Stability of networked control systems under communication delays
Summer School on Automatic Control

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Outline

1. Introduction
2. Input delay
3. Wirtinger-based inequality
4. Impulsive system
5. Looped functionals
6. Conclusions
Classical problems in automatic control

Objectives

Design a continuous-time control law such that the closed-loop system $\dot{x} = f(x(t), u(t))$ is asymptotically stable
Due to the emergence of powerful processors, computers, communications media, closing the control loop introduces additional constraints:

New objectives [HNX07, Zam08]

To take into account the *constraints induced by these digital components* in the control loop
Examples of networked control systems

Some examples:

1) Multiple control loops in a car:
   * Centralized controller (dedicated slots for each tasks)
   * Actuator loops (red ones)
   * Sensor loops (green ones)

2) Tele-operated systems:
   * Control over a reliable network (wire links)
   * Digital channels (discrete data)
   * 2 interconnected control loops (Master ↔ Slave)
Examples of networked control systems

Some examples:

4) Computer Controlled Systems:

- Certification of the control laws
- Processor Failure (packet loss)
- Digital computation (discrete data)

Application to flight control
(PhD. P Andrianiaina with D. Simon, NeCS/GIPSA-Lab)

Objectives of the PhD:

- Research of less conservative implementation
- Guaranty of stability wrt. dead-line miss
- End-to-end control quality
- Security of the controller
- Robustness of the closed-loop system
Some examples:

4) Computer Controlled Systems:

- Certification of the control laws
- Processor Failure (packet loss)
- Digital computation (discrete data)

Application to flight control
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$T = WCET$: Certification vs. poor performances
Some examples:

4) Computer Controlled Systems:
   * Certification of the control laws
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   * Digital computation (discrete data)

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\( T < WCET \): No certification vs. improved performances
Some examples:

4) Computer Controlled Systems:
   * Certification of the control laws
   * Processor Failure (packet loss)
   * Digital computation (discrete data)

Application to flight control
(PhD. P Andrianiaina with D. Simon, NeCS/GIPSA-Lab)

Context & objectives of the thesis
   * PhD in computer science (certification of processus)
   * Robustness of control laws to ensure certification

⇒ PhD in automatic control
Some examples:

3) Fleet of multiple mobile robots:

* Control over unreliable network (wireless communication)
* Multi-agent systems (≠ Master-Slave)
* Decentralized Control (no supervision)

Goal: Detect & follows gradients of concentration of the source flows by scientific sensors located on-board,
Networked Control Systems (NCS):

**Systems wherein the control loops are closed through a real-time network.**

The defining feature of an NCS is that control and feedback signals are exchanged among the system’s components in the form of information packages through a network.

NCS can be categorized into the three major fields:
Objectives:

To design feedback control strategies while control data are exchanged through a unreliable communication link.
Objectives:
To design feedback control strategies while control data are exchanged through a unreliable communication link.

Key Problems:
To ensure stability of the system while exchanged data are corrupted by components of a network.
1) Quantization & Coding:

Numerical approximation.

\[ \hat{x}(t) = q(x(t)) \]

(the real information is truncated)

Example:

\[ x = \pi \quad \Rightarrow \quad \hat{x} = 3.14 \]
2) Communication Delay:

Transfert of information.

\[ \hat{x}(t) = x(t - \delta) \]

(the real information is not available before \( h \) second.)

Example: \((\delta = \pi/2)\)

\[ x(t) = \cos(t) \quad \rightarrow \quad \hat{x}(t) (= \cos(t - \pi/2)) = \sin(t). \]
3) Sampling of the information flow:

\[ \hat{x}(t) = x(t_k) \]

- \( t_k \): sampling instants

\((\text{the real information is held})\)

**Example:** \( t_k = k\pi \)

\[ x(t) = \cos(t) \quad \Rightarrow \quad \hat{x}(t) = (-1)^k \]
4) Quantification + Delay + Sampling

\[ \hat{x}(t) = q(x(t_k - \delta_k)) \]

- \( t_k \): sampling instants
- \( \delta_k \): delay of the \( k^{th} \) packet
- \( q(\cdot) \): quantification

Objectives:

Design control laws which are **robust with respect to the effects of the network**.
Hybrid dynamical systems review: dynamics

\[ \mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G) \]

- \( n \in \mathbb{N} \) (state dimension)
- \( \mathcal{C} \subseteq \mathbb{R}^n \) (flow set)
- \( \mathcal{D} \subseteq \mathbb{R}^n \) (jump set)
- \( F : \mathcal{C} \to \mathbb{R}^n \) (flow map)
- \( G : \mathcal{D} \to \mathbb{R}^n \) (jump map)

\[ \mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases} \]
Hybrid dynamical systems review: continuous dynamics

\( \mathcal{H} = (C, D, F, G) \)

- \( n \in \mathbb{N} \) (state dimension)
- \( C \subseteq \mathbb{R}^n \) (flow set)
- \( D \subseteq \mathbb{R}^n \) (jump set)
- \( F : C \Rightarrow \mathbb{R}^n \) (flow map)
- \( G : D \Rightarrow \mathbb{R}^n \) (jump map)

\[ \begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + x_2(1 - x_1^2)
\end{aligned} \]

Van der Pol
Hybrid dynamical systems review: **discrete dynamics**

\[ H = (\mathcal{C}, \mathcal{D}, F, G) \]

- \( n \in \mathbb{N} \) (state dimension)
- \( \mathcal{C} \subseteq \mathbb{R}^n \) (flow set)
- \( \mathcal{D} \subseteq \mathbb{R}^n \) (jump set)
- \( F : \mathcal{C} \Rightarrow \mathbb{R}^n \) (flow map)
- \( G : \mathcal{D} \Rightarrow \mathbb{R}^n \) (jump map)

\[ H \ni \begin{cases} 
\dot{x} \in F(x), & x \in \mathcal{C} \\
x^+ \in G(x), & x \in \mathcal{D}
\end{cases} \]

\[ x^+ \in \begin{cases} 
\{0, 1\} & \text{if } x = 0 \\
\{0, 2\} & \text{if } x = 1 \\
\{1, 2\} & \text{if } x = 2
\end{cases} \]

A possible sequence of states from \( x_0 = 0 \) is:

\[(0 \cdot 1 \cdot 2 \cdot 1)^i, \quad i \in \mathbb{N}\]
5) Multi-agent systems: neighborhood communication

\[
\hat{x}_i(t) = \begin{bmatrix}
\gamma_1 & 0 & \ldots & 0 \\
0 & \gamma_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \gamma_n \\
\end{bmatrix} x(t), \quad \gamma_i = 0, 1.
\]

(Agent \( i \) has only a partial vision of its neighborhood)

Consensus algorithm

\[
x(\infty) = \frac{\sum_i^5 x_i(0)}{5}
\]

\[
\dot{x} = -Lx
\]
5) Multi-agent systems: neighborhood communication

\[ \hat{x}_i(t) = \begin{bmatrix} \gamma_1 & 0 & \ldots & 0 \\ 0 & \gamma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \gamma_n \end{bmatrix} x(t), \quad \gamma_i = 0, 1. \]

(Agent \(i\) has only a partial vision of its *neighborhood*)
Topic of the lecture: Control over networks

- Motivations
- First approach: stability analysis based on the discretized model
- Impulsive system approach
- Looped functionals approaches
- Event-triggered control
Consider \( \{t_k\}_{k \in \mathbb{N}} \) be an increasing sequence of sampling instants and two positive scalars \( T_1 \leq T_2 \) such that

\[
\bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}) = [0, +\infty)
\]

and

\[
\forall k \in \mathbb{N}, \quad T_k = t_{k+1} - t_k \in [T_1, T_2].
\]
Consider the following sampled-data system

\[ \forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + Bu(t_k), \tag{1} \]

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) represent the state and the input vectors.

The control law is a linear state feedback, \( u = Kx \) with a given gain \( K \in \mathbb{R}^{m \times n} \). The system is governed by

\[ \forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k). \tag{2} \]
The objective is to guarantee robust stability of system

\[ \forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k). \]

- w.r.t. possible variations of \( T_k = t_{k+1} - t_k \)
- w.r.t. uncertainties in the matrices \( A \) and \( B \).
∀ \, t \in [t_k, \, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k).

If the matrices $A$, $B$ and $K$ are constant and known, one can easily compute the transition matrix $\Phi$, as follows

$$
\Phi(s) = e^{As} + \int_0^s e^{A(s-\theta)} d\theta B
$$

which guarantees that

$$
x(t_k + s) = \Phi(s)x(t_k), \quad \forall s \in [t_k, \, t_{k+1}],
$$
∀t ∈ [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k).

