Set-theoretic Approach to Analysis and Control for Nonlinear Hybrid Systems.

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Outline

1. Introduction, motivations and objectives
   - Set-methods in control

2. Set-theory for nonlinear systems
   - Convex Difference Inclusions for DT systems
   - Viability theory for CT systems
   - Nonlinear hybrid systems
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Aim:
Analysis and control design for nonlinear and uncertain systems using set-theoretic methods, invariance in particular.

Consider the discrete-time system $x^{+} = f(x)$, where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition
Set $\Omega \subseteq \mathbb{R}^n$ is invariant if every trajectory with $x_0 \in \Omega$ remains in $\Omega$.

Geometric condition: $\Omega \subseteq \mathbb{R}^n$ invariant iff $f(\Omega) \subseteq \Omega$. 
Given $\lambda \in [0, 1)$:

- $\Omega \subseteq \mathbb{R}^n$ is $\lambda$-contractive if $f(\Omega) \subseteq \lambda \Omega$.
- $\lambda$-contractiveness $\Rightarrow$ invariance.

For linear (parametric uncertain) systems, any $\lambda$-contractive set induces a Lyapunov function, (Blanchini 1994).

Such property does not hold for generic nonlinear systems!
Nagumo theorem

**Nagumo theorem - Continuous-time system: \( \dot{x} = f(x) \)**

A set is **invariant** if and only if the velocity vector is directed towards the **interior** (or tangent to the boundary) of the set at any point of the **boundary**.

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*Blanchini and Miani (2008), Set-Theoretic Methods in Control*
Invariance in control

- Stability
- Convergence
- Constraints satisfaction
- Lyapunov functions
- λ-Contractiveness

MPC

Set-theory for nonlinear systems
Set-methods in control
Invariance for nonlinear systems

For linear systems:
- well established theoretical and computational results,
- iterative procedures (mainly for discrete-time systems),
- boundary-type condition for invariance, also for discrete-time systems,
- set-induced Lyapunov functions.

Problem
When moving from linear to nonlinear systems, useful properties related to linearity are lost \( \Rightarrow \) adaptation of tools for linear systems to the nonlinear case is not trivial.

Few general results for generic nonlinear (and hybrid) systems!
Motivation and objectives

- Invariance and set-theory are very important for analysis and control.
- Many well established results for linear systems.
- Few practical results for nonlinear and hybrid systems, mainly based on LDI approximations.

Objective

Contribute to reduce the gap between the importance of invariance and set-theory and the practical applicability of the obtained results, especially for nonlinear and hybrid systems.

Key idea: The "missing" ingredient to adapt methods and properties to nonlinear and hybrid systems is convexity.
1 Introduction, motivations and objectives
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Set-valued maps $F : \mathbb{R}^n \rightarrow \mathcal{J}(\mathbb{R}^n)$, i.e. $F(x) \subseteq \mathbb{R}^n$.

Difference and differential inclusions (Aubin et al.)

$x^+ \in F(x)$, discrete-time,
$\dot{x}(t) \in F(x(t))$, continuous-time,

where $F : \mathbb{R}^n \rightarrow \mathcal{J}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$.

Modeling scenarios:
- uncertain systems,
- constrained controlled system,
- approximation of nonlinear systems.
Given $\Omega \subseteq \mathbb{R}^n$, the support function of $\Omega$ at $\eta \in \mathbb{R}^n$ is

$$\phi_{\Omega}(\eta) = \sup_{x \in \Omega} \eta^T x.$$ 

- **Geometrically:** signed “distance” of the point of $\Omega$ further from the origin.
- For a convex, closed set $\Omega \subseteq \mathbb{R}^n$, $x \in \Omega$ iff

$$\eta^T x \leq \phi_{\Omega}(\eta), \quad \forall \eta \in \mathbb{R}^n.$$
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2 Set-theory for nonlinear systems
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Convex Difference Inclusions: CDI systems

A CDI system is given by

\[ x^+ \in \mathcal{F}(x), \]

where \( x \in \mathbb{R}^n \) and \( \mathcal{F}(\cdot) \) satisfies the assumption.

**Assumption**

The set valued map \( \mathcal{F} : \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n) \) is such that, for every \( \eta \in \mathbb{R}^n \), function \( \check{\mathcal{F}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) defined as

\[ \check{\mathcal{F}}(x, \eta) = \sup_{z \in \mathcal{F}(x)} \eta^T z, \]

is convex w.r.t. \( x \) on \( \mathbb{R}^n \), and \( \check{\mathcal{F}}(0, \eta) = 0 \).

**Equivalently**: CDI system if \( \mathcal{F} : \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n) \) is such that

\[ \mathcal{F}(\alpha x^1 + (1 - \alpha)x^2) \subseteq \alpha \mathcal{F}(x^1) \oplus (1 - \alpha) \mathcal{F}(x^2), \]

for every \( \alpha \in [0, 1] \) and every \( x^1, x^2 \in \mathbb{R}^n \), and \( \mathcal{F}(0) = \{0\} \).
Linear Difference Inclusions (LDI) and Linear Parameter Varying (LPV) systems are CDI systems, for instance:

\[ x(k + 1) = A(k)x(k) \in \{\Delta x(k) : \|\Delta\| \leq 1\} \]

The dynamics could be defined by a finite number of functions.
Convex Difference Inclusions: CDI systems

CDI systems enclose and approximate many classes of nonlinear systems.

