Stability and output regulation for a cascaded network of $2 \times 2$ hyperbolic systems with PI control

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1. Introduction

2. Statement of the problem and main result

3. Lyapunov techniques and the proof of the main result

4. Application for Saint Venant model

5. Conclusions
Introduction

- PDE hyperbolic systems and cascaded networks
- Boundary control problem
- Output regulation problem
- PI control design
PDE hyperbolic systems and cascaded networks

Engineering applications of PDE hyperbolic systems

- Hydraulic engineering - Saint Venant models
- Road traffic - Burgers equation
- Gas pipeline
- Heat exchanger process
- ...

Homogeneous first-order hyperbolic systems

Let $\phi \in \mathbb{R}^n$, $A(\phi) \in \mathbb{R}^{n \times n}$, $x \in [0, L], t \in \mathbb{R}^+$,

$$\phi_t + A(\phi) \phi_x = 0 \ , \ \phi(0, x) = \phi^0(x)$$

A has n real eigenvalues, i.e $\lambda_i \in \mathbb{R} \ \forall i = 1, 2, ..n$. If A is independent on $\phi$, system is linear. If not, it is quasi-linear.
PDE hyperbolic systems and cascaded networks

Cascaded network

- Popular in practical applications (channels of rivers, gas, ...)
- \( n \) PDE hyperbolic sub-systems
- \( n + 1 \) junctions, 2 free junctions and \( n - 1 \) mixed junctions.

**Figure**: Cascaded network of \( n \) systems

A cascaded network can be considered an **large PDE hyperbolic system with complex boundary conditions**!
Boundary control problem

Boundary conditions

\[ f\left(\phi(0, t), \phi(L, t), U(t)\right) = 0 \]

- Static control, i.e \( U(t) = g(\phi(0, t), \phi(L, t)) \).
- Dynamic control, i.e \( U(t) = g(\phi(0, t), \phi(L, t)) + \text{other dynamic parts} \).

Boundary control problem

Find boundary conditions such that:
- The PDE hyperbolic system has a unique solution in the corresponding state space.
- The PDE hyperbolic system is (globally/locally) asymptotically/exponentially stable w.r.t some equilibrium point.
Introduction

Boundary control problem

**Static control laws**

- **Literatures:**

- **Limits:** Not robust with constant perturbations.

**Dynamic control laws with integral actions**

- **Literatures with works of Pohjolainen, Xu, Dos Santos, C. Prieur, D. Georges,...**

- **Advantages:** Robust to constant perturbations.

- **Limits:** Become a coupling systems of PDE and ODE, difficult to prove stability.
### Output regulation problem

Given a system one wants to ensure that outputs $y(t)$ follow references $y_r$ despite disturbances, i.e $y(t) \rightarrow y_r$

#### Figure: Example of Disturbances

Disturbances in real model: error of the modelisation, linearisation, sensors, · · ·

⇒ Static error between the measurement output and the set-point.

Solution: using the integral action to eliminate the static error.
Output regulation problem

Example: A very trivial system:
\[
\dot{\phi} = u + d \\
y = \phi
\]
State \(\phi \in \mathbb{R}\), control \(u \in \mathbb{R}\), unknown constant disturbance \(d \in \mathbb{R}\), measure \(y \in \mathbb{R}\).

Objective: Given a reference \(y_r\) in \(\mathbb{R}\), design \(u\) such that \(y \to y_r\).

- If \(u = -(y - y_r)\) \(\Rightarrow\) equilibrium is stable but \(y \nrightarrow y_r\).
- If \(u = -(y - y_r) - z\), where \(\dot{z} = y - y_r\) \(\Rightarrow\) equilibrium is stable and \(y \to y_r\).

Conclusion: The integral term added rejects the constant disturbance.
PI control design

- PI controller is a type of dynamic boundary control law:
  
  \[ u(t) = K_P(y(t) - y_r) + K_Iz(t), \quad \dot{z} = y(t) - y_r \]

- Measured output on the boundary \( y(t) = g(\phi(0, t), \phi(L, t)) \)
- Input \( u(t) \), reference \( y_r \)
- Gain parameter matrices \( K_p, K_I \).

