}_i, \forall i = 1, \dots, p$$

- Practical criteria aim at minimizing the square sum of off diagonal entries of the matrices $\mathbf{U}\mathbf{R}_i\mathbf{U}^T$, $\forall i = 1, \dots, p$.
- Criteria usually write like:

$$\mathcal{C}(\mathbf{R}_1, \dots, \mathbf{R}_p; \mathbf{U}) = \sum_{i=1}^p \sum_{k \neq l} [\mathbf{U}\mathbf{R}_i\mathbf{U}^T]_{kl}^2 = \sum_{i=1}^p \text{offdiag}(\mathbf{U}\mathbf{R}_i\mathbf{U}^T)^2$$

Joint diagonalization: practical issues

In frequency domain

- Since Fourier transform preserves linearity:

$$\mathcal{F}\{\mathbf{x}(t)\} = \mathcal{F}\{\mathbf{A}\mathbf{s}(t)\} = \mathbf{A}\mathcal{F}\{\mathbf{s}(t)\}$$

- Practically, with short term Fourier transform on a time window W_i :

$$\mathcal{F}_{W_i}\{\mathbf{x}(t), T_i\} = \mathcal{F}_{W_i}\{\mathbf{A}\mathbf{s}(t), T_i\} = \mathbf{A}\mathcal{F}_{W_i}\{\mathbf{s}(t), T_i\}$$

- In the frequency domain, it is then possible to jointly diagonalizing $\mathbf{S}_{\mathbf{x}}(\nu) = E\left[\left|\mathbf{X}_{W_i}(\nu)\mathbf{X}_{W_i}^T(\nu)\right|\right]$

For both colored and non stationary sources

- One can jointly diagonalize a double set of matrices:
 - a set $\{\mathbf{R}_{\mathbf{x}}(\tau), \tau = 0, \dots, K_1\}$, using coloration property,
 - a set $\{\mathbf{R}_{\mathbf{x}, W_i}(0), i = 0, \dots, K_2\}$, using nonstationarity.

Conclusions on second order method

- Can be used for Gaussian non iid sources
 - Non iid means colored sources or nonstationary sources
 - Gaussian sources can be separated (contrary to HOS-based methods), but Gaussianity is not required
- Algorithms based either on ML using Gaussian assumptions, or on joint diagonalization algorithms
- Huge number of simple and fast algorithms for Approximate Joint Diagonalisation
- Joint diagonalization can be applied on matrices computed in the time or in the frequency domains, mixing coloration and non stationarity properties
- Other priors can be used for computing spatial filtering leading to source separation, and estimated using SOS through GEVD problem (e.g. Common Spatial Pattern, Periodic Component Analysis)

IV. Algorithms with prewhitening



Contents of course IV

- 12** Introduction
- 13** Problem in dimension 2
- 14** Orthogonal transform
 - Pair sweeping
 - CoM
 - JAD
 - STO
 - Limits
 - Other orthogonal approaches
- 15** Algorithms based on Deflation
 - Intro
 - Algos
 - FastICA
 - RobustICA
 - SAUD
- 16** Discussion, Bibliography

Algorithms

- Batch (off-line):
 - algebraic (eg. in dim 2)
 - semi-algebraic (eg. pair sweeping)
 - iterative (eg. descent of order r)
- Adaptive (on-line):
stochastic versions of the above, and in particular
 - order 0 (eg. stochastic iteration)
 - order 1 (eg. stochastic gradient)
 - order 2 (eg. stochastic Newton)
- Non stationarities: block- vs sample-wise updates

Spatial whitening

- Batch algorithms: already seen in slides [QR-PCA](#)
- Adaptive algorithms:
 - Adaptive QR [[Com89a](#)]
 - Adaptive SVD [[CG90](#)]
 - Gradient [[CA02](#)]
 - Relative gradient [[CL96](#)] [[CA02](#)]
- In this course IV, data are assumed whitened: $\mathbf{R}_x = \mathbf{I}$.

Direct vs Inverse

Two formulations in terms of cumulants:

- 1 Direct: look for \mathbf{A} so as to fit the input-output multilinearity cumulant relation:

$$\min_{\mathbf{A}} \left\| C_{\mathbf{y},ijk..l} - \sum_{p=1}^P A_{ip} A_{jp} A_{kp..} A_{lp} C_{\mathbf{S},ppp..p} \right\|^2$$

i.e. decompose $C_{\mathbf{y}}$ into a sum of P rank-one terms

Direct vs Inverse

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- 1 Direct: look for \mathbf{A} so as to fit the input-output multilinearity cumulant relation:

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i.e. decompose $C_{\mathbf{y}}$ into a sum of P rank-one terms

- 2 Inverse: look for \mathbf{B} :

$$\min_{\mathbf{B}} \sum_{mnp..q \neq ppp..p} \left| \sum_{ijk..l} C_{\mathbf{y},ijk..l} B_{im} B_{jn} B_{kp..} B_{lq} \right|^2$$

i.e. try to diagonalize $C_{\mathbf{y}}$ by linear transform, $\mathbf{z} = \mathbf{B}\mathbf{y}$

Orthogonal decomposition

If Q orthogonal, the two problems are equivalent:

Orthogonal decomposition

If \mathbf{Q} orthogonal, the two problems are equivalent:

1 Direct:

$$\min_{\mathbf{Q}, \Lambda} \left\| C_{ijkl} - \sum_{p=1}^P Q_{ip} Q_{jp} Q_{kp} Q_{lp} \Lambda_{pppp} \right\|^2$$

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1 Direct:

$$\min_{\mathbf{Q}, \Lambda} \left\| C_{ijkl} - \sum_{p=1}^P Q_{ip} Q_{jp} Q_{kp} Q_{lp} \Lambda_{pppp} \right\|^2$$

2 Inverse: $\min_{\mathbf{Q}, \Lambda} \left\| \sum_{ijkl} Q_{ip} Q_{jq} Q_{kr} Q_{ls} C_{ijkl} - \Lambda_{pppp} \delta_{pqrs} \right\|^2$ or

e.g. $\max_{\mathbf{Q}} \sum_p \left| \sum_{ijkl} Q_{ip} Q_{jp} Q_{kp} Q_{lp} C_{ijkl} \right|^2$

Orthogonal decomposition

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$$\text{e.g. } \max_{\mathbf{Q}} \sum_p \left| \sum_{ijkl} Q_{ip} Q_{jp} Q_{kp} Q_{lp} C_{ijkl} \right|^2$$

Proof.

The Frobenius norm is invariant under orthogonal change of coordinates.

Divide to conquer

Well posed problem: the set of orthogonal matrices is closed

But difficulty: many unknowns, in real or complex field

- 1st idea: address a sequence of problems of smaller dimension instead of a single one in larger dimension. (deflation, pair sweeping, fix descent directions...)

Divide to conquer

Well posed problem: the set of orthogonal matrices is closed

But difficulty: many unknowns, in real or complex field

- 1** 1st idea: address a sequence of problems of smaller dimension instead of a single one in larger dimension. (deflation, pair sweeping, fix descent directions...)
- 2** 2nd idea: decompose \mathbf{A} into two factors, $\mathbf{A} = \mathbf{L}\mathbf{Q}$, and compute \mathbf{L} so as to *exactly* standardize the data. Look for the best \mathbf{Q} in a second stage.

Algorithms: pair Sweeping (1)

Pairwise processing

Split the orthogonal matrix into a product of plane *Givens* rotations:

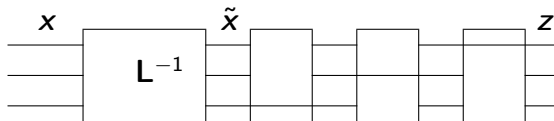
$$\mathbf{G}[i,j] \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+\theta^2}} \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

acting in the subspace defined by (z_i, z_j) .

➡ the dimension has been reduced to 2, and we have a single unknown, θ , that can be imposed to lie in $(-1, 1]$.

Algorithms: pair Sweeping (2)

Cyclic sweeping with fixed ordering: Example in dimension $P = 3$



Carl Jacobi, 1804-1851

Algorithms: pair Sweeping (3)

Sweeping a $3 \times 3 \times 3$ symmetric tensor

$$\begin{pmatrix} \color{red}{X} & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \rightarrow \begin{pmatrix} \color{red}{X} & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \rightarrow \begin{pmatrix} . & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \\
 \begin{pmatrix} x & x & x \\ x & \color{red}{X} & x \\ x & x & x \end{pmatrix} \rightarrow \begin{pmatrix} x & x & x \\ x & . & x \\ x & x & x \end{pmatrix} \rightarrow \begin{pmatrix} x & x & x \\ x & \color{red}{X} & x \\ x & x & x \end{pmatrix} \\
 \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & . \end{pmatrix} \rightarrow \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & \color{red}{X} \end{pmatrix} \rightarrow \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & \color{red}{X} \end{pmatrix}$$



$\color{red}{X}$: maximized
 x : minimized
 $.$: unchanged

} by last Givens rotation

Algorithms: pair Sweeping (4)

- Criteria Υ_{CoM} , Υ_{STO} and Υ_{JAD} , are rational functions of θ , and their absolute maxima can be computed algebraically.
- To prove this, consider the elementary 2×2 problem

$$\mathbf{z} = \mathbf{G} \tilde{\mathbf{x}},$$

denote C_{ijkl} the cumulants of \mathbf{z} , and

$$\mathbf{G} \stackrel{\text{def}}{=} \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+\theta^2}} \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

Solution for Υ_{CoM}

- We have $\Upsilon_{CoM} \stackrel{\text{def}}{=} (C_{1111})^2 + (C_{2222})^2$
- Denote $\xi = \theta - 1/\theta$. Then it is a rational function in ξ :

$$\psi_4(\xi) = (\xi^2 + 4)^{-2} \sum_{i=0}^4 b_i \xi^i$$

- Its stationary points are roots of a polynomial of degree 4:

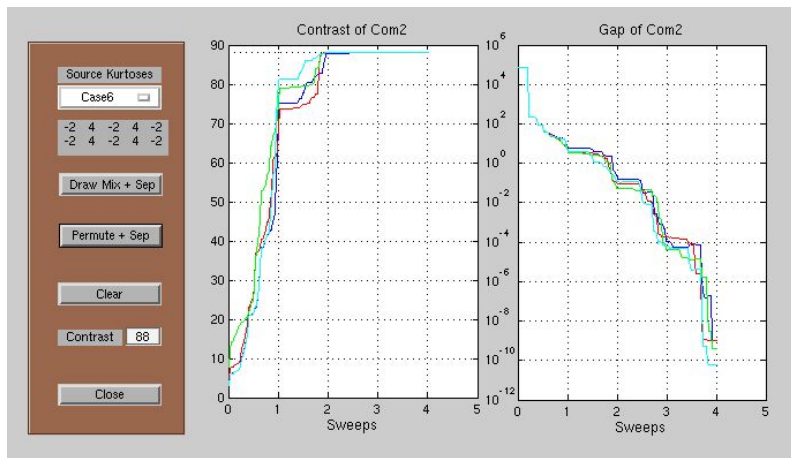
$$\omega_4(\xi) = \sum_{i=0}^4 c_i \xi^i$$

obtainable *algebraically* via Ferrari's technique. Coefficients b_i and c_i are given functions of cumulants of $\tilde{\mathbf{x}}$.

- θ is obtained from ξ by rooting a 2nd degree trinomial.

Separation of a noisy mixture by maximization of contrast

CoM2: Influence of ordering



Solution for Υ_{JAD}

- Goal: maximize squares of diagonal terms of $\mathbf{G}^H \mathbf{N}(r) \mathbf{G}$, where matrix slices are denoted $\mathbf{N}(r) = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}$ and are cumulants of $\tilde{\mathbf{x}}$
- Let $\mathbf{v} \stackrel{\text{def}}{=} [\cos 2\beta, \sin 2\beta]^T$. Then this amounts to maximizing the quadratic form $\mathbf{v}^T \mathbf{M} \mathbf{v}$ where

$$\mathbf{M} \stackrel{\text{def}}{=} \sum_r \begin{bmatrix} a_r - d_r \\ b_r + c_r \end{bmatrix} [a_r - d_r, b_r + c_r]$$

- Thus, 2β is given by the dominant eigenvector of \mathbf{M}
- and \mathbf{G} is obtained by rooting a 2nd degree trinomial.

Solution for Υ_{STO}

- Goal: maximize squares of diagonal terms of 3rd order tensors

$$\mathbf{T}[\ell]_{pqr} \stackrel{\text{def}}{=} \sum_{ijk} G_{pi} G_{qj} G_{rk} C_{ijkl}$$

- If $\mathbf{G} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix}$ then denote $\mathbf{v} \stackrel{\text{def}}{=} [\cos 2\beta, \sin 2\beta]^T$.

- Angle 2β is given by vector \mathbf{v} maximizing a quadratic form $\mathbf{v}^T \mathbf{B} \mathbf{v}$, where \mathbf{B} is 2×2 symmetric and contains sums of products of cumulants of $\tilde{\mathbf{x}}$

- θ is obtained from ξ by rooting a 2nd degree trinomial.

First conclusions

- Pair sweeping can be executed thanks to the equivalence between pairwise and mutual independence
- The cumulant tensor can be diagonalized iteratively via a Jacobi-like algorithm
- For each pair, there is a closed-form solution for the optimal Givens rotation (absolute maximum of the contrast criterion).

Questions:

- But what about *global* convergence?
- Pairwise processing holds valid if model is exact

Stationary points: symmetric matrix case

- Given a matrix \mathbf{m} with components m_{ij} , it is sought for an orthogonal matrix \mathbf{Q} such that Υ_2 is maximized:

$$\Upsilon_2(\mathbf{Q}) = \sum_i M_{ii}^2; \quad M_{ij} = \sum_{p,q} Q_{ip} Q_{jq} m_{pq}.$$

- Stationary points of Υ_2 satisfy for any pair of indices $(q, r), q \neq r$:

$$M_{qq}M_{qr} = M_{rr}M_{qr}$$

- Next, $d^2\Upsilon_2 < 0 \Leftrightarrow M_{qr}^2 < (M_{qq} - M_{rr})^2$, which proves that
 - $M_{qr} = 0, \forall q \neq r$ yields a maximum
 - $M_{qq} = G_{rr}, \forall q, r$ yields a minimum
 - Other stationary points are saddle points

Stationary points: symmetric tensor case

- Similarly, one can look at relations characterizing local maxima of criterion Υ

$$\begin{aligned}
 T_{qqqq} T_{qqqr} - T_{rrrr} T_{qrrr} &= 0, \\
 4T_{qqqr}^2 + 4T_{qrrr}^2 - (T_{qqqq} - \frac{3}{2} T_{qqrr})^2 \\
 - (T_{rrrr} - \frac{3}{2} T_{qqrr})^2 &< 0.
 \end{aligned}$$

for any pair of indices (p, q) , $p \neq q$.

- As a conclusion, contrary to Υ_2 in the matrix case, Υ might have theoretically spurious local maxima in the tensor case (order > 2).

Open problem

- 1 At each step, a plane rotation is computed and yields the global maximum of the objective Υ restricted to one variable



Open problem

- 1 At each step, a plane rotation is computed and yields the global maximum of the objective Υ restricted to one variable
- 2 There is no proof that the sequence of successive plane rotations yields the global maximum, in the general case (tensors that are not necessarily diagonalizable)



Open problem

- 1** At each step, a plane rotation is computed and yields the global maximum of the objective Υ restricted to one variable
- 2** There is no proof that the sequence of successive plane rotations yields the global maximum, in the general case (tensors that are not necessarily diagonalizable)
- 3** Yet, no counter-example has been found since 1991



Other algorithms for orthogonal diagonalization

The 3 previous criteria summarize most ways to address tensor approximate diagonalization.

But the orthogonal matrix can be treated differently, e.g. via *other parameterizations*

- express an orthogonal matrix as the exponential of a skew-symmetric matrix: $\mathbf{Q} = \exp \mathbf{S}$
- or use the Cayley parameterization: $\mathbf{Q} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}$,
- ...

Joint Approximate Diagonalization s

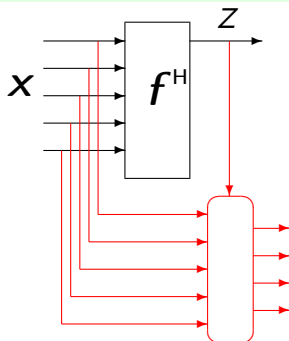
The idea is to consider the symmetric tensor of dimension K and order d as a collection of K^{d-2} symmetric $K \times K$ matrices

This is the *Joint Approximate Diagonalization* (JAD) problem

Algorithms based on Deflation

- Principle: Joint extraction vs Deflation
- Unitary adaptive deflation
- A so-called “fixed point” algorithm: FastICA
- RobustICA
- Deflation without spatial prewhitening, algebraic deflation
- Discussion on MISO criteria

Joint extraction vs Deflation



Deflation:

- Advantage: (a) reduced complexity at each stage, (b) simpler to understand
- Drawbacks: (i) accumulation of regression errors, limitation of number of extracted sources, (ii) possibly larger final complexity

Adaptive algorithms

Deflation by Kurtosis Gradient Ascent

Again same idea

After standardization, it is equivalent to maximize 4th order moment criterion, $\mathcal{M}_{z(4)} = \mathbb{E}\{|z|^4\}$, whose gradient is:

$$\nabla \mathcal{M} = 4 \mathbb{E}\{\mathbf{x} (\mathbf{f}^H \mathbf{x}) (\mathbf{x}^H \mathbf{f})^2\}$$

Overview

- Fixed step gradient on angular parameters: [DL95, CJ10]
- Locally optimal step gradient on filter taps: FastICA [CJ10, ch.6]
- Globally optimal step gradient on filter taps: RobustICA [CJ10, ZC10]
- Semi-Algebraic Unitary Deflation (SAUD) [ACX07]

Adaptive algorithms

Adaptive implementation

- Fully adaptive solutions (update at every sample arrival) nowadays little useful
- Always easy to devise fully adaptive, or block-adaptive solutions from block semi-algebraic algorithms (but reverse is not true!)

Unitary adaptive deflation (1)

■ Extraction

- To extract the first source, find a unitary matrix \mathbf{U} so as to maximize the kurtosis of the first output
- Matrix \mathbf{U} can be iteratively determined by a sequence of Givens rotations
- At each step, determine the best angle of the Givens rotation, e.g. by a gradient ascent [DL95]

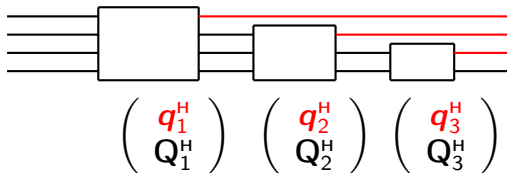
NB: only $P - 1$ Givens rotations are involved

■ Deflation

- After convergence, the first output is extracted, and the $P - 1$ remaining outputs of \mathbf{U} can be processed in the same way

Unitary adaptive deflation (2)

At stage k , $\mathbf{Q} = \begin{pmatrix} \mathbf{q}_k^H \\ \mathbf{Q}_k^H \end{pmatrix}$ is unitary of size $P - k + 1$, and only its first row is used to extract source k , $1 \leq k \leq P - 1$



A so-called fixed point: FastICA (1)

- Any gradient ascent of a function $\mathcal{M}_\rho = \mathbb{E}\{\rho(\mathbf{f}^H \mathbf{x})\}$ under unit-norm constraint $\|\mathbf{f}\|^2 = 1$ admits the Lagrangian formulation:

$$\mathbb{E}\{\mathbf{x}\dot{\rho}(\mathbf{f}^H \mathbf{x})\} = \lambda \mathbf{f}$$

- **Convergence:** when $\nabla \mathcal{C}$ and \mathbf{f} collinear (and *not* when gradient is null, because of constraint $\|\mathbf{f}\| = 1$).
- One can take $\rho(z) = |z|^4$ (but this choice makes FastICA uninteresting)

A so-called fixed point: FastICA (2)

Details of the algorithm proposed in [Hyv99] in the real field; only difference compared to [Tug97] is fixed step size.

- **Gradient:** $\nabla \mathcal{M} = 4 \mathbb{E}\{\mathbf{x} (\mathbf{f}^T \mathbf{x})^3\}$
- **Hessian:** $12 \mathbb{E}\{\mathbf{x} \mathbf{x}^T (\mathbf{f}^T \mathbf{x})^2\}$
- **Heavy approximation** of Hyvarinen [Hyv99]:

$$\mathbb{E}\{\mathbf{x} \mathbf{x}^T (\mathbf{f}^T \mathbf{x})^2\} \approx \mathbb{E}\{\mathbf{x} \mathbf{x}^T\} \mathbb{E}\{(\mathbf{f}^T \mathbf{x})^2\}$$

- If \mathbf{x} standardized and \mathbf{f} unit norm, then Hessian equals Identity.
- This yields an “approximate Newton iteration”: *a mere fixed step gradient!*

$$\begin{aligned} \mathbf{f} &\leftarrow \mathbf{f} - \frac{1}{3} \mathbb{E}\{\mathbf{x} (\mathbf{f}^T \mathbf{x})^3\} & \text{or} & & \mathbf{f} &\leftarrow \mathbb{E}\{\mathbf{x} (\mathbf{f}^T \mathbf{x})^3\} - 3 \mathbf{f} \\ \mathbf{f} &\leftarrow \mathbf{f} / \|\mathbf{f}\| \end{aligned}$$

FastICA: weaknesses

This is a mere fixed step-size projected gradient algorithm, inheriting problems such as:

- Saddle points (slow/ill convergence)
- Flat areas (slow convergence)
- Local maxima (ill convergence)

NB: slow convergence may mean high complexity to reach the solution, or stopping iterations before reaching convergence (depends on stopping criterion).

Polynomial rooting

Theorem (1830). A polynomial of degree higher than 4 cannot in general be rooted algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and radicals (root extractions).



Niels Abel, 1802-1829



Evariste Galois 1811-1832

How to fix most drawbacks: RobustICA

Principle: Cheap exhaustive Line Search of a criterion \mathcal{J}

- Look for *absolute maximum* in the gradient direction (1-dim search)
- *Not costly* when criteria are polynomials or rational functions of low degree (same as AMiSRoF [GC98]: polynomial to root, but here *at most of degree 4*)
- Applies to Kurtosis Maximization (KMA), Constant-Modulus (CMA), Constant-Power (CPA) Algorithms...

This yields corresponding Optimal-Step (OS) algorithms: OS-KMA, OS-CMA, OS-CPA... [ZC08] [ZC05]

RobustICA

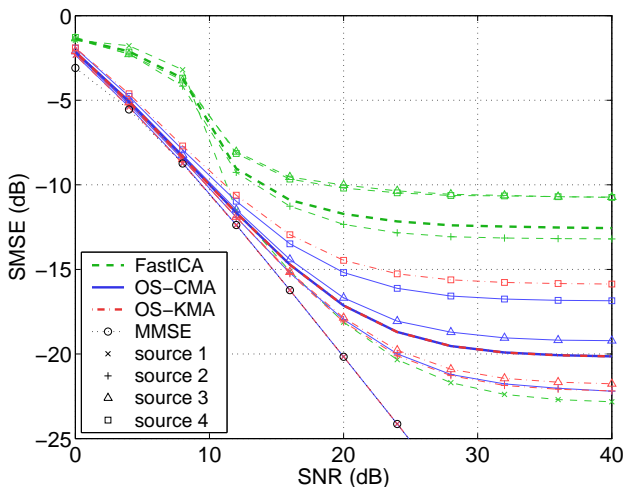
Algorithm

- compute coefficients of polynomial $\frac{\partial}{\partial \mu} \mathcal{J}(\mathbf{f} + \mu \nabla)$ for fixed \mathbf{f} and ∇
- compute all its roots $\{\mu_i\}$
- select μ_{opt} among those roots, which yields the absolute maximum
- set $\mathbf{f} \leftarrow \mathbf{f} + \mu_{opt} \nabla$

See: [ZC10, ZC08]

RobustICA vs FastICA

Signal Mean Square error



Semi-Algebraic Unitary Deflation

CoM1 [Com01]

```

Loop on sweeps
  for  $i = 1$  to  $P - 1$ 
    for  $j = i$  to  $P$ 
      Algebraic  $2 \times 2$  separ.
    end
  end
end
Extraction

```

SAUD [ACX07]

```

for  $i = 1$  to  $P - 1$ 
  Loop on sweeps
    for  $j = i$  to  $P$ 
      Algebraic  $2 \times 2$  separ.
    end
  end
  Extraction
end

```


Equivalence between KMA and CMA

- Recall the 2 criteria:

$$\Upsilon_{KMA} = \frac{\text{Cum}\{z, z, z^*, z^*\}}{[\mathbb{E}\{|z|^2\}]^2}, \quad \mathcal{J}_{CMA} = \mathbb{E}\{[|z|^2 - R]^2\}$$

- Assume 2nd Order circular sources: $\mathbb{E}\{s^2\} = 0$
- Then KMA and CMA are equivalent [Reg02] [Com04]

Discussion on Deflation (MISO) criteria

Let $z \stackrel{\text{def}}{=} \mathbf{f}^H \mathbf{x}$. Criteria below stationary iff differentials of p and q are collinear:

- **Ratio:** $\text{Max}_f \frac{p(\mathbf{f})}{q(\mathbf{f})}$

Example: **Kurtosis**, with $p = E\{|z|^4\} - 2E\{|z|^2\}^2 - |E\{z^2\}|^2$
and $q = E\{|z|^2\}^2$

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- **Difference:** $\text{Min}_{\mathbf{f}} p(\mathbf{f}) - \alpha q(\mathbf{f})$

Examples: Constant Modulus, with $p = E\{|z|^4\}$ and
 $q = 2 a E\{|z|^2\} - a^2$ or

Constant Power, with $q = 2 a \Re(E\{z^2\}) - a^2$

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 $q = 2 a \text{E}\{|z|^2\} - a^2$ or

Constant Power, with $q = 2 a \Re(\text{E}\{z^2\}) - a^2$

■ **Constrained:** $\text{Max}_{q(\mathbf{f})=1} p(\mathbf{f})$

Example: Cumulant, with

$$p = \text{E}\{|z|^4\} - 2\text{E}\{|z|^2\}^2 - |\text{E}\{z^2\}|^2$$

Example: Moment, with $p = \text{E}\{|z|^4\}$, if standardized and
with either $q = \|\mathbf{f}\|^2$ or $q = \text{E}\{|z|^2\}^2$

Bibliographical comments s

- Orthogonal diagonalization of symmetric tensors:
 - Direct diago: CoM without slicing [Com92a] [Com94b]
 - JAD in 2 modes: [Lee78] (\mathbb{R}), [CS93] (\mathbb{C}), with matrix exponential [TF07]
 - JAD 2 modes with positive definite matrices: [FG86]
 - JAD in 3 modes: [LMV01]
- Orthogonal diagonalization of non symmetric tensors:
 - ALS type [Kro83] [Kie92]
 - ALS on pairs: [MvL08] [Sør10]
 - JAD in 2 modes (\mathbb{R}): [PPPP01]
 - Matrix exponential [SICD08]
- Deflation, FastICA, RobustICA: [CJ10, ch.6] [ZC07] [ZCK06] [ZC10]

V. Algorithms without prewhitening



Contents of course V

17 Invertible transform

18 Iterative algorithms

- HJ algorithm
- InfoMax
- Relative gradient
- Probabilistic approach
- RobustICA

19 Semi algebraic algorithms

- Alternating Least Squares
- Diagonally dominant matrices
- Joint Schur
- ICAR

20 Concluding remarks

Approximate diagonalization by invertible transform

One view of the problem: tensor (approximate) diagonalization by invertible transform

- Joint Approximate Diagonalization (JAD) of a collection of:
 - symmetric matrices
 - symmetric diagonally dominant matrices
 - symmetric positive definite matrices
 - for a collection of matrices, not necessarily symmetric (2 invertible transforms)
 - ...
- Direct approaches without slicing the tensor into a collection of matrices: algorithms devised for *underdetermined mixtures* apply (eg. ICAR)

NB: Ill-posedness of optimization over set of invertible matrices

Hérault-Jutten algorithm (1/2)

- **Recursive structure:** $z = x - \mathbf{C}z$, i.e. $\mathbf{F} = (\mathbf{I} + \mathbf{C})^{-1}$, with $\text{Diag}\{\mathbf{C}\} = 0$.