- **Generalized saturated systems.** *(Tarbouriech et al. AUT11)*

- **DC systems:** \( x^+ = g(x) - h(x) \), with \( g(\cdot), h(\cdot) \) convex *(Fiacchini, Alamo, Camacho AUT10).*

- **LDI and LPV systems.**

- **Systems** \( x^+ = f(x) \), with \( f(\cdot) \) twice differentiable:

\[
\dot{F}(x, \eta) = \sum_{j=1}^{n} \{ \eta_j f_j(x_0) + (x - x_0)^T \nabla f_j(x_0)) + \rho_j |\eta_j|(x - x_0)^T (x - x_0) \}.
\]
Necessary and sufficient condition for invariance

- For $\Omega \subseteq \mathbb{R}^n$ convex and closed, $x \in \Omega$ iff
  \[ \eta^T x \leq \phi_\Omega(\eta), \quad \forall \eta \in \mathbb{R}^n. \]

- For CDI, the set $\Omega \in \mathcal{K}^0(\mathbb{R}^n)$ is $\lambda$-contractive iff
  \[ \eta^T z \leq \lambda \phi_\Omega(\eta), \quad \forall z \in \mathcal{F}(x), \quad \forall x \in \Omega, \quad \forall \eta \in \mathbb{R}^n, \]
  in fact, it is equivalent to:
  \[ \mathcal{M}_{\mathcal{F}}(\Omega) = \bigcup_{x \in \Omega} \mathcal{F}(x) \subseteq \lambda \Omega. \]

**Problem:** involves every $x \in \Omega$, **not a boundary-type** condition!
Theorem (Fiacchini, Alamo, Camacho SCL12)

For a CDI system, a convex, compact set $\Omega \in \mathcal{K}^0(X)$ is a $\lambda$-contractive set, with $\lambda \in [0, 1]$, if and only if

$$\tilde{F}(x, \eta) \leq \lambda \phi_\Omega(\eta), \quad \forall x \in \partial \Omega, \quad \forall \eta \in \mathbb{R}^n.$$ 

It is a boundary-type condition given by convex constraints!
Induced Lyapunov functions for CDI systems

Proposition

Given a CDI system and a $\lambda$-contractive set $\Omega \in \mathcal{K}^0(X)$, also the set $\alpha \Omega \in \mathcal{K}^0(X)$ is $\lambda$-contractive, with the same contracting factor, for all $\alpha \in [0, 1]$.

This leads to set-induced Lyapunov functions for CDI systems.

Definition: Given $\Omega \in \mathcal{K}^0(\mathbb{R}^n)$ and $D \in \mathcal{S}(\mathbb{R}^n)$, define $\mathcal{V}_\Omega : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{R}$:

$$\mathcal{V}_\Omega(D) = \min_{\alpha \geq 0} \{ \alpha : D \subseteq \alpha \Omega \}.$$
Induced Lyapunov functions for CDI systems

**Proposition** *(Fiacchini, Alamo, Camacho SCL12)*

Given a CDI system, any \( \lambda \)-contractive set \( \Omega \in \mathbb{X}^0(X) \) with contracting factor \( \lambda \in [0, 1) \) induces a local Lyapunov function, \( \mathcal{V}_\Omega(\cdot) \), on \( \mathcal{S}(\Omega) \).

Given \( X_0 \in \mathcal{S}(\Omega) \), we have

\[
\mathcal{V}_\Omega(X_{k+1}) \leq \lambda \mathcal{V}_\Omega(X_k) < \mathcal{V}_\Omega(X_k),
\]

\[\Rightarrow \mathcal{V}_\Omega(X_k) \leq \lambda^k, \quad \forall k \in \mathbb{N}.\]

Geometrically, \( X_0 \subseteq \Omega \) implies

\[X_k \subseteq \lambda^k \Omega, \quad \forall k \in \mathbb{N}.\]
Convexity and maximal invariant set

**Proposition**

Given a CDI system and two $\lambda$-contractive sets $\Omega_1 \in \mathcal{K}^0(X)$ and $\Omega_2 \in \mathcal{K}^0(X)$ and contracting factors $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [0, 1]$, respectively, then $\Omega_3 = \text{co}(\Omega_1, \Omega_2) \in \mathcal{K}^0(X)$ is a $\lambda$-contractive set with contracting factor $\lambda_3 = \max\{\lambda_1, \lambda_2\}$.