- Schema of closed-loop system:

- Objective: Design PI controller (determine \( K_P \) and \( K_I \)) such that:
  - Stability of closed-loop system
  - Output regulation: \( y(t) \rightarrow y_r \)
Plan

1. Introduction

2. Statement of the problem and main result

3. Lyapunov techniques and the proof of the main result

4. Application for Saint Venant model

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Network model

- $n$ PDE hyperbolic systems

\[
\begin{align*}
\partial_t \phi_{i1}(x, t) + \lambda_{i1} \partial_x \phi_{i1}(x, t) &= 0, \quad x \in [0, L], \ t \in [0, \infty), \ i = 1, n \\
\partial_t \phi_{i2}(x, t) - \lambda_{i2} \partial_x \phi_{i2}(x, t) &= 0
\end{align*}
\]

where two states $\phi_{i1}, \phi_{i2} : [0, L] \times [0, \infty) \to \mathbb{R}$ and $\lambda_{i1} > 0, \lambda_{i2} > 0$.

- Boundary conditions defined at junctions

\[
\begin{align*}
\phi_{i2}(L, t) &= R_{i2} \phi_{i1}(L, t) + u_i(t) \\
\phi_{i1}(0, t) &= R_{i1} \phi_{i2}(0, t) + \alpha_i \phi_{(i-1)1}(L, t) + \delta_i \phi_{(i-1)2}(L, t)
\end{align*}
\]

where $\phi_{01} = \phi_{02} = 0$.

- $n$ measured outputs

\[y_i(t) = a_i \phi_{i1}(L, t) + b_i \phi_{i2}(L, t) + y_{ir}\]
PI structure and state space

- Design $n$ PI controllers at each juctions

\[ u_i(t) = K_{iP}(y_i(t) - y_{ir}) + K_{il}z_i(t) , \dot{z}_i = y_i(t) - y_{ir} \]

$K_{iP} \in \mathbb{R}$ and $K_{il} \in \mathbb{R}$ to be designed.

- Consider the state space of closed-loop network:

\[ E = \left((L^2(0, L))^2 \times \mathbb{R}\right)^n \]

with the norm associated

\[ ||Y||_E^2 = \sum_{i=1}^{n} \left(||\phi_{i1}(\cdot, t)||_{L^2(0,L)}^2 + ||\phi_{i2}(\cdot, t)||_{L^2(0,L)}^2 + z_i^2(t)\right) \]

where $Y = (\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n) \in E$
Statement of the problem and main result

Main result

Two hypotheses

- \( H_1 : a_i \neq 0 \ \forall \ i = 1, n \)
- \( H_2 : a_i + b_i R_i R_2 \neq 0 \ \forall \ i = 1, n \)

Theorem (Trinh-Andrieu-Xu 2017)

There exists \( \mu^* > 0 \) such that, if two hypotheses \( H_1 \) and \( H_2 \) are satisfied, for each \( \mu \in (0, \mu^*) \) and

\[
K_{iP} = \frac{-R_i R_2}{a_i} , \quad K_{iI} = -\mu \frac{(b_i + a_i R_i e^{\mu L})(a_i + b_i R_i R_2)}{a_i} , \ \forall i = 1, n
\]

Then, we have:

- Existence and uniqueness of solutions in \( \mathbb{E} \)
- The exponential stability of 'zero' point in \( \mathbb{E} \).
- With initial conditions in \( ((H^1(0, L))^2 \times \mathbb{R})^n \), output regulation, i.e

\[
\lim_{t \to \infty} |y_i(t) - y_{ir}| = 0 , \ \forall i = 1, n.
\]
About the theorem

\[ y_i(t) = a_i \phi_{i1}(L, t) + b_i \phi_{i2}(L, t) + y_{ir} \]

\[ K_iP = \frac{-R_{i2}}{a_i}, \quad K_{il} = -\mu \left( \frac{b_i + a_i R_{i1} e^{\mu L}}{a_i} \right) (a_i + b_i R_{i2}) , \quad \forall i = \overline{1, n} \]