If the matrices A, B and K are constant and known, one can easily compute the transition matrix Φ, as follows

\[ \Phi(s) = e^{As} + \int_0^s e^{A(s-\theta)} d\theta B \]

which guarantees that

\[ x(t_k + s) = \Phi(s)x(t_k), \quad \forall s \in [t_k, t_{k+1}], \]

and in particular

\[ x(t_{k+1}) = \Phi(T_k)x(t_k). \]

⇒ How to ensure stability stability of such systems?
If the sampling period is constant, i.e. \( T_k = T > 0 \) for all \( k \geq 0 \), then the following theorem holds.

**Schur Criteria**

The discrete-time system \( x(t_{k+1}) = \Phi(T)x(t_k) \) is asymptotically stable if the transition matrix \( \Phi(T) \) is Schur, i.e.

\[
|\lambda(\Phi(T))| < 1
\]
If the sampling period is constant, i.e. $T_k = T > 0$ for all $k \geq 0$. then the following theorem holds

**Lyapunov Theorem**

The discrete-time system $x(t_{k+1}) = \Phi(T)x(t_k)$ is asymptotically stable if there exists a matrix $P > 0$ such that the transition matrix $\Phi(T)$ satisfies the following LMI

$$\Phi^T(T)P\Phi(T) - P < 0$$

**Proof**

Consider the Lyapunov function $V(x(t_k)) = x^T(t_k)Px(t_k)$. The the increment of the Lyapunov function leads to

$$\Delta V(x(t_k)) = x^T(t_k)\left(\Phi^T(T)P\Phi(T) - P\right)x(t_k) < 0$$
If the sampling period is constant, i.e. \( T_k = T > 0 \) for all \( k \geq 0 \), then the following theorem holds

In this situation, the two previous theorems are equivalent, i.e.

\[
|\lambda(\Phi(T))| < 1 \quad \iff \quad \Phi^T(T)P\Phi(T) - P \prec 0
\]

Asymptotic Stability
If the sampling period is constant, i.e. $T_k = T > 0$ for all $k \geq 0$, then the following theorem holds

In this situation, the two previous theorems are equivalent, i.e.

\[ |\lambda(\Phi(T))| < 1 \iff \Phi^T(T)P\Phi(T) - P \prec 0 \]

Asymptotic Stability

⇒ Does this equivalence still hold for time-varying samplings?
If the sampling is non-uniform (\(T_k\) is time-varying),...

*Do the previous theorems hold??*
If the sampling is non-uniform ($T_k$ is time-varying),...
Consider the following example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} x(t_k)$$

$\Rightarrow |\lambda(\Phi(T))| < 1$ has become neither necessary nor sufficient.
If the sampling is non-uniform ($T_k$ is time-varying),...

Hopefully, the Lyapunov Theorem still holds

**Lyapunov Theorem**

The discrete-time system $x(t_{k+1}) = \Phi(T)x(t_k)$ is asymptotically stable if there exists a matrix $P \succ 0$ such that the transition matrix $\Phi$ satisfies the following LMI

$$\Phi^T(T_k)P\Phi(T_k) - P \preceq 0, \quad \forall T_K$$

- If $T_k$ varies within a *finite set of possible values*, this conditions can be easily solved.
- The matrix $P$ is the same for all the possible values of $T_k$. (conservatism???)
If the sampling is non-uniform ($T_k$ is time-varying),...

Additionally, the Lyapunov Theorem can be precised by entering into
the framework of **Hybrid Systems** and **Switched Systems**

**Lyapunov Theorem**

The discrete-time system $x(t_{k+1}) = \Phi(T)x(t_k)$ is asymptotically
stable if there exists a matrix $P \succ 0$ such that the transition matrix
$\Phi$ satisfies the following LMI

$$\Phi^T(T_{\sigma(k)})P_{\sigma(k)}\Phi(T_{\sigma(k)}) - P_{\sigma(k-1)} \prec 0, \quad \forall \sigma, \forall k$$

where $\sigma : \mathbb{N} \to \{1, \ldots, N\}$

- More stability conditions can be obtained using the discretized
model using LPV or Polytopic approaches
Limitations of the discrete-time approaches:

- If the matrices $A$ and $B$ which define the systems are subject to parametric uncertainties, the extensions of the discrete-time approach are complicated

$$
\Phi(s) = e^{(A+\Delta A)s} + \int_0^s e^{(A+\Delta A)(s-\theta)} d\theta (B + \Delta B)
$$

- If the matrices $A$ and $B$ are time-varying, the computation of $\Phi$ is a tight problem

$$
\Phi(s) \neq e^{A('t')s} + \int_0^s e^{A('t')(s-\theta)} d\theta B(t)
$$

since it requires $A(t_1)$ and $A(t_2)$ commutes for all $t_1, t_2$ in $\mathbb{R}$ (i.e. $A(t_1)A(t_2) = A(t_2)A(t_1)$).
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$\Rightarrow$ no "efficient" extensions to include robustness with respect to parameter uncertainties.
Limitations of the discrete-time approaches:

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$\Rightarrow$ no "efficient" extensions to include robustness with respect to parameter uncertainties.

$\Rightarrow$ There is a need for an alternative approach to take into account these problems.
The objective is to guarantee robust stability of system

\[ \forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k). \]

- w.r.t. possible variations of \( T_k = t_{k+1} - t_k \)
- w.r.t. uncertainties in the matrices \( A \) and \( B \).
The objective is to guarantee robust stability of system

\[ \forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = Ax(t) + BKx(t_k). \]

- w.r.t. possible variations of \( T_k = t_{k+1} - t_k \)
- w.r.t. uncertainties in the matrices \( A \) and \( B \).

The main idea is to see this differential equation as a functional differential equation

\[ x(t_k) \text{ is a past value of } x(t). \]

and see if the methods usually employed to assess stability of systems with time-varying delays are still applicable.
Basics of the input delay approach

Sampled signal (here with a uniform sampling)

Delayed signal with the particular delay function

\[ \tau(t) = t - t_k, \quad \forall t \in [t_k, t_{k+1}) \]
Basics of the input delay approach

Indeed, by a smart introduction of the current-time the sampled signal can be rewritten as

\[ x(t_k) = x(t - t + t_k) = x(t - (t - t_k)) = x(t - \tau(t)) \]

However analysis stability of systems with such delays was not trivial

- Input-to state stability (Nesic, Teel 1998??)
- Classical results from the literature required

\[ \dot{\tau}(t) < 1 \]

while in the case of sampled-data systems

\[ \dot{\tau}(t) = 1, \quad \forall t \neq t_k \]
Basics of the input delay approach

Stability analysis of time-delay systems

Consider a linear time delay system given by

\[
\begin{cases}
\dot{x}(t) = Ax(t) + BKx(t - \tau(t)) & \forall t \geq 0, \\
x(t) = \phi(t) & \forall t \in \left[-\tau_{\text{max}}, 0\right],
\end{cases}
\]  
(3)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( \phi \) is the initial condition and \( A, BK \in \mathbb{R}^{n \times n} \) are constant matrices. The delay \( h \) is a positive scalar and satisfies the constraints:

\[
\tau(t) \in [0, \tau_m], \quad \dot{\tau}(t) \in [d_{\text{min}}, 1]
\]  
(4)

where \( \tau_m \) are given positive constants.
Basics of the input delay approach
⇒ Exhibit Lyapunov-Krasovskii functionals to assess stability.

Lyapunov-Krasovskii approach: A first approach

One of the most popular/classical terms of LK functionals is [FS02a],...

\[ V(x_t) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Sx(s)ds \]

Its derivative can be expressed as

\[ \dot{V}(x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} PA + A^TP + S & PBK \\ (PBK)^T & (1-\dot{\tau}(t))S \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \]

Stability results from the Lyapunov-Krasovskii Theorem and is expressed using the LMI framework...
Basics of the input delay approach
⇒ Exhibit Lyapunov-Krasovskii functionals to assess stability.

Lyapunov-Krasovskii approach: A first approach

The system is asymptotically stable if there exist $P \succ 0$ and $S \succ 0$ such that the following LMI is satisfied

$$
\begin{bmatrix}
PA + A^T P + S & PBK \\
(PBK)^T & (1 - \dot{\tau}(t))S
\end{bmatrix} \prec 0
$$

Remark

Since $\dot{\tau} = 1$, the previous LMI can not be satisfied.
Basics of the input delay approach
⇒ Exhibit Lyapunov-Krasovskii functionals to assess stability.

**Lyapunov-Krasovskii approach: A popular term**

One of the most popular/classical terms of LK functionals to derive delay-dependent stability conditions is [FS02a],...

\[
V(x_t) = \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta
\]

Its derivative is

\[
\dot{V}(x_t) = \tau_m \dot{x}^T(t)R\dot{x}(t) - \int_{t-\tau_m}^{t} \dot{x}^T(s)R\dot{x}(s)ds
\]

This functional requires the use relevant inequality such as
→ the Jensen’s inequality
Let us recall this famous lemma [Gu00].