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- **Update rule:** $C_{ij}[k + 1] = C_{ij}[k] + \mu f(z_i[k]) g(z_j[k])$

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- **Update rule:** $C_{ij}[k+1] = C_{ij}[k] + \mu f(z_i[k]) g(z_j[k])$
- The iteration $\mathbf{C}[k+1] = \mathbf{C}[k] + \mu \Phi(\mathbf{C}[k])$ actually searches for zeros of function Φ having negative derivatives if $0 < \mu < 1$ (resp. positive if $-1 < \mu < 0$).

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- **Stochastic iteration** Robbins-Monro (1951) [CJH91]:
If $\Phi(\mathbf{C})$ is the expectation of some $\varphi(\mathbf{C}, z)$, then stochastic version

$$\mathbf{C}[k + 1] = \mathbf{C}[k] + \mu \varphi(\mathbf{C}[k], z[k])$$

Hérault-Jutten algorithm (2/2)

- **Choice of functions** $f(\cdot)$ and $g(\cdot)$.
 - At least one should be non linear
 - Different
 - Odd
 - Link with Estimating Equations if $f(\mathbf{z}) = \psi(\mathbf{z})$ and $g(\mathbf{z}) = \mathbf{z}$
- **Convergence** is slow at the end, and the limiting zero depends on f and g . Permuting f and g changes the limiting zero [Sor91].
- **Direct implementations** exist [MM97]
- **Normalization** improves behavior [DAC02]

InfoMax

- **Idea:** maximize output Mutual Information [BS95] [Car99]

$$\Upsilon_{MI}(z) = \sum_i H(z_i) - H(z)$$

But may tend to infinity!

- Impose as constraints $H(z_i)$ constant \Rightarrow well posed
Now just maximize output joint entropy $H(\mathbf{y})$
- **But how?**
Build $z = g(\mathbf{B}\mathbf{x})$, where $g(\cdot)$ is a c.d.f mapping \mathbb{R} to $[0, 1]$.
- Maximize the output entropy $H(g(\mathbf{B}\mathbf{x}))$ w.r.t. \mathbf{B} by a stochastic version of:

$$\mathbf{B} \leftarrow \mathbf{B} + \mu \nabla H(g(\mathbf{B}\mathbf{x})).$$

Relative gradient

Standard gradient:

$$\Upsilon(\mathbf{x} + \mathbf{h}) - \Upsilon(\mathbf{x}) \approx \langle \mathbf{g}(\mathbf{x}), \mathbf{h} \rangle$$

Relative gradient: Useful when \mathbf{x} belongs to a continuous multiplicative group. Take $\mathbf{h} = \epsilon \mathbf{x}$:

$$\Upsilon(\mathbf{x} + \epsilon \mathbf{x}) - \Upsilon(\mathbf{x}) \approx \langle \mathbf{g}^R(\mathbf{x}), \epsilon \rangle$$

Relationship: because $\langle \mathbf{g}(\mathbf{x}), \epsilon \mathbf{x} \rangle = \langle \mathbf{g}(\mathbf{x}) \mathbf{x}^T, \epsilon \rangle$, we have:

$$\mathbf{g}^R(\mathbf{x}) = \mathbf{g}(\mathbf{x}) \mathbf{x}^T$$

Estimating Equations in BSS

- **Mutual Information:**

$$\Upsilon_{MI}(\mathbf{z}) = \sum_i H(z_i) - H(\mathbf{z})$$

- If $\mathbf{z} = \mathbf{B}\mathbf{x}$, then $H(\mathbf{z}) = H(\mathbf{x}) + \log |\det \mathbf{B}|$

- **Gradient:** $\frac{\partial H(\mathbf{z})}{\partial \mathbf{B}} = \mathbf{B}^{-T}$ because $H(\mathbf{x})$ constant, and
 $\frac{\partial}{\partial \mathbf{B}} \sum_i H(z_i) = \mathbb{E}\{\sum_i \frac{\partial}{\partial \mathbf{B}} \log p_i(z_i)\} = \mathbb{E}\{\sum_i \frac{\partial \log p_i(z_i)}{\partial z_i} \frac{\partial z_i}{\partial \mathbf{B}}\}$
 Hence

$$\frac{\partial \Upsilon_{MI}(\mathbf{z})}{\partial \mathbf{B}} = \mathbb{E}\{\psi_{\mathbf{z}}(\mathbf{z})\mathbf{x}^T\} + \mathbf{B}^{-T}$$

where $\psi_{\mathbf{z}}(\mathbf{z})$ is the score function.

- **Relative gradient:** $\nabla^R \Upsilon = \nabla \Upsilon \cdot \mathbf{B}^T$ yields
 the *Estimating Equation*:

$$\nabla^R \Upsilon_{MI}(\mathbf{z}) = \mathbb{E}\{\psi_{\mathbf{z}}(\mathbf{z})\mathbf{z}^T\} + \mathbf{I} \quad (34)$$


Equivariance

Assume we want to maximize $\Upsilon(\mathbf{B})$

- $\Upsilon(\mathbf{B} + \epsilon \mathbf{B}) - \Upsilon(\mathbf{B}) \approx \langle \nabla^R \Upsilon(\mathbf{B}), \epsilon \mathbf{B} \rangle = \langle \nabla^R \Upsilon(\mathbf{B}) \mathbf{B}^T, \epsilon \rangle$
- Maximal if $\epsilon = \nabla^R \Upsilon(\mathbf{B}) \mathbf{B}^T$
- Update rule: $\mathbf{B} \leftarrow (\mathbf{I} + \mu \epsilon) \mathbf{B}$
- Now denote $\mathbf{G} = \mathbf{B} \mathbf{A}$ the global filter. Post-multiplying by \mathbf{A} shows that update rule is equivalent to

$$\mathbf{G} \leftarrow (\mathbf{I} + \mu \epsilon) \mathbf{G}$$

which does not depend on $\mathbf{A} \Rightarrow$ uniform performance

 For more details, see [CJ10, ch.4].

Probabilistic approach (1)

Let $\mathbf{T}(q)$ be a collection of symmetric *positive semidefinite* matrices
 Look for \mathbf{B} such that $\mathbf{M}(q) \stackrel{\text{def}}{=} \mathbf{B} \mathbf{T}(q) \mathbf{B}^T$ are as diagonal as possible. Let $\alpha_k \in [0, 1]$ and $\sum_k \alpha_k = 1$.

1 Criterion to maximize: $\Upsilon \stackrel{\text{def}}{=} \sum_q \alpha_q \log \frac{\det \mathbf{M}(q)}{\det \text{Diag}\{\mathbf{M}(q)\}}$

We have $\Upsilon \leq 0$ from Hadamard's inequality.

- Avoids singularity
- Linked to Maximum Likelihood

2 Use multiplicative update as $\mathbf{B}^{(\ell+1)} = \mathbf{U} \mathbf{B}^{(\ell)}$

3 Criterion after update: $\Upsilon = \sum_q \alpha_q \log \frac{\det \mathbf{M}(q) \det^2 \mathbf{U}}{\det \text{Diag}\{\mathbf{U} \mathbf{M}(q) \mathbf{U}^T\}}$

Probabilistic approach (2)

- 4 Variation of Υ during one update:

$$\sum_q \alpha_q \left[2 \log \det \mathbf{U} - \log \det \mathbf{Diag}\{\mathbf{U}\mathbf{M}(q)\mathbf{U}^T\} + \log \det \mathbf{Diag}\mathbf{M}(q) \right]$$

- 5 Update two rows at a time, i.e. \mathbf{U} is equal to Identity except for entries $(i, i), (i, j), (j, i), (j, j)$.

By concavity of log, get a lower bound on variation:

$$\sum_q \alpha_q \left[2 \log \det \mathbf{U} - \log(\mathbf{U}\mathbf{P}\mathbf{U}^T)_{11} - \log(\mathbf{U}\mathbf{Q}\mathbf{U}^T)_{22} \right]$$

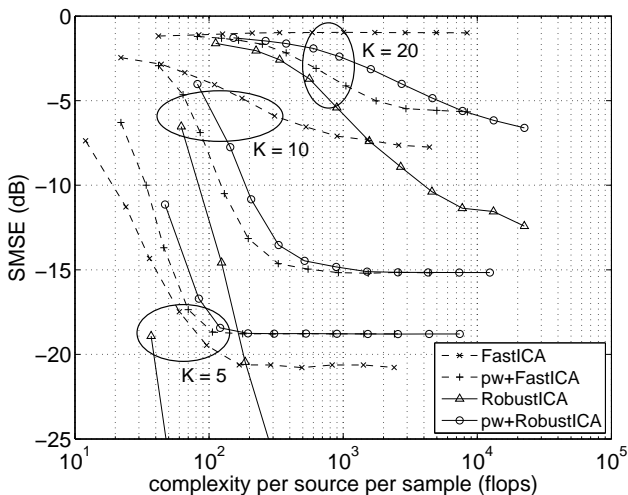
where \mathbf{P} and \mathbf{Q} are the 2×2 matrices:

$$\mathbf{P} = \sum_q \frac{\alpha_q}{M(q)_{ii}} \mathbf{M}(q)[i, j] \text{ and } \mathbf{Q} = \sum_q \frac{\alpha_q}{M(q)_{jj}} \mathbf{M}(q)[i, j]$$

- 6 Maximize this bound instead. This leads to rooting a *2nd degree trinomial*. Sweep all the pairs in turns

RobustICA again, vs FastICA

Signal Mean Square error [ZC07]



Survey of some semi-algebraic algorithms

- 1 An algorithm of ALS type: ACDC
- 2 An algorithm with a multiplicative update, valid if \mathbf{A} is diagonally dominant
- 3 An algorithm based on Joint triangularization
- 4 An SVD-based algorithm: ICAR (Biome4)

Alternating Least Squares

1 Two writings of the criterion:

$$\Upsilon = \sum_q \|\mathbf{T}(q) - \mathbf{B}\boldsymbol{\Lambda}(q)\mathbf{B}^H\|^2$$

$$\Upsilon = \sum_q \|\mathbf{t}(q) - \mathcal{B}\boldsymbol{\lambda}(q)\|^2$$

Alternating Least Squares

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- 2 Stationary values for $\text{Diag}\boldsymbol{\Lambda}(q)$: $\boldsymbol{\lambda}(q) = \{\mathbf{B}^H \mathbf{B}\}^{-1} \mathbf{B}^H \mathbf{t}(q)$

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- 2 Stationary values for $\text{Diag}\boldsymbol{\Lambda}(q)$: $\boldsymbol{\lambda}(q) = \{\mathbf{B}^H \mathbf{B}\}^{-1} \mathbf{B}^H \mathbf{t}(q)$
- 3 Stationary value for each column $\mathbf{b}[\ell]$ of matrix \mathbf{B} is the dominant eigenvector of the Hermitean matrix

$$\mathbf{P}[\ell] = \frac{1}{2} \sum_q \lambda_\ell(q) \{ \tilde{\mathbf{T}}[q; \ell]^H + \tilde{\mathbf{T}}[q; \ell] \}$$

where $\tilde{\mathbf{T}}[q; \ell] \stackrel{\text{def}}{=} \mathbf{T}(q) - \sum_{n \neq \ell} \lambda_n(q) \mathbf{b}[n] \mathbf{b}[n]^H$.

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where $\tilde{\mathbf{T}}[q; \ell] \stackrel{\text{def}}{=} \mathbf{T}(q) - \sum_{n \neq \ell} \lambda_n(q) \mathbf{b}[n] \mathbf{b}[n]^H$.

- 4 ALS: calculate $\boldsymbol{\Lambda}(q)$ and \mathbf{B} *alternately*. Use LS solution when matrices are singular.

Diagonally dominant matrices (1)

One wishes to minimize iteratively $\sum_q \|\mathbf{T}(q) - \mathbf{A} \mathbf{\Lambda}(q) \mathbf{A}^T\|^2$
Assume \mathbf{A} is strictly diagonally dominant: $|A_{ii}| > \sum_{j \neq i} |A_{ij}|$
(cf. Levy-Desplanques theorem)

- 1 Initialize $\mathbf{A} = \mathbf{I}$
- 2 Update \mathbf{A} multiplicatively as $\mathbf{A} \leftarrow (\mathbf{I} + \mathbf{W})\mathbf{A}$, where \mathbf{W} is zero-diagonal
- 3 Compute the best \mathbf{W} assuming that it is small and that $\mathbf{T}(q)$ are almost diagonal (first order approximation)

Diagonally dominant matrices (2)

Computational details:

- We have: $\mathbf{T}^{(\ell+1)}(q) \leftarrow (\mathbf{I} + \mathbf{W})\mathbf{T}^{(\ell)}(q)(\mathbf{I} + \mathbf{W})^T$
- $\mathbf{T}^{(\ell)}(q) \stackrel{\text{def}}{=} (\mathbf{D} - \mathbf{E})$, where \mathbf{D} is diagonal, and \mathbf{E} zero-diagonal
- If \mathbf{W} and \mathbf{E} are small:

$$\mathbf{T}^{(\ell+1)}(q) \approx \mathbf{D}(q) + \mathbf{W}\mathbf{D}(q) + \mathbf{D}(q)\mathbf{W}^T - \mathbf{E}(q)$$
- Hence minimize, wrt \mathbf{W} :

$$\sum_q \sum_{i \neq j} |W_{ij} D_{jj}(q) + W_{ji} D_{ii}(q) - E_{ij}(q)|^2$$

- This is of the form $\min_{\mathbf{W}} \|\mathbf{J}\mathbf{w} - \mathbf{e}\|^2$, where \mathbf{J} is sparse: $\mathbf{J}^T\mathbf{J}$ is block diagonal

Thus one gets at each iteration a collection of *decoupled* 2×2 linear systems

Joint triangularization of matrix slices

- 1 From \mathbf{T} , determine a collection of matrices (e.g. matrix slices), $\mathbf{T}(q)$, satisfying $\mathbf{T}(q) = \mathbf{A} \mathbf{D}(q) \mathbf{B}^T$, $\mathbf{D}(q) \stackrel{\text{def}}{=} \mathbf{diag}\{C_{q,:}\}$.

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- 2 Compute the Generalized Schur decomposition $\mathbf{Q} \mathbf{T}(q) \mathbf{Z} = \mathbf{R}(q)$, where $\mathbf{R}(q)$ are upper-triangular

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- 2 Compute the Generalized Schur decomposition $\mathbf{Q} \mathbf{T}(q) \mathbf{Z} = \mathbf{R}(q)$, where $\mathbf{R}(q)$ are upper-triangular
- 3 Since $\mathbf{Q}^T \mathbf{R}(q) \mathbf{Z}^T = \mathbf{A} \mathbf{D}(q) \mathbf{B}^T$, matrices $\mathbf{R}' \stackrel{\text{def}}{=} \mathbf{Q} \mathbf{A}$ and $\mathbf{R}'' \stackrel{\text{def}}{=} \mathbf{B}^T \mathbf{Z}$ are upper triangular, and can be assumed to have a unit diagonal. Hence \mathbf{R}' and \mathbf{R}'' can be computed by solving from the bottom to the top the triangular system, two entries R'_{ij} and R''_{ij} at a time:

$$\mathbf{R}(q) = \mathbf{R}' \mathbf{D}(q) \mathbf{R}''$$

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$$\mathbf{R}(q) = \mathbf{R}' \mathbf{D}(q) \mathbf{R}''$$

- 4 Compute $\mathbf{A} = \mathbf{R}' \mathbf{Q}^T$ and $\mathbf{B}^T = \mathbf{R}'' \mathbf{Z}^T$

Joint triangularization of matrix slices

- 1 From \mathbf{T} , determine a collection of matrices (e.g. matrix slices), $\mathbf{T}(q)$, satisfying $\mathbf{T}(q) = \mathbf{A} \mathbf{D}(q) \mathbf{B}^T$, $\mathbf{D}(q) \stackrel{\text{def}}{=} \text{diag}\{C_{q,:}\}$.
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$$\mathbf{R}(q) = \mathbf{R}' \mathbf{D}(q) \mathbf{R}''$$

- 4 Compute $\mathbf{A} = \mathbf{R}' \mathbf{Q}^T$ and $\mathbf{B}^T = \mathbf{R}'' \mathbf{Z}^T$
- 5 Compute matrix \mathbf{C} from \mathbf{T} , \mathbf{A} and \mathbf{B} by solving the over-determined linear system $\mathbf{C} \cdot \{(\mathbf{B} \odot \mathbf{A})^T\} = \mathbf{T}_{K \times JI}$

ICAR (1/2)

Source kurtosis signs are known and stored in diagonal matrix Δ .

- 1 Compute 4th order cumulants of x and store them in a $N^2 \times N^2$ “quadricovariance” matrix, \mathbf{Q}_x .
- 2 Upon appropriate ordering, we have:

$$\mathbf{Q}_x = [\mathbf{A} \odot \mathbf{A}] \mathbf{D} \Delta \mathbf{D} [\mathbf{A} \odot \mathbf{A}]^H \quad (35)$$

where $\mathbf{D} \Delta \mathbf{D}$ is the $N \times N$ diagonal matrix of source marginal cumulants, $D_{ii} \in \mathbb{R}^+$.

- 3 Compute a square root of $\mathbf{Q}_x = \mathbf{E} \Lambda \mathbf{E}^H$ of rank N as $\mathbf{Q}_x^{1/2} = \mathbf{E} (\Lambda \Delta)^{1/2}$. Then $\mathbf{Q}_x^{1/2} \Delta \mathbf{Q}_x^{H/2} = \mathbf{Q}_x$ and

$$\exists \mathbf{V} : \quad \mathbf{Q}_x^{1/2} = [\mathbf{A} \odot \mathbf{A}] \mathbf{D} \mathbf{V}^H, \text{ with } \mathbf{V}^H \Delta \mathbf{V} = \Delta$$

ICAR (2/2)

- 4 $\mathbf{Q}_x^{1/2}$ has N^2 rows and N columns. Denote $\mathbf{\Gamma}_n$ each $N \times N$ block. Then

$$\mathbf{\Gamma}_n = \mathbf{A}^* \mathbf{\Phi}_n \mathbf{D} \mathbf{V}^H$$

where $\mathbf{\Phi}_n = \text{Diag}\{\mathbf{A}(n, :)\}$

- 5 Compute the products $\mathbf{\Theta}_{m,n} = \mathbf{\Gamma}_m^{-1} \mathbf{\Gamma}_n$. Then we have

$$\mathbf{\Theta}_{m,n} = \mathbf{V}^{-H} \mathbf{\Phi}_m^{-1} \mathbf{\Phi}_n \mathbf{V}^H$$

- 6 Compute \mathbf{V} by jointly diagonalizing $\mathbf{\Theta}_{m,n}$, $1 \leq m < n \leq N$.
- 7 Compute an estimate of \mathbf{A} from the N blocks of $\mathbf{Q}_x^{1/2} \mathbf{V}^{-H} = \mathbf{A} \odot \mathbf{A}$, and eventually an estimate of the sources

For further details, see [\[AFCC05\]](#).

Bibliographical comments s

For collection of symmetric matrices:

- maximizes iteratively a lower bound to the decrease on a probabilistic objective [Pha01]
- alternately minimize $\sum_q \|\mathbf{T}(q) - \mathbf{A} \mathbf{\Lambda}(q) \mathbf{A}^T\|^2$ wrt \mathbf{A} and $\mathbf{\Lambda}(q)$ [Yer02]. See also: [LZ07] [VO06]
- minimizes iteratively $\sum_q \|\mathbf{T}(q) - \mathbf{A} \mathbf{\Lambda}(q) \mathbf{A}^T\|^2$ under the assumption that \mathbf{A} is diagonally dominant [ZNM04].

For collection of non symmetric matrices:

- factor \mathbf{A} into orthogonal and triangular parts, and perform a joint Schur decomposition [LMV04].
- others: algorithms applicable to underdetermined case work here, e.g. ICAR [AFCC05]. Also: [CL13].

Summary

For over-determined mixtures:

- Solving the invertible problem is sufficient
- Orthogonal framework:
 - Fully use 2nd order statistics, and then decompose approximately the tensor under *orthogonal constraint*.
Easier to handle singularity, but arbitrary.
 - Several (contrast) criteria and algorithms for orthogonal decomposition
- Invertible framework:
 - Decompose the higher order cumulant tensor directly under invertible constraint
 - Ill-posed
 - Several algorithms, mainly working with matrix slices

Principles & algorithms dedicated to under-determined mixtures can also apply (cf. subsequent course).

VI. Convolutive Mixtures



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- Algorithms for ARMA channels

24 Methods in the frequency domain

- Introduction
- SOS
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- Applications

Why convolutive mixtures ?

- Limitation of linear instantaneous (or memoryless) mixtures
- source signal may arrive to the sensors with different delays, following different paths
- source signal can be filtered due to the channel propagation
- such situations occurs in:
 - telecommunications due to multiple paths,
 - audio and speech processing due to sound propagation in air, reverberation in room.

For better modelization of mixtures in such situations, one has to consider mixtures of filtered sources, i. e. convolutive mixtures.

Notations

Sources and mixtures

- Vector of unknown source signals: $\mathbf{s}(n) = (s_1(n), \dots, s_P(n))^T$
- Vector of observable mixtures: $\mathbf{x}(n) = (x_1(n), \dots, x_K(n))^T$
- Delayed samples of sources contribute to observations at a given time n , which is modelled by a MIMO (multi-input multi-output) linear time invariant (LTI) filter with impulse response $\mathbf{A}(n)$.
- Relation between sources and mixtures is

$$\mathbf{x}(n) = \sum_{k \in \mathbb{Z}} \mathbf{A}(k) \mathbf{s}(n - k)$$

Notations (con't)

Separation structure and estimated sources

- Estimated sources: $\mathbf{y}(n) = (y_1(n), \dots, y_P(n))^T$
- $\mathbf{y}(n)$ is obtained through a MIMO LTI filter $\mathbf{B}(n)$, objective of which is to “inverse” the channel mixture $\mathbf{A}(n)$:

$$\mathbf{y}(n) = \sum_{k \in \mathbb{Z}} \mathbf{B}(k) \mathbf{x}(n - k)$$

z-transform notations

Mixing and separating filters

$$\mathbf{A}[z] = \sum_{k \in \mathbb{Z}} \mathbf{A}(k)z^{-k} \quad \text{and} \quad \mathbf{B}[z] = \sum_{k \in \mathbb{Z}} \mathbf{B}(k)z^{-k}$$

Global filter

- The global filter denoted $\mathbf{G}(n)$, relies $\mathbf{y}(n)$ and $\mathbf{s}(n)$:

$$\mathbf{G}(n) = \sum_{k \in \mathbb{Z}} \mathbf{G}(k)\mathbf{s}(n - k), \forall n \in \mathbb{Z}$$

- The global filter is depending on mixing and separating filters:

$$\mathbf{G}(n) = \sum_{k \in \mathbb{Z}} \mathbf{B}(n - k)\mathbf{A}(k), \forall n \in \mathbb{Z} \quad \text{and}$$

$$\mathbf{G}[z] = \mathbf{B}[z]\mathbf{A}[z].$$

Models of sources

The model of sources is important for modelling practical situations. It also plays an important role in the indeterminacies.

iid sources

- The signal $\{s(n)\}$, $n \in \mathbb{Z}$, is an independent and identically distributed (iid) sequence if the sample distributions are identical and mutually statistically independent.

Definition of linear sources

- The signal $\{s(n)\}$, $n \in \mathbb{Z}$, is a linear signal if it can be written as the output of a filter fed by an iid sequence.

Definition of nonlinear sources

- The signal $\{s(n)\}$, $n \in \mathbb{Z}$, is a nonlinear signal if it is not linear, i.e. if it cannot be written as the output of a filter feed by an iid sequence.

Other properties on sources

Other priors on sources, encountered in various situations or application domains, can be exploited, e.g.:

- Nonstationarity, for instance, in music or speech processing
- Cyclo-stationarity, for instance in telecommunications
- Colored sources
- Etc.

Assumptions

Global filter

Three main assumptions can be done on the filters, leading to different situations and level of difficulty

- H1: The mixing filter is stable
- H2: The mixing filter is causal
- H3: The mixing filter is finite impulse response (FIR) filter

General inversibility results

Stable mixing filter

- *Proposition 1:* For $P = K$), the stable $P \times P$ filter $\mathbf{A}[z]$ is invertible by a stable filter iff $\det(\mathbf{A}[z]) \neq 0, \forall z, |z| = 1$.

Stable and causal mixing filter

- *Proposition 2:* For $P = K$), the stable $P \times P$ filter $\mathbf{A}[z]$ is invertible by a stable and causal filter iff $\det(\mathbf{A}[z]) \neq 0$ in a domain $\{z \in \mathbb{C}, |z| > r\} \cup \{\infty\}$, where $0 \leq r < 1$.

Stable $P \neq K$) mixing filter

We now assume that $P \neq K$.

- *Proposition 3:* The stable $K \times P$ filter $\mathbf{A}[z]$ is invertible by a stable filter iff $\text{rank}\{\mathbf{A}[z]\} = P, \forall z, |z| = 1$.

Proofs can be found in [CJ10, ch.8]

Contrast criteria (1) P

Indeterminacies in convolutive mixtures

- **White sources:** the only filters preserving both strong whiteness (i.i.d) and independence are

$$\mathbf{T}[z] = \mathbf{\Lambda}[z] \mathbf{P}$$

where $\mathbf{\Lambda}[z]$ is diagonal, and $\lambda_{ii}[z] = \alpha_i z^{\beta_i}$, $\beta \in \mathbb{Z}$, i.e. pure delays integer multiples of sampling period.

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- **Colored sources:** sole mutual independence cannot allow to recover source color. Trivial filters are of same form, where $\mathbf{\Lambda}[z]$ is any diagonal matrix.
- **Linear processes:** When each source is the output of a linear filter fed by a white process, then it is then equivalent to assume that sources are white.

Contrast criteria (2)

Contrasts based on output cumulants: Proofs derived in the static case hold true in the convolutive case when sources are iid.