This implies:

- no loss of generality assuming convexity of invariant sets,
- the maximal invariant set is convex.
Computational issues

Proposition

A polytope \( \Omega = \{ x \in \mathbb{R}^n : Hx \leq 1 \} \), whose vertices are \( v_j \), is a \( \lambda \)-contractive set with \( \lambda \in [0, 1] \) for a CDI system iff

\[
\dot{\mathcal{F}}(v_j, H_i^T) \leq \lambda, \quad \forall j \in \mathbb{N}_n, \quad \forall i \in \mathbb{N}_h,
\]

and for an uncertain CDI system iff

\[
\dot{\mathcal{F}}(v_j, H_i^T) \leq \lambda - \phi_W (H_i^T), \quad \forall j \in \mathbb{N}_n, \quad \forall i \in \mathbb{N}_h,
\]

- The conditions involve only a finite number of points and constraints!
- Computational affordable algorithms for obtaining polytopic \( \lambda \)-contractive sets and polyhedral Lyapunov functions for CDI systems.
Numerical example

**Generalized saturated system**

\[ x_{k+1} = Ax_k + B \varphi(Fx_k, k), \]

where

\[
A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.1 \end{bmatrix},
B = \begin{bmatrix} 0.5 \\ 1.1 \end{bmatrix},
\]

\[
F = \begin{bmatrix} -0.5236 & -1.1264 \end{bmatrix},
\]

with \( \Gamma(y) = \max\{\mu(y + \sigma), -y_0\} \), \( \mu = 1 \), \( \sigma = 0.2 \) and \( y_0 = 1.8 \).

Constraints: \( |x_1| \leq 15 \), \( |x_2| \leq 6 \):
Results on set-theory and invariance for CDI systems *(Fiacchini thesis)*:

- **Necessary and sufficient** condition for invariance,
- Induced Lyapunov functions,
- **Robust** invariance for uncertain CDI systems,
- One-step operator,
- Domain of attraction,
- Computational issues.

**Key idea:** it is convexity, not linearity, the missing ingredient!
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Problem statement

Given the system

\[ \dot{x}(t) = g(x(t)) + h(x(t))u(t), \]

where \( u(t) \in U(x(t)) \subseteq \mathbb{R}^m \) with \( U(x) = \{ u \in \mathbb{R}^m : Mu \leq N(x) \} \).

- **Objectives:**
  - design a computation-oriented method for obtaining polytopic control invariant sets and polyhedral Lyapunov functions;
  - extend the result to a particular class of nonconvex sets and functions.

- **Proposal:** apply viability theory to

\[ \dot{x}(t) \in F(x(t)) = g(x(t)) \oplus h(x(t))U(x(t)), \]

and \( \Omega = \{ x \in \mathbb{R}^n : Hx \leq 1 \} \).

- **Extension:** consider nonconvex sets and functions.
Given the convex closed set $K \subseteq \text{dom}(F)$,

$$C_k(x) = \bigcup_{h>0} \frac{K-x}{h},$$

is the (Clarke) tangent cone of $K$ at $x \in K$. 

**Viability Theorem (Control invariance)**

Consider a Marchaud map $F : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R}^n)$ and a closed, convex subset $K \subseteq \text{dom}(F)$. If

$$\forall x \in K, \quad F(x) \cap C_K(x) \neq \emptyset,$$

then for any state $x_0 \in K$, there exists a viable solution on $[0, \infty)$ to the differential inclusion $\dot{x} \in F(x)$.

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$^a$ the graph and the domain of $F(\cdot)$ are closed, $F(x)$ are convex, the growth of $F(\cdot)$ is linear.
Given a polytopic $\Omega = \{ x \in \mathbb{R}^n : Hx \leq 1 \}$ the Minkowski function is

$$\Psi_{\Omega}(x) = \min_{\alpha \geq 0} \{ \alpha \in \mathbb{R} : x \in \alpha \Omega \} = \max_{j \in \mathbb{N}_{nh}} \{ H_j x \}.$$ 

Given $x \in \mathbb{R}^n$ and $k \in \mathbb{N}_{nh}$, define:

- $\Omega(x) = \Psi_{\Omega}(x)\Omega$, (notice: $x \in \partial \Omega(x)$),
- $k_{\Omega}(x) = \arg \max_{j \in \mathbb{N}_{nh}} \{ H_j x \} \subseteq \mathbb{N}_{nh}$,
- $C_{\Omega(x)}(x) = \{ v \in \mathbb{R}^n : H_k v \leq 0, \forall k \in k_{\Omega}(x) \}$. 

**Objective**

The necessary and sufficient viability condition of $\Omega(x)$ at $x \in \Omega$:

$$H_k (g(x) + h(x)u) \leq 0, \quad \forall k \in k_{\Omega}(x).$$
Polytopic control invariant sets

Given $k \in \mathbb{N}_{n_h}$, $x \in R_k = \{x \in \mathbb{R}^n : H_i x < H_k x, \forall i \neq k\}$.

Convex optimization problem

$$L^*_k(x) = \max_{\beta, \delta, \sigma} \ L_k(\beta, \delta, \sigma; x),$$

s.t.

$$\sum_{i=1}^{n_h} \beta_i + \delta \leq 1, \quad \beta \geq 0, \quad \delta \geq 0, \quad \sigma \geq 0,$$

$$\delta \tau H_k h(x) + \sum_{j=1}^{n_u} \sigma_j M_j = 0.$$

where $L_k(\beta, \delta, \sigma; x) = \sum_{i=1}^{n_h} \beta_i H_i x + \delta \tau H_k g(x) + \delta H_k x - \sum_{j=1}^{n_u} \sigma_j N_j(x)$

Proposition

The optimal solution $L^*_k(x)$ is such that

- $L^*_k(x) \leq H_k x \iff$ there exists $u \in U(x)$ such that viability holds at $x$,
- $L^*_k(x) > H_k x \iff$ viability is not satisfied at $x$ for any $u \in U(x)$. 
Polytopic control invariant sets