**Jensen’s Inequality**

For given symmetric positive definite matrices $R \succ 0$ and for any differentiable signal $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$
\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du \geq \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \quad (5)
$$

**Proof:** For all $\varepsilon \leq 1$, $\dot{\omega}^T(u)R\dot{\omega}(u) - \varepsilon \dot{\omega}^T(u)R\dot{\omega}(u) \geq 0$. Since $R \succ 0$, the Schur Complement yields

$$
\begin{bmatrix}
\dot{\omega}^T(u)R\dot{\omega}(u) & \dot{\omega}^T(u) \\
\dot{\omega}(u) & (\varepsilon R)^{-1}
\end{bmatrix} \succ 0
$$
Let us recall this famous lemma [Gu00].

**Jensen’s Inequality**

For given symmetric positive definite matrices $R \succ 0$ and for any differentiable signal $\omega$ in $[a, b] \to \mathbb{R}^n$, the following inequality holds:

$$\int_{a}^{b} \dot{\omega}^T(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} (\omega(b) - \omega(a))^T R(\omega(b) - \omega(a))$$  \hspace{1cm} (5)

**Proof**: Compute the integral of the previous LMI

$$\begin{bmatrix} \int_{a}^{b} \dot{\omega}^T(u) R \dot{\omega}(u) du & \int_{a}^{b} \dot{\omega}^T(u) du \\ \int_{a}^{b} \dot{\omega}(u) du & \int_{a}^{b} du (\varepsilon R)^{-1} \end{bmatrix} = \begin{bmatrix} \int_{a}^{b} \dot{\omega}^T(u) R \dot{\omega}(u) du & (\omega(b) - \omega(a))^T \\ (\omega(b) - \omega(a))^T & (b-a)(\varepsilon R)^{-1} \end{bmatrix} \succeq 0$$

Finally the Schur complement ensures

$$\int_{a}^{b} \dot{\omega}^T(u) R \dot{\omega}(u) du - \varepsilon \frac{1}{b-a} (\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \geq 0$$

Finally, $\varepsilon \to 1$ yields the result.
Let us recall this famous lemma [Gu00].

**Jensen’s Inequality**

For given symmetric positive definite matrices $R \succ 0$ and for any differentiable signal $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$\int_a^b \dot{\omega}(u)^T R \dot{\omega}(u) du \geq \frac{1}{b-a} (\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \quad (5)$$

- Popular result for stability analysis of time-delay systems [...]
- Provides appropriate bound for DD stability conditions
- The Jensen inequality is usually combine with other efficient lemma for time-varying delay system [PKJ11]...
We are now in position to provide stability conditions for sampled-data systems: Consider the functional given by

\[ V(x_t) = x^T(t)Px(t) + \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)R\dot{x}(s)ds d\theta. \]

Differentiating \( V \) along the trajectories of the systems leads to

\[ \dot{V}(x_t) = 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) - \int_{t-\tau(t)}^t \dot{x}^T(s)R\dot{x}(s)ds. \]

**Step 1:** Since \( \tau(t) \leq \tau_m \), it yields

\[ \dot{V}(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) - \int_{t-\tau(t)}^t \dot{x}^T(s)R\dot{x}(s)ds. \]

**Step 2:** Thanks to the **Jensen’s inequality**, the following bound is derived

\[ \dot{V}(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) - \frac{1}{\tau(t)}(x(t) - x(t - \tau(t)))^T R(x(t) - x(t - \tau(t))). \]
This leads to the first stability theorem:

**Theorem 1 [FSR04]**

Assume that there exists $P > 0$ and $R > 0$, such that $\Psi_0 \prec 0$ where

\[
\Psi_0 = \left[ \begin{array}{cc}
PA + A^T P & \frac{1}{\tau_m} R \\
(PBK)^T & PBK + \frac{1}{\tau_m} R
\end{array} \right] + \tau_m \left[ \begin{array}{c}
A \\
BK
\end{array} \right]^T R \left[ \begin{array}{c}
A \\
BK
\end{array} \right]
\]

Then the sampled-data system is asymptotically stable for all samplings that satisfy $\tau(t) \leq \tau_m$. 
This leads to the first stability theorem:

**Theorem 1 [FSR04]**

Assume that there exists $P > 0$ and $R > 0$, such that $\Psi_0 \prec 0$ where

$$
\Psi_0 = \begin{bmatrix}
PA + A^T P - \frac{1}{\tau_m} R & PBK + \frac{1}{\tau_m} R \\
(PBK)^T & -\frac{1}{\tau_m} R
\end{bmatrix} + \tau_m \begin{bmatrix}
A \\
BK
\end{bmatrix}^T R \begin{bmatrix}
A \\
BK
\end{bmatrix}
$$

Then the sampled-data system is asymptotically stable for all samplings that satisfy $\tau(t) \leq \tau_m$.

**Comments:**

- Originally, based on the descriptor representation [FS02b],
This leads to the first stability theorem:

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\end{bmatrix}
+ \tau_m \begin{bmatrix}
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BK
\end{bmatrix}^T R \begin{bmatrix}
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\end{bmatrix}
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A \\
BK
\end{bmatrix}^T R \begin{bmatrix}
A \\
BK
\end{bmatrix}
$$

Then the sampled-data system (20) is asymptotically stable for all samplings that satisfy $\tau(t) \leq \tau_m$. [FS02b]

**Comments:**

- Originally, based on the descriptor representation [FS02b],
- **Delay-dependent** stability conditions,
- By use of the Schur complement, the conditions can be made linear w.r.t. the matrices $A$ and $B$. $\Rightarrow$ Robustness.
- Extensions to the stabilization problem.
Consider the linear time-delay system with the matrices

\[ E_1 : A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

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**Table:** The maximal allowable sampling period \( \tau_m \).
Consider the linear time-delay system with the matrices

\[ E_1 \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

**Theorems**  
\begin{array}{|c|c|c|}
\hline
Theorem 1 & \tau_m & NDV \\
\hline
              & 0.999 & \text{n}^2 + \text{n} \\
\hline
\end{array}

**Table**: The maximal allowable sampling period \( \tau_m \).

\[ E_2 \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

**Theorems**  
\begin{array}{|c|c|c|}
\hline
Theorem 1 & \tau_m & NDV \\
\hline
              & \emptyset & \text{n}^2 + \text{n} \\
\hline
\end{array}

**Table**: The maximal allowable sampling period \( \tau_m \).
Regarding the previous examples, the conditions from Theorem 1 have introduced a large conservatism.

Where does this conservatism come from?

**Step 1:** Since $\tau(t) \leq \tau_m$, it yields

$$V(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds.$$

**Step 2:** Thanks to the **Jensen’s inequality**, the following bound is derived

$$V(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) - \frac{1}{\tau(t)}(x(t) - x(t - \tau(t)))^T R(x(t) - x(t - \tau(t))).$$
Let us first focus our attention to Step 1.

**Step 1bis:** Since $\tau(t) \leq \tau_m$, it yields

$$
\dot{V}(x_t) = 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t)
- \int_t^{t-\tau(t)} \dot{x}^T(s)R\dot{x}(s)ds
- \int_{t-\tau_m}^{t-\tau(t)} \dot{x}^T(s)R\dot{x}(s)ds.
$$

**Step 2bis:** Again, applying the Jensen’s inequality leads ($\tau(t) = \tau$)

$$
\dot{V}(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t)
- \frac{1}{\tau}(x(t) - x(t - \tau))^T R(x(t) - x(t - \tau))
- \frac{1}{\tau_m - \tau}(x(t - \tau) - x(t - \tau_m))^T R(x(t - \tau) - x(t - \tau_m)).
$$

*How can we take into account the two last terms?*

*More particularly $\frac{1}{\tau}$ and $\frac{1}{\tau_m - \tau}$?*
Reciprocally convex combination lemma [PKJ11] (reformulated to our purpose)

**Lemma**

Let $n_1, n_2 \in \mathbb{N}$, $R_1 \succ 0$, $R_2 \succ 0$ and $\alpha$ in $(0, 1)$. Define:

$$\Theta(\alpha, R_1, R_2) = \begin{bmatrix} \frac{1}{\alpha} R_1 & 0 \\ \ast & \frac{1}{1-\alpha} R_2 \end{bmatrix}.$$  

If there exists a matrix $X$ in $\mathbb{R}^{n_1 \times n_2}$ such that $\begin{bmatrix} R_1 & X \\ \ast & R_2 \end{bmatrix} \succ 0$, then

$$\min_{\alpha \in (0, 1)} \Theta(\alpha, R_1, R_2) \geq \begin{bmatrix} R_1 & X \\ \ast & R_2 \end{bmatrix}.$$

The proof refers to [PKJ11].
Reciprocally convex combination lemma

**Proof:** Following [PKJ11], it is possible to show that

\[
\Theta(\alpha, R_1, R_2) = \begin{bmatrix}
R_1 & X \\
\ast & R_2
\end{bmatrix} + \begin{bmatrix}
\beta I & 0 \\
0 & -\frac{1}{\beta} I
\end{bmatrix} \begin{bmatrix}
R_1 & X \\
\ast & R_2
\end{bmatrix} \begin{bmatrix}
\beta I & 0 \\
0 & -\frac{1}{\beta} I
\end{bmatrix}.
\]

where \( \beta = \sqrt{\frac{1-\alpha}{\alpha}} \).