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Examples: The following are contrasts, if (a) at most one source cumulant is null, (b) $p + q > 2$ and (c) $r \geq 1$:

$$\Upsilon_{p,q}^{(r)} = \sum_i |\text{Cum}\{p, q\}\{z_i\}|^r \quad (36)$$

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More generally, if ϕ is convex strictly increasing on \mathbb{R}^+

$$\Upsilon_{p,q}^{\phi} = \sum_i \phi(|\text{Cum}\{p, q\}\{z_i\}|) \quad (37)$$

For more details, see [CJ10, ch.3] [Com04] [MP97] [Com96] and refs therein.

Para-unitarity

What are the filters preserving second-order whiteness?

The so-called *Para-unitary* filters:

$$\mathbf{U}[z] \mathbf{U}[1/z^*]^H = \mathbf{I}$$

Or in time domain:

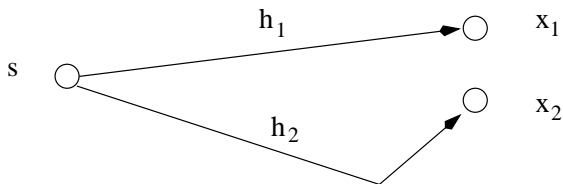
$$\sum_k \mathbf{U}(k) \mathbf{U}(k-n)^H = \delta(n) \mathbf{I}$$

Contrast criteria (3)

- Also possible to devise new families of contrasts for *para-unitary equalizers* after prewhitening [CM97] [Com96].
For instance at order 4:

$$\Upsilon(\mathbf{y}) = \sum_i \sum_{j p} \sum_{k q} |\text{Cum}\{y_i[n], y_i[n]^*, y_j[n-p], y_k[n-q]^*\}|^2 \quad (38)$$

- In the above, one can conjugate any of the variables y_ℓ 's provided at most one source cumulant is null
- Holds true for almost any cumulants of order ≥ 3
- Only two indices need to be identical with same delay

SIMO mixture with diversity $K = 2$ (1)

Disparity condition:

$$h_1[z] \wedge h_2[z] = 1 \Rightarrow x_1[z] \wedge x_2[z] = s[z]$$

Bézout:

$$\begin{aligned} \exists v_1[z], v_2[z] / \quad v_1[z] h_1[z] + v_2[z] h_2[z] &= 1 \\ \Rightarrow v_1[z] x_1[z] + v_2[z] x_2[z] &= s[z] \end{aligned}$$

Thus

FIR filter $\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ admits the FIR inverse $\mathbf{v} = (v_1, v_2)$.

SIMO mixture with diversity $K = 2$ (2)**Theorem**

If two polynomials $p(z) = \sum_{i=0}^m a_i z^i$ and $q(z) = \sum_{i=0}^n b_i z^i$ are prime, then the resultant below is non zero:

$$\mathcal{R}(p, q) = \begin{vmatrix} a_0 & \dots & a_m & 0 & \dots \\ 0 & \ddots & & \ddots & 0 \\ 0 & 0 & a_0 & \dots & a_m \\ b_0 & \dots & b_n & 0 & \dots \\ 0 & \ddots & & \ddots & 0 \\ 0 & 0 & b_0 & \dots & b_n \end{vmatrix} \stackrel{\text{def}}{=} \det \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$$

Use of time diversity

Time diversity

If channel bandwidth exceeds symbol rate $\frac{1}{T_s}$ (excess bandwidth), then a sampling faster than $\frac{1}{T_s}$ brings extra information on channel.

[Sl094] [TXK94]

How to build a SIMO channel from a SISO?

- sample twice faster: $x[k] = x(k T_s/2)$
- denote odd samples $x_1[k] = x[2k + 1]$, and even samples $x_2[k] = x[2k]$
- then

$$\begin{pmatrix} x_1[k] \\ x_2[k] \end{pmatrix} = \begin{pmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{pmatrix} s[k] \stackrel{\text{def}}{=} \mathbf{H} s[k]$$

Matrix \mathbf{H} is full rank (well conditioned) if sufficient *excess bandwidth*

MISO Dynamic Extractor: Deflation

- Fixed step gradient Deflation [Tug97]
- No spurious minima of contrast criterion [CJ10, ch.6] [DL95]
- Optimal Line search along a descent direction, OS-KMA [ZC05] [Com02]

After Prewhitening: PAJOD (1)

- Technique applied *after* space-time prewhitening
- Then one looks for a *para-unitary* equalizer, by maximizing the contrast

$$\mathcal{J}_{2,r}(\mathbf{m}y) = \sum_{\mathbf{b}} \sum_{\beta} \|\mathbf{Diag}\{\mathcal{H}^H \mathbf{M}(\mathbf{m}\mathbf{b}, \mathbf{m}\beta) \mathcal{H}\}\|^2$$

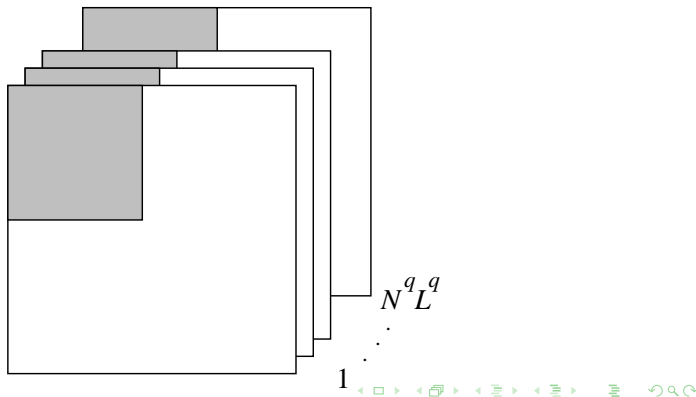
Matrix \mathcal{H} is now defined differently, and is semi-unitary.
 Matrices $\mathbf{M}(\cdot)$ contain cumulants of whitened observations

- Contrast (38) is maximized again by a sweeping technique

PAJOD (2)

PAJOD: Partial Approximate Joint Diagonalization of matrix slices

One actually attempts to diagonalize only a portion of the tensor



MIMO Blind Equalization

- linear prediction after BI [Com90]
- linear prediction [AMML97] [GL99]
- subspace [MDCM95] [GDM97] [CL97] [AMLM97]
- identifiability issues by subspace techniques [LM00] [Des01]

Equalization after prior Blind Identification

Assume channel $\mathbf{H}[z]$ has been identified, with:

$$\mathbf{x}[z] = \mathbf{H}[z] \cdot \mathbf{s}[z] + \mathbf{v}[z]$$

An estimate of $\mathbf{s}[z]$ is obtained with $\mathbf{F}[z] \star \mathbf{x}[z]$.

Possible equalizers $\mathbf{F}[z]$:

- Zero-Forcing: $\mathbf{F}[z] = \mathbf{H}[z]^{-1}$
- Matched Filter: $\mathbf{F}[z] = \mathbf{H}[1/z^*]^H$
(used in MLSE; optimal if channel AWGN; maximizes output SNR)
- Minimum Mean Square Error (MSE):
 $\mathbf{F}[z] = (\mathbf{H}[z]\mathbf{H}[1/z^*]^H + \mathbf{R}_v[z])^{-1}\mathbf{H}[1/z^*]^H$

➔ One can insert soft or hard decision to stabilize the inverse, or to reduce noise, e.g. decision Feedback Equalizers (DFE).

Overview

- Interest
- MA identifiability (second order vs hOS)
- SISO: Cumulant matching
- MIMO: Cumulant matching and linear prediction (non monic MA)
- Algebraic approaches, Quotient Ring
- SIMO: Subspace approaches
- MIMO: Subspace, IIR, ...

BE vs BI

If sources $s_p[k]$ are discrete, it is:

- rather easy to define a BE optimization criterion in order to match an output alphabet
- difficult to exploit a source alphabet in BI

Example the property of constant modulus of an alphabet is mainly used in Blind Equalization: CMA (Constant Modulus Algorithm)

Interest of Blind Identification

- When the mixture does not have a stable inverse
 - ➡ When may want to control stability by soft/hard decision in a Feedback Equalizer
- When sources are not of interest (e.g. channel characteristics, localization only)

SISO Cumulant matching (1)

- Consider first the **SISO** case

$$x[n] = \sum_{k=0}^L h[k] s[n-k] + v[k]$$

where $v[k]$ is Gaussian stationary, and $s[n]$ is 4th order white stationary.

- Then, by the multilinearity property of cumulants (cf. slide):

$$C_x(i, j) \stackrel{\text{def}}{=} \text{Cum}\{x[t+i], x[t+j], x[t+L], x[t]\} = h[i] h[j] h[L] h[0] c_s$$

with $c_s \stackrel{\text{def}}{=} \text{Cum}\{s[n], s[n], s[n], s[n]\}$.

- By substitution of the unknown $h[L] h[0] c_s$, one gets a whole family of equations [SGS94] [Com92b]:

$$h[i] h[j] C_x(k, \ell) = h[k] h[\ell] C_x(i, j), \quad \forall i, j, k, \ell \quad (39)$$

SISO Cumulant matching (2)

- A solution to the subset of (39) for which $j = \ell$ can be easily obtained:

$$h[i] C_x(k, j) - h[k] C_x(i, j) = 0, \quad 0 \leq i < k \leq L, \quad 0 \leq j \leq L \quad (40)$$

- This is a *linear system* of $L(L + 1)^2/2$ equations in $L + 1$ unknowns
 \Rightarrow **Least Square (LS) solution**, up to a scale factor (e.g. $h(0) = 1$).
- Since 4th order only, asymptotically (for large samples) insensitive to Gaussian noise.
- Total Least Squares (TLS) solution possible as well

MIMO Cumulant matching (1)

Inteterminacy

- Scale (scalar) factor for SISO, but $\mathbf{\Lambda P}$ factor for MIMO

Reduction to a monic model [Com94b]

- If $\mathbf{H}[0]$ is invertible, one has

$$\mathbf{y}[n] = \mathbf{H}[0] \mathbf{s}[n], \quad (41)$$

$$\mathbf{x}[n] = \sum_{k=0}^L \mathbf{B}[k] \mathbf{y}[n-k] + \mathbf{w}[k] \quad (42)$$

where $\mathbf{B}[k] \stackrel{\text{def}}{=} \mathbf{H}[k] \mathbf{H}[0]^{-1}$.

- Because $\mathbf{B}[0] = \mathbf{I}$, MA model (42) is said to be *monic*.
- Indeterminacy* is only in (41), which is solved by ICA if $\mathbf{s}[n]$ is spatially white

MIMO Cumulant matching (2)

Kronecker notation

- Store 4th order cumulant tensors in vector form:

$$\mathbf{c}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \stackrel{\text{def}}{=} \text{vec}\{\text{Cum}\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}\}$$

- Then, we have the property (where \square denotes term-wise *Hadamard* product):

$$\begin{aligned} \mathbf{c}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} = & \mathbb{E}\{\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}\} - \mathbb{E}\{\mathbf{a} \otimes \mathbf{b}\} \otimes \mathbb{E}\{\mathbf{c} \otimes \mathbf{d}\} \\ & - \mathbb{E}\{\mathbf{a} \otimes \mathbb{E}\{\mathbf{b} \otimes \mathbf{c}\} \otimes \mathbf{d}\} \\ & - \mathbb{E}\{\mathbf{a} \otimes \mathbf{1}_\beta \otimes \mathbf{c} \otimes \mathbf{1}_\delta\} \square \mathbb{E}\{\mathbf{1}_\alpha \otimes \mathbf{b} \otimes \mathbf{1}_\gamma \otimes \mathbf{d}\} \end{aligned}$$

MIMO Cumulant matching (3)

Assume monic MA model (42) where $s[n]$ white in time and L fixed, and denote

$$\mathbf{c}_x(i, j) \stackrel{\text{def}}{=} \mathbf{vec}\{\text{Cum}\{\mathbf{x}[t+i], \mathbf{x}[t+j], \mathbf{x}[t+L], \mathbf{x}[t]\}\}$$

- Then we can prove [Com92b]:

$$C_x(i, j) = C_x(0, j) \mathbf{B}[i]^T, \quad \forall j, 0 \leq j \leq L$$

where $C_x(i, j) \stackrel{\text{def}}{=} \mathbf{Unvec}_P(\mathbf{c}_x)$ is $P^3 \times P$

- For every fixed i , $\mathbf{B}[i]$ is obtained by solving the system of $(L+1)P^4$ equations in P^2 unknowns in LS sense:

$$[\mathbf{I}_P \otimes C_x(0, j)] \mathbf{vec}\{\mathbf{B}[i]\} = \mathbf{c}_x(i, j) \quad (43)$$

MIMO Cumulant matching (4)

■ Summary of the algorithm

- Choose a maximum L
- Estimate cumulants of observation, $\mathbf{c}_x(i, j)$ for $i, j \in \{0, \dots, L\}$
- Solve the $(L + 1)$ systems (43) in $\mathbf{B}[i]$
- Compute the residue $\mathbf{y}[t]$ (*Linear Prediction*)
- Solve the ICA problem $\mathbf{y}[t] = \mathbf{H}[0] \mathbf{s}[t]$

■ Weaknesses

- $\mathbf{H}[0]$ must be invertible
- FIR model needs to have a stable inverse

Algebraic Blind identification (1)

Types of discrete source studied

- BPSK: $b[k] \in \{-1, 1\}$, *i.i.d.*
- MSK: $m[k + 1] = j m[k] b[k]$
- QPSK: $p[k] \in \{-1, -j, 1, j\}$, *i.i.d.*
- $\frac{\pi}{4}$ -DQPSK: $d[k + 1] = e^{j\pi/4} d[k] p[k]$
- 8-PSK: $q[k] \in \{e^{jn\pi/4}, n \in \mathbb{Z}\}$, *i.i.d.*
- etc...

Algebraic Blind identification (2)

Input/Output relations:

- For $s[k]$ **BPSK**: $E\{x[n] x[n - \ell]\} = s[0]^2 \sum_{m=0}^L h[m] h[m + \ell]$
- For $s[k]$ **MSK**:
 $E\{x[n] x[n - \ell]\} = s[0]^2 \sum_{m=0}^L (-1)^m h[m] h[m + \ell]$
- For $s[k]$ **QPSK**:
 $E\{x[n]^2 x[n - \ell]^2\} = s[0]^2 \sum_{m=0}^L h[m]^2 h[m + \ell]^2$
- etc..

Algebraic Blind identification (3)

Principle:

- Compute all roots of the polynomial system in $h[n]$.
For instance for MSK sources and a channel of length 2
[GCMT02]:

$$h[0]^2 - h[1]^2 + h[2] = \beta_0$$

$$h[0] h[1] - h[1] h[2] = \beta_1$$

$$h[0] h[2] = \beta_2$$

- Choose among these roots the one that best matches the I/O correlation:

$$E\{x[n] x[n - \ell]^*\} = \sum_{m=0}^L h[m] h[m + \ell]^*$$

SIMO mixture (1)

- FIR of length L and dimension K :

$$\mathbf{x}(n) = \sum_{i=1}^L \mathbf{h}(i)s(n-i) + \mathbf{b}(n)$$

with: $\mathbb{E}\{\mathbf{b}(m)\mathbf{b}(n)^H\} = \sigma_b^2 \mathbf{I} \delta(m-n)$
 and $\mathbb{E}\{\mathbf{b}(m)s(n)^*\} = \mathbf{0}$

- For T successive values:

$$\begin{pmatrix} \mathbf{x}(n) \\ \mathbf{x}(n-1) \\ \vdots \\ \mathbf{x}(n-T) \end{pmatrix} = \begin{pmatrix} \mathbf{h}(0) & \mathbf{h}(1) & \dots & \mathbf{h}(L) & 0 & \dots & 0 \\ 0 & \mathbf{h}(0) & \dots & \dots & \mathbf{h}(L) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{h}(0) & \mathbf{h}(1) & \dots & \mathbf{h}(L) \end{pmatrix} \begin{pmatrix} s(n) \\ \vdots \\ s(n-T-L) \end{pmatrix}$$

Or in compact form:

$$\mathbf{X}(n:n-T) = \mathcal{H}_T \mathbf{S}(n:n-T-L) \quad (44)$$

Here, \mathcal{H}_T is of size $(T+1)K \times (T+L+1)$

SIMO mixture (2)

- **Condition of “column” matrix**

\mathcal{H} has strictly more rows than columns **iff**

$$(T + 1)K > T + L + 1$$

$$\Leftrightarrow T > L/(K - 1) - 1 \Leftrightarrow T \geq L$$

It suffices that T exceeds channel memory.

- **Disparity condition**

Columns of \mathcal{H} are linearly independent **iff**

$$\mathbf{h}[z] \neq \mathbf{0}, \forall z$$

- **Noise subspace**

Under these 2 conditions, there exists a “noise subspace”:

$$\exists \mathbf{v} / \mathbf{v}^H \mathcal{H}_T = \mathbf{0}$$

SIMO mixture (3)

Properties of vectors in the null space

- Since $\mathbf{R}_x \stackrel{\text{def}}{=} \mathbb{E}\{\mathbf{X}\mathbf{X}^H\} = \mathcal{H}_T \mathcal{H}_T^H + \sigma_b^2 \mathbf{I}$, vectors $\mathbf{v}^{(p)}$ of noise space can be computed from \mathbf{R}_x :

$$\mathbf{R}_x \mathbf{v}^{(p)} = \sigma_b^2 \mathbf{v}^{(p)}$$

- And since convolution is commutative:

$$\mathbf{v}^{(p)H} \mathcal{H}_T = \mathbf{h}^H \mathcal{V}^{(p)}$$

where $\mathcal{V}^{(p)}$ block Töplitz, built on $\mathbf{v}^{(p)}$.

- Thus $\mathbf{h}^H = [\mathbf{h}(0)^H, \mathbf{h}(1)^H, \dots, \mathbf{h}(L)^H]$ are obtained by computing the left singular vector common to $\mathcal{V}^{(p)}$.

SIMO mixture (4)

Summary of the SIMO Subspace Algorithm

- Choose $T \geq L$
- Compute \mathbf{R}_x , correlation matrix of size $(T + 1)K$
- Compute the $d = T(K - 1) + K - L - 1$ vectors $\mathbf{v}^{(p)}$ of the noise space
- Compute vector \mathbf{h} minimizing the quadratic form

$$\mathbf{h}^H \left[\sum_{p=1}^d \mathbf{v}^{(p)} \mathbf{v}^{(p)H} \right] \mathbf{h}$$

- Cut \mathbf{h} into $L + 1$ slices $\mathbf{h}(i)$ of length K

Under the assumed hypotheses, the solution is unique up to a scalar scale factor [MDCM95]

SIMO mixture (5)

Summary of the SIMO Subspace Algorithm when $K = 2$

- Choose $T = L$. There is a single vector \mathbf{v} in the noise space
- Compute \mathbf{R}_x , correlation matrix of size $(T + 1)K$
- Compute the vector \mathbf{v} of the noise space
- Cut \mathbf{v} into $L + 1$ slices $\mathbf{v}(i)$ of length $K = 2$
- Compute $\mathbf{h}(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{v}(i)$

In fact $x_i = h_i \star s \Rightarrow h_2 \star x_1 - h_1 \star x_2 = 0$

Approach called **SRM** (*Subchannel Response Matching*)

[XLTK95] [GN95]

SISO Identifiability

- Second order statistics

- $\alpha_\ell = E\{x[n] x[n - \ell]^*\}$ ➡ allow to estimate $|h[m]|$

- $\beta_\ell = E\{x[n] x[n - \ell]\}$ ➡ allow to estimate $h[m]$ if $E\{s^2\} \neq 0$

- Fourth order statistics ➡ many (polynomial) additional equations

- $\gamma_{0jkl} = \text{Cum}\{x[n], x[n - j], x[n - k], x[n - \ell]\}$

- $\gamma_{0j}^{k\ell} = \text{Cum}\{x[n], x[n - j], x[n - k]^*, x[n - \ell]^*\}$

If some sources are 2nd order circular, sample Statistics of order higher than 2 are mandatory, but *otherwise not* [GCMT02] !

SIMO Identifiability

With a receive diversity, (deterministic) identifiability conditions are weaker [Hua96] [XLTK95]

- **Definition** A length- N input sequence $s[n]$ has P modes iff the Hankel matrix below is full row rank:

$$\begin{pmatrix} s[1] & s[2] & \dots & s[N-p+1] \\ s[2] & s[3] & \ddots & s[N-p+2] \\ \vdots & \vdots & & \vdots \\ s[p] & s[p+1] & \dots & s[N] \end{pmatrix}$$

- **Theorem** A $K \times L$ FIR channel \mathbf{h} is identifiable if:
 - Channels $h_k[z]$ do not have common zeros
 - The observation length of each $x_k[n]$ must be at least $L + 1$
 - The input sequence should have at least $L + 1$ modes (sufficiently exciting)

Subspace algorithm for MIMO mixtures

- Similarly to the SIMO case, we have the compact form:

$$\mathbf{X}(n) = \mathcal{H}_T \mathbf{S}(n) + \mathbf{B}(n)$$

where \mathcal{H} is now built on matrices $\mathbf{H}(k)$, $1 \leq k \leq L$, and is of size $(T+1)K \times (T+L+1)P$.

- For large enough T , this matrix is "column shaped"
- Again $\mathbf{R}_x = \mathcal{H}_T \mathcal{H}_T^H + \sigma_b^2 \mathbf{I}$
- But now, vectors of the noise space characterize $\mathbf{H}[z]$ only up to a constant post-multiplicative matrix \Rightarrow *ICA must be used afterwards*
- Foundations of the MIMO subspace algorithm are more complicated [Loubaton'99]

In the MIMO case, HOS are in general mandatory.

MIMO Subspace parameterization c

The basic idea is to expand the filtering operations, in which matrix entries are z polynomials, in large matrices with scalar entries.

Source and observation vectors

- Vector of sources:

$$\mathbf{S}(n) = (s(n), s(n-1), \dots, s(n - (L + L') + 1)), \text{ where}$$
$$s(k) = (s_1(k), s_2(k), \dots, s_P(k))$$

- Vector of observations:

$$\mathbf{X}(n) = (x(n), x(n-1), \dots, x(n - L' + 1)), \text{ where}$$
$$x(k) = (x_1(k), x_2(k), \dots, x_K(k))$$

Subspace parameterization

Mixing matrix

- L and L' correspond to channel length and window size, respectively; they must satisfy $KL' \geq P(L + L')$
- The above condition ensures a determined model, i.e. with an over-determined matrix \mathbf{H} , with size $KL' \times P(L + L')$
- With the above notations, the model $\mathbf{x}(n) = \mathbf{A}[z]\mathbf{s}(n)$ can be written $\mathbf{X}(n) = \mathbf{H}\mathbf{S}(n)$, where \mathbf{H} is a huge Sylvester block-Toeplitz matrix:

$$\begin{pmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-L'+1) \end{pmatrix} = \begin{pmatrix} \mathbf{A}(0) & \mathbf{A}(1) & \dots & \mathbf{A}(L) & 0 & \dots & 0 \\ 0 & \mathbf{A}(0) & \mathbf{A}(1) & \dots & \mathbf{A}(L) & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \mathbf{A}(0) & \mathbf{A}(1) & \dots & \mathbf{A}(L) \end{pmatrix} \begin{pmatrix} s(n) \\ s(n-1) \\ \vdots \\ s(n-(L+L')+1) \end{pmatrix}$$

Subspace parameterization

Source separation

- If L and L' satisfy the above condition and the [channel condition](#), it exists a left inverse of \mathbf{H} which provides source separation.
- More details on algorithms for estimating the left inverse can be found in [MJL00]
- Main drawbacks of the method are: huge matrix; observation number (K) must be strictly larger than the source number

SISO ARMA mixtures P

What are the tools when the channel is IIR?

- In general, just consider it as a FIR (truncation) → already seen
- But also possible to assume presence of a recursive part
 - Define I/O relation: $\sum_{i=0}^p a_i x[n-i] = \sum_{j=0}^q b_j w[n-i]$
where $w[\cdot]$ is i.i.d. and $a_0 = b_0 = 1$
 - Second order $c_x(\tau) \stackrel{\text{def}}{=} E\{x[n]x[n+\tau]\}$ can be used to identify a_k :

$$\sum_{k=0}^p a_k c_x(\tau - k) = 0, \quad \forall \tau > q$$

- Then compute the residue and identify b_ℓ with HOS (cf. slide)
- Also possible with HOS only for AR part [SGS94]

MIMO ARMA mixtures (1)

Results of SISO case can be extended [Com92b]

- Take a K -dimensional ARMA model: Define I/O relation:

$$\sum_{i=0}^p \mathbf{A}_i \mathbf{x}[n-i] = \sum_{j=0}^q \mathbf{B}_j \mathbf{w}[n-j]$$

where $w[\cdot]$ is i.i.d. and $\mathbf{A}_0 = \mathbf{I}$ and \mathbf{B}_0 invertible

- For instance at order 4, AR identification is based on:
 - $\sum_{j=1}^p \mathbf{A}_j \bar{\mathbf{c}}_x(t, \tau - j) = -\bar{\mathbf{c}}_x(t, \tau), \forall \tau > q, \forall t$
 - with $\bar{\mathbf{c}}_x(i, j) \stackrel{\text{def}}{=} \mathbf{Unvec}_K(\text{Cum}\{\mathbf{x}[n], \mathbf{x}[n], \mathbf{x}[n+i], \mathbf{x}[n+j]\})$

MIMO ARMA mixtures (2)

Limitations

- Sources need to be *linear processes*
- \mathbf{B}_0 needs to be invertible
- AR residuals need to be computed (MA filtering) to compute \mathbf{B}_i
- One can compute MA residuals (AR filtering) if input $s[n]$ is requested \rightarrow but might be unstable

Representation in frequency domain c

Fourier domain ?

- One difficulty in the convolutive model: $x(t) = \mathbf{A}(t) * s(t)$
related to $*$
- Using Plancherel theorem, $*$ becomes a simple product in the Fourier domain
- Moreover:
 - Fourier transform preserves linearity,
 - information is the same in time and frequency domain.

Representation in frequency domain (con't)

BSS equations in Fourier domain

- Taking the Fourier transform, $\mathbf{x}(t) = \mathbf{A}(t) * \mathbf{s}(t)$ becomes:

$$\mathbf{X}(\nu) = \mathbf{A}(\nu)\mathbf{S}(\nu)$$

- $*$ is cancelled; but, one just have one equation: not sufficient for estimating $\mathbf{A}(\nu)$ or its inverse !

Separation problem in the frequency domain

BSS equations in Fourier domain with STFT

- Applying short term Fourier transform (STFT), on sliding windows:

$$\mathbf{X}(\nu, t) = \mathbf{A}(\nu)\mathbf{S}(\nu, t)$$

- $\mathbf{A}(\nu)$ is not modified, if channel impulse response is time invariant. Of course, it is if channel is changing, e.g. if sources are moving.
- For each value ν_0 of the variable ν , one has an instantaneous mixture, with complex-valued entries:

$$\mathbf{X}(\nu_0, t) = \mathbf{A}(\nu_0)\mathbf{S}(\nu_0, t)$$

- In the Fourier domain, the convolutive mixing (real valued) equation is then transformed in a (infinite) set of instantaneous (complex-valued) mixing equations

Separation problem in the discrete frequency domain

BSS equations in discrete Fourier domain

- With discrete data in time, similar equations are obtained by SF discrete FT

$$\mathbf{X}(\nu_k, t) = \mathbf{A}(\nu_k)\mathbf{S}(\nu_k, t)$$

where ν_k corresponds to the k -th frequency bins in the DFT.

