From Local Approximation to a G^1 Global Representation

Cédric Gérot, Dominique Attali, and Annick Montanvert

Abstract. To represent a complex surface, it is useful to describe it as a set of simple parametric primitives such as quadrics. But if one wants to use few primitives, these have to be smoothly blended. To define this blending, we propose to describe the initial global surface with *charts*. The blending surfaces result from a convex combination of primitives whose weights are defined on open sets of $\mathbb{R}^2$ given by the charts. We have established the properties that the weight functions must satisfy to obtain a G^1 representation of the global surface, and we have constructed such functions.

§1. Introduction

The abundance of high quality volumetric image data and new performant segmentation methods for multidimensional image data make 3-D objects ready for analysis. Volumetric objects are basically represented by a binary voxel representation or by a triangulation of the surface. Because they are based on huge lists of voxels or surface elements, they are not efficient for capturing global and local shape features with a view to characterizing shape properties. The spline surfaces can be very useful, but become difficult to use for topologically arbitrary surfaces modeling because they require a rectangular parameterization. On the contrary, any surface can be approximated using quadric surface patches as in [2]. While they lead to a good shape description, the quadric patches do not define an overall continuous surface. Ideally, a surface representation for image analysis should allow us to represent continuously any complex surface with few parameters, and to extract shape properties as well.

We propose to represent a surface with *charts*. A chart is composed of a patch U lying on the surface and a homeomorphism of U onto a 2D-domain. This notion has already been used, but essentially for image synthesis. Thus, in [14], it allows texture mapping on a triangulated surface which is too

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complex to be described by only one chart. It is also used in [9] to design a surface with B-splines on any topological polyhedra, and in [15] to generalize the B-splines for constructing surfaces from irregular control meshes that can be embedded in the plane. Then, Eck et al. [5] use this notion to design a subdivision mesh from any triangular mesh.

We use it for image analysis because it allows to unfold a complex surface (for instance the surface of a brain). It is then an appropriate tool to extract surface features. Before using it, we first have to construct it. To do so, we begin by representing the surface by means of a set of simple surfaces called primitives (quadrics for instance) which approach it locally. (We currently study new processes to extract primitives from 3D objects). As mentioned above, the primitives do not define a globally continuous surface in general. So they have to be smoothly blended. This paper is focused on a solution to the blending problem.

Several different approaches to surface blending have been suggested. Firstly to fill a hole on a surface, one can interpolate a position and tangency conditions network [18,7,16,4], or construct a rational patch to fill a polygonal hole [10,8]. Our blending problem is not to fill a hole. But our approach solve this problem too. Secondly, to blend two surfaces, one can apply a rolling-ball algorithm [1,3,6]. But one cannot blend more than three surfaces at the same time. One can also meld isopotentials if the primitives are implicitly defined, [17,11,12]. But this seems to be a too restrictive condition. Our approach differs from these methods in that we blend any number of primitives at the same time, provided they can be parameterized.

This paper begins with mathematical definitions which are necessary to define our representation with charts. Then we present our approach for surface blending. Next we illustrate the different steps of our approach with some examples. Finally, we conclude with future work.

§2. Surface Representation with Charts

We begin with some mathematical definitions coming from differential geometry [13].

Definition 1. *A* n-dimensional manifold *is a topological space such that each point admits a neighborhood homeomorphic to* $\mathbb{R}^n$.

Definition 2. *A* chart (U, ψ) *is composed of an open set* U *of an n-manifold and a homeomorphism* ψ *of* U *onto an open set of* $\mathbb{R}^n$.

Definition 3. *Two charts* (U_i, ψ_i) *and* (U_j, ψ_j) *agree with each other if their* transition function

$$\psi_{ij} = \psi_j \circ \psi_i^{-1} \colon \psi_i(U_i \cap U_j) \to \psi_j(U_j \cap U_i)$$

is a diffeomorphism.

Definition 4. *Such a collection of maps charting all of the manifold is called an* atlas.

Remark. Every 2-dimensional manifold admits an atlas.

To give an atlas describing a surface is to give a representation of it with charts. This representation possesses two main advantages. Firstly, it combines local information with global information (ψ^{-1} is a local parameterization of the surface). That means one can work locally on the surface without undesirable consequences on the global surface because the atlas maintains this consistency by definition. Secondly, an atlas allows to translate problems given on any 2-dimensional manifold into problems given on $\mathbb{R}^2$.