Indeed developing the second term, we get

\[
\Theta(\alpha, R_1, R_2) = \begin{bmatrix}
(1 + \frac{1-\alpha}{\alpha})R_1 & (1 - 1)X \\
\ast & (1 + \frac{\alpha}{1-\alpha})R_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\alpha} R_1 & 0 \\
\ast & \frac{1}{1-\alpha} R_2
\end{bmatrix}.
\]

Finally, providing that \( \begin{bmatrix}
R_1 & X \\
\ast & R_2
\end{bmatrix} \succ 0 \), we get that

\[
\Theta(\alpha, R_1, R_2) \geq \begin{bmatrix}
R_1 & X \\
\ast & R_2
\end{bmatrix}, \quad \forall \alpha \in (0, 1).
\]
Relation with the first stability theorem:

From Step 2bis we have

\[
\dot{V}(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) \\
- \frac{1}{\tau}(x(t) - x(t - \tau))^T R(x(t) - x(t - \tau)) \\
- \frac{1}{\tau_m - \tau}(x(t - \tau) - x(t - \tau_m))^T R(x(t - \tau) - x(t - \tau_m))
\]

which can be rewritten as

\[
\dot{V}(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) \\
- \frac{1}{\tau_m}\xi^T \begin{bmatrix} I & 0 \\ -I & I \\ 0 & -I \end{bmatrix} \Theta\left(\frac{\tau}{\tau_m}, R, R\right) \begin{bmatrix} I & 0 \\ -I & I \\ 0 & -I \end{bmatrix}^T \xi.
\]

where \(\xi = \begin{bmatrix} x^T(t) & x^T(t - \tau) & x^T(t - \tau_m) \end{bmatrix}\).

We can then use the previous lemma to derive the following theorem.
Theorem 2

Assume that there exist scalars $\tau_m > 0$ and some matrices $P \succ 0$, $S \succ 0$, $R \succ 0$, and $X$ in $\mathbb{R}^{n \times n}$ such that $\Theta_0 = \begin{bmatrix} R & X \\ X^* & R \end{bmatrix} \succ 0$ and $\Psi_1(\tau_m) \prec 0$ where

$$
\Psi_1(\tau_m) = \begin{bmatrix}
\psi_0 & PBK & 0 \\
(PBK)^T & 0 & 0 \\
0 & 0 & -S \\
\end{bmatrix} + \tau_m \begin{bmatrix}
A \\
BK \\
0 \\
\end{bmatrix}^T \begin{bmatrix}
A \\
BK \\
0 \\
\end{bmatrix} - \frac{1}{\tau_m} \begin{bmatrix}
I & 0 \\
-I & I \\
0 & -I \\
\end{bmatrix} \Theta_0 \begin{bmatrix}
I & 0 \\
-I & I \\
0 & -I \\
\end{bmatrix},
$$

and

$$
\psi_0 = PA + A^T P + S
$$

Then the system (20) is asymptotically stable for all samplings that satisfy $\tau(t) \leq \tau_m$. 
Proof: Consider a Lyapunov-Krasovskii functional of the form

\[
V(\tau, x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{t-\tau_m}^{t} x^T(s)Sx(s)ds \\
+ \int_{t-\tau_m}^{t} \int_{\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta.
\]  

(6)

Differentiating \( V \) along the trajectories of (20) leads to:

\[
\dot{V}(\tau, x_t, \dot{x}_t) \leq \xi^T \begin{bmatrix}
\psi_0 & PBK & 0 \\
(PBK)^T & 0 & 0 \\
0 & 0 & -S
\end{bmatrix} \xi \\
+ \xi^T \tau_m \begin{bmatrix}
A \\
BK \\
0
\end{bmatrix}^T R \begin{bmatrix}
A \\
BK \\
0
\end{bmatrix} \xi \\
- \int_{t-\tau_m}^{t} \dot{x}^T(s)R\dot{x}(s)ds,
\]  

(7)

where \( \xi = \begin{bmatrix} x^T(t) & x^T(t-\tau) & x^T(t-\tau_m) \end{bmatrix}^T \).

The end of the proof relies on the previous developments...
Consider the linear time-delay system (20) with the matrices

\[ E \begin{bmatrix} 1 \end{bmatrix} A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

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Table: The maximal allowable sampling period \( \tau_m \).
Consider the linear time-delay system (20) with the matrices 

\[ E_2 : 1 \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \]

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**Table:** The maximal allowable sampling period \( \tau_m \).

\[ E_2 : 2 \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

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**Table:** The maximal allowable sampling period \( \tau_m \).
The previous theorem is taken from [PKJ11], which is one of the most performant stability theorem form the literature, adapted to the situation where the delay function satisfies

\[ \tau(t) \in [0, \tau_m]. \]
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The potential improvements regarding the stability analysis of such systems using an LMI formulation are related to

- Complexity (i.e. number of decision variables)
- Conservatism reduction
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$$\tau(t) \in [0, \tau_m].$$

The potential improvements regarding the stability analysis of such systems using an LMI formulation are related to

- Complexity (i.e. number of decision variables)
- Conservatism reduction

⇒ **Open questions:**

- P1) Can we reduce the number of decision variables?
  → Alternative reciprocally convex combination lemma
- P2) Can we reduce the conservatism?
  → Wirtinger-based integral inequality
In the particular case $R_1 = R_2 = R$, the following lemma is derived to reduce the number of decisions variables.

**Modified reciprocally convex combination lemma**

Let $n \in \mathbb{N}$, $R$ in $\mathbb{R}^{n \times n}$ such that $R \succ 0$ and $\alpha$ in $(0, 1)$. Then the following inequality holds

$$\min_{\alpha \in (0, 1)} \Theta(\alpha, R, R) \geq \begin{bmatrix} R & -R \\ * & R \end{bmatrix}.$$ 

**Proof:** Following [PKJ11], it is possible to show that

$$\Theta(\alpha, R, R) = \begin{bmatrix} R & -R \\ * & R \end{bmatrix} + \begin{bmatrix} \beta I & 0 \\ 0 & -\frac{1}{\beta} I \end{bmatrix} \begin{bmatrix} R & -R \\ * & R \end{bmatrix} \begin{bmatrix} \beta I & 0 \\ 0 & -\frac{1}{\beta} I \end{bmatrix}.$$ 

where $\beta = \sqrt{\frac{1-\alpha}{\alpha}}$.

The result follows from $\begin{bmatrix} R & -R \\ * & R \end{bmatrix} = \begin{bmatrix} I \\ -I \end{bmatrix} R \begin{bmatrix} I \\ -I \end{bmatrix}^T$ and $R \succ 0$. 


In the particular case $R_1 = R_2 = R$, the following lemma is derived to reduce the number of decision variables.

**Modified reciprocally convex combination lemma**

Let $n$ in $\mathbb{N}$, $R$ in $\mathbb{R}^{n \times n}$ such that $R \succ 0$ and $\alpha$ in $(0, 1)$. Then the following inequality holds

$$
\min_{\alpha \in (0, 1)} \Theta(\alpha, R, R) \geq \begin{bmatrix} R & -R \\ * & R \end{bmatrix}.
$$

**Recall of the original lemma**

Let $n_1, n_2$ in $\mathbb{N}$, $R_1 \succ 0, R_2 \succ 0$ and $\alpha$ in $(0, 1)$. If there exists a matrix $X$ in $\mathbb{R}^{n_1 \times n_2}$ such that

$$
\begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix} \succ 0,
$$

then

$$
\min_{\alpha \in (0, 1)} \Theta(\alpha, R_1, R_2) \geq \begin{bmatrix} R_1 & X \\ * & R_2 \end{bmatrix}.
$$
Theorem

Assume that there exist scalars $\tau_m > 0$ and some matrices $P \succ 0$, $S \succ 0$, $R \succ 0$, and $X$ in $\mathbb{R}^{n \times n}$ such that $\Theta_0 = \begin{bmatrix} R & X \\ \ast & R \end{bmatrix} \succ 0$ and $\Psi_1(\tau_m) \prec 0$ where

$$
\Psi_1(\tau_m) = \begin{bmatrix}
\psi_0 & P \sqrt{K} & 0 \\
(P \sqrt{K})^T & 0 & 0 \\
0 & 0 & -S \\
\end{bmatrix} + \tau_m \begin{bmatrix}
A \\
B \sqrt{K} \\
0 \\
\end{bmatrix}^T \begin{bmatrix}
A \\
B \sqrt{K} \\
0 \\
\end{bmatrix}
$$

$$
-\frac{1}{\tau_m} \begin{bmatrix}
I \\
-2I \\
-2I \\
I \\
\end{bmatrix} \Theta_0 \begin{bmatrix}
I \\
-2I \\
-2I \\
I \\
\end{bmatrix}^T,
$$

and

$$
\psi_0 = PA + A^T P + S
$$

Then the system (3) is asymptotically stable for all samplings that satisfy $\tau(t) \leq \tau_m$. 
Example 1: $A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$

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Table: The maximal allowable sampling period $\tau_m$. 

