Pole and Zero Assignment by Proportional Feedback

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Abstract—**A procedure is described which enables specified poles and zeros of a transfer function of a linear system to be obtained by using proportional state feedback to two inputs in a restricted class of problem.**

Introduction

The problem considered is that in which it is desired to achieve a specified transfer function between one input variable and one output variable of a time-invariant linear system by the use of proportional state feedback. The system has two inputs, the second input being used only for positioning the zeros. The procedure is restricted to systems in which the inner product of the relevant input and output vectors is nonzero. For both zero and pole positioning, it uses established results in modal control theory.

System Description

The system is described by the equations

$$\dot{x} = Ax + Bu + Bu', \; y = c^T x \tag{1}$$

where x is an n-dimensional state vector, u and u' are two-dimensional input vectors, and

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where u_1 and u_2 are scalar inputs, y is a scalar output, A and B are constant matrices, and

$$B = [b_1 \,\vdots\, b_2]$$

where b_1 and b_2 are n-vectors, and c^T is a constant measurement vector.

The case considered is that in which $c^T b_1 \neq 0$. The system (1) is controllable and observable.

Problem Statement

The problem is to find the state feedback gain vectors k_1^T and k_2^T where

$$u' = \begin{bmatrix} k_1^T \\ \cdots \\ k_2^T \end{bmatrix} x,$$

so that the transfer function relating y to u_1 has specified poles and zeros.

Procedure

Step 1

We first determine k_2^T to locate the zeros. Let k_1^T be a zero vector at this stage.

Using a result obtained by Brockett [1], the zeros of the transfer function relating y to u_1 are eigenvalues of the matrix

$$\left(I - \frac{b_1 c^T}{c^T b_1}\right)(A + b_2 k_2^T) \tag{2}$$

which may be written as

$$A_0 + b_0 k_2^T \tag{3}$$

where

$$A_0 = \left(I - \frac{b_1 c^T}{c^T b_1}\right) A \tag{4}$$

$$b_0 = \left(I - \frac{b_1 c^T}{c^T b_1}\right) b_2. \tag{5}$$

We now check the pair (A_0, b_0) for controllability, using Gilbert's method, so as to reveal which eigenvalues are uncontrollable. A_0 has the eigenvalue 0, which is uncontrollable through b_0. However, provided that the other eigenvalues are controllable, we may use the results of modal control theory, e.g., [2], [3], to determine k_2^T such that the matrix $(A_0 + b_0 k_2^T)$ has any $(n - 1)$ specified eigenvalues and the eigenvalue 0. The $(n - 1)$ eigenvalues will be the zeros of the transfer function. The eigenvalue 0 has no physical significance, and arises only because the degree of the numerator of the transfer function is $(n - 1)$.

Step 2

The system poles will have been changed by the application of feedback k_2^T, and we now determine k_1^T to locate the poles as required.

It is first necessary to check the pair $((A + b_2 k_2^T), b_1)$ for controllability. If this test is satisfied, k_1^T may be found by again using the results of modal control theory [2], [3] to move the poles to any desired locations. The feedback k_1^T will have no effect on the zeros which were established in Step 1 because this feedback is applied to the input from which the transfer function is taken.

Conclusion

For a restricted class of problem, the procedure enables the poles and zeros of the transfer function to be given specified values, and it provides a check at each stage for the existence of a solution. Failure of the test for controllability at Step 1 does not necessarily imply that a satisfactory solution cannot be obtained if the uncontrollable zeros have acceptable values. An observer may be used, if necessary, to provide the feedback, and this will not affect the transfer function.

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On Determining the Zeros of State-Space Systems

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Abstract—**The zeros of a linear, time-invariant, state-space system are defined, and a matrix rank test for determining their locus is formally established and illustrated by example.**

Introduction

The primary purpose of this correspondence is to present a state-space matrix rank test which can be employed to determine the location of the zeros of a linear, time-invariant, multivariable system. The importance of this test in locating system zeros at $s = 0$ has already been demonstrated in a variety of applications, e.g., in step disturbance elimination studies [1], static decoupling [2], the design of integral feedback and feedforward regulators [3], and in classifying multivariable systems according to "type" [4].