§3. Smooth Blend

We have given the mathematical definition of the representation with charts. We now discuss how to use it to construct a G^1 global representation of a surface. Our aim is to represent a 2-dimensional manifold V by means of a set of simple surfaces called primitives (quadrics for instance) which approach V locally. To be more precise, we assume a family $\{U_i\}$ of open sets on V the union of which covers V and such that each U_i is approximated by a primitive P_i in such a way that there is a bijection b_i of U_i onto P_i: $P_i = b_i(U_i)$. The blend we want to construct between the primitives P_i must be a smooth surface S which overlaps a closed set of each P_i, called pure area and defined by $b_i(U_i \setminus \bigcup_{j \neq i} U_j)$. So, to be able to construct a smooth blend, the primitives must overlap sufficiently (see §4.1).

The blend is defined as a convex combination of the primitives P_i which approximate overlapping open sets U_i. The surface S is defined by an atlas and is a representation of V.

Hypotheses.

- We suppose that P_i are 2-dimensional manifolds parametrized by p_i, homeomorphisms which are C^1 on an open set Ω_i of $\mathbb{R}^2$: $P_i = p_i(\Omega_i)$.
- Let Ω_{ij} be the open set of Ω_i defined by $\Omega_{ij} = p_i^{-1}(b_i(U_i \cap U_j))$.
- We suppose that there exist some bijective transition functions $\varphi_{ij} : \Omega_{ij} \to \Omega_{ji}$, such that $\varphi_{kj} \circ \varphi_{ik} = \varphi_{ij}$. In particular, $\Omega_{ii} = \Omega_i$ and φ_{ii} is the identity. We write $\mathcal{P}(\mathbb{N})$ for the set of subsets of $\mathbb{N}$, and define for all i

$$\begin{aligned} \mathcal{I}_i : &\Omega_i \to \mathcal{P}(\mathbb{N}) \\ &m \mapsto \{j \in \mathbb{N} : \ m \in \Omega_{ij}\} \end{aligned}$$

- Let the weight functions α_i be defined on Ω_i and satisfying the following:

  Property 1. Convexity

  1a) $\forall i, \forall m \in \Omega_i, 0 \leq \alpha_i(m) \leq 1$

  1b) $\forall i, \forall m \in \Omega_i, \sum_{j \in \mathcal{I}_i(m)} \alpha_j(\varphi_{ij}(m)) = 1$

  Property 2. Regularity

  2a) $\alpha_i(m) = 1$ if $p_i(m)$ belongs to the pure area

  2b) $\alpha_i(m) = 0$ if m does not belong to Ω_i

- We define for all i

$$\begin{aligned}\varphi_i : &\Omega_i \to \mathbb{R}^3\\ &m \mapsto \sum_{j\in\mathcal{I}_i(m)} \alpha_j(\varphi_{ij}(m))p_j(\varphi_{ij}(m))\end{aligned}$$

Remark. This definition is consistent: $\varphi_j(\varphi_{ij}(m)) = \varphi_i(m)$.

Proposition. With these hypotheses we get:

- If φ_i is bijective, φ_{ij} is C^0 and α_i is C^0, then S is a 2-dimensional manifold for which an atlas is $\{(\varphi_i^{-1}(\Omega_i), \varphi_i^{-1})\}$.
- If φ_{ij} is C^1 and α_i is C^1, then S is described by a C^1-atlas (S is then a G^1-surface).

Property 2a guarantees that $\varphi_i(m) = p_i(m)$ if $p_i(m)$ belongs to the pure area. Properties 2a and 2b can be inconsistent with each other if the pure area is not strictly included in $p_i(\Omega_i)$. But in this case, another primitive can be introduced, which overlaps locally P_i.

On one hand, our representation is more efficient if few primitives are used. On the other hand, S is closer to V when more primitives are used. So the appropriate balance must be found with regard to these needs. But, if the blend is not defined specifically to perform the approximation of V by S, the approximation error is on the same order of magnitude as that due to the local approximation by each primitive. This last property is due to the convexity property followed by the weight functions.