**Reciprocally convex combination lemma**
Reciprocally convex combination lemma

\[ AX : 1 \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

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Table: The maximal allowable sampling period \( \tau_m \).

\[ AX : 2 \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

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Table: The maximal allowable sampling period \( \tau_m \).
Can we reduction of the conservatism of Step 2 (Step 2bis)?

Step 2bis: Again, applying the Jensen’s inequality leads ($\tau(t) = \tau$)

$$\dot{V}(x_t) \leq 2x^T(t)P\dot{x}(t) + \tau_m\dot{x}^T(t)R\dot{x}(t) - \frac{1}{\tau}(x(t) - x(t - \tau))^T R(x(t) - x(t - \tau)) - \frac{1}{\tau_m - \tau}(x(t - \tau) - x(t - \tau_m))^T R(x(t - \tau) - x(t - \tau_m)).$$

**Jensen’s Inequality**

For given symmetric positive definite matrices $R \succ 0$ and for any differentiable signal $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du \geq \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \quad (8)$$

Where does this conservatism come from?

From the Jensen inequality as a robust analysis shows →
Stability of time-delay systems can be deduced by robust analysis. Delay elements are embedded into a norm bounded uncertainty. The use of Small Gain Theorem allows to conclude.

Introducing the Lyapunov functional

\[ \int_{t}^{\infty} \int_{\nu}^{t} \dot{x}(\omega)^T R \dot{x}(\omega) d\omega d\nu \]

and using Jensen inequality is equivalent to consider that \( s \in \mathbb{C}^+, |\delta_1(s)| < 1, \delta_1(s) = \frac{1-e^{-hs}}{hs} \), i.e.

\[ \begin{bmatrix} 1_n \\ \delta_1(s)1_n \end{bmatrix}^* \begin{bmatrix} -R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1_n \\ \delta_1(s)1_n \end{bmatrix} < 0. \]

\( \delta_1 \) is embedded in a norm-bounded uncertainty.

\( \Rightarrow \) There is room for improvements!
Jensen Inequality

For given symmetric positive definite matrices $R \succ 0$ and for any differentiable signal $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du \geq \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a))$$  \hspace{1cm} (9)

**Alternative Proof:** Consider the following integral:

$$I = \int_a^b (\dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b-a})^T R(\dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b-a})du \geq 0$$

This implies that

$$I = \int_a^b \dot{\omega}(u)R\dot{\omega}(u)du - \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \geq 0$$
Objective

Following the alternative proof, it has been noticed that the Jensen’s inequality can be rewritten as

\[ I = \int_a^b (\dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a})^T R(\dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a}) \, du \geq 0 \]

This leads to the idea of finding less conservative inequalities of the same form. This means

\[ I \geq g(\dot{\omega}, \omega, b - a, ...) \]

This can be provided by looking at Wirtinger’s inequalities.
Wirtinger’s inequality

Let $z$ be a differentiable function of $[a \ b] \rightarrow \mathbb{R}^n$ such that $z(a) = z(b) = 0$. Then for any $n \times n$ matrix $R \succ 0$, the following inequality holds

$$
\int_a^b \dot{z}^T(u)R\dot{z}(u)du \geq \frac{\pi^2}{(b - a)^2} \int_a^b z^T(u)Rz(u)du
$$

(10)

This inequality and related ones have been already employed in the context of time-delay systems ([FO09, LF12])
Wirtinger’s inequality

Let $z$ be a differentiable function of $[a \ b] \rightarrow \mathbb{R}^n$ such that $z(a) = z(b) = 0$. Then for any $n \times n$ matrix $R \succ 0$, the following inequality holds

$$
\int_{a}^{b} \dot{z}(u) R \dot{z}(u) du \geq \frac{\pi^2}{(b - a)^2} \int_{a}^{b} z^T(u) R z(u) du
$$

(11)

This inequality and related ones have been already employed in the context of time-delay systems ([FO09, LF12])

⇒ How to relate this inequality to Jensen?

$$
\int_{a}^{b} \left( \dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a} \right)^T R \left( \dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a} \right) du \geq 0
$$
Wirtinger’s inequality [SG13]

Let $z$ be a differentiable function of $[a \ b] \to \mathbb{R}^n$ such that $z(a) = z(b) = 0$. Then for any $n \times n$ matrix $R \succ 0$, the following inequality holds

$$
\int_a^b \dot{z}^T(u) R \dot{z}(u) \, du \geq \frac{\pi^2}{(b - a)^2} \int_a^b z^T(u) R z(u) \, du \tag{12}
$$

This inequalities have been already employed in the context of time-delay systems ([FO09, LF12])

⇒ How to relate this inequality to Jensen?

$$
\int_a^b (\dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a})^T R (\dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a}) \, du \geq 0.
$$

⇒ Idea: Take $\dot{z}(u) = \dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b - a}$. 
Construction of $z$

If $\dot{z}$ is chosen such that $\dot{z}(u) = \dot{\omega}(u) - \frac{\omega(b) - \omega(a)}{b-a}$. Then $z$ is given by

$$z(u) = \omega(u) - \left(\frac{u - a}{b - a}\right) (\omega(b) - \omega(a)) + C.$$  

where $C$ is a constant vector that has to be defined.

In order to apply the Wirtinger’s inequality, the function $z$ has to meet the condition $z(a) = z(b) = 0$. This implies that

$$z(a) = \omega(a) + C = 0$$
$$z(b) = \omega(b) - (\omega(b) - \omega(a)) + C = \omega(a) + C = 0$$

Then the conditions of apply the Wirtinger’s inequality can be satisfies if $C = -\omega(a)$ and $z$ is rewritten as

$$z(u) = \omega(u) - \left(\frac{u - a}{b - a}\right) \omega(b) - \left(\frac{b - u}{b - a}\right) \omega(a).$$
Lemma [SG12]

For all \( R \succ 0 \), and all function \( \omega \) in \([a, b] \rightarrow \mathbb{R}^n\), it holds:

\[
\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{\pi^2/4}{b-a} \Omega_2^T R \Omega_2
\]

where \( \Omega_1 = \omega(b) - \omega(a) \) and \( \Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du \).

Proof: Consider the previous function \( z(u) \). Then the Wirtinger’s and Jensen’s inequalities ensure that

\[
\int_a^b z^T(u) R z(u) du \geq \frac{\pi^2}{(b-a)^2} \int_a^b z^T(u) R z(u) du
\]

\[
\geq \frac{\pi^2}{(b-a)^3} \left( \int_a^b z(u) du \right)^T R \left( \int_a^b z(u) du \right)
\]

with

\[
\int_a^b z(u) du = -\frac{(b-a)}{2} \Omega_2
\]
Another construction of $z$

Regarding the construction of the previous function, can we consider a additional polynomial term. Consider then $\tilde{z}$ given by

$$\tilde{z}(u) = \omega(u) - \left(\frac{u - a}{b - a}\right)\omega(b) - \left(\frac{b - u}{b - a}\right)\omega(a) + \frac{(u - a)(b - u)}{(b - a)^2} \Theta$$

where $\Theta$ is a vector to be defined.

By construction, this function satisfies the boundary conditions $\tilde{z}(a) = \tilde{z}(b) = 0$. We can then apply the Wirtinger inequality.
Lemma [SG13]

For all $R > 0$, and all function $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, it holds:

$$\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2$$

where $\Omega_1 = \omega(b) - \omega(a)$ and $\Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du$.

Proof: Consider the previous function $\tilde{z}(u)$. Then the Wirtinger’s and Jensen’s inequalities ensure that

$$\int_a^b \dot{\tilde{z}}^T(u) R \dot{\tilde{z}}(u) du \geq \frac{\pi^2}{(b-a)^3} \left( \int_a^b \tilde{z}(u) du \right)^T R \left( \int_a^b \tilde{z}(u) du \right)$$

Some computation shows that $\frac{1}{b-a} \int_a^b \tilde{z}(u) du = \frac{1}{6} (\Theta - 3\Omega_2)$
Lemma [SG13]

For all \( R > 0 \), and all function \( \omega \) in \([a, b] \to \mathbb{R}^n\), it holds:

\[
\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2
\]

where \( \Omega_1 = \omega(b) - \omega(a) \) and \( \Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du \).