Notice that:
- the feasibility region of the dual problem is a polyhedron.
- if $h(x)$ is constant:
  - it does not depend on $x$, only on $H_k$!
  - the $n_v$ vertices $(\beta^p, \delta^p, \sigma^p)$, with $p \in \mathbb{N}_{n_v}$, are easily obtainable.
- The optimal $L_k^*(x)$ is attained at one of its vertex, then:

$$L_k(\beta, \delta, \sigma; x) \leq \max_{p \in \mathbb{N}_{n_v}} L_k(\beta^p, \delta^p, \sigma^p; x) = L_k^*(x),$$

Proposition

- The set of $R_k$ such that viability is satisfied at $x \in R_k$ by a $u(x) \in U(x)$ is

$$\mathcal{V}_k = \bigcap_{p \in \mathbb{N}_{n_v}} \{ x \in \mathbb{R}^n : L_k(\beta^p, \delta^p, \sigma^p; x) \leq H_k x \}.$$ 

- The region of $R_k$ for which viability is violated for all $u \in U(x)$ is

$$\bar{\mathcal{V}}_k = \bigcup_{p \in \mathbb{N}_{n_v}} \{ x \in \mathbb{R}^n : L_k(\beta^p, \delta^p, \sigma^p; x) > H_k x \}.$$ 

Polytopic control invariant sets

Results (Fiacchini, Tarbouriech, Prieur ACC11):

- necessary and sufficient condition for general sectors, given $\mathcal{K} \subseteq \mathbb{N}_n^h$:
  \[ R_{\mathcal{K}} = \{ x \in \mathbb{R}^n : H_i x < H_k x, \ \forall i \notin \mathcal{K}, \ \forall k \in \mathcal{K} \} \]

- exact characterization of the regions in which viability holds;
- computation of the extremes of the feasibility region;
- contractiveness and induced Lyapunov functions;
- optimal input computation.

Open problems:

- a priori knowledge of the shape;
- generality yields computational issues (naturally...), assumptions required;
- continuity of the input: existence, computability;
- nonconvex sets and nonconvex induced Lyapunov functions. (Fiacchini, Tarbouriech, Prieur)
Consider the closed polytope:

$$\tilde{\Theta} = \{ x \in \mathbb{R}^n : \tilde{H}_i x \leq -1, \forall i \in \mathbb{N}_n \},$$

Then:

$$\bar{\Theta} = \text{cl} (\mathbb{R}^n / \tilde{\Theta}) = \{ x \in \mathbb{R}^n : \exists j \in \mathbb{N}_n, \text{ s.t. } \bar{H}_j x \leq 1 \},$$

where $\bar{H} = -\tilde{H}$, is possibly nonconvex and $0 \in \bar{\Theta}$.

Function:

$$\Phi_{\bar{\Theta}}(x) = \max_{\alpha} \{ \alpha \in \mathbb{R} : \exists j \in \mathbb{N}_n, \text{ s.t. } \bar{H}_j x \leq \alpha \},$$

- is homogeneous;
- $\Phi_{\bar{\Theta}} \leq 1$ iff $x \in \bar{\Theta}$;
- is such that $\Phi_{\bar{\Theta}}(x) = \min_{i \in \mathbb{N}_n} \{ \bar{H}_i x \}$.

Analogous of the Minkowski function to particular sets $\Theta$, possibly nonconvex.
Nonconvex sets and functions

Family of nonconvex sets: $\mathcal{C}(\mathbb{R}^n)$

Denote with $\mathcal{C}(\mathbb{R}^n)$ the subsets of $\mathbb{R}^n$ that can be expressed using the intersection and the union operators and closed half-spaces in $\mathbb{R}^n$ containing the origin in their interior.

Clearly any closed half-space containing the origin in its interior belongs to $\mathcal{C}(\mathbb{R}^n)$ and for all $B, C \in \mathcal{C}(\mathbb{R}^n)$, then $B \cup C \in \mathcal{C}(\mathbb{R}^n)$ and $B \cap C \in \mathcal{C}(\mathbb{R}^n)$.

Definition

Given the half-space $A = \{x \in \mathbb{R}^n : Hx \leq 1\}$, with $H \in \mathbb{R}^{1 \times n}$, define $\Phi_A(x) = Hx$. Given two sets $B, C \in \mathcal{C}(\mathbb{R}^n)$ define

$$\Phi_{B \cup C}(x) = \min \{\Phi_B(x), \Phi_C(x)\}, \quad \Phi_{B \cap C}(x) = \max \{\Phi_B(x), \Phi_C(x)\}.$$ 

Given $A = \{x \in \mathbb{R}^n : H_i^A x \leq 1, \forall i \in \mathbb{N}_A\}$ and $B = \{x \in \mathbb{R}^n : \exists j \in \mathbb{N}_B, \text{ s.t. } H_j^B x \leq 1\}$ then

$$\Phi_{A \cap B}(x) = \max \left\{ \max_{i \in \mathbb{N}_A} \{H_i^A x\}, \min_{j \in \mathbb{N}_B} \{H_j^B x\} \right\},$$ 

$$\Phi_{A \cup B}(x) = \min \left\{ \max_{i \in \mathbb{N}_A} \{H_i^A x\}, \min_{j \in \mathbb{N}_B} \{H_j^B x\} \right\}.$$
Nonconvex sets and functions

Given $\Theta \in \mathcal{C}(\mathbb{R}^n)$, the function $\Phi_{\Theta} : \mathbb{R}^n \to \mathbb{R}$ is:

- continuous;
- positively homogeneous of degree 1;
- such that: $x \in \Theta \iff \Phi_{\Theta}(x) \leq 1$;
- positive definite iff $\Theta$ is bounded.