Two output conditions (two hypotheses) for our PI control design:

- \( H_1 \) for existence of our PI controller.
  \[ a_i \neq 0 \quad \forall i = \overline{1, n} \]

- \( H_2 \) for having dynamic feedback (by integral action), i.e. \( K_{il} \neq 0 \).
  \[ a_i + b_i R_{i2} \neq 0 \quad \forall i = \overline{1, n} \]
1 Introduction

2 Statement of the problem and main result

3 Lyapunov techniques and the proof of the main result

4 Application for Saint Venant model

5 Conclusions
Lyapunov techniques and the proof of the main result

Lyapunov candidate functional

Use Lyapunov techniques ⇔ construct a candidate Lyapunov function.

\[ V(φ_{11}, φ_{12}, z_1, \ldots, φ_{n1}, φ_{n2}, z_n) = \sum_{i=1}^{n} p_i V_i \]

where

\[ V_i(φ_{i1}, φ_{i2}, z_i) = \int_{0}^{L} \begin{pmatrix} φ_{i1} e^{-\frac{μx}{2}} \\ φ_{i2} e^{\frac{μx}{2}} \\ z_i \end{pmatrix}^T P_i \begin{pmatrix} φ_{i1} e^{-\frac{μx}{2}} \\ φ_{i2} e^{\frac{μx}{2}} \\ z_i \end{pmatrix} dx \]

with

\[ P_i = \begin{pmatrix} 1 & 0 & q_i3 \\ 0 & q_i1 & q_i4 \\ q_i3 & q_i4 & q_i2 \end{pmatrix} \]

Here \( p_i > 0 \) and \( q_{i1}, q_{i2}, q_{i3}, q_{i4} \) need to be designed.
Lyapunov candidate functional

\[ V = \sum_{i=1}^{n} p_i \int_0^L \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ z_i \end{pmatrix}^T \begin{pmatrix} 1 & 0 & q_{i3} \\ 0 & q_{i1} & q_{i4} \\ q_{i3} & q_{i4} & q_{i2} \end{pmatrix} \begin{pmatrix} \phi_{i1} e^{\frac{\mu x}{2}} \\ \phi_{i2} e^{-\frac{\mu x}{2}} \\ z_i \end{pmatrix} \, dx \]

- If \( q_{i2} = q_{i3} = q_{i4} = 0 \), this is the Lyapunov functionnal of Bastin, Coron and Andréa Novel 2009 for a cascaded network.
- If \( n = 1 \) and \( q_{i3} = q_{i4} = 0 \), this is the Lyapunov functionnal of Bastin and Coron 2016 for a single system.
- By adding the new terms \( (q_{i3}, q_{i4} \neq 0) \) and \( n \) positive parameters \( p_i \), it allows to deal with dynamic feedback of cascaded network of \( n \) systems.
Design of Lyapunov functional

\[ V_i(\phi_1, \phi_2, z_i) = \int_0^L \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ z_i \end{pmatrix}^T \begin{pmatrix} 1 & 0 & q_{i3} \\ 0 & q_i & q_{i4} \\ q_{i3} & q_{i4} & q_{i2} \end{pmatrix} \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ z_i \end{pmatrix} dx \]

Lemma (For sub-functional \( V_i \))

Let \( q_{i1}, q_{i2}, q_{i3}, q_{i4} \) be defined as follows:

\[ q_{i1} > \frac{3\lambda_{i1} R_{i1}^2}{\lambda_{i2}}, \quad q_{i2} = \mu e^{\mu L} \lambda_{i2} q_{i1}, \quad q_{i3} = \mu e^{\frac{3\mu L}{2}} \frac{a_i \lambda_{i2} q_{i1}}{\lambda_{i1}}, \quad q_{i4} = \mu e^{\frac{3\mu L}{2}} a_i R_{i1} q_{i1}. \]

Then there exists \( \mu^* > 0, M_i > 0 \) and \( \gamma_i > 0 \) such that for all \( \mu \in (0, \mu^*) \)