To construct weight functions which satisfy the convexity properties, we first construct functions β_i satisfying the following

Property 3.

3a) $\forall i, \forall m \in \Omega_i, 0 \le \beta_i(m) \le 1$

3b) $\forall i, \forall m \in \Omega_i, \sum_{j\in\mathcal{I}_i(m)} \beta_j(\varphi_{ij}(m)) \neq 0$

3c) $\beta_i(m) = 1$ if $p_i(m)$ belongs to the pure area

3d) $\beta_i(m) = 0$ if m does not belong to Ω_i

Then, the weight functions α_i defined by the following expression have all the desired properties:

$$\alpha_i(m) = \frac{\beta_i(m)}{\sum_{j\in\mathcal{I}_i(m)} \beta_j(\varphi_{ij}(m))},$$

where φ_{ij} is a C^1 transition function.

§4. Applications

In this paper, we detail the construction of the open sets Ω_i, the weight functions α_i, and the functions φ_i. Further work will focus on the construction of domains U_i and the transition functions φ_{ij}.

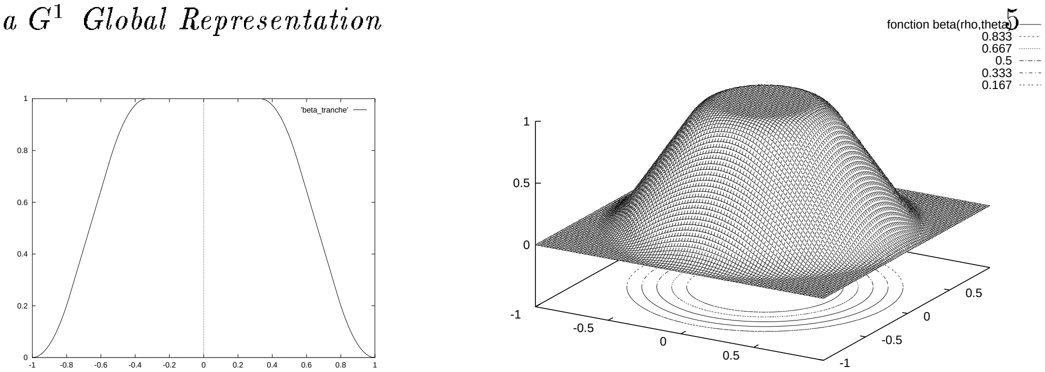

Fig. 1. Functions $b(\rho)$ and $\beta(\rho, \theta)$.

4.1. Weight Functions

As shown in §3, to construct satisfactory weight functions, we first construct function β_i satisfying Property 3. We suppose the open set Ω is a disc whose radius is R. Let r be the radius of the smaller disc having the same center as Ω and including the set of points $\{m \in \Omega : \ p(m) \text{ belongs to the pure area}\}$. To simplify the notations, we call this set of points the pure area too. Then we can give a cylindric definition of β, where t is a shape parameter:

$$\beta(\rho, \theta) = b(\rho)$$

with

$$b(\rho) = \begin{cases} 1 & \text{if } \rho \leq r, \\ P(\rho) & \text{if } r < \rho \leq r + t, \\ L(\rho) & \text{if } r + t < \rho \leq R - t, \\ 1 - P(R + r - \rho) & \text{if } R - t < \rho \leq R, \\ 0 & \text{if } R < \rho. \end{cases}$$

where $L(x) = Dx + E$, $P(x) = Ax^2 + Bx + C$ and $D = \frac{-1}{R-r-t}$, $E = \frac{1-(R+r)D}{2}$, $A = \frac{D}{2t}$, $B = -2Ar$, $C = 1 - Ar^2 - Br$.

Fig. 2 shows the weight function α after normalization, in a case where P is combined with five other primitives.