Moreover:

**Proposition**

For every bounded $A, B \in \mathcal{C}(\mathbb{R}^n)$ then $B \subseteq A$ if and only if $\Phi_A(x) \leq \Phi_B(x)$, for all $x \in \mathbb{R}^n$.

**Definition**

Given $\Theta \in \mathcal{C}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

- $I_{\Theta}(x) = \{i \in \mathbb{N}_G : H_i^{\Theta}x = \Phi_{\Theta}(x)\}$;
- $\bar{I}_{\Theta}(x) = \mathbb{N}_{n_G} / I_{\Theta}(x)$;
- $\Theta_I(x) \in \mathcal{C}(\mathbb{R}^n)$ the set obtained by removing constraints with $j \in \bar{I}_{\Theta}(x)$.
Nonconvex sets and functions

**Proposition**

For every $\Theta \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

- there exists a neighborhood $N(x)$ such that $\Phi_\Theta(z) = \Phi_{\Theta I}(x)(z)$, $\forall z \in N(x)$.
- there exists a neighborhood of the origin $N^0(x)$ such that $v \in N^0(x)$ $\Phi_\Theta(x + v) = \Phi_\Theta(x) + \Phi_{\Theta I}(x)(v)$.
- for every $v \in \mathbb{R}^n$: $\Phi_{\Theta I}(x)(x + v) = \Phi_{\Theta I}(x)(x) + \Phi_{\Theta I}(x)(v)$.

**Proposition**

Given $\Theta \in C(\mathbb{R}^n)$, there exists $u(x) \in U(x)$ such that

$$
\Phi_{\Theta I}(x)(g(x) + h(x)u(x) + \lambda x) \leq 0,
$$

for all $x \in \mathbb{R}^n$, with $\lambda > 0$, if and only if there exists $u(t) \in U(x(t))$ for $t \geq 0$, such that

$$
\Phi_\Theta(x(t)) \leq \Phi_\Theta(x_0)e^{-\lambda t}, \quad \forall x_0 \in \mathbb{R}^n.
$$
Nonconvex sets and functions

Theorem

Given \( \Theta \in C(\mathbb{R}^n) \) and \( \mathcal{I} \subseteq 2^{n_G} \), suppose that \( \Theta_I(x) \) is convex in \( R_\mathcal{I} \). The subset of \( R_\mathcal{I} \) where viability holds is

\[
\mathcal{V}_\mathcal{I}^\lambda = \bigcap_{p \in \mathbb{N}_{n_v}} \{ x \in \mathbb{R}^n : L_{\mathcal{I}}^\lambda (\beta, \delta, \sigma; x) \leq H_\mathcal{I}^\Theta x, \forall i \in \mathcal{I} \},
\]

with \( L_{\mathcal{I}}^\lambda (\beta, \delta, \sigma; x) = \sum_{k=1}^{n_G} \beta_k H_k^\Theta x + \sum_{i \in \mathcal{I}} \delta_i \tau H_i^\Theta g(x) + \sum_{i \in \mathcal{I}} \delta_i (\tau \lambda + 1) H_i^\Theta x - \sum_{j=1}^{n_u} \sigma_j N_j(x) \), and

\( (\beta^p, \delta^p, \sigma^p) \) with \( p \in \mathbb{N}_{n_v} \) are the \( n_v \) extremal solutions of

\[
\sum_{k=1}^{n_G} \beta_k + \sum_{i \in \mathcal{I}} \delta_i \leq 1, \quad \beta \geq 0, \quad \delta \geq 0, \quad \sigma \geq 0,
\]

\[
\sum_{i \in \mathcal{I}} \delta_i \tau H_i^\Theta h(x) + \sum_{j=1}^{n_u} \sigma_j M_j = 0.
\]

If \( \Theta_I(x) \) is nonconvex in \( R_\mathcal{I} \), the problem is solved by the Theorem and the fact

\[
\max \{ \min_{i \in \mathbb{N}_y} \{ y^i \}, z \} \leq \alpha \iff \exists j \in \mathbb{N}_y \text{ s.t. } \max \{ y^j, z \} \leq \alpha.
\]
Brockett’s integrator: (Brockett, Sontag, Prieur)

\[
\begin{align*}
\dot{z}_1 &= v_1, \\
\dot{z}_2 &= v_2, \\
\dot{z}_3 &= z_1 v_2 - z_2 v_1,
\end{align*}
\Rightarrow
g(z) = 0, \quad h(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -z_2 & z_1 \end{bmatrix}
\]

in \( \dot{z} = g(z) + h(z)v \).

- **Input:** \( V(z) = V = \{ v \in \mathbb{R}^2 : \|v\|_\infty \leq 1 \} \), then

\[
M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}^T, \quad N(z) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T,
\]

- **Level set:** \( \Omega = \{ z \in \mathbb{R}^3 : Hz \leq 1 \} \), with

\[
H^T = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},
\]
Brockett’s integrator

Consider the upper horizontal facet: \( H_k = [0, 0, 1] \).