- Denoting N_ν this number, the convolutive mixing (real valued) equation is then transformed in a set of N_ν instantaneous (complex-valued) mixing equations

Solving the separation problem in frequency domain

Solving in any frequency bin

- Based on observations $\mathbf{X}(\nu_k, t)$,

$$\mathbf{X}(\nu_k, t) = \mathbf{A}(\nu_k)\mathbf{S}(\nu_k, t)$$

can be solved using any method for instantaneous linear mixture.

Source reconstruction

- For reconstructing the wideband sources, we just have to add estimated sources in each frequency bins...
- Just ??? No, due to permutation (and scale) indeterminacies !

$$\mathbf{B}(\nu_k) = \mathbf{P}(\nu_k)\mathbf{A}(\nu_k)^{-1}$$

- Problem if $\mathbf{P}(\nu_k) \neq \mathbf{P}(\nu_l)$, $k \neq l$

SOS in frequency domain

SOS in each frequency bin

- If source are not iid, we can used SOS methods, exploiting either coloration or nonstationarity,
- In each frequency band ν_k , the DSP (Fourier transform of the covariance matrix):

$$E[\mathbf{X}(\nu_k, t)\mathbf{X}^H(\nu_k, t-\tau)] = \mathbf{A}(\nu_k)E[\mathbf{S}(\nu_k, t)\mathbf{S}^H(\nu_k, t-\tau)]\mathbf{A}^H(\nu_k), \forall \tau$$

- With the notation $\mathbf{S}_{\mathbf{X}}(\nu_k, \tau) = E[\mathbf{X}(\nu_k, t)\mathbf{X}^H(\nu_k, t - \tau)]$, one obtains similar equations than in the time domain (due to FT linearity):

$$\mathbf{S}_{\mathbf{X}}(\nu_k, \tau) = \mathbf{A}(\nu_k)\mathbf{S}_{\mathbf{S}}(\nu_k, \tau)\mathbf{A}^H(\nu_k), \forall \tau$$

- Joint diagonalization can be used on set of spectral matrices:
 - $\mathbf{S}_{\mathbf{X}}(\nu_k, \tau)$, $\tau = 0, 1, \dots, K$ for colored sources,
 - $\mathbf{S}_{\mathbf{X}, W_i}(\nu_k, 0) = E_{W_i}[\mathbf{X}(\nu_k, t)\mathbf{X}^H(\nu_k, t)]$, computed on time windows W_i , exploiting source nonstationarity

Solving the permutation indeterminacies

Many methods and criteria, exploiting two mains ideas:

Continuity of filters

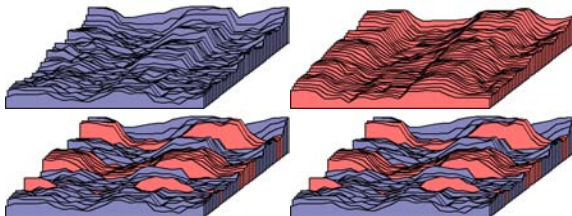
- Smooth variation of $\mathbf{B}(\nu)$:
 - short impulse response [PS00]
 - initialisation of $\mathbf{B}(\nu_k)$ estimation by $\hat{\mathbf{B}}(\nu_{k-})$ [PSB03]
- Problem with impulse response with echoes, e.g. if reverberation

Continuity of sources

- Time continuity of DSP of the estimated sources, on neighbor frequency bins [SP03]

Results of permutation regularization

- Most of the regularization criteria perform well when source power or channel coefficients are large enough
- Basically, it remains
 - global permutation (not important),
 - and frequency-bloc permutation, since a regularization failure in a frequency bins implies usually failure on a block !



Reprinted from PhD defense talk of Bertrand Rivet

Applications

- Main applications domains of convolutive mixture are:
 - Telecommunications
 - Acoustics, audio with speech and music processing
 - For speech application, refer to the lecture of L. Daudet and the book [MLS07]

VII. Non Linear Mixtures



Contents of course VII

25 Identifiability using ICA

- Problem
- General results

26 ICA for NL mixtures

- General NL
- PNL mixtures
- Inversion of Wiener system

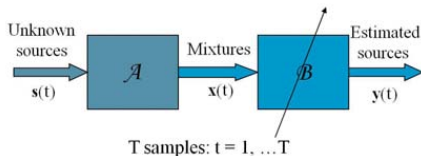
27 Discussion

Model and question

Model

- K noiseless nonlinear mixtures of P independent sources

$$\mathbf{x}(t) = \mathcal{A}(\mathbf{s}(t))$$



Question

- Assuming the mixing mapping \mathcal{A} is invertible, is it possible to estimate an inverse mapping \mathcal{B} using independence ?
- In other words: output independence \Leftrightarrow s source separation ?

Darmon's results

Undeterminacy

- Let s_i and s_j be 2 independent random variables, $f_i(s_i)$ and $f_j(s_j)$ are independent too
- Source separation is achieved if $y_i = h_i(s_j)$, i.e. the global mapping $\mathcal{G} = \mathcal{B} \circ \mathcal{A}$ is diagonal, up to a permutation.
- Such diagonal (up to a permutation) mappings \mathcal{G} will be defined as *trivial* mappings

Nonlinear mixtures are non identifiable using ICA

- It always exists non diagonal nonlinear mixing mappings which preserve independence
- Darmon [Dar53] proposed a general method for constructing such mappings. The idea has then been used by Hyvärinen and Pajunen [HP99].

Darmon's construction

Assumptions

- Consider the mapping $\mathbf{y} = \mathcal{G}(\mathbf{x})$ where \mathcal{G} is invertible
- Then : $p_{\mathbf{y}}(\mathbf{y}) = p_{\mathbf{x}}(\mathbf{x}) |\det J_{\mathcal{G}}(\mathbf{x})|^{-1}$
- Without loss of generality, one can assume \mathbf{y} are uniform in $[0, 1]$

Mapping \mathcal{G} preserving independence

- We are looking for \mathcal{G} such that the random vector \mathbf{y} is independent, i.e.

$$|\det J_{\mathcal{G}}(\mathbf{x})|^{-1} = p_{\mathbf{x}}(\mathbf{x})$$

A possible mapping (1/2)

Choosing the non diagonal mapping \mathcal{G}

$$g_1(\mathbf{x}) = g_1(x_1)$$

$$g_2(\mathbf{x}) = g_2(x_1, x_2)$$

$$\vdots$$

$$g_K(\mathbf{x}) = g_K(x_1, \dots, x_K)$$

leads to a diagonal Jacobian

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & 0 & \dots & 0 \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & 0 & \dots \\ \vdots & \dots & & \vdots \\ \frac{\partial g_K}{\partial x_1} & \frac{\partial g_K}{\partial x_2} & \dots & \frac{\partial g_K}{\partial x_K} \end{vmatrix} = \prod_i \frac{\partial g_i}{\partial x_i}$$

A possible mapping (2/2)

The last condition implies

$$\prod_i \frac{\partial g_i}{\partial x_i} = p_{\mathbf{x}}(\mathbf{x}) = p(x_1)p(x_2/x_1) \dots p(x_K/x_1 \dots x_{K-1})$$

A possible solution is:

$$\begin{aligned} g_1(\mathbf{x}) &= F_{x_1}(x_1) \\ g_2(\mathbf{x}) &= F_{x_2/x_1}(x_1, x_2) \\ &\vdots \\ g_K(\mathbf{x}) &= F_{x_K/x_1 \dots x_{K-1}}(x_1, \dots, x_K) \end{aligned}$$

where F_x is the cumulative probability density function of x

A simple example [JBZH04]

Consider 2 independent Gaussian variables x_1 and x_2 with joint pdf

$$p_{\mathbf{x}}(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right)$$

Consider the following mapping and its Jacobian

$$\begin{cases} x_1 = r \cos \theta \\ x_2 = r \sin \theta \end{cases} \quad J = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

The joint pdf of r and θ is

$$p_{r\theta}(r, \theta) = \begin{cases} \frac{r}{2\pi} \exp(-r^2) & \text{if } (r, \theta) \in \mathbb{R}^+ \times [0, 2\pi] \\ 0 & \text{otherwise} \end{cases}$$

Although r and θ are depending both of x_1 and x_2 , they are statistically independent!

Conclusions

General results

- Statistical independence is not sufficient for insuring identifiability of NL mixtures
- For any sources, it exists invertible mappings with non diagonal Jacobian (i.e. mixing mappings) which preserve statistical independence
- if the mapping can be identified, source can be recovered up to a NL mapping (and permutation)

For overcoming the problem,

- one can consider approaches which reduce the set of nontrivial mappings which preserve independence
-

Regularization for avoiding nontrivial mappings preserving independence

Smooth mappings

- Restrict \mathcal{B} to be a smooth mapping [MA99, TWZ01]

Structurally constrained mappings

- Restrict \mathcal{B} by structural constraints: post-nonlinear mixtures [TJ99b], mixtures satisfying addition theorem [KLR73, EK02], bilinear and linear-quadratic mixtures [HD03, DH09]

Priors on sources

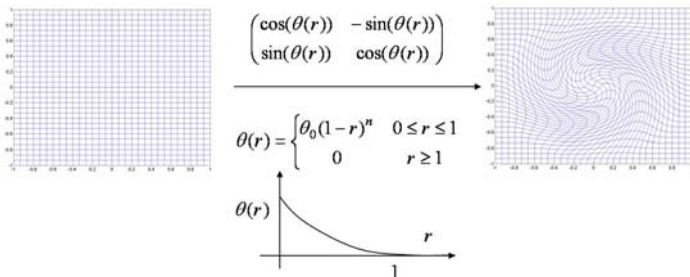
- Consider priors on sources : bounded sources [BZJN02], Markov [LJH04] or colored sources [HJ03]

Smooth mappings

Almeida's claim

- For smooth mappings (e.g. mappings provided by multi-layer perceptrons (MLP)), ICA can lead to separation

Counterexample [JBZH04]



Smoothness of the mappings is not a sufficient prior

Structural constraints: general results

Trivial mappings: definition

- Definition: \mathcal{H} is a trivial mapping if it transforms *any* random vector with independent components in another random vector with independent components.
- The set of the trivial mappings will be denoted \mathcal{Z}

Trivial mappings: properties

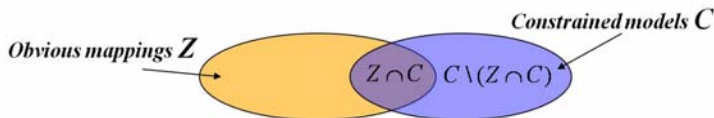
- A trivial mapping is then a mapping preserving independence for *any* random vector
- It can be shown that a trivial mapping satisfies
$$H_i(s_1, \dots, s_{nbs}) = h_i(s_{\sigma(i)}), \forall i = 1, \dots, K$$
- The Jacobian matrix of a trivial mapping is a diagonal matrix up to a permutation

Structural constraints

There is an infinity of nontrivial mappings preserving independence

Constrained model of mixtures

- If the mapping $\mathcal{G} = \mathcal{B} \circ \mathcal{A}$ is constrained to belong to a model \mathcal{C} , the undeterminacies can be reduced, and hopefully cancelled
- Consider $\Omega = \{F_{s_1}, \dots, F_{s_p}\}$, the set of signal distributions such that $\exists \mathcal{G} \in \mathcal{C} - \mathcal{Z}$ which preserves independence for any $\omega \in \Omega$
- Ω then contains all the (particular) source distributions which cannot be separated by mapping belonging to \mathcal{C} .



Separation is then possible

- (1) for source distributions which do not belong to Ω , (2) with

Structural constraints

Case of linear memoryless mixtures

- \mathcal{C} is the set of square matrices
- $\mathcal{Z} \cap \mathcal{C}$ is the set of square matrices which are the product of a diagonal matrix and a permutation matrix
- Ω is the set of distributions which contain at least 2 Gaussian sources (consequence of the Darmois-Skitovich theorem)

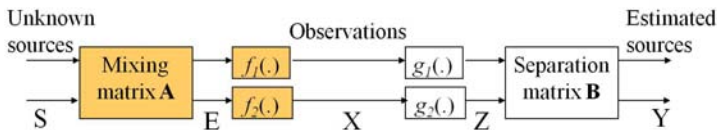
Conclusions

- For linear memoryless mixtures, source separation is possible using statistical independence (1) for sources which are not in Ω (i.e. at most one Gaussian) and (2) with scale and permutation undeterminacies.

Structural constraints: PNL mixtures

Post-nonlinear (PNL) mixtures

- PNL are particular nonlinear mixtures, which structural constraints : linear part, following by NL componentwise mappings.
- PNL are realistic enough : linear channel, nonlinear sensors



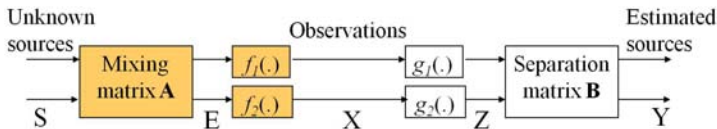
PNL identifiability [TJ99a, AJ05] with suited \mathcal{B}

- if (1) at most one source is Gaussian, (2) the mixing matrix has at least two nonzero entries per row and per column, and (3) the NL mappings f_i are invertible and satisfy $f_i'(0) \neq 0$, then y is independent iff $g_i \circ f_i$ is linear and $\mathbf{BA} = \mathbf{DP}$

Structural constraints: PNL mixtures

Comments

- PNL has the same indeterminacies that linear mixtures !
- It does not work if \mathbf{A} does not mix the sources !
- PNL is a special case of mixtures satisfying addition theorem (see below)



Constrained NL mappings

Extension of the D-S theorem

- D-S theorem has been extended to NL mappings \mathcal{A} , continuous at least separately on each variable, satisfying an *addition theorem* [KLR73]

$$f(x + y) = \mathcal{A}[f(x), f(y)] = f(x) \circ f(y)$$

- Example:

$$\begin{aligned} f(.) &= \tan(.) \\ \mathcal{A}(u, v) &= (u + v)/(1 + uv) \end{aligned} \quad \tan(x+y) = \frac{\tan(x) + \tan(y)}{1 + \tan(x)\tan(y)}$$

- If (E, \circ) is an Abelian group, one can define a multiplication, denoted $*$, and satisfying

$$\begin{aligned} f(cx) &= c * f(x) \quad \text{or} \quad cf^{-1}(u) = f^{-1}(c * u) \\ c_1 f^{-1}(u_1) + \dots + c_k f^{-1}(u_k) &= f^{-1}(c_1 * u_1 \circ \dots \circ c_k * u_k) \end{aligned}$$

Constrained NL mappings

Theorem

- Let X_1, \dots, X_n be independent variables such that

$$\begin{aligned} E_1 &= a_1 * X_1 \circ \dots \circ a_n * X_n \\ E_2 &= b_1 * X_1 \circ \dots \circ b_n * X_n \end{aligned}$$

are independent and where $*$ and \circ satisfy the above conditions, then $f^{-1}(X_i)$ is normally distributed if $a_i b_i \neq 0$.

- The proof is based on applying the D-S theorem to $f^{-1}(E_1)$ and $f^{-1}(E_2)$:

$$f^{-1}(E_1) = f^{-1}(a_1 * X_1 \circ \dots \circ a_n * X_n) = a_1 f^{-1}(X_1) + \dots + a_n f^{-1}(X_n)$$

$$f^{-1}(E_2) = f^{-1}(b_1 * X_1 \circ \dots \circ b_n * X_n) = b_1 f^{-1}(X_1) + \dots + b_n f^{-1}(X_n)$$

Constrained NL mappings

Comments

- These NL mappings are basically linearisable mappings
- PNL are particular NL mappings satisfying the addition theorem

$$\mathcal{A}(u_1, u_2) = \begin{cases} f_1(a_{11}f_1^{-1}(u_1) + a_{12}f_2^{-1}(u_2)) \\ f_2(a_{21}f_1^{-1}(u_1) + a_{22}f_2^{-1}(u_2)) \end{cases}$$

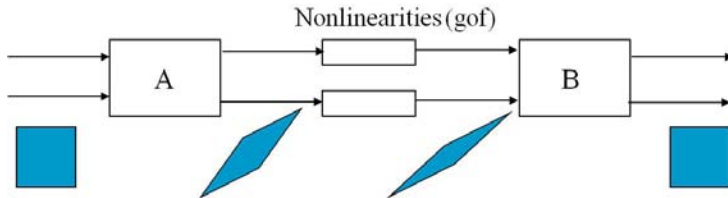
- In fact, denoting $s_1 = f_1^{-1}(u_1)$ and $s_2 = f_2^{-1}(u_2)$, one gets PNL mixtures:

$$\mathcal{A}(f(s_1), f(s_2)) = \begin{cases} f_1(a_{11}s_1 + a_{12}s_2) \\ f_2(a_{21}s_1 + a_{22}s_2) \end{cases}$$

Constraints on sources: bounded sources

PNL with bounded sources [BZJN02]

- Theorem: A component-wise mapping preserves boundary linearity iff it is a linear mapping



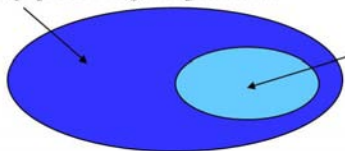
- New proof for identifiability and separability of PNL mixtures, restricted to bounded sources
- Algorithm with independent criteria and estimations for linear and nonlinear parts of PNL separating structure

Constraints on sources: temporally correlated sources

Temporally correlated sources

- Modelled by AR or Markov models [HJ03]
- The set of NL mapping preserving independence is reduced if sources are non iid, due to stronger independence criterion (independence of random processes)

NL mappings preserving independence



NL mappings preserving independence for colored sources

Constrained on sources: temporally correlated sources

Darmon's construction

- For 2 mixtures of 2 sources

$$\begin{cases} y_1(t) = g_1(x_1(t), x_2(t)) = F_{X_1}(x_1(t)) \\ y_2(t) = g_2(x_1(t), x_2(t)) = F_{X_2/X_1}(x_1(t), x_2(t)) \end{cases}$$

- If $y_1(t)$ and $y_2(t)$ are independent,
 $p_{Y_1 Y_2}(y_1(t), y_2(t)) = p_{Y_1}(y_1(t))p_{Y_2}(y_2(t))$
- Since sources are temporally correlated, one can prove that generally:

$$\begin{aligned} p_{Y_1 Y_2}(y_1(t+1), y_2(t)) &\neq p_{Y_1}(y_1(t+1))p_{Y_2}(y_2(t)) \\ \text{i.e. } p(F_{X_1}(x_1(t+1)), F_{X_2/X_1}(x_1(t), x_2(t))) &\neq \\ p(F_{X_1}(x_1(t+1)))p(F_{X_2/X_1}(x_1(t), x_2(t))) & \end{aligned}$$

Constrained on sources: temporally correlated sources

After some computations, one can show the relation becomes:

$$\int p_{A/BC}(a, b, c)p_B(b)db \neq p_A(a)$$

- In fact, left-side is a function of variables a and c , while right side only depends on b
- Consequently, the above mapping is generally not a mixing mappings preserving independence for colored sources.
- The set of mixing mappings preserving independence is then reduced for colored sources.

Conclusions

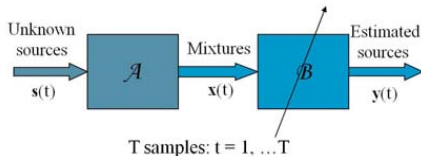
- For NL mixtures, generally independence does not insure identifiability and separability
- Regularization is required for reducing solutions to trivial mappings:
 - Generally, smoothness is not a sufficient constraint
 - Structural constraints leads to particular NL identifiable and separable mixtures
 - Priors on the sources can reduce the set of solutions
- Extention to NL ICA
- Are NL mixtures useful ? Theoretical curiosity or practical applications ?

Mutual information in NL mixtures

Mutual information and its derivative

- $I(\mathbf{Y}) = \sum_i H(Y_i) - H(\mathbf{Y}) = \sum_i H(Y_i) - H(\mathbf{X}) + E[\log |\det J_B|]$
- Estimating equation: $\frac{\partial I(\mathbf{Y})}{\partial \Theta} = 0$, where Θ represents the parameter vector
- It then leads to the following equations:

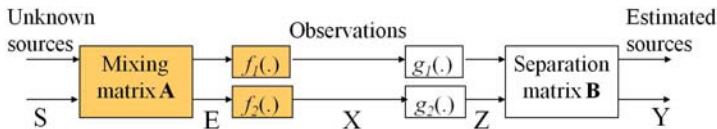
$$\begin{aligned} \frac{\partial I(\mathbf{Y})}{\partial \Theta} &= \sum_i \frac{\partial H(Y_i)}{\partial \Theta} + \frac{\partial E[\log |\det J_B|]}{\partial \Theta} \\ &= \sum_i E\left[\frac{d \log p_{Y_i}(y_i)}{dy_i} \frac{\partial y_i}{\partial \Theta}\right] + \frac{\partial E[\log |\det J_B|]}{\partial \Theta} = 0 \end{aligned}$$



Mutual information for PNL mixtures

Mutual information

$$\begin{aligned}
 I(\mathbf{Y}) &= \sum_i H(Y_i) - H(\mathbf{Y}) \\
 &= \sum_i H(Y_i) - H(\mathbf{X}) + E[\log |\det J_{\mathcal{G}}|] + E[\log |\det J_{\mathbf{B}}|] \\
 &= \sum_i H(Y_i) - H(\mathbf{X}) + E\left[\log \left| \prod_i \frac{\partial g_i(x_i, \theta_i)}{\partial x_i} \right| \right] + \log |\det \mathbf{B}|
 \end{aligned} \tag{45}$$



Mutual information for NL mixtures

Derivatives of the mutual information

- With respect to \mathbf{B}

$$\begin{aligned} \frac{\partial I(\mathbf{Y})}{\partial \mathbf{B}} &= \sum_i \frac{\partial H(Y_i)}{\partial \mathbf{B}} - \frac{\partial}{\partial \mathbf{B}} E \left[\log \left| \prod_i \frac{\partial g_i(x_i, \theta_i)}{\partial x_i} \right| \right] - \frac{\partial \log |\det \mathbf{B}|}{\partial \mathbf{B}} = 0 \\ &= \sum_i E \left[\frac{d \log p_{Y_i}(y_i)}{dy_i} \frac{\partial y_i}{\partial \mathbf{B}} \right] - \mathbf{B}^T = 0 \end{aligned}$$

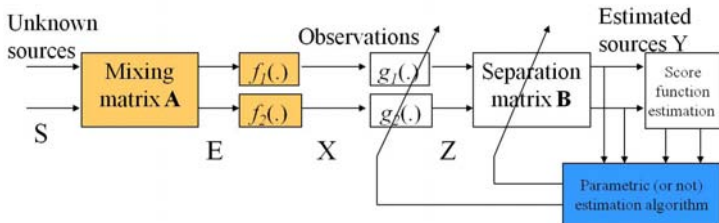
- With respect to parameter of g_i 's

$$\begin{aligned} \frac{\partial I(\mathbf{Y})}{\partial \theta_k} &= \sum_i \frac{\partial H(Y_i)}{\partial \theta_k} - \frac{\partial}{\partial \theta_k} E \left[\log \left| \prod_i \frac{\partial g_i(x_i, \theta_i)}{\partial x_i} \right| \right] - \frac{\partial \log |\det \mathbf{B}|}{\partial \theta_k} = 0 \\ &= \sum_i E \left[\frac{d \log p_{Y_i}(y_i)}{dy_i} \frac{\partial y_i}{\partial \theta_k} \right] - E \left[\frac{\partial \log |g'_i(x_i, \theta_i)|}{\partial \theta_k} \right] = 0 \end{aligned}$$

Separation of PNL mixtures

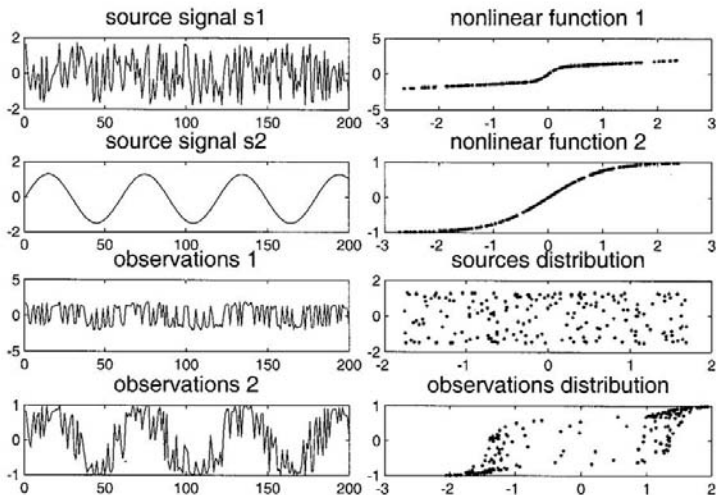
The algorithm consists in 3 steps:

- estimation of marginal score functions of estimated sources,
- estimation of the nonlinear functions g_i ,
- estimation of the separating matrix \mathbf{B} ,



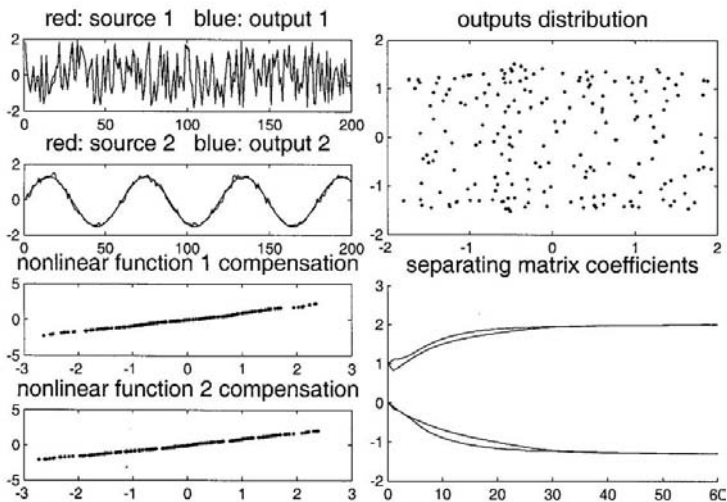
Separation of PNL mixtures

Sources and mixtures



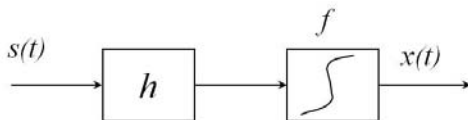
Separation of PNL mixtures

Estimated sources

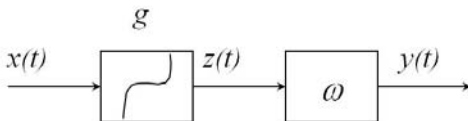


PNL mixtures and NL deconvolution

Wiener systems



Hammerstein systems



Blind inversion of Wiener system

Wiener system is a usual NL model in biology, in satellite communications, etc.