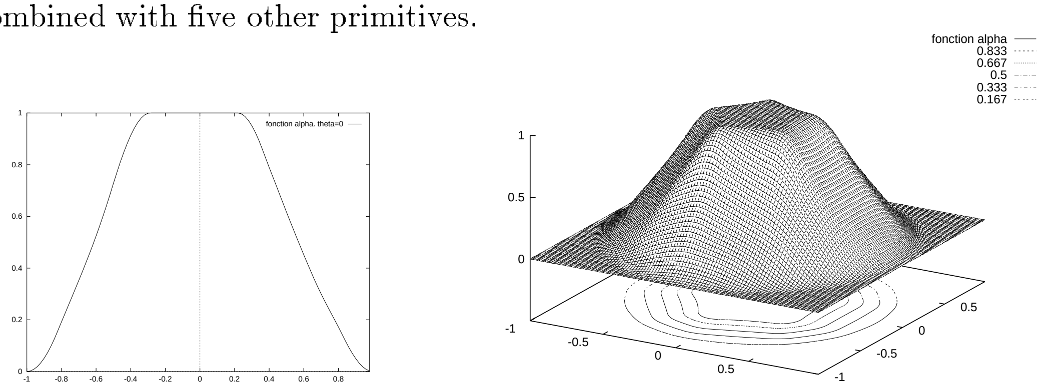

Fig. 2. Function α for a fixed θ and for all θ.

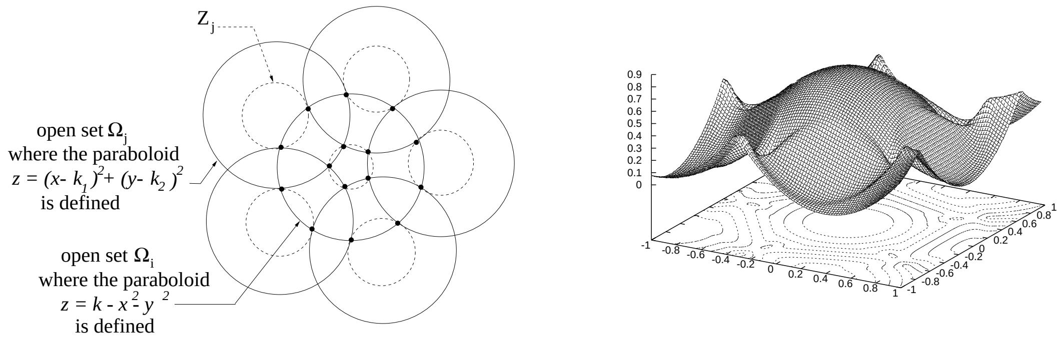

Fig. 3. Blending paraboloids: an example satisfying the hypotheses ($r = 0.71R$).

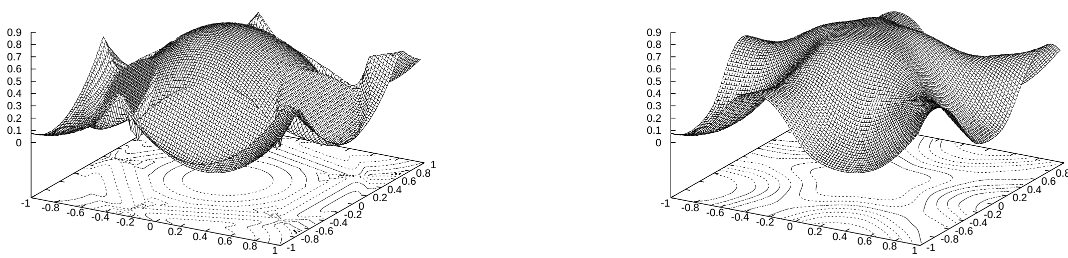

Fig. 4. Blendings with too large ($r = 0.85R$) and too small ($r = 0.07R$) pure areas.

In order that the small disc whose radius is r better fits the pure area, two modifications can be easily implemented. Firstly, we can define the small disc containing the pure area with different center than the center of Ω. β will have the same definition but with R depending on θ. Secondly, we can use ellipses rather than discs.

The parameter t, which belongs to $(0, 0.5)$, controls the nonlinear part of b. The smaller t, the smaller this part is. To avoid a final surface which is visually too sharp, t must be neither too small nor too large.

The size of the pure area also plays an important role in the surface smoothness. As shown in Fig. 4, if the pure area is too large, then the transitions between the primitives are too sharp in regards with the resolution of a visualization process. On the contrary, a pure area which is too small causes smooth transitions, but the shape of primitives is lost. In the example shown in Fig. 3, we have implemented a case where the pure area is half the area of Ω. This balance gives a good solution.