Feasibility region: \( \sum_{i=1}^{n_h} \beta_i + \delta \leq 1, \beta \geq 0, \delta \geq 0, \sigma \geq 0 \) s.t.

\[
\delta H_k h(z) + \sum_{j=1}^{n_u} \sigma_j M_j = 0 \quad \Rightarrow \quad \begin{cases} 
-\delta z_2 + \sigma_1 - \sigma_3 = 0, \\
\delta z_1 + \sigma_2 - \sigma_4 = 0.
\end{cases}
\]

(Non-trivial) extreme for \( \delta = 1 \) and \( \beta = 0 \), then

\[
\begin{cases} 
-z_2 + \sigma_1 - \sigma_3 = 0, \\
z_1 + \sigma_2 - \sigma_4 = 0.
\end{cases}
\]

and the extreme depends on \( z \).

For \( z_1 \geq 0 \) and \( z_2 \geq 0 \):

- extreme \( \sigma^p = [z_2, 0, 0, z_1]^T \), then \( v = [1, -1]^T \).
- from:

\[
\sum_{i=1}^{n_h} \beta_i^p H_i z + \delta^p H_k g(z) + \delta^p (\lambda + 1) H_k z - \sum_{j=1}^{n_u} \sigma_j^p N_j(z) \leq H_k z,
\]

\( (\lambda + 1) H_k z - \sum_{j=1}^{n_u} \sigma_j^p N_j(z) \leq H_k z, \)

then contractivity in \( \lambda z_3 \leq z_1 + z_2 \).
Brockett’s integrator

New shape: if $z > 0$ and $[1, 1, -\lambda]z \leq 0$ then active constraint:

$$H_k z = [-1, -1, 2\lambda] z \leq \lambda,$$

(Non-trivial) extreme for $\delta = 1$ and $\beta = 0$:

$$\begin{cases} -1 - 2\lambda z_2 + \sigma_1 - \sigma_3 = 0, \\ -1 + 2\lambda z_1 + \sigma_2 - \sigma_4 = 0, \end{cases} \Rightarrow$$

- if $z_1 \geq \frac{1}{2\lambda}$ \Rightarrow $\sigma^p = [2\lambda z_2 + 1, 0, 0, 2\lambda z_1 - 1]^T$ and $v = [1, -1]^T$ then contractivity for

$$-\lambda z_1 - \lambda z_2 - 2\lambda^2 z_3 - 1 - 2\lambda z_2 + 1 - 2\lambda z_1 \leq 0, \quad \Leftrightarrow \quad \lambda z_3 \leq 1.5z_1 + 1.5z_2,$$

- if $z_1 < \frac{1}{2\lambda}$ \Rightarrow $\sigma^p = [2\lambda z_2 + 1, -2\lambda z_1 + 1, 0, 0]^T$ and $v = [1, 1]^T$ then contractivity for

$$-\lambda z_1 - \lambda z_2 - 2\lambda^2 z_3 - 1 - 2\lambda z_2 - 1 + 2\lambda z_1 \leq 0, \quad \Leftrightarrow \quad \lambda z_3 \leq \lambda^{-1} - 0.5z_1 + 1.5z_2,$$

Notice: same results imposing $H_k \dot{z} \leq -\lambda H_k z$. 
Brockett’s integrator

First quadrant, i.e. $z \geq 0$:

a) if $[-1, -1, \lambda]z \geq 0$ then:

$$v_1 = 1, \quad v_2 = \begin{cases} -1, & \text{if } z_1 \geq \frac{1}{2\lambda}, \\ 1, & \text{if } z_1 < \frac{1}{2\lambda}, \end{cases}$$


b) if $[-1, -1, \lambda]z < 0$ and $[-1, -1, 1]z > 0$ then contractiveness for $\lambda z_3 \leq z_1 + z_2$ with $v = [-1, 1]^T$,

c) if $[-1, -1, 1]z = 0$ then contractiveness for with $v = [-(z_1 - 1)\alpha, -(z_2 + 1)\alpha]^T$, where with $\alpha = \max(|z_1 - 1|, |z_2 + 1|)$, to slide,

d) if $[-1, -1, 1]z < 0$ then contractiveness for $z_1 + z_2 \leq 2\lambda^{-1}$ with $v = [-1, -1]^T$, 


Brockett’s integrator

(a) Trajectory

(b) Evolution of $z_1$ and $z_2$

(c) State: $z_1$ (blue), $z_2$ (green), $z_3$ (red)

(d) Lyapunov function
Conclusions and future work

Results (Fiacchini, Tabouriech, Prieur ACC11):
- Characterization of invariance of polytopes for nonlinear CT systems.
- Computation of the exact regions where viability holds.
- Dual feasibility region vertices characterization.

Further results (Fiacchini, Tabouriech, Prieur):
- Polytopic contractive set and polyhedral control Lyapunov functions.
- Nonconvex sets and induced Lyapunov functions.
- Control input design.
- Application to the Brockett’s example.