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Preliminaries

We begin by assuming complete knowledge of the dynamical behavior of an nth order, m-input, p-output (with m and p no greater than n), linear, time-invariant system in terms of the state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) + E(D)u(t) \tag{1}$$

with $D = d/dt$, the differential operator. In terms of (1), the $(p \times m)$ rational transfer matrix $T(s)$ of the system can readily be determined, i.e., given (1),

$$T(s) = C(sI - A)B^{-1} + E(s), \tag{2}$$

and given $T(s)$, any least order (n) state-space system of the form (1) which satisfies (2) is called a *minimal state-space realization* of $T(s)$.

We now recall [5], [6] that any $(p \times m)$ rational transfer matrix can be factored in either of two (nonunique) ways, i.e.,

$$T(s) = R(s)P_c^{-1}(s) = P_0^{-1}(s)Q(s) \tag{3}$$

with $R(s)$ and $P_c(s)$ *relatively right prime* polynomial matrices in the Laplace operator s and $P_0(s)$ and $Q(s)$ *relatively left prime* polynomial matrices in s [5]. In a recent report [7] it was shown that any *numerator of* $T(s)$, i.e., any $(p \times m)$ polynomial matrix $R(s)$ or $Q(s)$ resulting from a prime factorization of $T(s)$, as in (3), is *unimodular equivalent*[1] to any other numerator of $T(s)$ and, furthermore, that a *zero* of $T(s)$ is any scalar s^* in $\mathcal{C}$, the complex field, for which the rank of $R(s^*)$ (or $Q(s^*)$), denoted as $\rho\{R(s^*)\}$, over $\mathcal{C}$ is less than its *normal rank* [8], [9], defined as $\rho\{R(s)\}$ over $\mathcal{P}(s)$, the field of rational functions in s. This notion of the zeros of $T(s)$ will now be extended to include state-space systems as well, i.e., s^* is a *zero* of (1) if and only if it is a zero of $T(s)$, as given by (2). In view of these preliminaries, we can now state and establish our main result.

The Main Result

Theorem: Any s^* in $\mathcal{C}$ is a zero of a minimal state-space system of the form (1) if and only if

$$\rho\left\{\begin{bmatrix} -s^*I + A, & B \\ C, & E(s^*) \end{bmatrix}\right\} \text{ over } \mathcal{C} < \rho\left\{\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}\right\} \text{ over } \mathcal{P}(s). \tag{4}$$

Proof: The proof of this theorem follows rather easily from the results given in [7] and the following lemma which is due to Rosenbrock [5].

Lemma: If (1) is a minimal realization of $T(s)$, as given by (2) and (3), then

$$\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}, \quad \begin{bmatrix} I_{n-m}, & 0, & 0 \\ 0, & P_c(s), & I_m \\ 0, & R(s), & 0 \end{bmatrix} \text{ and } \begin{bmatrix} I_{n-p}, & 0, & 0 \\ 0, & P_0(s), & Q(s) \\ 0, & I_p, & 0 \end{bmatrix}$$

are all unimodular equivalent.

Proof: See [5, theorem 2.1, chapter 3, section 2].

By Rosenbrock's lemma, it now follows that over $\mathcal{P}(s)$,

$$\rho\left\{\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}\right\} = n - m + \rho\left\{\begin{bmatrix} P_c(s), & I_m \\ R(s), & 0 \end{bmatrix}\right\}, \tag{5}$$

or that

$$\rho\left\{\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}\right\} = n + \rho\{R(s)\}, \tag{6}$$

[1] Two polynomial matrices of the same dimensions, such as $R(s)$ and $Q(s)$' are said to be (*unimodular*) *equivalent* if and only if $U_L(s)R(s)U_R(s) = Q(s)$ for some pair $\{U_L(s), U_R(s)\}$ of *unimodular matrices*, i.e., nonsingular polynomial matrices whose determinants are nonzero scalars in the real field $\mathcal{R}$.

i.e., the rank of the minimal system matrix $\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}$ is solely dependent on the rank of a numerator of $T(s)$. Since s^* is a zero of (1) if and only if $\rho\{R(s^*)\}$ over $\mathcal{C} < \rho\{R(s)\}$ over $\mathcal{P}(s)$, it is clear in view of (6) that s^* is a zero of (1) if and only if (4) holds, thus establishing the theorem.

Remarks

A number of remarks are now in order. In particular, we have the following.