Classical approaches

- Classical identification methods for nonlinear systems are based on higher-order cross-correlations
- Usually, input signal is assumed to be iid Gaussian
- If the distortion input is available, the compensation of the nonlinearities is almost straightforward, after identification of the NL
- However, in a real world situation, we don't know either the nonlinear system input or the input distribution

Blind inversion of Wiener system

Parameterization

- source: $\mathbf{S}(t) = [\dots s(t-k+1)s(t-k)s(t-k-1)\dots]^T$
- observation: $\mathbf{X}(t) = [\dots x(t-k+1)x(t-k)x(t-k-1)\dots]^T$
- channel:

$$H = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & h(k-1) & h(k) & h(k+1) & \dots \\ \dots & h(k-2) & h(k-1) & h(k) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

- Due to iid assumption, \mathbf{S} has independent components, Wiener systems is nothing but an infinite-dimension PNL mixture, $\mathbf{X}(t) = f[\mathbf{H}\mathbf{S}(t)]$, with a particular (Toeplitz) mixing matrix H .

Blind inversion of Wiener system

Identifiability

- If the Wiener system satisfies:
 - Subsystems h and f are unknown and invertible ; h can be a nonminimum phase filter
 - The input $s(t)$ is an unknown (a priori) non Gaussian iid process

it is a PNL mixtures, with a particular Toeplitz mixture matrix \mathbf{H} and with the same NL function f on each channel.
- PNL identifiability implies Wiener systems inversibility based on (temporal) sample statistical independence.
- PNL separability is only proved for finite dimensions. It is conjectured for infinite dimensions.
- Practically, the filter $h(k)$ and its inverse $w(k)$ are truncated.

Blind inversion of Wiener system

Criterion for inversion

- Output of the inversion (Hammerstein) structure:

$$y(t) = w * z(t) \text{ with } z(t) = g(x(t))$$

- $\mathbf{Y}(t)$ spatially independent \Leftrightarrow sequence $\{y(t)\}$ iid
- The mutual information for infinite dimensional stationary random vectors is defined from entropy rates [CT91]:

$$H(\mathbf{Y}) = \lim_{T \rightarrow \infty} \frac{1}{2T+1} H(y(-T), \dots, y(T))$$

$$I(\mathbf{Y}) = \lim_{T \rightarrow \infty} \frac{1}{2T+1} \left\{ \sum_{t=-T}^{+T} H(y(t)) - H(y(-T), \dots, y(T)) \right\}$$

- $I(\mathbf{Y})$ is always positive and vanishes if and only if $\{y(t)\}$ is an iid sequence
- For more details, see [TSJ01]

Other approaches for NL mixtures

Algorithms for PNL

- Relatively many algorithms have been designed for PNL mixtures. See a few references in [CJ10, ch.14].

Methods for smooth mappings

- MISEP proposed by Marques and Almeida [MA99]: the nonlinearity is modelled by multi-layer perceptrons (MLP) trained for insuring output independence.
- Tan et al. [TWZ01] used radial basis functions.

Bayesian approaches

- Nonlinear Factor Analysis (NFA) proposed by Helsinki team (Valpola, Karhunen, Honkela, etc.). See details and references in [CJ10, ch.14]
- Bayesian approach also developed by Duarte et al. [DJM10] for chemical NL mixtures, with positivity and other constraints

NL mixtures: conclusions and perspectives

Conclusions

- Generally, independence is not sufficient for insuring identifiability and separability
- Independence can insure identifiability and separability in structured NL mixtures. PNL mixtures are particular separable NL mixtures.
- Deconvolution and Wiener system inversion are related to BSS: time independence (iid) is then related to spatial independence.
- Mutual information or Information rate can then be used for measuring spatial independence or time independence (iid)
- Independence is powerful enough for blindly compensate strong NL distortions (1) in PNL multichannel systems: satellite antenna, sensors array, etc., or (2) in Wiener systems (NL dynamic SISO)

NL mixtures: conclusions and perspectives

Other contributions

- Extension to MIMO dynamic NL systems: Convolutional PNL mixtures [BZJN01]
- Bilinear and multi-linear models [HD03]
- Separation of NL mixtures with positivity constraints [DJM10]
- Practical applications for chemical sensor arrays [DJM10, DJTB⁺10] or scanned image processing [Alm05, MBBZJ08] (see in 339 of this course)

Perspectives and open issues

- Noisy NL mixtures (problem of noise amplification)
- Identifiability for bilinear and multi-linear models
- Identifiability based on other priors, criteria and mixing structures
- NL dynamic problem, e.g. encountered in gas sensor arrays

VIII. UnderDetermined mixtures



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28 Problem formulation

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- Cumulants

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- Symmetric tensors of dim. 2
- BIOME
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- Deterministic approaches

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Characteristic functions

Again

First c.f.

Characteristic functions

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First c.f.

- Real Scalar: $\Phi_x(t) \stackrel{\text{def}}{=} E\{e^{jt^x}\} = \int_u e^{jt^u} dF_x(u)$

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Second c.f.

- $\Psi(\mathbf{t}) \stackrel{\text{def}}{=} \log \Phi(\mathbf{t})$
- Properties:
 - Always exists in the neighborhood of 0
 - Uniquely defined as long as $\Phi(\mathbf{t}) \neq 0$

Characteristic functions (cont'd)

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 - if a c.f. $\Psi_x(t)$ is a polynomial, then its degree is at most 2 and x is Gaussian (*Marcinkiewicz'1938*) [Luka70]

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Proof.

$$\begin{aligned} \Psi_{x,y}(u, v) &= \log[\mathbb{E}\{\exp(ux + vy)\}] \\ &= \log[\mathbb{E}\{\exp(ux)\} \mathbb{E}\{\exp(vy)\}]. \end{aligned}$$



Problem posed in terms of Characteristic Functions

- If s_p independent and $\mathbf{x} = \mathbf{A} \mathbf{s}$, we have the *core equation*:

$$\Psi_{\mathbf{x}}(\mathbf{u}) = \sum_p \Psi_{s_p} \left(\sum_q u_q A_{qp} \right) \quad (47)$$

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- Taking the log concludes.

□

Problem: Decompose a multivariate function into a sum of univariate ones

Darמוש-Skitovich theorem (1953)

Theorem

Let s_i be statistically *independent* random variables, and two linear statistics:

$$y_1 = \sum_i a_i s_i \quad \text{and} \quad y_2 = \sum_i b_i s_i$$

If y_1 and y_2 are statistically independent, then random variables s_k for which $a_k b_k \neq 0$ are Gaussian.

NB: holds in both \mathbb{R} or \mathbb{C}

Sketch of proof

Let characteristic functions

$$\Psi_{1,2}(u, v) = \log \mathbb{E}\{\exp(j y_1 u + j y_2 v)\}$$

$$\Psi_k(w) = \log \mathbb{E}\{\exp(j y_k w)\}$$

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1 Independence between s_p 's implies:

$$\Psi_{1,2}(u, v) = \sum_{k=1}^P \varphi_k(u a_k + v b_k)$$

$$\Psi_1(u) = \sum_{k=1}^P \varphi_k(u a_k)$$

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2 Independence between y_1 and y_2 implies

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Does not restrict generality to assume that $[a_k, b_k]$ not collinear. To simplify, assume also φ_p differentiable.

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From now on, restrict the sum to terms $a_k b_k \neq 0$

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- 4** Write this at $u + \alpha/a_p$ and $v - \alpha/b_p$:

$$\sum_{k=1}^P \varphi_k \left(u a_k + v b_k + \alpha \left(\frac{a_k}{a_p} - \frac{b_k}{b_p} \right) \right) = f(u) + g(v)$$

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- 5 Subtract to cancel P th term, divide by α , and let $\alpha \rightarrow 0$:

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for some *univariate functions* $f^{(1)}(u)$ and $g^{(1)}(u)$.

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Conclusion: We have one term less

6 Repeat the procedure $(P - 1)$ times and get:

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NB: also holds if φ_p not differentiable

Equivalent representations

Let \mathbf{y} admit two representations

$$\mathbf{y} = \mathbf{A} \mathbf{s} \quad \text{and} \quad \mathbf{y} = \mathbf{B} \mathbf{z}$$

where \mathbf{s} (resp. \mathbf{z}) have independent components, and \mathbf{A} (resp. \mathbf{B}) have pairwise noncollinear columns.

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- These representations are *equivalent* if every column of \mathbf{A} is proportional to some column of \mathbf{B} , and vice versa.
- If all representations of \mathbf{y} are equivalent, they are said to be *essentially unique* (permutation & scale ambiguities only).

Identifiability & uniqueness theorems s

Let \mathbf{y} be a random vector of the form $\mathbf{y} = \mathbf{A} \mathbf{s}$, where s_p are independent, and \mathbf{A} has non pairwise collinear columns.

- **Identifiability theorem** \mathbf{y} can be represented as $\mathbf{y} = \mathbf{A}_1 \mathbf{s}_1 + \mathbf{A}_2 \mathbf{s}_2$, where \mathbf{s}_1 is non Gaussian, \mathbf{s}_2 is Gaussian independent of \mathbf{s}_1 , and \mathbf{A}_1 is essentially unique.
- **Uniqueness theorem** If in addition the columns of \mathbf{A}_1 are linearly independent, then the distribution of \mathbf{s}_1 is unique up to scale and location indeterminacies.

Remark 1: if \mathbf{s}_2 is 1-dimensional, then \mathbf{A}_2 is also essentially unique

Remark 2: the proofs are not constructive [KLR73]

Example of uniqueness s

Let s_i be independent with no Gaussian component, and b_i be independent Gaussian. Then the linear model below is identifiable, but \mathbf{A}_2 is not essentially unique whereas \mathbf{A}_1 is:

$$\begin{pmatrix} s_1 + s_2 + 2b_1 \\ s_1 + 2b_2 \end{pmatrix} = \mathbf{A}_1 \mathbf{s} + \mathbf{A}_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{A}_1 \mathbf{s} + \mathbf{A}_3 \begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix}$$

with

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hence the distribution of \mathbf{s} is essentially unique.

But $(\mathbf{A}_1, \mathbf{A}_2)$ not equivalent to $(\mathbf{A}_1, \mathbf{A}_3)$.

Example of non uniqueness s

Let s_i be independent with no Gaussian component, and b_i be independent Gaussian. Then the linear model below is identifiable, but the distribution of s is not unique because a 2×4 matrix cannot be full column rank:

$$\begin{pmatrix} s_1 + s_3 + s_4 + 2b_1 \\ s_2 + s_3 - s_4 + 2b_2 \end{pmatrix} = \mathbf{A} \begin{pmatrix} s_1 \\ s_2 \\ s_3 + b_1 + b_2 \\ s_4 + b_1 - b_2 \end{pmatrix} = \mathbf{A} \begin{pmatrix} s_1 + 2b_1 \\ s_2 + 2b_2 \\ s_3 \\ s_4 \end{pmatrix}$$

with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

CP of Cumulant tensor

If we take the d th derivative of the *core equation* (47) at $\mathbf{u} = 0$, we get a relation between cumulants of order d .

For instance at order $d = 4$ in the real case:

$$C_{x,ijkl} = \sum_p A_{ip} A_{jp} A_{kp} A_{lp} \kappa_p \quad (48)$$

➡ This can be noticed to be of the same form as the CP (20) in slides 55-60.

Link with polynomials

- **Symmetric tensors of order d and dim N** are bijectively mapped to homogeneous polynomials of degree d in N variables. For instance if $d = 3$:

$$p(\mathbf{x}) = \sum_{ijk} T_{ijk} x_i x_j x_k$$

- **Third order tensors of dimension $I \times J \times K$** are bijectively related to trilinear forms in $I + J + K$ variables:

$$p(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i,j,k} T_{ijk} x_i y_j z_k$$

- ➔ Hence, decomposing a polynomial is equivalent to decomposing a tensor.

Tenseurs symétriques de dim.2

Binary forms of degree d



James Joseph Sylvester (1814–1897)

Sylvester's theorem

Sylvester's theorem in \mathbb{R} (1886)

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- A binary quantic $t(x_1, x_2) = \sum_{i=0}^d c(i) \gamma_i x_1^i x_2^{d-i}$ can be written in $\mathbb{R}[x_1, x_2]$ as a sum of d th powers of r distinct linear forms:

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Proof of Sylvester's theorem (1)

Lemma

- For homogeneous polynomials of degree d parameterized as $p(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{|\mathbf{i}|=d} c(\mathbf{i}) \gamma(\mathbf{i}; p) \mathbf{x}^{\mathbf{i}}$, define the *apolar scalar product*:

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- Then $L(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{a}^T \mathbf{x} \Rightarrow \langle p, L^d \rangle = \sum_{|\mathbf{i}|=d} c(\mathbf{i}) \gamma(\mathbf{i}; p) \mathbf{a}^{\mathbf{i}} = p(\mathbf{a})$

Proof of Sylvester's theorem (2)

- 1** Assume the r distinct linear forms $L_j = \alpha_j x_1 + \beta_j x_2$ are given.
Let $q(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{j=1}^r (\beta_j x_1 - \alpha_j x_2)$. Then $q(\alpha_j, \beta_j) = 0, \forall j$.

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- 5 Reasoning goes also backwards

Decomposition of maximal rank: $x_1 x_2^{d-1}$

- 1** Maximal rank $r = d$ when (49) reduces to a 1-row matrix:

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- 4 Compute λ_j 's by solving the Van der Monde linear system:

$$\begin{bmatrix} 1 & \dots & 1 \\ \beta_1 & \dots & \beta_d \\ \beta_1^2 & \dots & \beta_d^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \dots & \beta_d^d \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Summary

Algorithm

- Sylvester's theorem allows to compute the decomposition, even in non generic cases: [BCMT10] [CGLM08] [CM96].

Rank of binary quantics

- A binary quantic of odd degree $2n + 1$ has *generic rank* $n + 1$
- A binary quantic of even degree $2n$ *generic rank* $n + 1$
- A binary quantic of degree d may reach *maximal rank* d .
Orbit of maximal rank is $x_1 x_2^{d-1}$.

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- In that case, we have

$$\mathcal{C}_{x,ijk}^{\ell mn} = \sum_{p=1}^P A_{ip} A_{jp} A_{kp} A_{\ell p}^* A_{mp}^* A_{np}^* \Delta_p$$

where $\Delta_p \stackrel{\text{def}}{=} \text{Cum}\{s_p, s_p, s_p, s_p^*, s_p^*, s_p^*\}$ denote the diagonal entries of a $P \times P$ diagonal matrix, $\mathbf{\Delta}^{(6)}$

Writing in matrix form

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- **Idea:** Use redundancy existing between blocks of $(\mathbf{C}_x^{(6)})^{1/2}$.

Using the invariance to estimate \mathbf{V}

- 1** Cut the $K^3 \times P$ matrix $(\mathbf{C}_x^{(6)})^{1/2}$ into K blocks of size $K^2 \times P$. Each of these blocks, $\mathbf{\Gamma}[n]$, satisfies:

$$\mathbf{\Gamma}[n] = (\mathbf{A} \odot \mathbf{A}^H) \mathbf{D}[n] (\mathbf{\Delta}^{(6)})^{1/2} \mathbf{V}$$

where $\mathbf{D}[n]$ is the $P \times P$ diagonal matrix containing the n th row of \mathbf{A} , $1 \leq n \leq K$.

Hence matrices $\mathbf{\Gamma}[n]$ share the same common right singular space

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- 2 Compute the joint EVD of the $K(K-1)$ matrices

$$\mathbf{\Theta}[m, n] \stackrel{\text{def}}{=} \mathbf{\Gamma}[m]^\dagger \mathbf{\Gamma}[n]$$

as: $\mathbf{\Theta}[m, n] = \mathbf{V} \mathbf{\Lambda}[m, n] \mathbf{V}^H$.

Estimation of \mathbf{A}

Matrices $\mathbf{\Lambda}[m, n]$ cannot be used directly because $(\mathbf{\Delta}^{(6)})^{1/2}$ is unknown. But we use \mathbf{V} to obtain the estimate of $\mathbf{A}^{\odot 3}$ up to a scale factor:

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Estimation of \mathbf{A}

Matrices $\mathbf{\Lambda}[m, n]$ cannot be used directly because $(\mathbf{\Delta}^{(6)})^{1/2}$ is unknown. But we use \mathbf{V} to obtain the estimate of $\mathbf{A}^{\odot 3}$ up to a scale factor:

$$\widehat{\mathbf{A}^{\odot 3}} = (\mathbf{C}_x^{(6)})^{1/2} \mathbf{V} \quad (53)$$

One possibility to get \mathbf{A} from $\mathbf{A}^{\odot 3}$ is as follows:

- 3 Build K^2 matrices $\Xi[m]$ of size $K \times P$ form rows of $\widehat{\mathbf{A}^{\odot 3}}$
- 4 From $\Xi[m]$ find $\widehat{\mathbf{A}}$ and diagonal matrices $\mathbf{D}[m]$, in the LS sense:

$$\Xi[m] \mathbf{D}[m] \approx \widehat{\mathbf{A}}, \quad 1 \leq m \leq K^2$$

Estimation of \mathbf{A} (details) s

Stationary values of criterion $\sum_{m=1}^M \|\Xi_m \mathbf{D}_m - \mathbf{A}\|_F^2$, $M \stackrel{\text{def}}{=} K^2$, yield the solution below

- Obtain vectors $\mathbf{d}_p \stackrel{\text{def}}{=} [\mathbf{D}_1(p, p), \mathbf{D}_2(p, p), \dots, \mathbf{D}_M(p, p)]^T$, by solving the linear systems:

$$\mathbf{F}_p \mathbf{d}_p = 0$$

where matrices \mathbf{F}_p are defined as

$$\mathbf{F}_p(m_1, m_2) = \begin{cases} (M-1) \{\Xi_{m_1}^H \Xi_{m_1}\}(p, p) & \text{if } m_1 = m_2 \\ -\{\Xi_{m_1}^H \Xi_{m_2}\}(p, p) & \text{otherwise} \end{cases}$$

- Deduce the estimate $\hat{\mathbf{A}} = \frac{1}{M} \sum_{m=1}^M \Xi_m \mathbf{D}_m$

Conditions of identifiability of BIOME($2r$) s

- Source cumulants of order $2r > 4$ are nonzero and have the same sign
- Columns vectors of mixing matrix \mathbf{A} are not collinear
- Matrix $\mathbf{A}^{\odot(r-1)}$ is full column rank.
- This last condition implies that tensor rank must be at most K^{r-1} (e.g. $P \leq K^2$ for order $2r = 6$).

FOOBI algorithms

Again same problem: Given a $K^2 \times P$ matrix $\mathbf{A}^{\odot 2}$, find a real orthogonal matrix \mathbf{Q} such that the P columns of $\mathbf{A}^{\odot 2} \mathbf{Q}$ are of the form $\mathbf{a}[p] \otimes \mathbf{a}[p]^*$

FOOBI algorithms

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- **FOOBI:** use the K^4 determinantal equations characterizing rank-1 matrices $\mathbf{a}[p] \mathbf{a}[p]^H$ of the form:

$$\phi(\mathbf{X}, \mathbf{Y})_{ijkl} = x_{ij}y_{lk} - x_{ik}y_{lj} + y_{ij}x_{lk} - y_{ik}x_{lj}$$

FOOBI algorithms

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- **FOOBI2:** use the K^2 equations of the form:
 $\Phi(\mathbf{X}, \mathbf{Y}) = \mathbf{X} \mathbf{Y} + \mathbf{Y} \mathbf{X} - \text{trace} \mathbf{X} \mathbf{Y} - \text{trace} \mathbf{Y} \mathbf{X}$
 where matrices \mathbf{X} and \mathbf{Y} are $K \times K$ Hermitian.

FOOBI s

- 1 Normalize the columns of the $K^2 \times P$ matrix $\mathbf{A}^{\odot 2}$ such that matrices $\mathbf{A}[r] \stackrel{\text{def}}{=} \mathbf{Unvec}_K(\mathbf{a}^{\odot 2}[r])$ are Hermitian
- 2 Compute the $K^2(K-1)^2 \times P(P-1)/2$ matrix \mathbf{P} defined by $\phi(\mathbf{A}[r], \mathbf{A}[s])$, $1 \leq r \leq s \leq P$.
- 3 Compute the P weakest right singular vectors of \mathbf{P} , Unvec them and store them in P matrices $\mathbf{W}[r]$
- 4 Jointly diagonalize $\mathbf{W}[r]$ by a real orthogonal matrix \mathbf{Q}
- 5 Then compute $\mathbf{F} \stackrel{\text{def}}{=} (\mathbf{C}_x^{(4)})^{1/2} \mathbf{\Delta Q}$ and deduce $\hat{\mathbf{a}}[r]$ as the dominant left singular vectors of $\mathbf{Unvec}(\mathbf{f}[r])$.

FOOBI2 s

- 1 Normalize the columns of the $K^2 \times P$ matrix $\mathbf{A}^{\odot 2}$ such that matrices $\mathbf{A}[r] \stackrel{\text{def}}{=} \mathbf{Unvec}_K(\mathbf{a}^{\odot 2}[r])$ are Hermitian
- 2 Compute the $K(K+1)/2$ Hermitian matrix $\mathbf{B}[r, s]$ of size $P \times P$ defined by:

$$\Phi(\mathbf{A}[r], \mathbf{A}[s])|_{ij} \stackrel{\text{def}}{=} \mathbf{B}[i, j]|_{rs}$$

- 3 Jointly cancel diagonal entries of matrices $\mathbf{B}[i, j]$ by a real congruent orthogonal transform \mathbf{Q}
- 4 Then compute $\mathbf{F} \stackrel{\text{def}}{=} (\mathbf{C}_x^{(4)})^{1/2} \mathbf{\Delta Q}$ and deduce $\hat{\mathbf{a}}[r]$ as the dominant left singular vectors of $\mathbf{Unvec}(\mathbf{f}[r])$.

NB: Better bound than FOOBI and BIOME(4), but iterative algorithm sensitive to initialization

Algorithms based on characteristic functions

Fit with a model of exact rank

1 Back to the core equation (47):

$$\psi_x(\mathbf{u}) = \sum_p \psi_{s_p} \left(\sum_q u_q A_{qp} \right)$$

Algorithms based on characteristic functions

Fit with a model of exact rank

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$$\Psi_x(\mathbf{u}) = \sum_p \Psi_{s_p} \left(\sum_q u_q A_{qp} \right)$$

- 2 **Goal:** Find a matrix \mathbf{A} such that the K -variate function $\Psi_x(\mathbf{u})$ decomposes into a sum of P univariate functions $\psi_p \stackrel{\text{def}}{=} \Psi_{s_p}$.

Algorithms based on characteristic functions

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- 3 **Idea:** Fit both sides on a grid of values $\mathbf{u}[\ell] \in \mathcal{G}$

Equations derived from the CAF

- Assumption: functions ψ_p , $1 \leq p \leq P$ admit finite derivatives up to order r in a neighborhood of the origin, containing \mathcal{G} .

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- Then, Taking $r = 3$ as a working example:

$$\frac{\partial^3 \Psi_x}{\partial u_i \partial u_j \partial u_k}(\mathbf{u}) = \sum_{p=1}^P A_{ip} A_{jp} A_{kp} \psi_p^{(3)}\left(\sum_{q=1}^K u_q A_{qp}\right)$$

Equations derived from the CAF

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- If $L > 1$ point in grid \mathcal{G} , then yields another mode in tensor

Putting the problem in tensor form

- A decomposition into a sum of rank-1 terms:

$$T_{ijkl} = \sum_p A_{ip} A_{jp} A_{kp} B_{lp}$$

or equivalently

$$\mathbf{T} = \sum_p \mathbf{a}(p) \otimes \mathbf{a}(p) \otimes \mathbf{a}(p) \otimes \mathbf{b}(p)$$

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or equivalently

$$\mathbf{T} = \sum_p \mathbf{a}(p) \otimes \mathbf{a}(p) \otimes \mathbf{a}(p) \otimes \mathbf{b}(p)$$

- Tensor \mathbf{T} is $K \times K \times K \times L$, symmetric in all modes except the last.

Joint use of different derivative orders

Example [CR06]

- Derivatives of order 3:

$$T_{ijkl}^{(3)} = \sum_p A_{ip} A_{jp} A_{kp} B_{lp}$$

Joint use of different derivative orders

Example [CR06]

- Derivatives of order 3:

$$T_{ijkl}^{(3)} = \sum_p A_{ip} A_{jp} A_{kp} B_{lp}$$

- Derivatives of order 4:

$$T_{ijkml}^{(4)} = \sum_p A_{ip} A_{jp} A_{kp} A_{mp} C_{lp}$$

Joint use of different derivative orders

Example [CR06]

- Derivatives of order 3:

$$T_{ijkl}^{(3)} = \sum_p A_{ip} A_{jp} A_{kp} B_{lp}$$

- Derivatives of order 4:

$$T_{ijkml}^{(4)} = \sum_p A_{ip} A_{jp} A_{kp} A_{mp} C_{lp}$$

- Derivatives of orders 3 and 4:

$$T_{ijkl}[m] = \sum_p A_{ip} A_{jp} A_{kp} D_{lp}[m]$$

with $D_{lp}[m] = A_{mp} C_{lp}$ and $D_{lp}[0] = B_{lp}$.

Iterative algorithms

Goal: Minimize $\|\mathbf{T} - \sum_{p=1}^r \mathbf{u}_p \otimes \mathbf{v}_p \otimes \dots \otimes \mathbf{w}_p\|^2$

- Gradient with fixed or variable (ad-hoc) stepsize
- Alternating Least Squares (ALS)
- Levenberg-Marquardt
- Newton
- Conjugate gradient
- ...

Remarks

- Hessian is generally huge, but sparse
- Problem of *local minima*: ELS variants for all of the above

Warning: special constraints needed to ensure existence of best low rank approximate

Tensor package

- Matlab code are freely downloadable at:
<http://www.i3s.unice.fr/~pcomon/TensorPackage.html>
- Codes include:
 - Alternating Least Squares (ALS) at any order
 - Gradient
 - Quasi-Newton (BFGS update)
 - Levenberg-Marquardt
 - Conjugate gradient
 - Enhanced Line Search (ELS) for any of the above
- Special codes for NonNegative tensors

Non symmetric tensors

Tensors with partial symmetries, or no symmetry at all, often need to be approximated by decompositions of lower rank:

- 1 Characteristic functions: one mode corresponds to \mathbf{u}
- 2 INDSCAL: tensors with positive definite matrix slices
- 3 **Deterministic approaches:** no symmetry at all (cf. course on Applications)

Bibliography

- Existence: [LC09] [CGLM08] [Ste08] [CL11] [LC10]
- Uniqueness: [SB00] [BS02]
- Computation: [AFCC04] [Ste06] [RCH08] [CL13] [Paa99]
[Paa97]
- General: [SBG04] [CLDA09]

Sparse Component Analysis

- Review paper (see [GL06] and Chapter 10 in [CJ10]),

Main ideas

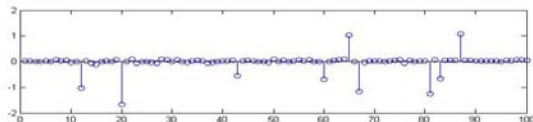
- Initial underdetermined source separation problem:
 $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t)$, nbr sources $P \gg K$ nbr d'observations
- Transform with a sparsifying mapping \mathcal{T} , which preserves linearity (e.g. wavelet transform, ST Fourier transform, etc.):

$$\tilde{\mathbf{x}}(t) = \mathcal{T}(\mathbf{x}(t)) = \mathcal{T}(\mathbf{A}\mathbf{s}(t)) = \mathbf{A}\mathcal{T}(\mathbf{s}(t)) = \mathbf{A}\tilde{\mathbf{s}}(t)$$

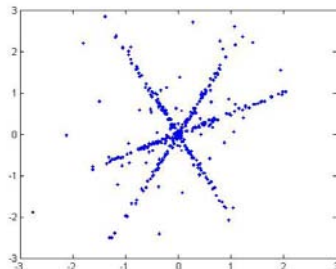
- Solve the source separation problem in the sparse space: $\hat{\tilde{\mathbf{s}}}(t)$
- Come back to the initial space, with inverse of \mathcal{T} :
 $\hat{\mathbf{s}}(t) = \mathcal{T}^{-1}(\hat{\tilde{\mathbf{s}}}(t))$

Examples of sources and mixtures

- Sources may be sparse in time or in any domain, preserving the linear nature of mixtures



- 2 mixtures of 3 sparse sources



Solving the under-determined source separation problem

Since \mathbf{A} is non invertible, indirect approach is not possible. One has to consider (1) identification of \mathbf{A} and then (2) restauration of \mathbf{s} , i.e.:

A 3-step approach

- Step 0 (representation space): if necessary apply sparsifying mapping \mathcal{T}
- Step 1 (identification): Estimate \mathbf{A} , e.g. by clustering.
- Step 2 (source restoration): At each instant n_0 , find the sparsest solution of

$$\mathbf{A}\mathbf{s}(n_0) = \mathbf{x}(n_0), n_0 = 0, \dots, N$$

Main question: HOW to find the sparsest solution of an Underdetermined System of Linear Equations (USLE)?