Future research:
- Robust invariance.
- Continuity of the input.
- Hybrid systems.
- Zeno problem.
- Computational issues.
1 Introduction, motivations and objectives
   - Set-methods in control

2 Set-theory for nonlinear systems
   - Convex Difference Inclusions for DT systems
   - Viability theory for CT systems
   - Nonlinear hybrid systems
Problem statement

Hybrid systems. Continuous-time and discrete-time dynamics:

\[
\begin{cases}
\dot{x} = \hat{g}(x), & \text{if } x \in \mathcal{F}, \\
x^+ = \tilde{g}(x), & \text{if } x \in \mathcal{J}.
\end{cases}
\]

- **Problem**: substantial inapplicability of classical control theories.
- **New theory** for control: \((Beker, Daafouz, Goebel, Hespanha, Liberzon, Nesić, Prieur, Sanfelice, Teel, Zaccarian)\).
- **Saturations** ⇒ additional nonlinearities.

Aims and contributions:

- **geometrical** characterization of contractiveness,
- ellipsoidal estimations of the basin of attraction and local Lyapunov functions for nested saturated hybrid systems,
- results for continuous and discrete-time systems recovered or extended,
- (parametric) convex optimization problem.
Consider the closed-loop nested saturated hybrid system

- **Continuous-time nested saturated dynamics** is
  \[
  \begin{cases}
  \dot{x} = \hat{g}(x) = \hat{A}x + \hat{B}\varphi(\hat{K}x + \hat{E}\varphi(\hat{F}x)), \\
  \tau = 1,
  \end{cases}
  \]
  if \( x \in \mathcal{F} \) or \( \tau < \rho \).

- **Discrete-time nested saturated dynamics** is
  \[
  \begin{cases}
  x^+ = \tilde{g}(x) = \tilde{A}x + \tilde{B}\varphi(\tilde{K}x + \tilde{E}\varphi(\tilde{F}x)) \\
  \tau^+ = 0,
  \end{cases}
  \]
  if \( x \in \mathcal{J} \) and \( \tau \geq \rho \).

- **The flow and jump sets**
  \[
  \mathcal{F} = \{ x \in \mathbb{R}^n : x^T M x \geq 0 \}, \\
  \mathcal{J} = \{ x \in \mathbb{R}^n : x^T M x \leq 0 \}.
  \]
  where \( M = M^T \in \mathbb{R}^{n \times n} \).

- **Temporal regularization**: \( \tau \) is the time passed from the last jump.
- **Bound \( \rho \geq 0 \)**: prevent Zeno effect, more general Lyapunov functions.
Inclusion condition for saturated functions

- **Objective**: Geometrical characterization of saturated functions.
- Applies to continuous-time and discrete-time systems ⇒ Hybrid ones.

**Theorem (Fiacchini et al. ACC 11)**

Given:

- the **saturated function**: \( g(x) = Ax + B\varphi(Kx) \),
- the **ellipsoid**: \( \Omega = \mathcal{E}(P) \),
- \( H(i,J) \in \mathbb{R}^{1 \times n} \) such that: \( |H(i,J)x| \leq 1 \), \( \forall x \in \Omega \), \( \forall J \subseteq \mathbb{N}_m \), \( \forall i \in J^a \),

then

\[
g(x) \in G(x) = \text{co} \{N(J)x \in \mathbb{R}^n : J \subseteq \mathbb{N}_m\},
\]

with

\[
N(J) = A + \sum_{i \in \bar{J}} B(i)K_i + \sum_{i \in J} B(i)H(i,J).
\]

\( a\mathbb{N}_m = \{x \in \mathbb{N} : 1 \leq x \leq m\} \), \( \bar{J} = \mathbb{N}_m \setminus J \)

Bounding set \( G(x) \) **convex** and with **known** vertices.
The Theorem leads to:

- estimations of the **domain** of attraction and local quadratic **Lyapunov** functions for **saturated** hybrid systems,

- **saturated continuous-time** systems: **reCOVERs** the results in (Alamo, Cepeda, Limon, CDC-ECC05), which **generalizes** those in (Hu, Lin, TAC02).

- **saturated discrete-time** systems: **imPROVES** the results in (Hu, Lin, Chen, SCL02).

- **saturated hybrid** systems: **extends** the results to the hybrid context (Fiacchini, Tarbouriech, Prieur, ACC11).
Objective: Geometrical characterization of nested saturated functions.

Theorem (Fiacchini, Tarbouriech, Prieur TAC12)

Given:
- the nested saturated function: \( g(x) = Ax + Bφ(Kx + Eφ(Fx)) \),
- the ellipsoid: \( Ω = \mathcal{E}(P) \),
- \( H(j,J) \in \mathbb{R}^{1 \times n} \) s.t.: \( |H(j,J)x| \leq 1 \), \( \forall x \in Ω \), \( \forall J \subseteq \mathbb{N}_m \), \( j \in J \)
- \( L(i,I(k)) \in \mathbb{R}^{1 \times n} \) s.t.: \( |L(i,I(k))x| \leq 1 \), \( \forall x \in Ω \), \( k \in \mathbb{N}_m \), \( I(k) \subseteq \mathbb{N}_p \), \( i \in I(k) \),

then \( g(x) \in S(x) = \text{co} \left( \{ Q(J,I)x \in \mathbb{R}^n : J \subseteq \mathbb{N}_m, I(k) \subseteq \mathbb{N}_p, k \in \mathbb{N}_m \} \right) \),

where \( I = \{ I(1), I(2), \ldots, I(m) \} \) and

\[
Q(J,I) = A + \sum_{j \in J} B(j) \left( K_j + \sum_{i \in I(j)} E_{j,i} F_i + \sum_{i \in I(j)} E_{j,i} L(i,I(j)) \right) + \sum_{j \in J} B(j) H(j,J).
\] (1)

Bounding set \( G(x) \) convex and with known vertices.
Hybrid systems contractiveness

For nested saturated hybrid systems, given $\Omega = \mathcal{E}(P)$, we have to impose that:

- set $\alpha \Omega$ with $\alpha \in [0, 1]$, is contractive for the continuous-time dynamics.
- the value of $V(x) = x^TPx$ decreases between two successive jumps.