The derivative of \( \tilde{z} \) is \( \dot{\tilde{z}}(u) = \dot{\omega}(u) - \frac{1}{b-a} \Omega_1 + \frac{b+a-2u}{(b-a)^2} \Theta \).

Some calculations show that

\[
\int_a^b \dot{\tilde{z}}^T(u) R \dot{\tilde{z}}(u) du = \int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du - \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{1}{3(b-a)} \Theta^T R \Theta + \frac{2}{(b-a)} \Theta^T R \Omega_2. \tag{13}
\]
Lemma [SG13]

For all $R > 0$, and all function $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, it holds:

$$\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2$$

where $\Omega_1 = \omega(b) - \omega(a)$ and $\Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du$.

The derivative of $\tilde{z}$ is $\dot{\tilde{z}}(u) = \dot{\omega}(u) - \frac{1}{b-a} \Omega_1 + \frac{b+a-2u}{(b-a)^2} \Theta$.

Some calculations show that

$$\int_a^b \dot{\tilde{z}}^T(u) R \dot{\tilde{z}}(u) du = \int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du - \frac{1}{b-a} \Omega_1^T \frac{R}{(b-a)^2} \Omega_1$$

$$- \frac{3}{(b-a)} \Omega_2^T R \Omega_2$$

$$+ \frac{1}{3(b-a)} (\Theta - 3\Omega_2)^T R (\Theta - 3\Omega_2).$$

(14)

$$\left(\int_a^b \tilde{z}(u) du\right)^T R \left(\int_a^b \tilde{z}(u) du\right) = \frac{b-a}{36} (\Theta - 3\Omega_2)^T R (\Theta - 3\Omega_2).$$

(15)
Lemma [SG13]

For all \( R > 0 \), and all function \( \omega \) in \([a, b] \rightarrow \mathbb{R}^n\), it holds:

\[
\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2
\]

where \( \Omega_1 = \omega(b) - \omega(a) \) and \( \Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du \).

The derivative of \( \tilde{z} \) is \( \dot{\tilde{z}}(u) = \dot{\omega}(u) - \frac{1}{b-a} \Omega_1 + \frac{b+a-2u}{(b-a)^2} \Theta \).

Some calculations show that

\[
\int_a^b \dot{\tilde{z}}^T(u) R \dot{\tilde{z}}(u) du = \int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du - \frac{1}{b-a} \Omega_1^T \frac{R}{(b-a)^2} \Omega_1 \\
- \frac{3}{(b-a)} \Omega_2^T R \Omega_2 \\
+ \frac{12-\pi^2/4}{36(b-a)} (\Theta - 3\Omega_2)^T R (\Theta - 3\Omega_2).
\]

Finally, taking \( \Theta = 2\Omega_2 \) yields the result.
Lemma [SG13]

For all $R > 0$, and all function $\omega$ in $[a, b] \rightarrow \mathbb{R}^n$, it holds:

$$
\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2
$$

where $\Omega_1 = \omega(b) - \omega(a)$ and $\Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du$.

- The second term of (40) is positive definite. \(\rightarrow\) encompass Jensen inequality
- Less conservative than the first Wirtinger-based inequality.
- Introduction of the same signal $\frac{1}{b-a} \int_a^b \omega(u) du$. 


Theorem 3: [SGF13]

Assume that there exist $P \succ 0$, $S \succ 0$, $R \succ 0$ and $X$ in $\mathbb{R}^{2n \times 2n}$, such that the following LMIs are satisfied for $\tau$ in $\{0 \tau_m\}$

$$
\Phi(\tau) = \Phi_0(\tau) - \frac{1}{\tau_m} \left[ \begin{array}{c} G_2 \\ G_3 \end{array} \right]^T \Theta_0 \left[ \begin{array}{c} G_2 \\ G_3 \end{array} \right] \prec 0,
$$

(17)

where $\Theta_2 = \left[ \begin{array}{cc} \tilde{R} & X \\ * & \tilde{R} \end{array} \right] \succ 0$ and

$$
\Phi_0(\tau) = \text{He}(G_1^T(\tau)PG_0) + \hat{S} + \tau_m G_0^T \hat{R} G_0,
$$

$$
\hat{S} = \text{diag}(S, 0, -S, 0_{2n}), \quad \hat{R} = \text{diag}(R, 0_{3n}), \quad \tilde{R} = \text{diag}(R, 3R),
$$

$$
G_0 = \left[ \begin{array}{cccc} A & A_d & 0 & 0 \\ I & 0 & -I & 0 \\ I & 0 & 0 & 0 \\ I & -I & 0 & 0 \end{array} \right], \quad G_1(\tau) = \left[ \begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & \tau I & (\tau_m - \tau)I \end{array} \right],
$$

$$
G_2 = \left[ \begin{array}{cccc} I & -I & 0 & 0 \\ I & 0 & -2I & 0 \\ I & 0 & 0 & -2I \end{array} \right], \quad G_3 = \left[ \begin{array}{cccc} 0 & -I & I & 0 \\ 0 & I & I & 0 \end{array} \right].
$$

The system (20) is asymptotically stable for all time-varying delay $\tau$ in $[0, \tau_m]$. 
Consider the Lyapunov-Krasovskii functional given by

\[ V_2(\tau, x_t, \dot{x}_t) = \left[ \begin{array}{c} x(t) \\ \int_{t-\tau_m}^t x(s) ds \end{array} \right]^T P \left[ \begin{array}{c} x(t) \\ \int_{t-\tau_m}^t x(s) ds \end{array} \right] + \int_{t-\tau_m}^t x^T(s) S x(s) ds + \int_{t-\tau_m}^t (\tau_m - t + s) \dot{x}^T(s) R \dot{x}(s) ds, \tag{18} \]

By noting that

\[ \int_{t-\tau_m}^t x(s) ds = \tau \frac{1}{\tau} \int_{t-\tau}^t x(s) ds + (\tau_m - \tau) \frac{1}{\tau_m - \tau} \int_{t-\tau_m}^{t-\tau} x(s) ds \]
\[ \frac{d}{dt} \int_{t-\tau_m}^t x(s) ds = x(t) - x(t - \tau_m) \]

the differentiation of (18) along the trajectories of (20) leads to:

\[ \dot{V}_2(\tau, x_t, \dot{x}_t) = \zeta^T(t) \psi_0(\tau) \zeta(t) - \int_{t-\tau_m}^t \dot{x}^T(s) R \dot{x}(s) ds, \tag{19} \]

where \( \zeta(t) = \left[ \begin{array}{cccc} x^T(t) & x^T(t - \tau) & x^T(t - \tau_m) & \frac{1}{\tau} \int_{t-\tau}^t x^T(s) ds & \frac{1}{\tau_m - \tau} \int_{t-\tau_m}^{t-\tau} x^T(s) ds \end{array} \right]. \]
Following the same procedure as previously but applying the Wirtinger-based inequality, we get that

\[
\int_{t-\tau_m}^{t} \dot{x}^T(s) R \dot{x}(s) ds \geq \zeta^T(t) \left[ \frac{G_2^T \tilde{R} G_2}{\tau} + \frac{G_3^T \tilde{R} G_3}{\tau_m - \tau} \right] \zeta(t).
\]

Applying the improved reciprocally convex inequality

\[
\int_{t-\tau_m}^{t} \dot{x}^T(s) R \dot{x}(s) ds \geq \frac{1}{\tau_m} \zeta^T(t) \begin{bmatrix} G_2 \\ G_3 \end{bmatrix}^T \Theta_0 \begin{bmatrix} G_2 \\ G_3 \end{bmatrix} \zeta(t).
\]

Finally, we get

\[
\dot{V}_2(x_t, \dot{x}_t) \leq \zeta^T(t) \left[ \psi_0(\tau) - \frac{1}{\tau_m} \begin{bmatrix} G_2 \\ G_3 \end{bmatrix}^T \Theta_0 \begin{bmatrix} G_2 \\ G_3 \end{bmatrix} \right] \zeta(t).
\]
Finally using the alternative improved reciprocally convex combination, the following theorem is obtained.