1. \( \frac{1}{M_i} V_i(\phi_1, \phi_2, z_i) \leq ||\phi_1(., t)||^2_{L^2(0,L)} + ||\phi_2(., t)||^2_{L^2(0,L)} + z_i^2(t) \leq M_i V_i(\phi_1, \phi_2, z_i) \)

2. \( \dot{V}_i(t) \leq -\gamma_i V_i(t) - F_i(t) + G_{i-1}, \)

where \( F_i(t) = \frac{1}{4} z_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} + \phi_{i1}^2(L, t) \frac{\lambda_{i1} e^{-\mu L}}{2} \),

\[ G_{i-1} = \phi_{(i-1)1}^2(L, t) \lambda_{i1} \alpha_i^2 \left( 3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right) + z_{i-1}^2(t) \lambda_{i1} \beta_i^2 \left( 3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right) \)
Sketch of proof

**$V_i$ is definite positive**

$$V_i(\phi_{i1}, \phi_{i2}, z_i) = \int_0^L \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ z_i \end{pmatrix}^T P_i \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ z_i \end{pmatrix} \ dx$$

With $\mu$ small enough, prove that $P_i$ is symmetric positive definite (SDP)

**Consider $\dot{V}_i$**

$$\dot{V}_i = -\int_0^L \begin{pmatrix} \phi_{i1}(x, t)e^{-\frac{\mu x}{2}} \\ \phi_{i2}(x, t)e^{\frac{\mu x}{2}} \\ z_i(t) \\ \phi_{i1}(L, t) \end{pmatrix}^T Q_i \begin{pmatrix} \phi_{i1}(x, t)e^{-\frac{\mu x}{2}} \\ \phi_{i2}(x, t)e^{\frac{\mu x}{2}} \\ z_i(t) \\ \phi_{i1}(L, t) \end{pmatrix} \ dx - F_i(t) + G_{i-1}$$

With $\mu$ small enough, prove that $Q_i \in \mathbb{R}^{4 \times 4}$ is SDP

$$\Rightarrow \forall t \in \mathbb{R}_+, \exists \gamma_i > 0, \quad \dot{V}_i(t) \leq -\gamma_i V_i(t) - F_i(t) + G_{i-1}.$$
Lyapunov techniques and the proof of the main result

Design of Lyapunov functional

\[ \mathcal{V}(\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n) = \sum_{i=1}^{n} p_i V_i \]

Lemma (For global functional \( \mathcal{V} \))

Let \( q_{i1}, q_{i2}, q_{i3}, q_{i4} \) be defined in Lemma of sub functional \( V_i \), and \( p_i \) be defined as follows

\[ p_1 > 0, \quad p_{i+1} = \epsilon p_i \]

Then there exists \( \epsilon > 0 \) and \( \mu^* > 0 \) such that for every \( \mu \in (0, \mu^*) \), we have:

1. There exists \( M > 0 \) such that

\[
\frac{1}{M} \mathcal{V}(\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n) \leq \| (\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n) \|_E^2
\]

\[ \leq M \mathcal{V}(\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n). \]

2. There exists \( \gamma > 0 \) such that

\[ \dot{\mathcal{V}}(t) \leq -\gamma \mathcal{V}(t). \]
Sketch of proof

\[ \mathcal{V}(\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n) = \sum_{i=1}^{n} p_i V_i \]

\( \mathcal{V} \) is definite positive

\[ \sum_{i=1}^{n} \frac{p_i}{M_i} V_i \leq \| (\phi_{11}, \phi_{12}, z_1, \cdots, \phi_{n1}, \phi_{n2}, z_n) \|^2_E \leq \sum_{i=1}^{n} p_i M_i V_i \]

Employing the definite positive property of Lemma for sub functional \( V_i \), one finds the proof.