**Theorem (Fiacchini, Tarbouriech, Prieur TAC12)**

Given:

- the nested saturated hybrid system and the ellipsoid: $\Omega = \mathcal{E}(P)$,
- $\hat{H}(j, J) \in \mathbb{R}^{1 \times n}$ s.t.: $|\hat{H}(j, J)x| \leq 1$ for every $J \subseteq \mathbb{N}_{mc}$ and $j \in J$, for all $x \in \Omega$;
- $\hat{L}(i, I(k)) \in \mathbb{R}^{1 \times n}$ s.t.: $|\hat{L}(i, I(k))x| \leq 1$ for every $k \in \mathbb{N}_{mc}$, $I(k) \subseteq \mathbb{N}_{pc}$ and $i \in I(k)$, for all $x \in \Omega$;
- $\hat{H}(u, U) \in \mathbb{R}^{1 \times n}$ s.t.: $|\hat{H}(u, U)x| \leq 1$ for every $U \subseteq \mathbb{N}_{md}$ and $u \in U$;
- $\hat{L}(v, V(l)) \in \mathbb{R}^{1 \times n}$ s.t.: $|\hat{L}(v, V(l))x| \leq 1$ for every $l \in \mathbb{N}_{md}$, $V(l) \subseteq \mathbb{N}_{pd}$ and $v \in V(l)$, for all $x \in \Omega$,

if

$$
\hat{Q}(J, I)^TP + P\hat{Q}(J, I) \leq -2\lambda P, \\
\tilde{Q}(U, V)^Te^{-\rho \ln P}Pe^{-\rho \ln \tilde{Q}(U, V)} - \sigma M \leq P,
$$

with $\hat{Q}(\cdot, \cdot)$ and $\tilde{Q}(\cdot, \cdot)$ as in (1), then

- $\Omega$ is an ellipsoidal estimation of the basin of attraction,
- $V(x) = x^TPx$ yields to a local Lyapunov function in $\Omega$.

Function $V(x)$ is not necessarily decreasing but yields to a Lyapunov function.
Hybrid systems contractiveness

- Computationally affordable: convex problem (LMI) for fixed $\lambda$;

- conditions for global asymptotic stability;

- alternative criteria, e.g. decrease during jump, bounded increase during flow \((Hespanha, Liberzon, Teel AUT08)\), etc. ;

- results on simple saturated hybrid systems are extended;

- results extendable to general convex sets, polytopes for instance;

- not every combination is admissible, complexity reduction is possible \((Fiacchini, Tarbouriech, Prieur, CDC 11)\);

- general functions bounded in $\Omega$ yield more general conditions for contractiveness: also necessary? Convex? \((Fiacchini, Tarbouriech, Prieur, CDC 11)\).
Numerical example

- **Saturated** linear unstable system
  
  \[
  \begin{cases}
  \dot{x}_p(t) &= 0.1x_p(t) + \varphi(y_c(t)), \\
  y_p(t) &= x_p(t).
  \end{cases}
  \]

- **Stabilizing** PI controller:
  
  \[
  \begin{cases}
  \dot{x}_c(t) &= -0.2y_p(t), \\
  y_c(t) &= x_c(t) - 2y_p(t).
  \end{cases}
  \]

- Discrete-time dynamics, **saturated reset**:
  
  \[x_c(k + 1) = x_c(k) + \varphi(-x_c(k)).\]

- Temporal regularization: \(\rho = 2\).

- (Parametric) convex problem.
**Numerical example**

- **Nested saturated system:**
  \[
  \begin{aligned}
  \dot{x}_p &= 0.1 x_p + \varphi(x_c - 2 \varphi(x_p)), \\
  \dot{x}_c &= -0.2 \varphi(x_p),
  \end{aligned}
  \]

- **Discrete-time dynamics, saturated reset:**
  \[x_c(k + 1) = x_c(k) + \varphi(-x_c(k)).\]

- **Reduction** of the estimation of the domain of attraction.
Conclusions:

- **geometrical** approach to the analysis of (nested) saturated functions,
- application to continuous and discrete-time saturated systems: results recovered or extended,
- nested saturated hybrid systems: ellipsoidal estimations of the domain of attraction and local Lyapunov functions,
- conditions for global asymptotic stability,
- allowing increasing during the jumps: more general case,
- (parametric) convex problem: computational results.

Considerations and future works:

- **Extensions**: multiple jumps, increasing during flow.
- **More general cases**: nonlinear hybrid systems, generic convex sets, polyhedral Lyapunov functions.
- **Necessary and sufficient** conditions for contractiveness of saturated systems (?!).
...thanks!