Remark 1: We first note that *if $\rho\{B$ and $C\} \geq min(p,m) \triangleq r$ over $\mathcal{R}$, then the normal rank of $\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}$ will be $n + r$*, an observation which directly follows from the results given in [10], although the details associated with the formal establishment of this fact will not be presented here. Under these conditions, the zeros of a minimal state-space system can be found by simply determining those s^* which reduce the normal rank $(n + r)$ of $\begin{bmatrix} -s^*I + A, & B \\ C, & E(s^*) \end{bmatrix}$. To demonstrate, let us consider the following minimal state-space system of the form (1), with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}, \text{ and } E(D) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since B and C are both of rank $r = 2$, the normal rank of $\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}$ is $5 = n + r$, and the zeros of this system are given by those s^* for which the system matrix

$$\begin{bmatrix} -s^*I + A, & B \\ C, & E(s^*) \end{bmatrix} = \begin{bmatrix} -s^* & 1 & 0 & 0 & 0 \\ 1 & -s^* - 1 & 2 & 1 & 2 \\ 0 & 1 & -s^* - 3 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is singular. Therefore, since the determinant of the system matrix is equal to $-s^* - 3$, we conclude that the system has a single zero at $s = -3$. To verify this observation in light of the results given in [7], we first note that $T(s) = C(sI - A)B^{-1} + E(s)$ can be factored as $R(s)P^{-1}(s)$ with, for example, $R(s) = \begin{bmatrix} s + 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $P(s) = \begin{bmatrix} s^2 + 3s - 1, & -2s - 8 \\ -s, & s + 3 \end{bmatrix}$. Since $|R(s)| = s + 3$, $s = -3$ is the only scalar in $\mathcal{C}$ which reduces the normal rank of the numerator $R(s)$, which confirms the fact that -3 is the only zero of the system.

Remark 2: In view of Remark 1, it is now clear that *if $\rho\{B$ and $C\} \geq r$, then the minimal system (1) has a zero at the origin ($s^* = 0$) if and only if*

$$\rho\left\{\begin{bmatrix} A & B \\ C & E(0) \end{bmatrix}\right\} < n + r. \tag{7}$$

It is of interest and importance to note that the determination of the rank of $\begin{bmatrix} A & B \\ C & E(0) \end{bmatrix}$ (usually with $E(s) = E$, a constant matrix) is equivalent to the determination of the presence or absence of any system zeros at the origin of the complex plane, and as noted in the Introduction of this correspondence, this test plays a fundamental role in a variety of multivariable system applications [1]–[4].

Remark 3: We finally note that if a given state-space system is uncontrollable, unobservable, or both, then the uncontrollable and/or unobservable "modes" of the system will appear as "cancellable" pole-zero terms in the transfer matrix of the system [10]. Rosenbrock defines any such uncontrollable (unobservable) system modes as *input* (*output*) *decoupling zeros* [5], although we prefer to call these modes *nonminimal zeros* or *nonminimal poles* since they occur only in nonminimal systems and "cancel out" of both the numerator and denominator of the transfer matrix of the system. In view of this observation, we now note that *any nonminimal zeros of a system do not correspond to the defined (minimal) zeros* [7] *which characterize the transfer matrix of the system,* although both the minimal and nonminimal zeros of a system can be shown to reduce the normal rank of a general (not necessarily minimal) state-space system matrix $\begin{bmatrix} -sI + A, & B \\ C, & E(s) \end{bmatrix}$. While the formal establishment of this latter fact is relatively straightforward and not unlike the proof of our main result, it does involve certain additional steps and notions which would significantly lengthen this report. Furthermore, a more general result can actually be obtained, i.e., by combining the results given in [5]–[7], in view of the proof of our main result, it is not difficult to show that *the minimal and nonminimal zeros of any general differential operator representation* $P(D)z(t) = Q(D)u(t); y(t) = R(D)z(t) + W(D)u(t)$ *are equal to those* s^* *in* $\mathcal{C}$ *which reduce the normal rank of the system matrix* $\begin{bmatrix} -P(s) & Q(s) \\ R(s) & W(s) \end{bmatrix}$.

Conclusions

We have now shown that the zeros of a multivariable system of the form (1) are given by those s^* which reduce the normal rank of its system matrix. The significance of this state-space matrix rank test when $s^* = 0$ was noted, and an example was presented to illustrate the procedure. The distinction between minimal and nonminimal system zeros was also discussed.

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On the Numerators and Zeros of Rational Transfer Matrices

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Abstract—The notion of a "numerator" of a rational transfer matrix is defined. The fact that any two numerators of the same transfer matrix are equivalent is then formally established and employed in the development of a number of equivalent definitions of the zeros of a rational transfer matrix.