Uniqueness of the sparsest solution

- $\mathbf{x} = \mathbf{A}\mathbf{s}$, K equations (observations), P unknowns (sources):
 $P \gg K$
- Question: Is the sparse solution unique?
- *Theorem* [Don04, GN03]: If there is a solution $\hat{\mathbf{s}}$ with less than $K/2$ non-zero components, then it is unique with probability 1 (that is, for almost all matrices \mathbf{A}).

How to find the sparsest solution ?

- $x = \mathbf{A}s$, K equations (observations), P unknowns (sources):
 $P \gg K$
- Goal: Finding the sparsest solution (at least $P - K$ sources are zero)
- Direct method (Minimum L0 norm method):

$$\min \|s\|_0 \quad \text{s.t.} \quad x = \mathbf{A}s$$

- 1 Set $P - K$ (arbitrary) sources equal to zero
- 2 Solve the remaining system of K equations and K unknowns
- 3 Repeat steps 1 and 2 for all possible choices, and take the sparsest answer.

Not tractable

- Example: $P = 50$, $K = 30$, then $C_P^K \approx 5 \times 10^{13}$
- On a basic PC, 30×30 system of equations : $2 \times 10^{-4}s$
- Total time: $(5 \times 10^{13}) \times (2 \times 10^{-4}s) = 300$ years!

A few faster algorithms

- Method of Frames (MoF) [DDD04]: linear method, fast but not accurate
- Matching Pursuit [MZ93]: greedy method, fast but not accurate
- Basis Pursuit (minimal L_1 norm, Linear Programming) [CDS99]: accurate and tractable method, but still time consuming
- Smoothed L_0 [MBZJ09]: fast and accurate, based on GNC principle

Theorem [DE03]: under some (sparsity) conditions: $\operatorname{argmin} L_1 = \operatorname{argmin} L_0$

Applications of SCA

Among many others, applications in

- Audio and music (see the lecture of L. Daudet)
- Astrophysics (see the lecture of Starck)

IX. NonNegative Mixtures



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- Titre sous-section

33 Non-negative Matrix Factorization

- Condition for NN ICA
- Algorithm principles
- Discussion

34 NonNegative Tensor Factorization

Why non negative mixture ?

- In many real-world applications, sources are non-negative (NN): images, chemical concentrations, note or speech amplitude in audio, etc.
- In addition, mixtures can sometimes be non-negative too: superposition of images, mixtures of chemical molecules or ions, etc.

NN for linear source separation problem

- Mixing equation $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t)$; $t = 1, \dots, N$ can be written as:

$$\mathbf{X} = \mathbf{A}\mathbf{S}$$

where \mathbf{X} is $K \times N$, and \mathbf{S} is $P \times N$.

- Solving source separation consists in factorizing the observable mixtures \mathbf{X} in two matrices, with non-negativity constraints, on \mathbf{S} , on \mathbf{A} or on both \mathbf{S} and \mathbf{A} .

From matrices to tensors

Matrix and tensor factorization

- More complex data (3D, 3D+t, etc.), e.g. sequence of images (video, TEP or MRI, hyperspectral images) or multi-modal data, can be represented by N size arrays, $N > 2$
- Non-negative matrix factorization (NMF) can then be extended to Non-negative Tensor Factorization (NTF)

Ealier works

In NMF

- Paatero and Tapper [PT94] considered positivity constraints on \mathbf{A} and \mathbf{s} in environmental modelization,
- Lee and Seung [LS99] used NMF for feature extraction in face images

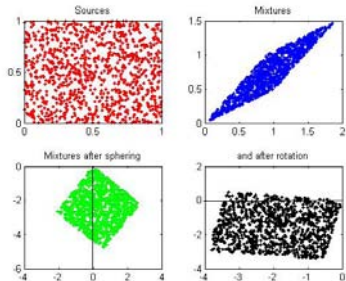
In Nonnegative ICA

- Plumbley et al addressed the problem for audio and especially music separation [PAB⁺02]
- Plumbey proposed theoretical conditions for nonnegative independent component analysis in 2002 [Plu02]
- First algorithms have been proposed by Plumbley [Plu03, Plu05] and Plumbley and Oja [PO04],

Conditions for NN ICA [Plu02]

Assumptions and pre-processing

- $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t)$, and sources $\mathbf{s}(t)$ are NN and well-grounded
- Definition: A source s is well-grounded if it has non-zero pdf in the positive neighborhood of $s = 0$.
- One then apply a whitening \mathbf{W} , computed on zero-mean data $(\mathbf{x}(t) - E[\mathbf{x}])$, but applied on \mathbf{x} : $\mathbf{z} = \mathbf{W}\mathbf{x}$



Conditions for NN ICA [Plu02]

Theorem [Plu02]

- Let \mathbf{z} be a random vector of real-valued, non-negative and well-grounded independent sources, each with unit variance and let $\mathbf{y} = \mathbf{U}\mathbf{z}$ be an orthogonal rotation of \mathbf{z} , then \mathbf{U} is a permutation matrix iff \mathbf{y} is nonnegative with probability 1.
- Practically, matrix \mathbf{U} can be estimated by minimizing a criterion:

$$J(\mathbf{U}) = \frac{1}{2}E\left[\left|\mathbf{z} - \mathbf{U}^T\mathbf{y}^+\right|^2\right]$$

where $\mathbf{y}^+ = (y_1^+, \dots, y_P^+)$, with $y_i^+ = \max(y_i, 0)$.

- In [Plu03], Plumbley propose a few algorithms, based
 - on NN PCA,
 - Givens rotations,
 - line search,
 - geodesic search.

Simple Gradient algorithm

Cost function and its differential

- It is the square error, written like the Frobenius norm:

$$J(\mathbf{A}, \mathbf{S}) = \frac{1}{2} \sum_{i,j} (x_{ij} - (\mathbf{AS})_{ij})^2 = \frac{1}{2} \|\mathbf{X} - \mathbf{AS}\|_F^2$$

- Its gradient with respect to \mathbf{S} entries writes:

$$\frac{\partial J(\mathbf{A}, \mathbf{S})}{\partial s_{ij}} = - \sum_{i,j} [\mathbf{A}^T \mathbf{X} - \mathbf{A}^T \mathbf{AS}]_{i,j}$$

Update equations

There are similar due to symmetry between \mathbf{A} and \mathbf{S} in the cost function:

$$\begin{cases} s_{ij} &= [s_{ij} + \mu_{ij}([\mathbf{A}^T \mathbf{X}]_{ij} - [\mathbf{A}^T \mathbf{AS}]_{ij})]_+ \\ a_{kl} &= a_{kl} + \mu_{kl}([\mathbf{XS}^T]_{kl} - [\mathbf{ASS}^T]_{kl}) \end{cases}$$

Gradient with multiplicative updates

For enhancing gradient algorithm, Lee and Seung [LS99] proposed multiplicative update:

Choice of the step size

- One chose μ_{ij} so that first and third right side terms of the update equation cancel:

$s_{ij} = [s_{ij} + \mu_{ij}([A^T X]_{ij} - [A^T AS]_{ij})]_+$, i.e. $s_{ij} = \mu_{ij}[A^T AS]_{ij}$, hence:

$$\mu_{ij} = \frac{s_{ij}}{[A^T AS]_{ij}}$$

New update equations

- The new (multiplicative) update equations are then::

$$s_{ij} = s_{ij} \frac{[A^T X]_{ij}}{[A^T AS]_{ij}} \text{ and } a_{ij} = a_{ij} \frac{[AS^T]_{ij}}{[ASS^T]_{ij}}$$

Alternating Least Square

Newton-like method

- The gradient of $J(\mathbf{A}, \mathbf{S})$ with respect to \mathbf{S} writes:

$$\frac{\partial J(\mathbf{A}, \mathbf{S})}{\partial \mathbf{S}} = -(\mathbf{A}^T \mathbf{X} - \mathbf{A}^T \mathbf{A} \mathbf{S})$$

- It must be equal to 0 at the minimum, i.e.

$$(\mathbf{A}^T \mathbf{A}) \mathbf{S} = \mathbf{A}^T \mathbf{X}$$

Newton-like update equations

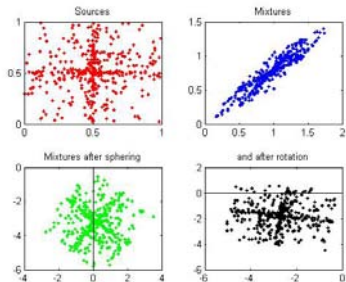
- The Newton-like update equations are then:

$$\mathbf{S} = [(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{X}]_+ \quad \text{and} \quad \mathbf{A} = [\mathbf{X} \mathbf{S}^T (\mathbf{S} \mathbf{S}^T)^{-1}]_+$$

Comments

Well-grounded condition

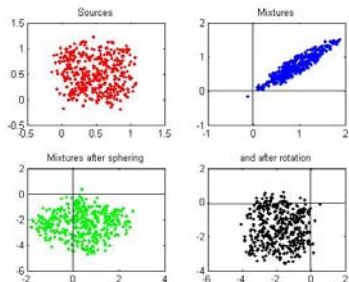
- This condition is very important, since one need samples on the edges of the scatter plot
- With uniform distribution: satisfied. But with a triangle pdf...
- Similar problem noted by Moussaoui (ICASSP 06) in chemical spectroscopy
- Moussaoui mentionned problem of tendancy in chemical spectra



Noisy mixtures of NN sources

Noisy mixtures

- Finally, NMF are not robust to additive noise
- Moreover, is additive noise realistic ?
- Relevant noise modelling in relation with the physics of the data



Beyond NMF

NMF pros and cons

- Simple formulation and algorithms
- No proof of unique factorization
- Require strict enough conditions
- Poor robustness to noisy mixtures

NN Bayesian approach

- NN priors is simple to consider
- Although Bayesian approaches are cost consuming, they are efficient for small samples
- See the lecture of A. Mohammad-Djafari in this course

NonNegative Tensor Factorization (1/2)

- In a CP decomposition

$$\mathcal{T} = (\mathbf{A}, \mathbf{B}, \dots, \mathbf{C}) \cdot \mathcal{L} \quad (54)$$

the solution is not unique: constraints needed (to avoid e.g. $\|\mathbf{A}\| \rightarrow \infty$)

- If \mathcal{T} is nonnegative, one can impose to $\{\mathbf{A}, \mathbf{B}, \dots, \mathbf{C}, \mathcal{L}\}$ to be all nonnegative \rightarrow concept of *nonnegative rank* [LC09]
- Impose unit-norm factors in (54): $\|\mathbf{A}\| = \|\mathbf{B}\| = \dots = \|\mathbf{C}\| = 1$
- Choice of norm?

NonNegative Tensor Factorization (2/2)

- Define $\|\mathcal{T}\|_E = \sum_{ij..k} |T_{ij..k}|$, better suited in \mathbb{R}^+ than Frobenius
- Then constraints

$$\|\mathbf{a}_p\|_1 = \|\mathbf{b}_p\|_1 = \dots = \|\mathbf{c}_p\|_1 = 1 \quad (55)$$

fixes the scale indeterminacy

- With that constraint, $\|\mathbf{u} \otimes \mathbf{v} \otimes \dots \otimes \mathbf{w}\| = 1$, and $\|\mathcal{T}\|_E = \|\boldsymbol{\lambda}\|_1$.
- Function $\Upsilon(\mathbf{A}, \mathbf{B}, \dots, \mathbf{C}, \boldsymbol{\lambda}) = \|\mathcal{T} - \sum_{i=1}^R \mathbf{a}_i \otimes \mathbf{b}_i \otimes \dots \otimes \mathbf{c}_i\|_E$ is *unbounded* but *coercive continuous*. Hence its infimum is reached, even if not defined on a compact.

Conclusion: The best *low-rank nonnegative tensor approximation* problem is well posed under constraint (55).

X. Applications



Contents of course X

35 Introduction

36 Deterministic approaches

- Fluorescence
- Antenna Array Processing
- Spread spectrum communications

37 Hyperspectral image of Mars

38 Back to Cumulant-based approaches

- Discrete sources
- Algorithms

39 Smart chemical sensor arrays

- Problem and model
- Method based on silent sources
- Bayesian approach

40 Removing show-through in Scanned images

A large range of applications

Source separation is a common problem

For multi-dimensional recordings...

- Biomedical signals processing: non invasive methods, localisation and artefacts cancellation in EEG, ECG, MEG, fMRI,
- Communications and antenna processing, sonar, radar, mobile phones,
- Audio and music processing,
- Image processing in astrophysics,
- Monitoring,
- Pre-processing for classification,
- Scanned image processing,
- Smart sensor design.

ICA: a solution for source separation

How to use ICA ?

- ICA is now a usual method for solving blind source separation problems
- ICA is an estimation method, which requires:
 - a separating model, suitable to the mixing model,
 - an independence criterion,
 - an optimisation algorithm.
- When running ICA algorithm, one obtains a solution, which is the best one under the above constraints.

Are the Independent Components (IC) relevant ?

ICA: relevance of independent component

- Relevance requires:
 - a good model,
 - the sources satisfy the independence assumption,
 - the optimisation algorithm does not stop in a bad local minima.
- With actual problems, basically, the relevance is not sure since:
 - the model is often an approximation of the physical system,
 - source independence assumption is not sure,
 - we don't know either the sources or their number.

Before applying ICA

Use all the possible priors

- Positivity, sparsity, temporal dependence, periodicity or cyclostationnarity, discrete-valued data, etc.
- Priors can lead to better criteria than independence.

Careful modelization of observations

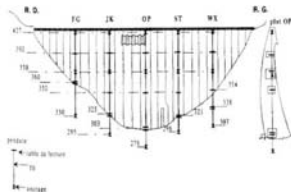
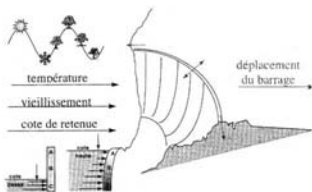
How the physical system provides the observations?

- Modelization leads to a mixing model, and hence to a suited model of separating structure.
- Modelization is important for showing what should be the estimated sources.
- Physics of the system is important for component interpretation.

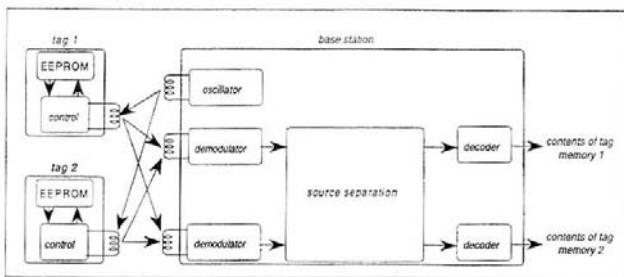
Choose the best representation domain for applying BSS

Dam monitoring

- The dam wall moves according to the water level, the temperature, etc.
- Simple pendular are hung on the wall, at different locations: the pendular deviations measure the wall motions, with different sensitivities for the water level, the temperature, etc.
- What is the relation between deviations and water level, temperature, etc. ?



Multi-access control



Y. Deville, L. Andry, Application of blind source separation techniques for multi-tag contactless identification systems (NOLTA 95; IEICE Trans. On Fundamentals of Electronics, 1996; French patent Sept. 1995 subsequently extended)

ICA: preprocessing for classification

Bayesian classification framework

- Relate each multi-dimensionnal observation \mathbf{x} to a class C_i
- Bayesian classification requires conditional densities $p(\mathbf{x}/C_i)$
- Multivariate density estimation (small sample, complexity)

A simple idea [Com95, AAG03]

- Transform observations \mathbf{x} in $\tilde{\mathbf{x}}$ by ICA, for all observations associated to C_i
- In the transform space, components are close to independance, so that one can write: $p(\tilde{\mathbf{x}}/C_i) \approx \prod_k p(\tilde{x}_k/C_i)$
- Then, using the inverse transform, one gets $p(\mathbf{x}/C_i)$
- The method replaces 1 multivariate (size N) density estimation by N scalar density estimations

Fluorescence Spectroscopy 1/4

An optical excitation produces several effects

Fluorescence Spectroscopy 1/4

An optical excitation produces several effects

- Rayleigh diffusion
- Raman diffusion
- Fluorescence

Fluorescence Spectroscopy 1/4

An optical excitation produces several effects

- Rayleigh diffusion
- Raman diffusion
- Fluorescence

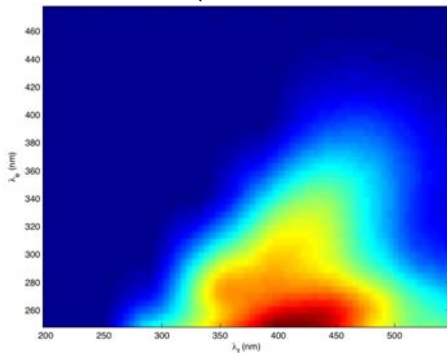
At low concentrations, Beer-Lambert law can be linearized
[LMRB09]

$$I(\lambda_f, \lambda_e, k) = I_o \sum_{\ell} \gamma_{\ell}(\lambda_f) \epsilon_{\ell}(\lambda_e) c_{k,\ell}$$

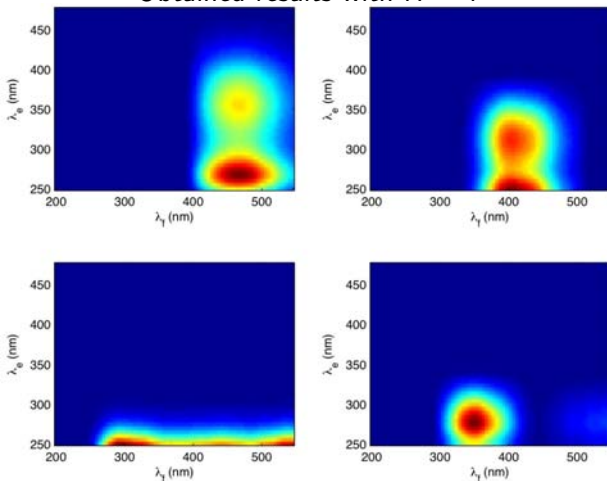
➡ Hence 3rd array decomposition with real *nonnegative factors* [SBG04].

Fluorescence Spectroscopy 2/4

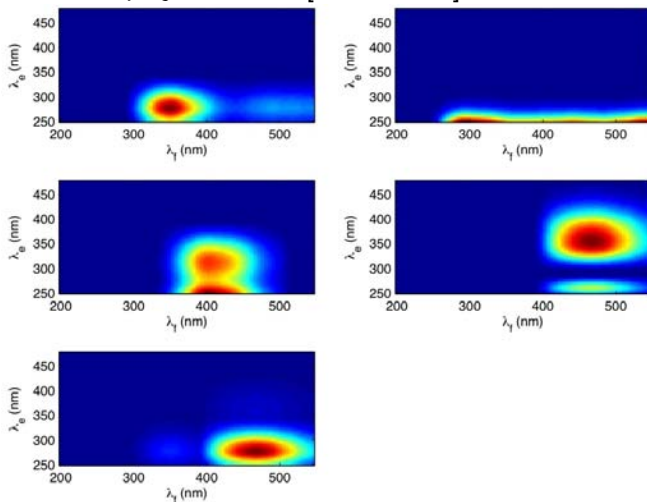
Mixture of 4 solutes (one concentration shown)



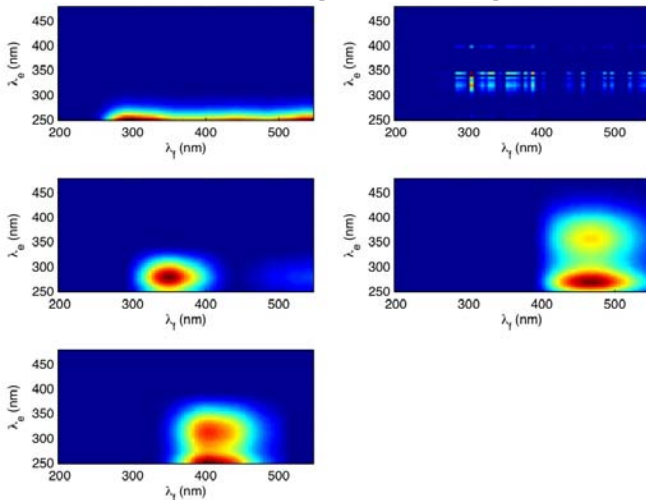
Fluorescence Spectroscopy 3/4

Obtained results with $R = 4$ 

With projected ALS [Cichocki'09] and $R = 5$



With nonnegative BFGS [Royer et al'10] and $R = 5$



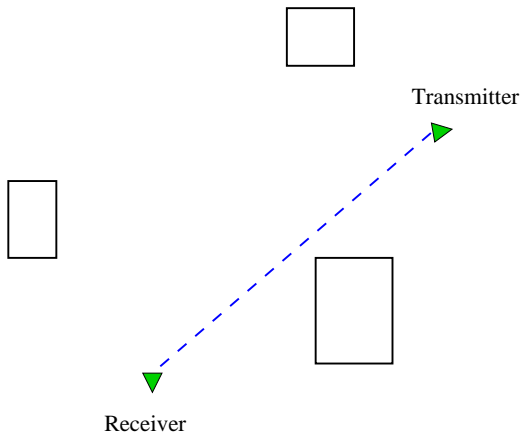
Link with Bayesian approaches

Similar Non Negative tensor decomposition in Bayesian approaches:

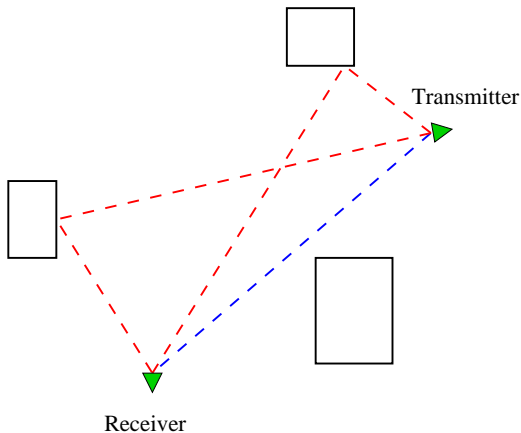
$$p(x_1, x_2, \dots, x_p) \approx \sum_{i=1}^R p(z_i) p(x_1|z_i) \dots p(x_p|z_i)$$

See [LC09], and course on Bayesian techniques.

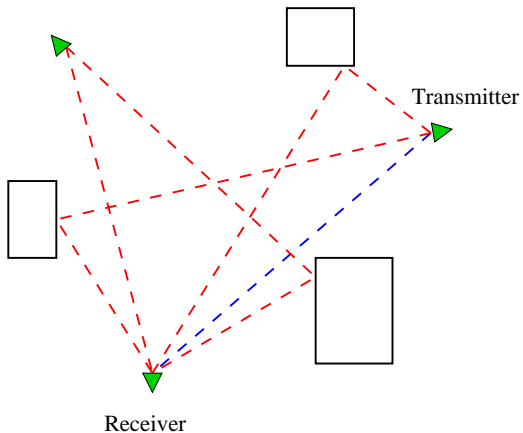
Antenna Array Processing



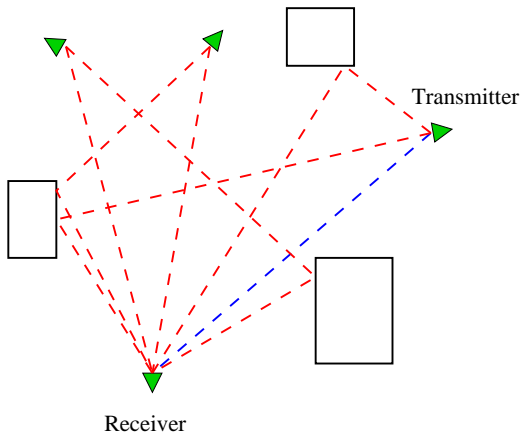
Antenna Array Processing



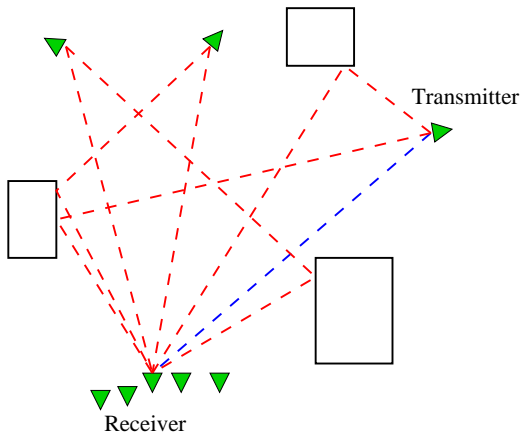
Antenna Array Processing



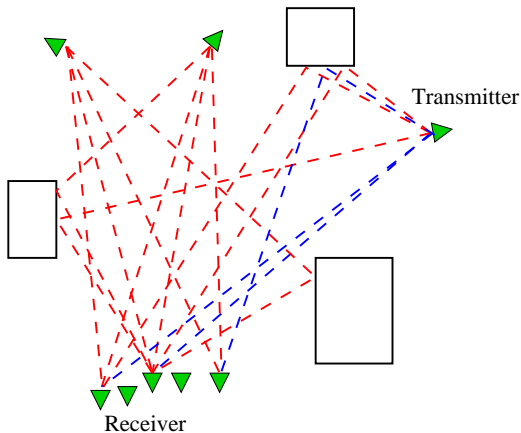
Antenna Array Processing



Antenna Array Processing



Antenna Array Processing



Narrow band model in the far field

Modeling the signals received on an array of antennas generally leads to a *matrix decomposition*:

$$T_{ij} = \sum_q a_{iq} S_{jq}$$

i : space q : path, source **A**: antenna gains
 j : time **S**: transmitted signals

Narrow band model in the far field

Modeling the signals received on an array of antennas generally leads to a *matrix decomposition*:

$$T_{ijp} = \sum_q a_{iq} s_{jq} h_{pq}$$

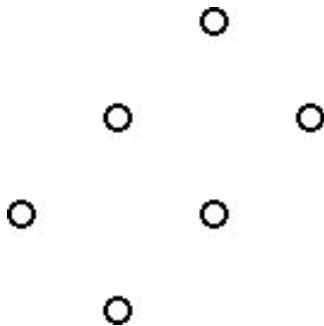
i : space q : path, source **A**: antenna gains
 j : time **S**: transmitted signals

But in the presence of additional *diversity*, a tensor can be constructed, thanks to new index p

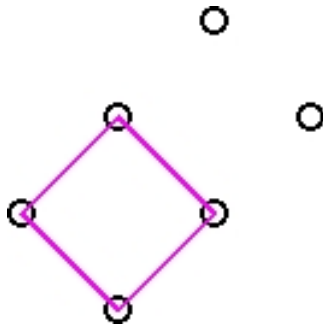
Possible diversities in Signal Processing

- space
- time
- space translation (array geometry)
- time translation (chip)
- frequency/wavenumber (nonstationarity)
- excess bandwidth (oversampling)
- cyclostationarity
- polarization
- finite alphabet
- ...