We have constructed satisfactory weight functions. To apply our representation we must define the functions $\mathcal{I}_i$, bijective and C^1 transition functions φ_{ij}, and check that φ_i is bijective to be sure that S is a G^1 surface.

4.2. A Simple Case

We first apply our representation in the case where the surface V to be represented can be described by $v(x,y) = [x, y, f(x,y)]$, $(x,y) \in D \subset \mathbb{R}^2$: a land surface in topography for example.

We suppose that a set of open discs Ω_i is defined by any local approximation strategy, and parameterizations $p_i(x,y) = [x, y, p_i^z(x,y)]$, $(x,y) \in \Omega_i$ are given such that $D \subset \bigcup_i \Omega_i$ and each p_i is a C^1 approximation of $\{v(x,y) : \ (x,y) \in \Omega_i \cap D\}$.

It is easy to check if a point (x,y) is inside a disc Ω_j, and so to define $\mathcal{I}_i(x,y)$. Besides, the transition functions φ_{ij} are, in this case, the identity, which is C^1 and bijective. Finally, the functions φ_i constructed by convex combinations of such p_i are bijective. So, the surface S described by the atlas $\{(\varphi_i^{-1}(\Omega_i), \varphi_i^{-1})\}$ is G^1.

Fig. 3 shows an example of this first case. We deal with six open discs. One of them, Ω_i, is centered on the origin. The others are centered on the vertices of a pentagon which encircles Ω_i. The primitives are paraboloids. The central one is defined by $z = k - x^2 - y^2$, and the others by $z = (x - k_j^1)^2 + (y - k_j^2)^2$ where k_*^* are constant.

This example displays the blend between two primitives quite similar locally around their parts to blend (a central and a peripheral), and between two dissimilar primitives (two peripherals). The surface is smooth even if adjacent primitives are strongly different from each other.

4.3. A More General Case

Most of the surfaces to be represented cannot be described by $[x, y, v^z(x,y)]$.

To deal with any surface V, we require a triangular mesh which is a first approximation of V. To simplify the notations, we name this mesh V too. We define on it a set of domains U_i. Each U_i is a set of vertices, edges and faces of V. It is isomorphic to an open disc, and well approximated by a primitive (a plane in Fig. 5).

In this case, we do not give an analytic expression to the functions p_i, φ_{ij} and the open set Ω_{ij}, but they are defined on a finite set of points. They are described by links between vertices of different meshes (see Fig. 5). For instance, we construct a mesh Ω_i lying on $\mathbb{R}^2$, using the bijective harmonic map presented in [5], on U_i. Therefore, each vertex u of U_i is linked to a vertex ω of Ω_i. Because of these links, $\mathcal{I}_i$, φ_{ij} and Ω_{ij} can be defined on the vertices of Ω_i as follows:

For every vertex u of V, we construct $l(u)$, the list of the vertices linked to u. Each of these vertices lies on a different Ω_j. $l(u)$ contains only one vertex ω if ω belongs to a pure area. Let ω be a vertex of an open set Ω_i. Let u be the vertex of U_i (and so V) linked to it. For all i, if one of the vertices of $l(u)$, ω', belongs to Ω_j, then $\mathcal{I}_i(\omega)$ includes j, ω belongs to Ω_{ij} and $\varphi_{ij}(\omega) = \omega'$; or else ω does not belong to Ω_{ij}.

The functions β_i are calculated as in §4.1, and thanks to $\mathcal{I}_i$, we calculate α_i on the vertices of Ω_i.