**Theorem 5**

Assume that there exist $P \succ 0$, $S \succ 0$, $Q \succ 0$, $R \succ 0$ such that the following LMIs are satisfied for $h$ in $\{\tau_m, \tau_m\}$ and for $\dot{h}$ in $\{d_m, d_M\}$

\[
\Psi_2(\tau) = \Phi_0(\tau) - \frac{1}{\tau_m} \left[ G_2 - G_3 \right]^T R \left[ G_2 - G_3 \right] \prec 0,
\]

where $\Psi_0(\tau)$, $\hat{S}$ have the same definition as in the previous theorem. The system (20) is asymptotically stable for all time-varying delay $\tau$ in $[0, \tau_m]$. 
Ex: 1 \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}

<table>
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<th>NDV</th>
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<td>Theorem 1</td>
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<td>( n^2 + n )</td>
<td>3, 26</td>
</tr>
<tr>
<td>Theorem 2</td>
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<tr>
<td>Theorem 4</td>
<td>2.128</td>
<td>( 7n^2 + 2n )</td>
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<tr>
<td>Theorem 5</td>
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<td>( 3n^2 + 2n )</td>
<td>3, 26</td>
</tr>
</tbody>
</table>

Table: The maximal allowable sampling period \( \tau_m \).
Ex: 2 \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},

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Table: The maximal allowable sampling period $\tau_m$. 

⇒ Where does the conservatism come from?
Sampled-data Systems vs. Time-delay systems

Are these two classes of systems equivalent?

**Initial conditions:**

Time-delay systems
\[ x_0(\theta) = \phi(\theta), \quad \forall \theta \in [-\tau_m, 0] \]

Sampled-data systems
\[ x(0) = x(t_0) \]

**Delay function:**

Time-delay systems
\[ \begin{cases} 
\tau(t) \in [0, \tau_m] \\
\dot{\tau}(t) = \text{??} \\
\tau(t_k) = \text{??} 
\end{cases} \]
(includes the case, \( \dot{\tau}(t) = 0 \ldots \))

Sampled-data systems
\[ \begin{cases} 
\tau(t) \in [0, \tau_m] \\
\dot{\tau}(t) = 1 \\
\tau(t_k) = 0 
\end{cases} \]
Sampled-data Systems vs. Time-delay systems

Are these two classes of systems equivalent?
Consider the following example

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1 \quad 0] x(t_k)
\]

\begin{align*}
\tau &= 0.15 \\
\tau &= 1 \\
\tau &= 1.7 \\
\tau &= 2.2
\end{align*}
Sampled-data Systems vs. Time-delay systems

Are these two classes of systems equivalent?

There is a need for a more dedicated approach to include the constraints on the delay function which represent the sampling delay...

This leads us to the *impulsive systems approach*
Basics of the impulsive systems approach

Impulsive system representation:

Consider a linear time delay system given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + BKx(t_k) \quad \forall t \geq 0, \\
x(t_0) &= x_0,
\end{align*}
\]  

(20)

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(x_0\) is the initial condition and \(A, BK \in \mathbb{R}^{n \times n}\) are constant matrices. The sampling instants are such that

\[
T_k = t_{k+1} - t_k \in [\tau_{min}, \tau_m],
\]

(21)

where \(\tau_{min}\) and \(\tau_m\) are given positive constants.
Basics of the impulsive systems approach

**Impulsive system representation:**

Consider a linear time delay system given by

\[
\begin{align*}
\begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
x(t^+) \\
y(t^+)
\end{bmatrix}
&= \begin{bmatrix}
A & BK \\
0 & 0 \\
I & 0 \\
I & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix}, \quad t \neq t_k, \\
\begin{bmatrix}
x(t_0) \\
y(t_0)
\end{bmatrix} = \begin{bmatrix} x_0 \\
y_0
\end{bmatrix}, \quad t = t_k,
\end{align*}
\]

(22)

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(y(t) \in \mathbb{R}^n\), is the sampled version of \(x(t)\), \(x_0\) is the initial condition and \(A, BK \in \mathbb{R}^{n \times n}\) are constant matrices. The sampling instants are such that

\[
T_k = t_{k+1} - t_k \in [\tau_{min}, \tau_m],
\]

(23)

where \(\tau_{min}\) and \(\tau_m\) are given positive constants.
One can use the tool dedicated tools to assess stability of *Hybrid Systems*.

**Lemma [LF12]**

Let there exist positive numbers $\alpha$, $\beta$ and a functional $V : \mathbb{R} \times \mathcal{W}[-\tau_m, 0]) \times L_2[-\tau_m, 0] \rightarrow \mathbb{R}$ such that

$$\alpha |\phi(0)|^2 \leq V(t, \phi, \dot{\phi}) \leq \beta \|\phi\|_{W}^2.$$ 

Let $\bar{V}(t) = V(t, x_t, \dot{x}_t)$, is continuous from the right for $x(t)$, solution of the system, absolutely continuous for $t = t_k$ and satisfies

$$\lim_{t \to t_k^-} \bar{V}(t) \geq \bar{V}(t_k).$$

If, along the solution of the system, $\dot{\bar{V}}(t) \leq -\gamma |x(t)|^2$ for $t = t_k$ and for some scalar $\gamma > 0$, then the system is asymptotically stable.
Stability of Impulsive systems.
What does it change?
1) New time-dependent terms can be considered in the Lyapunov-Krasovskii functionals:

\[ V_1(t, x_t) = (t_{k+1} - t)(x(t) - x(t_k))^T S(x(t) - x(t_k)) \]
\[ V_2(t, x_t) = 2(t_{k+1} - t)(x(t) - x(t_k))^T Q x(t_k) \]
\[ V_3(t, \dot{x}_t) = (t_{k+1} - t) \int_{t_k}^{t} \dot{x}^T(s) R \dot{x}(s) ds \]
\[ V_4(t, x(t_k)) = (t_{k+1} - t)(t - t_k)x^T(t_k) X x(t_k) \]

The positivity is ensured by \( R \succ 0 \) and

\[
\begin{bmatrix}
    S & -S + Q \\
    * & S - Q - Q^T
\end{bmatrix} \succ 0
\]

\[ V_1(t, x_t) + V_2(t, x_t) = (t_{k+1} - t) \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix}
    S & -S + Q \\
    * & S - Q - Q^T
\end{bmatrix} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix} \]
Theorem 6 [Seu12]

If there exist $P \succ 0$, $S \succ 0$, $Q$, $X \succ 0$, $R \succ 0$ and a matrix $N \in \mathbb{R}^{2n \times n}$ s.t.

$$
\Pi_1 + \tau_m \Pi_2 \prec 0,
\begin{bmatrix}
\Pi_1 & \tau_m N \\
* & -\tau_m R
\end{bmatrix} \prec 0,
\begin{bmatrix}
S & -S + Q \\
* & S - Q - Q^T
\end{bmatrix} \succ 0
$$

where

$$
\Pi_1 = \tilde{\Pi}_1 - NM_2 - M_2^T N^T,
\tilde{\Pi}_1 = M_1^T P M_3 + M_3^T P M_1 - M_2^T S M_2 - \tau_m M_2^T X M_2,
\Pi_2 = M_2^T S M_3 + M_3^T S M_2 + M_3^T R M_3 + M_2^T X M_2,
$$

and the matrices $M_i$, for $i = 1, 2, 3$ are given by:

$$
M_1 = \begin{bmatrix} I & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} I & -I \end{bmatrix}, \quad M_3 = \begin{bmatrix} A & A_d \end{bmatrix}.
$$

Then the sampled-data system is asymptotically for any time-varying period less than $\tau_m$. 

Sketch of the proof: Consider the functional given by

\[ V(t, x_t, \dot{x}_t) = x^T(t)Px(t) + V_1(t, x_t) + V_2(t, x_t) + V_3(t, \dot{x}_t) \]

Differentiating this functional with respect to time along the trajectories of the systems,

\[ \dot{V}(t, x_t, \dot{x}_t) = \xi^T \left[ \tilde{\Pi}_1 + (t_{k+1} - t)\Pi_2 - \int_{t_k}^{t} \dot{x}^T(s)R\dot{x}(s)ds \right] \xi \]

where \( \xi = \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix} \). Then the Jensen's inequality ensures that for all matrix \( N \)

\[ \dot{V}(t, x_t, \dot{x}_t) = \xi^T \left[ \Pi_1 + (t_{k+1} - t)\Pi_2 + (t - t_k)NR^{-1}N^T \right] \xi \]

Finally the results is obtained by ensuring that \( \Pi_1 + (t_{k+1} - t)\Pi_2 + (t - t_k)NR^{-1}N^T \prec 0 \), for all \( t \in [t_k, t_{K+1}] \).
Impulsive systems approach

\[ E_x : 1 \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

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**Table:** The maximal allowable sampling period \( \tau_m \).
\[ \text{Ex. 2} \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

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**Table:** The maximal allowable sampling period \(\tau_m\).