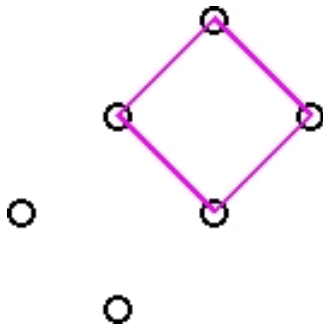
Space translation diversity (1/2)



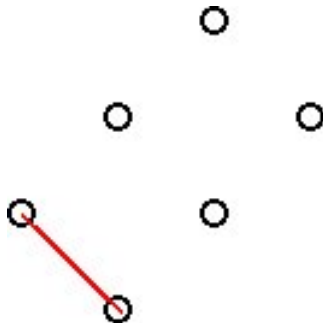
Space translation diversity (1/2)



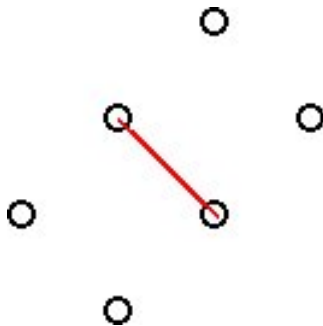
Space translation diversity (1/2)



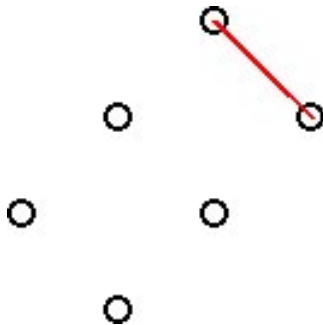
Space translation diversity (1/2)



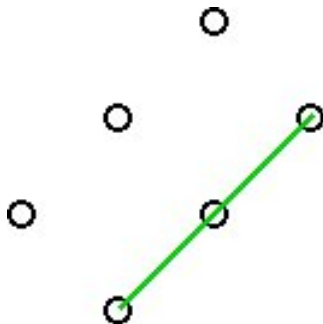
Space translation diversity (1/2)



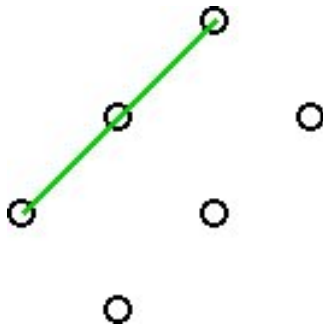
Space translation diversity (1/2)



Space translation diversity (1/2)



Space translation diversity (1/2)



Space translation diversity (2/2)

A_{iq} : gain between sensor i and source q

H_{pq} : transfer between reference and subarray p

S_{jq} : sample j of source q

β_q : path loss, \mathbf{d}_q : DOA, \mathbf{b}_i : sensor location

Tensor model (NB far field) [SBG00]

- Reference subarray: $A_{iq} = \beta_q \exp(j\frac{\omega}{c} \mathbf{b}_i^T \mathbf{d}_q)$

Space translation diversity (2/2)

A_{iq} : gain between sensor i and source q

H_{pq} : transfer between reference and subarray p

S_{jq} : sample j of source q

β_q : path loss, \mathbf{d}_q : DOA, \mathbf{b}_i : sensor location

Tensor model (NB far field) [SBG00]

- Reference subarray: $A_{iq} = \beta_q \exp(j\frac{\omega}{c} \mathbf{b}_i^T \mathbf{d}_q)$
- Space translation (from reference subarray):

$$\beta_q \exp(j\frac{\omega}{c} [\mathbf{b}_i + \mathbf{\Delta}_p]^T \mathbf{d}_q) \stackrel{\text{def}}{=} A_{iq} H_{pq}$$

Space translation diversity (2/2)

A_{iq} : gain between sensor i and source q

H_{pq} : transfer between reference and subarray p

S_{jq} : sample j of source q

β_q : path loss, \mathbf{d}_q : DOA, \mathbf{b}_i : sensor location

Tensor model (NB far field) [SBG00]

- Reference subarray: $A_{iq} = \beta_q \exp(j\frac{\omega}{c} \mathbf{b}_i^T \mathbf{d}_q)$

- Space translation (from reference subarray):

$$\beta_q \exp(j\frac{\omega}{c} [\mathbf{b}_i + \mathbf{\Delta}_p]^T \mathbf{d}_q) \stackrel{\text{def}}{=} A_{iq} H_{pq}$$

- Trilinear model:

$$T_{ijp} = \sum_q A_{iq} S_{jq} H_{pq}$$

p : index of subarray

Time translation diversity

DS-CDMA

A_{iq} : gain between sensor i and user q

$C_q(n)$: spreading sequence of user q (chip number: n)

$H_q(n)$: channel of user q

$B_{qp} = H_q \star C_q(p)$, after removal of ISI (guards)

S_{qj} : symbol j of user q

Time translation diversity

DS-CDMA

A_{iq} : gain between sensor i and user q

$C_q(n)$: spreading sequence of user q (chip number: n)

$H_q(n)$: channel of user q

$B_{qp} = H_q \star C_q(p)$, after removal of ISI (guards)

S_{qj} : symbol j of user q

Tensor model

$$T_{ijp} = \sum_q A_{iq} S_{qj} B_{qp}$$

p : index of chip with no ISI

Bibliographical comments

- NonNegative tensors: Projected ALS [CZPA09], Exact parameterization [RTMC11], Tensor package: [url](#).
- Antenna Array Processing: [SBG00], [CL11] [LC10]
- CDMA: Sidiropoulos [SGB00]
- Discrete sources and TCOM: [TVP96] [Com04] [ZC05] [ZC08]
- Audio (L.Daudet)
- Image (D.Fadili)

Hyperspectral image of Mars

Content

- Modeling the observations
- Aplying ICA
- IC interpretation
- Discussion

with H. Hauksdottir (MsSc thesis, Univ. of Iceland), J. Chanussot (GIPSA-lab, Grenoble), S. Moussaoui (IRCCyN, Nantes), F. Schmidt and S. Douté (Planetology Lab., Grenoble), D. Brie 5CRAN, Nancy), J. Benediksson (Univ. of Iceland)

Hyperspectral image of Mars: Data

Observations

- OMEGA instrument: a spectrometer on board of Mars Express
- 256 frequency bands in visible and IR, from 0.926 μm to 5.108 μm with a spectral resolution of 0.014 μm ,
- variable spatial resolution (elliptic orbit) from 300m to 4km.

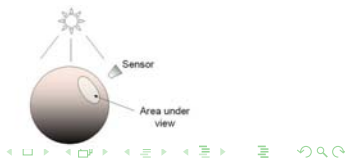
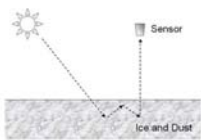
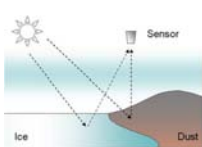
Data

- only 174 frequency bands (the other are too noisy),
- portion of image: about 256×256 pixels
- one hyperspectral image is a cube of 10,000,000 data
- South Polar Cap of Mars mainly contains 3 endmembers: CO₂ ice, H₂O ice and Dust.

Physical properties of observations

Reflected light on the spectrometer

- reflected light of sun incident light according to ground reflectance,
- atmospheric attenuation and direct sensor illumination,
- geometrical mixture occurs when a few different chemicals reflect directly the sun light according to their local abundance,
- intimate mixture corresponds to a (NL) mixture at very small scale,
- solar position with respect to the Polar cap leads to luminance gradient



Model of observations

Luminance in location (x, y) t wavelength λ

- Assuming geometrical mixture, the luminance writes as

$$L(x, y, \lambda) = \phi(\lambda) \left[L_a(\lambda) + \sum_{p=1}^P \alpha_p(x, y) L_p(\lambda) \right] \cos(\theta(x, y))$$

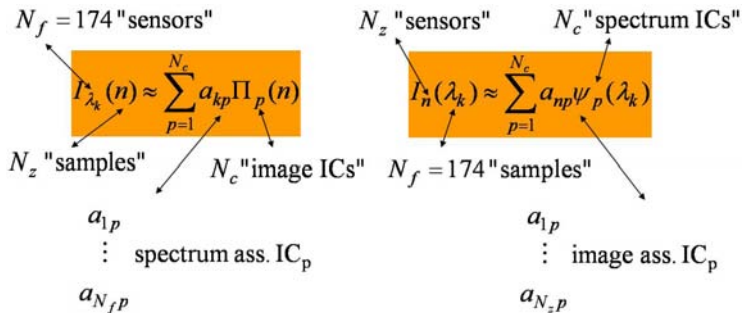
- With simple algebra, it leads to a linear model

$$L(x, y, \lambda) = \sum_{p=1}^P \alpha'_p(x, y) L'_p(\lambda) + E(x, y, \lambda) + \textit{noise}$$

- Estimated spectra different of original endmember spectra, since modified by gradient $(\cos \theta(x, y))$ and atmospheric effect $(\phi(\lambda))$
- What are the independent sources: abundances, spectra ?

Spectral or spatial ICA ?

ICA approximations: spatial or spectral decomposition ?



Spatial ICA

Choosing the number of ICs

- At least equal to the endmember number,
- sufficient for insuring an accurate approximation.

$$I_{\lambda_k}(n) \approx \sum_{p=1}^{N_c} a_{kp} \Pi_p(n)$$

Solutions

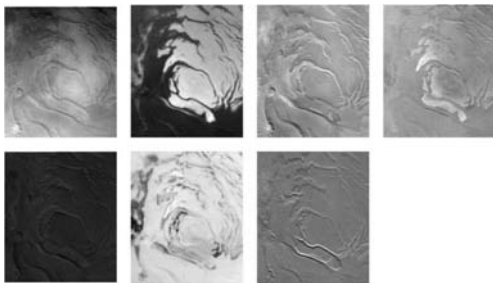
- 3 endmembers: H₂O, CO₂ ices and Dust.
- Using PCA, with $N_c = 7$ components, one recovers 98% of the initial image power.

Spatial ICA results

After PCA, JADE provides 7 ICs (images)

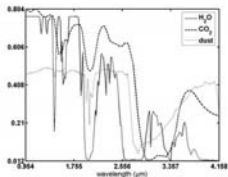
How to interpret the ICs ?

- For endmembers IC, one has ground truth with reference classification and reference spectra.
- For others ???

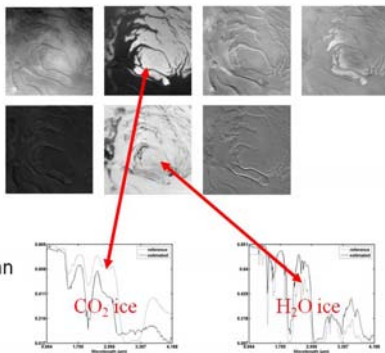


IC interpretation: using reference spectra

- How interpret ICs ?
 - reference spectra in Earth lab.

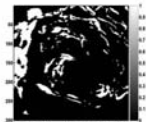


- Spectra associated to A column



IC interpretation: using reference classification

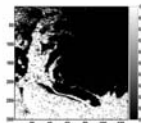
- How interpret ICs ?
 - reference classification



H₂O ice mask

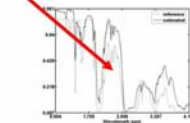
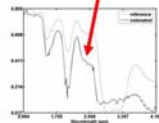
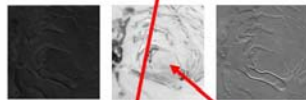
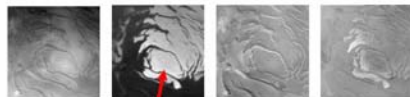


CO₂ ice mask



Dust

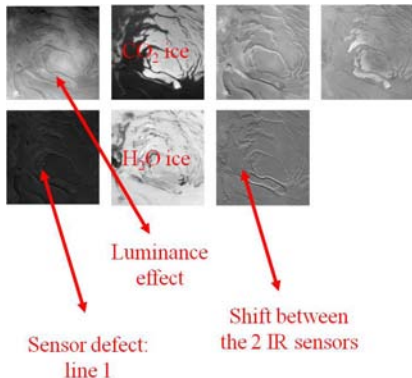
CO₂ ice and dust are strongly correlated!



Interpretation of other ICs

Labelling through expert interaction

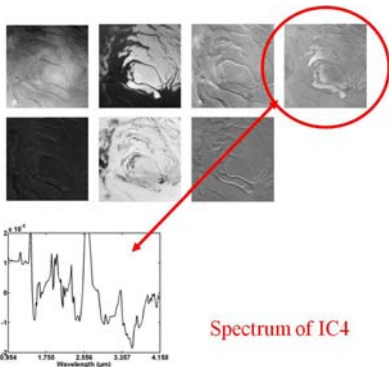
- Results with preprocessed data, with removal of luminance and atmospheric effects and defectuous sensors
- On these data, the power associated to IC1, IC3, IC4 and IC7 are very small
- We deduce that these ICs are related to effects removed by the preprocessing
- Spectra of these ICs are more or less flat



Interpretation of the last ICs

Labelling through expert interaction

- The spectrum looks like a nonlinear mixture of CO₂ ice and Dust spectra.
- Physicists suspect it could be associated to intimate mixtures.



Hyperspectral images of Mars: Discussion

Comments

- 5 ICs seems related to artefacts, and 2 like endmembers
- Artefacts cannot be explained with endmembers, and appear like ICs. The initial model should be improved.
- 2 endmembers are strongly correlated !
- Spectra are sometimes negative !

Conclusions

- ICA is not suited to this problem !
- Classification cannot be done with these ICs.
- One can design a method based on true assumptions, like positivity of spectra, e.g. based on Bayesian approaches [MHS⁺08].

Finite alphabets

- Back to contrast criteria: APF
- Approximation of the MAP estimate
- Semi-Algebraic Blind Extraction algorithm: AMiSRoF
- Blind Extraction by ILSP
- Convolutional model
- Presence of Carrier Offset (in Digital Communications)
- Constant Modulus

Contrast for discrete inputs (1)

- **Hypothesis H5** The sources take their value in a finite alphabet \mathcal{A} defined by the roots in \mathbb{C} of some polynomial $q(z) = 0$
- **Theorem [Com04]**
Under **H5**, the following is a contrast over the set \mathcal{H} of invertible $P \times P$ FIR filters.

$$\Upsilon(\mathbf{G}; \mathbf{z}) \stackrel{\text{def}}{=} - \sum_n \sum_i |q(z_i[n])|^2 \quad (56)$$

APF: Algebraic Polynomial Fitting

Contrast for discrete inputs (2)

- For given alphabet \mathcal{A} , denote \mathcal{G} the set of numbers c such that $c\mathcal{A} \subseteq \mathcal{A}$.
- **Lemma 1** Trivial filters satisfying **H5** are of the form:

$$\mathbf{P D}[z]$$

with $\mathbf{D}[z]$ diagonal and $D_{pp}[z] = c_p z^n$, for some $n \in \mathbb{Z}$ and some $c_p \in \mathcal{G}$.

- Because \mathcal{A} is finite, any $c \in \mathcal{G}$ must be of unit modulus, and we must have $c\mathcal{A} = \mathcal{A}, \forall c \in \mathcal{G}$.
Also any $c \in \mathcal{G}$ has an inverse c^{-1} in \mathcal{G} .

Contrast for discrete inputs : summary

Advantages

- The previous contrast allows to separate *correlated* sources
- But it needs all sources to have the same (known) alphabet
- If sources have *different* alphabets, one can extract sources in parallel with different criteria: *Parallel Extraction* [RZC05]
- By deflation with different criteria, one can extract more sources than sensors: *Parallel Deflation* [RZC05]

Equivalence between KMA and CMA

- **Kurtosis:**

$$\Upsilon_{KMA} = \frac{\text{Cum}\{z, z, z^*, z^*\}}{[E\{|z|^2\}]^2}$$

- **Constant modulus:**

$$\mathcal{J}_{CMA} = E\{[|z|^2 - R]^2\}$$

- Assume 2nd Order circular sources: $E\{s^2\} = 0$
- Then KMA and CMA are equivalent [Reg02] [Com04]
Proof.

Discussion on Deflation (MISO) criteria

Let $z \stackrel{\text{def}}{=} \mathbf{f}^H \mathbf{x}$. Criteria below stationary iff differentials \mathbf{p} and \mathbf{q} are collinear:

■ **Ratio:** $\text{Max}_{\mathbf{f}} \frac{p(\mathbf{f})}{q(\mathbf{f})}$

Example: Kurtosis, with $p = E\{|z|^4\} - 2E\{|z|^2\}^2 - |E\{z^2\}|^2$
and $q = E\{|z|^2\}^2$

■ **Difference:** $\text{Min}_{\mathbf{f}} p(\mathbf{f}) - \alpha q(\mathbf{f})$

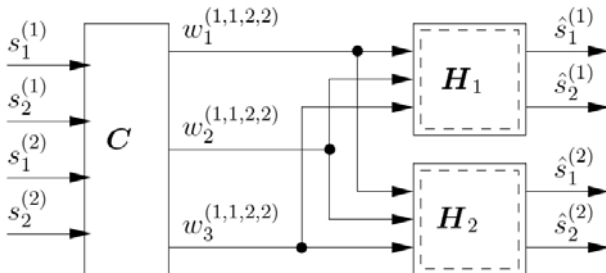
Example: Constant Modulus, with $p = E\{|z|^4\}$ and
 $q = 2a E\{|z|^2\} - a^2$ or **Constant Power**, with
 $q = 2a \Re(E\{z^2\}) - a^2$

■ **Constrained:** $\text{Max}_{q(\mathbf{f})=1} p(\mathbf{f})$

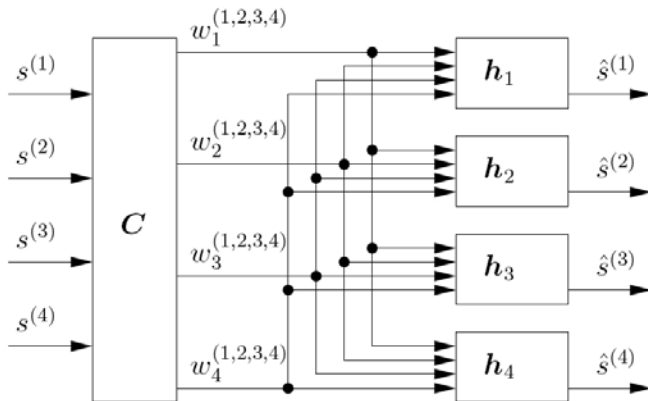
Example: Cumulant, with
 $p = E\{|z|^4\} - 2E\{|z|^2\}^2 - |E\{z^2\}|^2$

Example: Moment, with $p = E\{|z|^4\}$, if standardized and
with either $q = \|\mathbf{f}\|^2$ or $q = E\{|z|^2\}^2$

Parallel Deflation

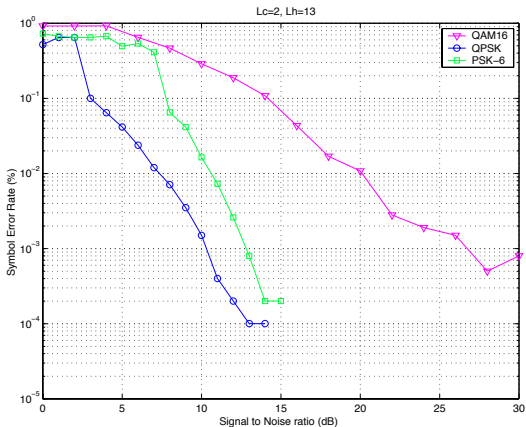


Parallel Extraction



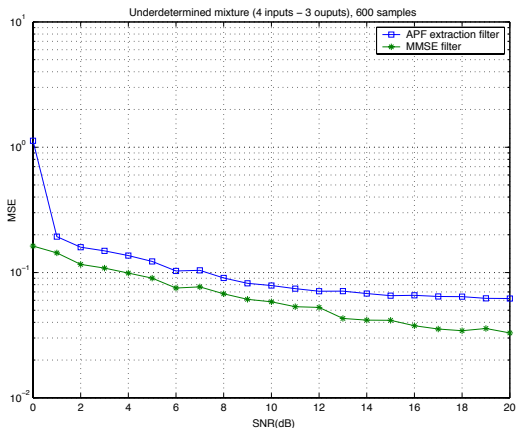
Parallel extraction

Parallel extraction of 3 sources (QPSK, QAM16, PSK6), from a 3-sensor length-3 random Gaussian channel [RZC05]



APF extraction

Parallel Deflation from a mixture of 4 sources (2 QPAK and 2 QAM16) received on 3 sensors. Extraction of a QPSK source in figure, compared to MMSE [RZC05]



MAP estimate

- Optimal solution

$$(\hat{\mathbf{H}}, \hat{\mathbf{s}})_{MAP} = \mathit{Arg} \mathit{Max}_{\mathbf{H}, \mathbf{s}} p_{s|x, H}(\mathbf{x}, \mathbf{s}, \mathbf{H})$$

- If $s_p \in \mathcal{A}$, and if noise is Gaussian, then

$$(\hat{\mathbf{H}}, \hat{\mathbf{s}})_{MAP} = \mathit{Arg} \mathit{Min}_{\mathbf{H}, \mathbf{s} \in \mathcal{A}^P} \|\mathbf{x} - \mathbf{H}\mathbf{s}\|^2$$

- Less costly to search (inverse filter when it exists)

$$(\hat{\mathbf{F}}, \hat{\mathbf{s}})_{MAP} = \mathit{Arg} \mathit{Min}_{\mathbf{F}, \mathbf{s} \in \mathcal{A}^P} \|\mathbf{F}\mathbf{x} - \mathbf{s}\|^2$$

- or by deflation:

$$(\hat{\mathbf{f}}, \hat{\mathbf{s}})_{MAP} = \mathit{Arg} \mathit{Min}_{\mathbf{f}, \mathbf{s} \in \mathcal{A}^P} \|\mathbf{f}^H \mathbf{x} - \mathbf{s}\|^2 \quad (57)$$

Approximation of the MAP estimate

For alphabet of constant modulus, MAP criterion (57) is asymptotically equivalent (for large samples of size T) to [GC98]:

$$\Upsilon_T(\mathbf{f}) = \frac{1}{T} \sum_{t=1}^T \prod_{j=1}^{\text{card}\mathcal{A}} |\mathbf{f}^H \mathbf{x}[t] - a_j[t]|^2$$

where $a_j[t] \in \mathcal{A}$

We have transformed an exhaustive search into a *polynomial* alphabet fit

Algorithm AMiSRoF

Absolute Minimum Search by Root Finding [GC98]

- Initialize $\mathbf{f} = \mathbf{f}_o$
- For $k = 1$ to k_{\max} , and while $|\mu_k| > \text{threshold}$, do
 - Compute gradient \mathbf{g}_k and Hessian \mathbf{H}_k at \mathbf{f}_{k-1}
 - Compute a search direction \mathbf{v}_k , e.g. $\mathbf{v}_k = \mathbf{H}_k^{-1} \mathbf{g}_k$
 - Normalize \mathbf{v}_k to $\|\mathbf{v}_k\| = 1$
 - Compute the *absolute* minimum μ_k of the rational function in μ :

$$\Phi(\mu) \stackrel{\text{def}}{=} \Upsilon_T(\mathbf{f}_{k-1} + \mu \mathbf{v}_k)$$

- Set $\mathbf{f}_k = \mathbf{f}_{k-1} + \mu_k \mathbf{v}_k$

Algorithm ILSP

Iterative Least-Squares with Projection [TVP96]

- Assumes that components $s_j[n] \in \mathcal{A}$, known alphabet
- Assumes columns of \mathbf{H} belong to a known *array manifold*
- Initialize \mathbf{H} , and start the loop
 - Compute LS estimate of matrix \mathbf{S} in equation $\mathbf{X} = \mathbf{H}\mathbf{S}$
 - Project \mathbf{S} onto \mathcal{A}
 - Compute LS estimate of \mathbf{H} in equation $\mathbf{X} = \mathbf{H}\mathbf{S}$
 - Project \mathbf{H} onto the array manifold

Smart chemical sensore arrays

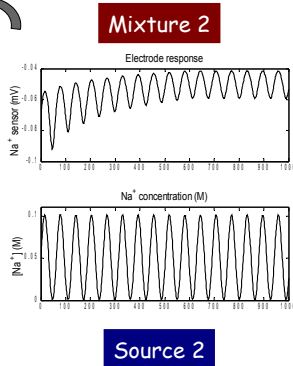
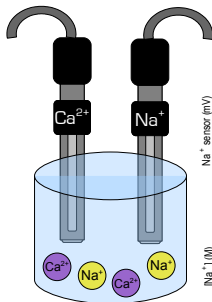
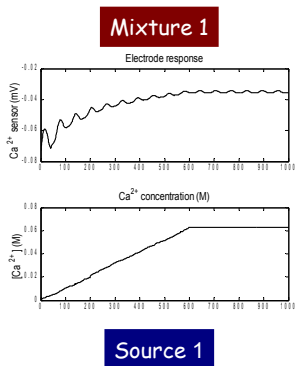
Content

- Nicolski-Eisenman model
- Method based on source silences (Eusipco 2008) [DJ08]
- Method based on positivity (ICA 2009) [DJM09]

These works were done by L. Duarte during his PhD thesis in GIPSA-lab (2006-2009)

Ion-selective electrodes: the interference problem

- Aim: to estimate the concentration of several ions in a solution.



Electrode description - The Nicolsky-Eisenman model

$$x_i(t) = c_i + d_i \log \left(s_i(t) + \sum_{j, j \neq i} a_{ij} s_j(t)^{\frac{z_i}{z_j}} \right), \quad (58)$$

- $s_i(t) \Rightarrow$ target ion concentration; $s_j(t) \Rightarrow$ interfering ions concentrations
- $c_i, d_i, a_{ij} \Rightarrow$ mixing model parameters;
- z_i and $z_j \Rightarrow$ valences of the ions i and j
- When $z_i = z_j \Rightarrow$ Post-nonlinear (PNL) mixing model.
- *We are interested in the case in which $z_i \neq z_j$.*
- *We consider a scenario with two ions and two electrodes.*

The Nicolsky-Eisenman model (cont.)

- In previous works, we assumed that the component-wise functions were known in advance.