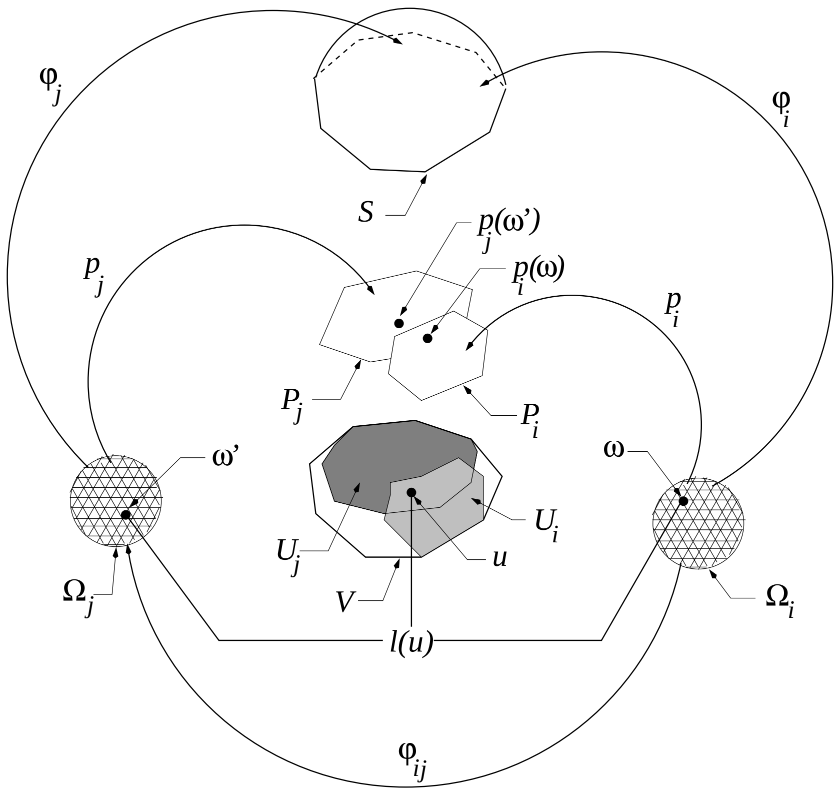

Fig. 5. Notation in the general case.

In the same way, we construct a mesh P_i, which is in bijection with U_i and whose vertices lie on the primitive which approximate U_i. Thanks to the links between the vertices of U_i and Ω_i, we define the links between the vertices of Ω_i and P_i. These links define the paramatrization p_i on the vertices of Ω_i.

We then construct a mesh S whose vertices are calculated by φ_i defined on the vertices of Ω_i as in §3.

Assuming there exist C^1 functions φ_{ij} and p_i which interpolate the values set on the vertices of Ω_i, and satisfy the hypotheses given in §3, the vertices of S lie on a G^1 surface.

Remark. Because we do not give an analytic expression for p_i and φ_{ij}, we have to store the meshes Ω_i and P_i. In future work, we will either have to give simple expressions for those functions, or decrease the size of the meshes.

§5. Conclusion

The representation with charts can be used to construct a useful surface model. But, before applying it to real data, we still have two crucial steps: the definition of the domains U, and the definition of transition functions. Then we will apply it to image analysis problems such as registration, surface feature extraction, texture mapping or animation. More precisely, we will begin with the visualization of S by a mesh hierarchy which offers different levels of detail.

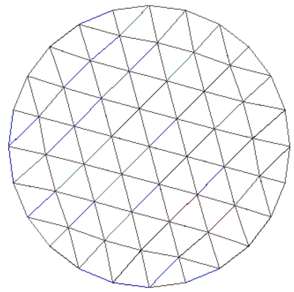

Fig. 6. An open set Ω_i.

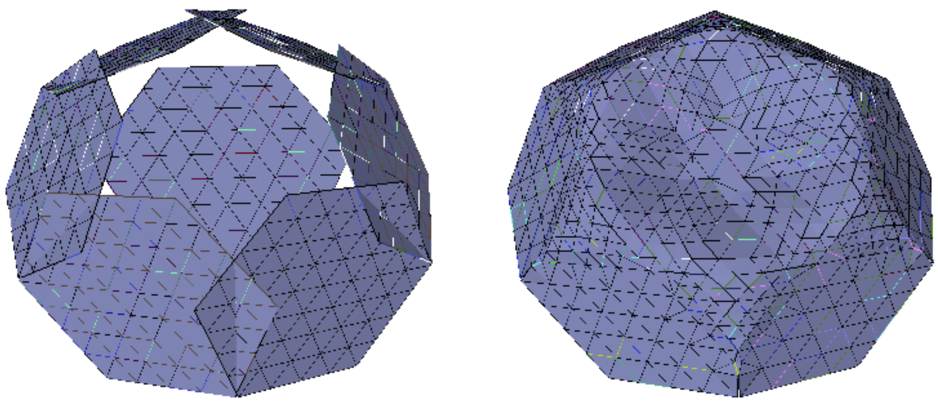

Fig. 7. The meshed primitives P_i and the mesh S.

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