(thanks to the contribution of\n\[ V_4(t, x(t_k)) = (t_{k+1} - t)(t - t_k)x^T(t_k)Xx(t_k). \]
2) New terms Include some relevant terms based on more involved inequalities. (Wirtinger inequality, again... but a different version)

Wirtinger inequality (3\textsuperscript{rd} version) [LF12]

Let \( z \in W[a, b] \) and \( z(a) = 0 \). Then the following inequality holds, for all \( R \succ 0 \).

\[
\int_a^b \dot{z}(u) R \dot{z}(u) du \geq \frac{\pi^2}{4(b-a)^2} \int_a^b z^T(u) R z(u) du
\]

Main idea:
Introduce a term of the form:

\[
V_5(x_t, \dot{x}_t) = \tau_m^2 \int_{t_k}^t \dot{x}^T(s) R \dot{x}(s) ds - \frac{\pi^2}{4} \int_{t_k}^t (x(s) - x(t_k))^T R (x(s) - x(t_k)) ds
\]

\[
\dot{V}_5(x_t, \dot{x}_t) = \tau_m^2 \dot{x}^T(t) R \dot{x}(t) - \frac{\pi^2}{4} (x(t) - x(t_k))^T R (x(t) - x(t_k))
\]

But it won’t be presented here...
A remark on the last term:

It may also deals with the case of sampling and communication delays with only light modifications

\[
V'_5(x_t, \dot{x}_t) = (\tau_m + h)^2 \int_{t_k-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds \\
- \frac{\pi^2}{4} \int_{t_k-h}^{t-h} (x(s) - x(t_k - h))^T R (x(s) - x(t_k - h)) ds
\]

and

\[
\dot{V}'_5(x_t, \dot{x}_t) = (\tau_m + h)^2 \dot{x}^T(t) R \dot{x}(t) \\
- \frac{\pi^2}{4} (x(t - h) - x(t_k - h))^T R (x(t - h) - x(t_k - h))
\]
An additional remark:

The previous theorem has been obtained by application of the Jensen inequality.
An additional remark:

The previous theorem has been obtained by application of the Jensen inequality.

As in the input delay approach, it is also possible to derive stability conditions using the Improved Wirtinger inequality... but it won’t be presented here...
An additional remark:

Is it not possible to consider a functional of the form

\[ V_6(t, x_t) = (t - t_k)(x(t) - x(t_{k+1}))^T S(x(t) - x(t_{k+1})) \]
An additional remark:

Is it not possible to consider a functional of the form

\[
V_6(t, x_t) = (t - t_k)(x(t) - x(t_{k+1}))^T S(x(t) - x(t_{k+1}))
\]

No, because there is no \( \theta_0 \) in \([-\tau_m, 0]\), such that

\[
x_t(\theta_c) = x(t + \theta_0) = x(t_{k+1})
\]

Indeed \( t_{k+1} \geq t \).
In the previous slide, we show that designing dedicated Lyapunov-Krasovskii functionals is a *crucial* step in the construction of stability conditions.

Looking at the terms defined in the previous theorem

\[
\begin{align*}
V_1(t, x_t) &= (t_{k+1} - t)(x(t) - x(t_k))^T S(x(t) - x(t_k)) \\
V_2(t, x_t) &= 2(t_{k+1} - t)(x(t) - x(t_k))^T Q x(t_k) \\
V_3(t, \dot{x}_t) &= (t_{k+1} - t) \int_{t_k}^{t} \dot{x}^T(s) R \dot{x}(s) ds \\
V_4(t, x(t_k)) &= (t_{k+1} - t)(t - t_k)x^T(t_k) X x(t_k)
\end{align*}
\]

one can see that they also satisfy the following property

\[
V_i(t_k, x_t) = V_i(t_{k+1}, x_t) = 0
\]

Can we use this property in the design of the functional?
The looped-functional based approach relies on the characterization of the trajectories of system in a lifted domain [Yam90, Seu12, BS12b, BS12a].

**Lifting**

Therefore, we view the entire state-trajectory as a sequence of functions

$$\{y(t_k + \tau), \ \tau \in (0, T_k]\}_{k \in \mathbb{N}},$$

with elements having a unique continuous extension to $[0, T_k]$ defined as

$$\chi_k(\tau) := y(t_k + \tau), \ \chi_k(0) = \lim_{s \downarrow t_k} y(s).$$

(25)
Assume that $T_k \in [\tau_{\text{min}}, \tau_m]$

**Looped functionals [BS12b, BS12a]**

A functional

$$f : [0, \tau_m] \times \mathbb{K}_{[\tau_{\text{min}}, \tau_m]} \times [\tau_{\text{min}}, \tau_m] \rightarrow \mathbb{R}$$

is said to be a **looped-functional** if the following conditions hold

1. the equality

$$f(0, z, T) = f(T, z, T), \quad (26)$$

holds for all functions $z \in C([0, T], \mathbb{R}^n) \subset \mathbb{K}_{[\tau_{\text{min}}, \tau_m]}$ and all $T \in [\tau_{\text{min}}, \tau_m]$, and

2. it is differentiable with respect to the first variable with the standard definition of the derivative.
Theorem [Seu12]

Let $0 < \varepsilon < \tau_{min} \leq \tau_m$ and $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a quadratic form verifying

$$\forall y \in \mathbb{R}^n, \quad \mu_1 \|x\|_2^2 \leq V(x) \leq \mu_2 \|x\|_2^2,$$

for some scalars $0 < \mu_1 \leq \mu_2$. Assume that one of the following equivalent statements hold:

(i) The sequence $\{V(\chi_k(T_k))\}_{k \in \mathbb{N}}$ is decreasing

(ii) There exists $V \in \mathcal{L}_f([\tau_{min}, \tau_m])$ such that

$$\mathcal{W}_k(\tau, \chi_k, \chi_{k-1}) := V(\chi_k(\tau)) + V(\tau, \chi_k, T_k),$$

has a derivative along the trajectories of system

$$\dot{\mathcal{W}}_k(\tau, \chi_k) := \dot{V}(\chi_k(\tau)) + \dot{V}(\tau, \chi_k, T_k) < 0,$$

for all $\tau \in (0, T_k), \ T_k \in [\tau_{min}, \tau_m], \ k \in \mathbb{N}$.

Then, the solutions of system are asymptotically stable for any sampling sequence $\{t_k\}_{k \in \mathbb{N}}$ satisfying $t_{k+1} - t_k \in [\tau_{min}, \tau_m], \ k \in \mathbb{N}$. 
Motivations

What are the benefits of the previous theorem:

This theorem is not based on the Lyapunov-Krasovskii theorem. (no ‘\(x_t\)’ but ‘\(\chi_k\)’)

If a term of the functional meets the boundary conditions, \(\Rightarrow\) No positivity constraints are required.

Use the same Lyapunov functional as in the previous Theorem

\[
\mathcal{W}(0, \chi_k) = V(t_k) \\
\mathcal{W}(I_k, \chi_k) = V(t_{k+1})
\]
Theorem 7

If there exist $P \succ 0$, $R \succ 0$, $S$, $X$, $Q$ and a matrix $N \in \mathbb{R}^{2n \times n}$ s.t.

\[
\begin{align*}
\Pi_1 + \tau_m \Pi_2 &< 0, \\
\begin{bmatrix}
\Pi_1 & \tau_m N \\
* & -\tau_m R
\end{bmatrix} &< 0,
\end{align*}
\]

(27)

where

\[
\begin{align*}
\Pi_1 &= \tilde{\Pi}_1 - NM_2 - M_2^T N^T, \\
\tilde{\Pi}_1 &= M_1^T PM_3 + M_3^T PM_1 - M_2^T SM_2 - \tau_m M_2^T XM_2, \\
\Pi_2 &= M_2^T SM_3 + M_3^T SM_2 + M_3^T RM_3 + M_2^T XM_2,
\end{align*}
\]

and the matrices $M_i$, for $i = 1, 2, 3$ are given by:

\[
M_1 = \begin{bmatrix} I & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} I & -I \end{bmatrix}, \quad M_3 = \begin{bmatrix} A & A_d \end{bmatrix}.
\]

Then the sampled-data system is asymptotically for any time-varying period less than $\tau_m$. 
\[ E_1 : A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

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**Table:** The maximal allowable sampling period \( \tau_m \).
Example

\[ Ex : 2 \quad A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

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Concluding remarks:

- A review of the most relevant method to address the stability of sampled-data systems (Input delay, impulsive system, looped functionals)
- A set of the most accurate inequalities to provide accurate LMI conditions (Jensen Inequality, Reciprocally convex combination, Wirtinger-based inequality)
- Discussion about the ratio Conservatism/ number of decision variables
Perspectives, extensions and open problems:

- Sum of square for sampled-data systems [?] 
- Sampled-data systems $\subset$ Hybrid systems (i.e. impulsive systems & switched systems) [?] 
- Is it possible to reduce the conservatism of the Wirtinger-based inequality? 
- Stability analysis of mixed sampled and communication delays 
- ...
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