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Introduction

In the case of a scalar (single input/output) system characterized by a rational transfer function,

$$t(s) = \frac{r(s)}{p(s)}, \tag{1}$$

with $r(s)$ and $p(s)$ relatively prime polynomials in the Laplace operator s; the *zeros* of the system [or $t(s)$] are defined as those scalars s_i belonging to the complex field $\mathcal{C}$, which "zero" $t(s)$; i.e., s_i is a *zero* of $t(s)$ if, and only if,

$$t(s_i) = \frac{r(s_i)}{p(s_i)} = 0 = r(s_i). \tag{2}$$

It is thus clear that the zeros of $t(s)$ are equal to the zeros of its *numerator* $r(s)$. Physically speaking, the zeros of a scalar system represent those dynamical "modes" of the system that will not appear at the system output when an appropriate set of initial conditions is placed on the internal state of the system and an appropriate input is applied. These are points that will be clarified in our subsequent discussions.

It is of general interest to extend the notion of the zeros of a scalar system to include the multivariable (multi-input/output) case as well, and while some recent results [1], [2] have essentially resolved this extension via "different" definitions of the zeros of a rational transfer matrix, some rather significant questions remain unresolved regarding not only the equivalence of these definitions but also alternative definitions and methods that can be employed to determine the zeros of a multivariable system. The primary purpose of this note will be therefore to resolve these questions, and we will begin by defining a "numerator" of a rational transfer matrix and formally establishing the "equivalence" of any two numerators of the same transfer matrix

The Numerators of $T(s)$

We first note that any $(p \times m)$ rational transfer matrix $T(s)$ can be factored in either of two (nonunique) ways [1]–[3]; i.e.,

$$T(s) = R(s)P_c^{-1}(s) = P_0^{-1}(s)Q(s) \tag{3}$$

with $R(s)$ and $P_c(s)$ relatively right prime (RRP) [1], [3], ($P_0(s)$ and $Q(s)$ relatively left prime) polynomial matrices of the appropriate dimensions and $P_c(s)(P_0(s))$ nonsingular over the rational field $\mathcal{P}$.

Definition: Any $(p \times m)$ polynomial matrix $R(s)$ or $Q(s)$ that satisfies (3) will be called a *numerator* of $T(s)$.

Theorem: Any two numerators of a $(p \times m)$ rational transfer matrix are equivalent; i.e., if $R(s)$ and $Q(s)$ are both numerators of $T(s)$, then

$$U_L(s)R(s)U_R(s) = Q(s) \tag{4}$$

for an appropriate pair $\{U_L(s), U_R(s)\}$ of unimodular matrices.[1]

Proof: If $T(s) = R(s)P_c^{-1}(s) = \bar{R}(s)\bar{P}_c^{-1}(s)$, both RRP factorizations, then $R(s)$ and $\bar{R}(s)$ are both numerators of $T(s)$ with

$$R(s) = \bar{R}(s)\bar{P}_c^{-1}(s)P_c(s). \tag{5}$$

Since $\bar{R}(s)$ and $\bar{P}_c(s)$ are RRP, there exists [1], [3] a polynomial matrix pair $\{M(s), N(s)\}$ such that

$$M(s)\bar{R}(s) + N(s)\bar{P}_c(s) = I_m. \tag{6}$$

If we now represent $\bar{P}_c^{-1}(s)$ as the quotient of its *adjoint* $\bar{P}_c^{+}(s)$ and its *determinant* $|\bar{P}_c(s)|$ and then postmultiply (6) by $\bar{P}_c^{+}(s)P_c(s)$, we obtain

$$M(s)\bar{R}(s)\bar{P}_c^{+}(s)P_c(s) + N(s)\bar{P}_c(s)\bar{P}_c^{+}(s)P_c(s) = \bar{P}_c^{+}(s)P_c(s). \tag{7}$$

Since $\bar{R}(s)\bar{P}_c^{+}(s)P_c(s) = R(s)|\bar{P}_c(s)|$ in view of (5) and $\bar{P}_c(s)\bar{P}_c^{+}(s)$

[1] A polynomial matrix $U(s)$ is a unimodular matrix [1], [2] if, and only if, $|U(s)| = \alpha$, a nonzero scalar belonging to the real field $\mathcal{R}$.