Resulting Model

In this work, we consider the complete model

$$\begin{aligned}x_1(t) &= d_1 \log \left(s_1(t) + a_{12} s_2(t)^k \right) \\x_2(t) &= d_2 \log \left(s_2(t) + a_{21} s_1(t)^{\frac{1}{k}} \right),\end{aligned}\tag{59}$$

with $k = z_1/z_2$.

The Nicolsky-Eisenman model (cont.)

- In previous works, we assumed that the component-wise functions were known in advance.

Resulting Model

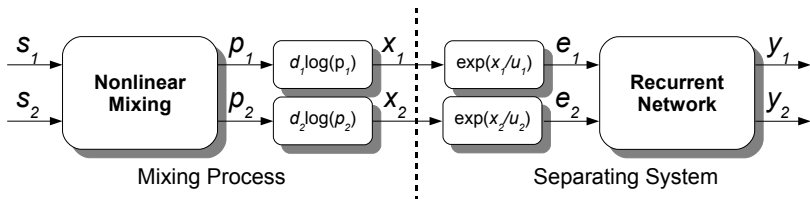
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Resulting Model

In this work, we consider the complete model

$$\begin{aligned} x_1(t) &= d_1 \log \left(s_1(t) + a_{12} s_2(t)^k \right) \\ x_2(t) &= d_2 \log \left(s_2(t) + a_{21} s_1(t)^{\frac{1}{k}} \right) \end{aligned}, \quad (59)$$

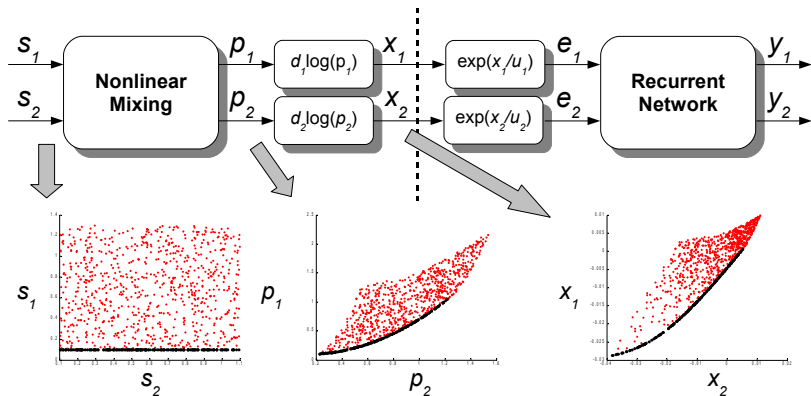
with $k = z_1/z_2$.

Assumptions

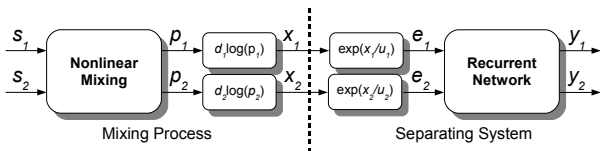
- 1 The sources are statistically independent;
- 2 The sources are positive and bounded, i.e.,
 $s_i(t) \in [S_i^{min}, S_i^{max}]$, where $S_i^{max} > S_i^{min} > 0$;
- 3 The mixing system is invertible in the region given by
 $[S_1^{min}, S_1^{max}] \times [S_2^{min}, S_2^{max}]$;
- 4 k (the ratio between the valences) is known and takes only positive integer values;

Basic idea

- Additional assumption: there is, at least, a period of time where one, and only one, source has zero-variance.



Basic idea in equations



- During the silent periods ($s_1(t) = S_1$):

$$\begin{aligned} p_1(t) &= S_1 + a_{12}s_2(t)^k \\ p_2(t) &= s_2(t) + a_{21}S_1^{\frac{1}{k}}. \end{aligned} \quad (60)$$

- In the (p_1, p_2) plane, we have a polynomial of order k :

$$p_1(t) = S_1 + a_{12}(p_2(t) - a_{21}S_1^{\frac{1}{k}})^k. \quad (61)$$

$$p_1(t) = \sum_{i=0}^k \varphi_i p_2(t)^i, \quad (62)$$

Recovering of polynomials

- Mapping between the (p_1, p_2) and (e_1, e_2) planes

$$\begin{aligned} e_1(t) &= \exp\left(\frac{d_1 \log(p_1(t))}{u_1}\right) = p_1(t)^{\frac{d_1}{u_1}} \\ e_2(t) &= \exp\left(\frac{d_2 \log(p_2(t))}{u_2}\right) = p_2(t)^{\frac{d_2}{u_2}} \end{aligned} \quad (63)$$

- Using this equation and the polynomial relation in the (p_1, p_2) plane

$$e_1(t) = \left[\sum_{i=0}^k \varphi_i e_2(t)^{\frac{u_2}{d_2} i} \right]^{\frac{d_1}{u_1}} \quad (64)$$

- For what values of u_1 and u_2 the function (64) is a polynomial of order k ?
 - Ideal solution $u_1 = d_1$ and $u_2 = d_2$;
 - When all the coefficients except φ_k in (64) are null, then $(u_1 = Dd_1, u_2 = Dd_2)$ also culminates in a polynomial of order k .

How to detect the silent periods?

- During the silent periods,

$$\begin{aligned}x_1 &= g_1(s_2) \\x_2 &= g_2(s_2)\end{aligned}\quad (65)$$

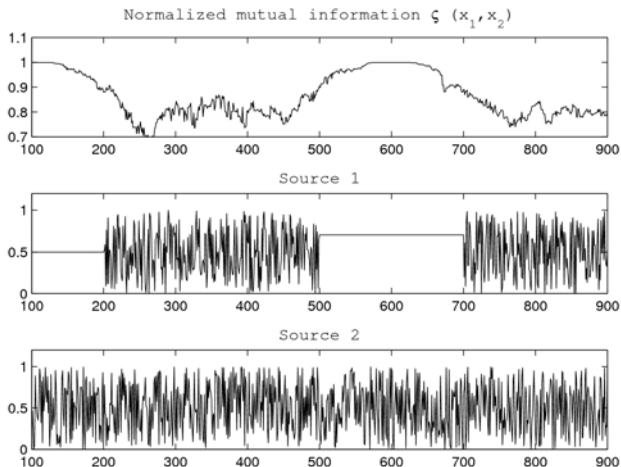
Maximum (nonlinear) correlation between x_1 and x_2

- Normalized mutual information

$$\zeta(x_1, x_2) = \sqrt{1 - \exp(-2I(x_1, x_2))} \quad (66)$$

- $\zeta(x_1, x_2) = 0$ when x_1 and x_2 are statistically independent;
- $\zeta(x_1, x_2) \rightarrow 1$ when there is a deterministic relation between x_1 and x_2 ;

Result for silent periods detection



Final Algorithm

1 Detection of silent periods

- Estimate the mutual information between the mixtures x_1 and x_2 for a moving-time window.
- Select the time window in which the mutual information is maximum.

2 Estimation of the component-wise functions by the polynomial recovering idea.

- For the selected time window, minimize the expression

$$\min_{u_1, u_2} \sum_t \left(e_1(t) - \sum_{i=0}^k \alpha_i (e_2(t))^i \right)^2, \quad (67)$$

3 Training the recurrent network

- Apply on e_i the algorithm for the simplified version of the NE model.

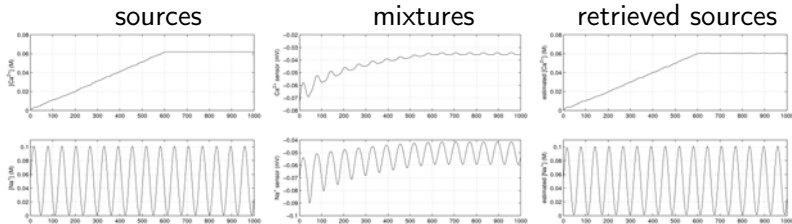
Results

- Estimation of the concentration of the ions Ca^{2+} and Na^{+} using two electrodes;
- Mixing parameters $a_{12} = 0.79$ and $a_{21} = 0.40$ (extracted from a IUPAC report) and $d_1 = 0.0129$ and $d_2 = 0.0258$;
- Sources artificially generated;
- Number of samples: 1000;
- Window length for silent period detection: 151;
- Performance index:

$$SIR_i = 10 \log \left(\frac{E\{s_i^2\}}{E\{(s_i - y_i)^2\}} \right). \quad (68)$$

Results (cont.)

$SIR_1 = 40.52dB$, $SIR_2 = 38.27dB$ and $SIR = 39.39dB$



Conclusions

- A complete nonlinear BSS method was developed for a chemical sensing application.
- The developed method can avoid time-demanding calibration stages.
- There are still several challenging points to be investigated:
 - The number of samples in a real application may be small.
 - The independence assumption may be rather strong, e.g. if a regulatory process between the ions exists.

Motivations for using the Bayesian approach

- Prior information is available

$$x_{it} = e_j + d_i \log_{10} \left(s_{it} + \sum_{j:j \neq i} a_{ij} s_{jt} \frac{z_i}{z_j} \right) + n_{it}, \quad (69)$$

- e_j takes value in the interval $[0.05, 0.35]$; [Gru07]
 - Theoretical value for the Nerstian slope $\Rightarrow d_i = RT \ln(10)/z_i F$ ($0.059V$ for room temperature); [FF03]
 - Always non-negative. Very often in the interval $[0, 1]$; [UBU⁺00]
 - The sources are positive.
- Takes noise into account;
 - In contrast to ICA, the statistical independence is rather a working assumption in the Bayesian approach [FG06];
 - May work even if the number of samples is small.

Bayesian source separation method: problem and notations

- Problem: given \mathbf{X} , estimate the unknown parameters $\theta = [\mathbf{S}, \mathbf{A}, \mathbf{d}, \mathbf{e}, \sigma, \phi]$;
- $\mathbf{S} \Rightarrow$ sources;
- $\phi \Rightarrow$ sources hyperparameters;
- $\mathbf{A} \Rightarrow$ selectivity coefficients;
- $\mathbf{d} \Rightarrow$ Nerstian slopes;
- $\mathbf{e} \Rightarrow$ offset parameters;
- $\sigma \Rightarrow$ noise variances.

Bayesian source separation method: an overview

- Problem: given \mathbf{X} , estimate the unknown parameters $\boldsymbol{\theta} = [\mathbf{S}, \mathbf{A}, \mathbf{d}, \mathbf{e}, \boldsymbol{\sigma}, \boldsymbol{\phi}]$;
- In the Bayesian approach, estimation of $\boldsymbol{\theta}$ is based on the posterior information

$$p(\boldsymbol{\theta}|\mathbf{X}) \propto p(\mathbf{X}|\boldsymbol{\theta})p(\boldsymbol{\theta}) \quad (70)$$

- The likelihood function is given by:

$$p(\mathbf{X}|\boldsymbol{\theta}) = \prod_{t=1}^{n_d} \prod_{i=1}^{n_c} \mathcal{N}_{x_{it}} \left(e_i + d_i \log \left(\sum_{j=1}^{n_s} a_{ij} s_{jt}^{z_i/z_j} \right), \sigma_i^2 \right), \quad (71)$$

since we assume an **i.i.d.** Gaussian noise which is **spatial independent**.

Prior definitions

- Log-normal prior distribution for the sources (non-negative distribution)

$$p(s_{jt}) = \frac{1}{s_{jt} \sqrt{2\pi\sigma_{s_j}^2}} \exp\left(-\frac{(\log(s_{jt}) - \mu_{s_j})^2}{2\sigma_{s_j}^2}\right) \mathbb{1}_{[0, +\infty[}(s_{jt}), \quad (72)$$

- Motivations

- The estimation of $\phi_j = [\mu_{s_j} \ \sigma_{s_j}^2]$ is not difficult, since we can define a conjugate pair.
 - Ionic activities are expected to have a small variation in the logarithmic scale.
- The sources are assumed i.i.d. and statistically mutually independent:

$$p(\mathbf{S}) = \prod_{j=1}^{n_s} \prod_{t=1}^{n_d} p(s_{jt}), \quad (73)$$

Prior definitions (cont.)

- Sources parameters $\phi_j = [\mu_{s_j} \ \sigma_{s_j}^2]$

$$p(\mu_{s_j}) = \mathcal{N}(\tilde{\mu}_{s_j}, \tilde{\sigma}_{s_j}^2), \quad p(1/\sigma_{s_j}^2) = \mathcal{G}(\alpha_{\sigma_{s_j}}, \beta_{\sigma_{s_j}}) \quad (74)$$

- Selectivity coefficients a_{ij} : very often within $[0, 1]$

$$p(a_{ij}) = \mathcal{U}(0, 1) \quad (75)$$

- Nernstian slopes d_i : ideally $0.059V$ at room temperature

$$p(d_i) = \mathcal{N}(\mu_{d_i} = 0.059/z_i, \sigma_{d_i}^2) \quad (76)$$

- Offset parameters e_i lie in the interval $[0.050, 0.350]V$

$$p(e_i) = \mathcal{N}(\mu_{e_i} = 0.20, \sigma_{e_i}^2) \quad (77)$$

- Noise variances σ_i :

$$p(1/\sigma_i^2) = \mathcal{G}(\alpha_{\sigma_i}, \beta_{\sigma_i}) \quad (78)$$

The posterior distribution

- The posterior distribution is given by

$$\begin{aligned}
 p(\boldsymbol{\theta}|\mathbf{X}) \propto & p(\mathbf{X}|\boldsymbol{\theta}) \cdot \prod_{j=1}^{n_s} \prod_{t=1}^{n_d} p(s_{jt}|\mu_{s_j}, \sigma_{s_j}^2) \cdot \prod_{j=1}^{n_s} p(\mu_{s_j}) \\
 & \cdot \prod_{j=1}^{n_s} p(\sigma_{s_j}) \cdot \prod_{i=1}^{n_c} \prod_{j=1}^{n_s} p(a_{ij}) \cdot \prod_{i=1}^{n_c} p(e_i) \cdot \prod_{i=1}^{n_c} p(d_i) \cdot \prod_{i=1}^{n_c} p(\sigma_i)
 \end{aligned} \tag{79}$$

- Bayesian MMSE estimator $\Rightarrow \boldsymbol{\theta}_{MMSE} = \int \boldsymbol{\theta} p(\boldsymbol{\theta}|\mathbf{X}) d\boldsymbol{\theta}$
(Difficult to calculate!)
- Given $\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \dots, \boldsymbol{\theta}^M$ (samples drawn from $p(\boldsymbol{\theta}|\mathbf{X})$), the Bayesian MMSE estimator can be approximated by:

$$\tilde{\boldsymbol{\theta}}_{MMSE} = \frac{1}{M} \sum_{i=1}^M \boldsymbol{\theta}^i \tag{80}$$

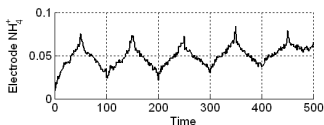
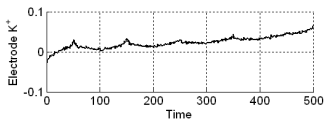
Results on synthetic data

- Synthetic sources (drawn from log-normal distributions);
- Noise level $\Rightarrow SNR_1 = SNR_2 = 18$ dB;
- Number of samples: 500;
- Performance index:

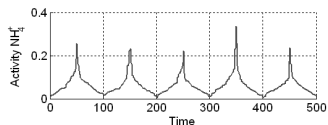
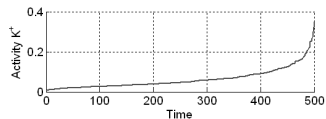
$$SIR_i = 10 \log \left(\frac{E\{s_i(t)^2\}}{E\{(s_i(t) - \hat{s}_i(t))^2\}} \right). \quad (81)$$

Results on synthetic data: mixtures

- Two ISEs ($n_c = 2$) for detecting the activities of NH_4^+ and K^+ ($n_s = 2$) \Rightarrow Post-nonlinear model;
- Mixing coefficients $a_{12} = 0.3$, $a_{21} = 0.4$, $d_1 = 0.056$, $d_2 = 0.056$, $e_1 = 0.090$, $e_2 = 0.105$;
- $SIR_1 = 19.7$ dB, $SIR_2 = 18.9$ dB, $SIR = 19.3$ dB



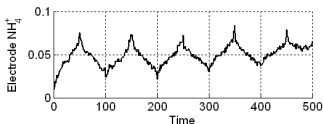
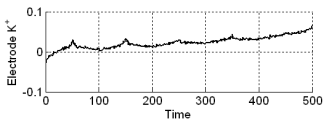
(a) Mixtures.



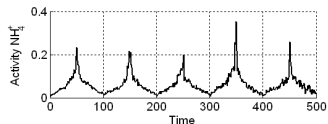
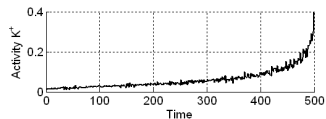
(b) Original sources.

Results on synthetic data: estimated sources

- Two ISEs ($n_c = 2$) for detecting the activities of NH_4^+ and K^+ ($n_s = 2$) \Rightarrow Post-nonlinear model;
- Mixing coefficients $a_{12} = 0.3$, $a_{21} = 0.4$, $d_1 = 0.056$, $d_2 = 0.056$, $e_1 = 0.090$, $e_2 = 0.105$;
- $SIR_1 = 19.7$ dB, $SIR_2 = 18.9$ dB, $SIR = 19.3$ dB



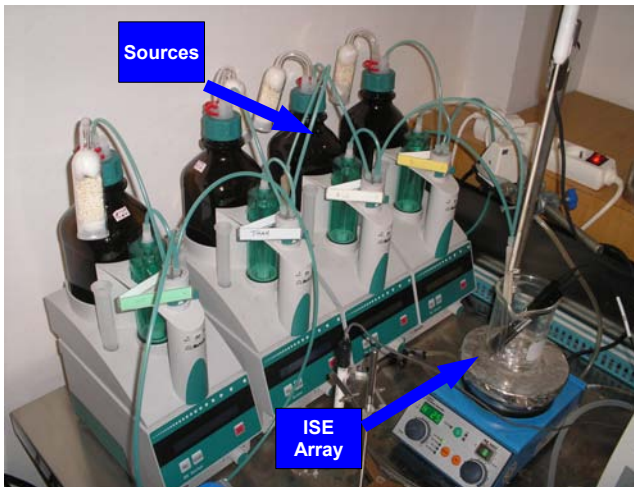
(a) Mixtures.



(b) Retrieved sources.

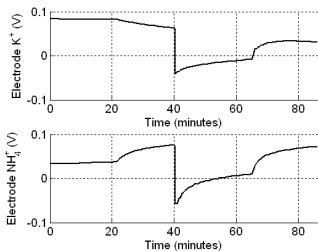
Results on real data

- ISE array ($NH_4^+ - ISE$ and $K^+ - ISE$)

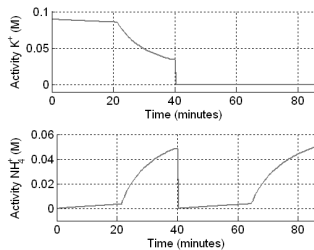


Results on real data (cont.)

$$n_d = 169, SIR_1 = 25.1 \text{ dB}, SIR_2 = 23.7 \text{ dB}, SIR = 24.4 \text{ dB}$$



(a) ISE array response.

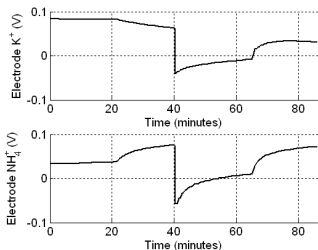


(b) Actual sources.

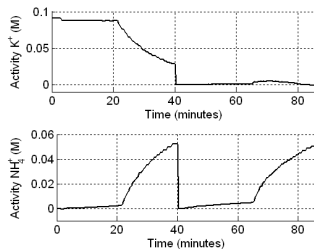
- Since the sources are clearly dependent here, an ICA-based method failed in this case.

Results on real data (cont.)

$$n_d = 169, SIR_1 = 25.1 \text{ dB}, SIR_2 = 23.7 \text{ dB}, SIR = 24.4 \text{ dB}$$



(a) ISE array response.



(b) Retrieved signals.

- Since the sources are clearly dependent here, an ICA-based method failed in this case.

Conclusions

- A Bayesian non-linear source separation was proposed for processing the outputs of an ISE array;
- Good results were obtained even in a complicated situation (dependent sources and reduced number of samples);
- Future works include
 - The development of a model that also takes into account the time-structure of the sources;
 - The case where neither the number of ions in the solution nor their valences are available;

Show-through effect

Cooperation with F. Merrikh-Bayat and M. Babaie-Zadeh, Sharif Univ. of Technology [MBBZJ08, MBBZJ11]

sous-titre

- Show-through, due to paper transparency and thickness,
- Pigment oil penetration,
- Vehicle oil component, due to loss of opacity,



State-of-the-art

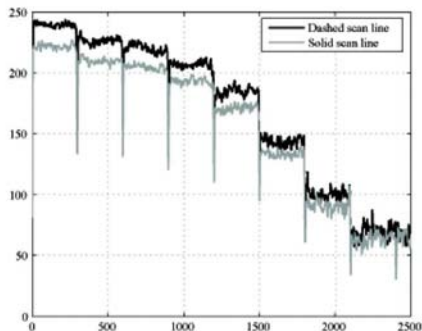
sous-titre

- Often applied for texts and handwriting documents: 1-side methods or 2-side methods,
- ICA assuming
 - Linear model of mixtures (Tonazzini et al., 2007 ; Ophir, Malah, 2007)
 - Nonlinear model of mixtures (Almeida, 2005 ; Sharma, 2001)
- In this work, we consider:
 - modelisation of the nonlinear mixture,
 - blurring effect.

Nonlinearity of show-through: experimental study

Evidence

- Sum of luminance is NL.
- Whiter the pixel, more important is show-through. More black than black is impossible !



Nonlinearity of show-through: mathematical model

Basic equation

- Show-through has a gain which depends of the grayscale of the front image
- It leads to the model of mixtures:

$$\begin{cases} f_r^s(x, y) = a_1 f_r^i(x, y) + b_1 f_v^i(x, y) g_1[f_r^i(x, y)] \\ f_v^s(x, y) = a_2 f_v^i(x, y) + b_2 f_r^i(x, y) g_2[f_v^i(x, y)] \end{cases}$$

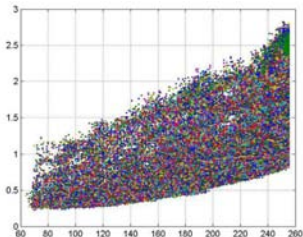
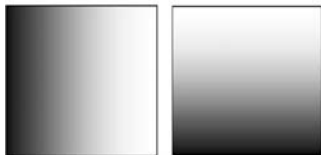
where i = initial, s = scanned, r = recto, v = verso, a_i and b_i denote unknown mixing parameters, and $g_i(\cdot)$ denote nonlinear gains

Nonlinearity of show-through: mathematical model

Shape of the gain

- The gain function can be estimated by computing

$$\begin{cases} g_1[f_r^i(x, y)] &= [f_v^s(x, y) - a_1 f_r^i(x, y)] / b_1 f_v^i(x, y) \\ g_2[f_v^i(x, y)] &= [f_v^s(x, y) - a_2 f_v^i(x, y)] / b_2 f_r^i(x, y) \end{cases}$$



Nonlinearity of show-through: mathematical model

Approximation of the gain function

- The gain function can be estimated by an exponential:

$$\begin{cases} g_1[f_r^i(x, y)] &= \gamma_1 \exp[\beta_1 f_r^i(x, y)] \approx \gamma_1(1 + \beta_1 f_r^i(x, y)) \\ g_2[f_v^i(x, y)] &= \gamma_2 \exp[\beta_2 f_v^i(x, y)] \approx \gamma_2(1 + \beta_2 f_v^i(x, y)) \end{cases}$$

- It leads to the approximated mixing model:

$$\begin{cases} f_r^s(x, y) &= a_1 f_r^i(x, y) + b'_1 f_v^i(x, y)[1 + \beta_1 f_r^i(x, y)] \\ f_v^s(x, y) &= a_2 f_v^i(x, y) + b'_2 f_r^i(x, y)[1 + \beta_2 f_v^i(x, y)] \end{cases}$$

- And finally:

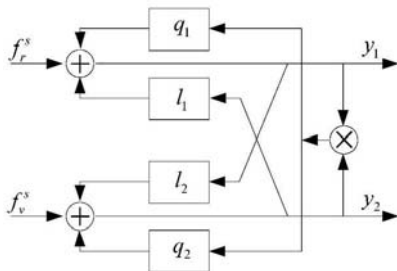
$$\begin{cases} f_r^s(x, y) &= a_1 f_r^i(x, y) - l_1 f_v^i(x, y) - q_1 f_v^i(x, y) f_r^i(x, y) \\ f_v^s(x, y) &= a_2 f_v^i(x, y) + l_2 f_r^i(x, y) - q_2 f_r^i(x, y) f_v^i(x, y) \end{cases}$$

Separation structure

Recursive structure

- Studied by Deville and Hosseini [HD03, DH09]

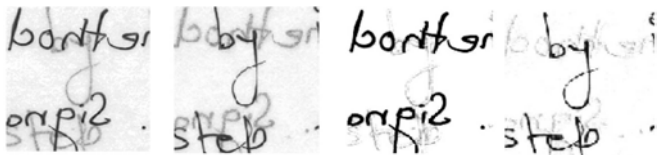
$$\begin{cases} f_r^s(x, y) = a_1 f_r^i(x, y) - l_1 f_v^i(x, y) - q_1 f_v^i(x, y) f_r^i(x, y) \\ f_v^s(x, y) = a_2 f_v^i(x, y) + l_2 f_r^i(x, y) - q_2 f_r^i(x, y) f_v^i(x, y) \end{cases}$$



Cancellation of show-through: preliminary results

Preliminary results with NL model

- Bilinear model neither always invertible, nor always stable.
- Parameters estimated by ML.



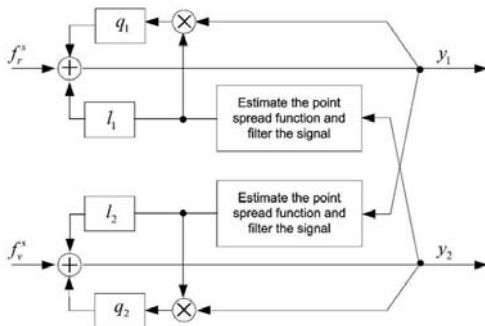
Comments

- The other side image never perfectly removed, especially when no superimposition.
- It means difference between verso image and recto image is not a simple gain
- Diffusion in the paper \Rightarrow blurring effect, modelled by 2D filter.

Improved model and recursive structure

Model with filtering

- The mixture is not the nonlinear superimposition of the recto image with the verso image, but with a *filtered* version of the verso image, hence the final separation structure



Cancellation of show-through

Final results with NL modeling and filtering



Scanned images: conclusions and perspectives

Summary

- Show-through is a NL phenomenon which can be modeled by bilinear mixtures ;
- In addition, it is necessary to consider the blurring effect, which can be viewed as a 2-D filtering effect.
- Experimental results show the mixture model has to take into account both NL and convolutive effects.

Perspectives

- For avoiding registration due to the 2-side scanning, one could explore if two scans of the same side (under different conditions) provide sufficient diversity;
- Other priors, like positivity of images, and of the coefficients could be exploited, e.g. by NMF or Bayesian approaches.

XI. Bibliography



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