\( \dot{\mathcal{V}} \) is definite negative

\[ \dot{\mathcal{V}}(t) \leq -\sum_{i=1}^{n} p_i \gamma_i V_i(t) - \sum_{i=1}^{n} z_i^2(t) (p_i A_i - p_{i+1} B_i) - \sum_{i=1}^{n} \phi_{i1}^2(L, t) (p_i C_i - p_{i+1} D_i) \]

With \( p_{i+1} = \epsilon p_i \), choosing \( \epsilon \) enough small, we have

\[ \forall t \in \mathbb{R}_+, \exists \gamma > 0, \dot{\mathcal{V}}(t) \leq -\gamma \mathcal{V}(t). \]
Proof of Theorem

Unique solution and 'zero' stability

1. (Existence and uniqueness of solutions)
   Choosing initial condition

   \[ (\phi_{11}^0(x), \phi_{12}^0(x), z_1^0, \ldots, \phi_{n1}^0(x), \phi_{n2}^0(x), z_n^0) \in \mathbb{E}, \quad \forall x \in [0, L] \]

   \[ \Rightarrow \] Closed-loop system with PI controller has a unique solution in \( \mathbb{E} \) (using idea in [Coron and Bastin 2008]).

2. (Exponential stability of 'zero' point in \( \mathbb{E} \))
   Directly deduced from the Lemma for global functional \( \nabla \).
Output regulation

With initial condition

\[(\phi_{11}^0(x), \phi_{12}^0(x), z_1^0, \cdots, \phi_{n1}^0(x), \phi_{n2}^0(x), z_n^0) \in ((H^1(0, L))^2 \times \mathbb{R})^n, \quad \forall x \in [0, L]\]

and the exponential stability of 'zero' in \(E\)

\rightarrow 'zero' stability in \(((H^1(0, L))^2 \times \mathbb{R})^n\) by closed graph theorem

With

\[\lim_{t \to \infty} ||\phi_{i1}||_{H^1(0, L)} = 0, \quad \lim_{t \to \infty} ||\phi_{i2}||_{H^1(0, L)} = 0\]

and Sobolev embedding theorem, we have

\[\lim_{t \to \infty} \phi_{i1}(x, t) = 0, \quad \lim_{t \to \infty} \phi_{i2}(x, t) = 0 \quad \forall x \in [0, L]\]

Therefore,

\[\lim_{t \to \infty} |y_i(t) - y_{ir}| = 0\]
Plan

1. Introduction

2. Statement of the problem and main result

3. Lyapunov techniques and the proof of the main result

4. Application for Saint Venant model

5. Conclusions
Cascade network of n Saint-Venant hydraulic systems

Cascade network

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} H_i \\ Q_i \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{B_i} \\ \frac{Q_i^2}{B_i H_i} + gB_i & \frac{2Q_i}{B_i H_i} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} H_i \\ Q_i \end{pmatrix} &= 0, \\
y_i(t) &= H_i(L, t) \hspace{1cm} \text{(output measurement)}
\end{align*}
\]

Boundary conditions:

\[Q_i^2(L, t) = \alpha_i \left( H_i(L, t) - U_i(t) \right) \quad \forall i = 1, n \text{ and } Q_1(0, t) = Q_0(\text{constant})\]

Conservation law of discharges:

\[Q_j(L, t) = Q_{j+1}(0, t) \hspace{1cm} j = 1, n-1\]
## Linearized model

- **Linearized network with** \( h_i = H_i - H_i^* \), \( q_i = Q_i - Q_0 \)

\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} (h_i q_i) + \begin{pmatrix}
0 \\
-\frac{Q_0^2}{B_i(H_i^*)^2} + g B_i H_i^* \\
\frac{2Q_0}{B_i H_i^*}
\end{pmatrix} \frac{1}{B_i} \frac{\partial}{\partial x} (h_i q_i) = 0 \\
y_i(t) = h_i(L, t) + H_i^*
\end{cases}
\end{align*}
\]

- **Boundary conditions**:

\[
2Q_0 q_i(L, t) = \alpha_i \left( h_i(L, t) - u_i(t) \right) \quad \forall i = 1, n
\]

\( q_1(0, t) = 0 \)

- **Conservation law of discharges**:

\[
q_j(L, t) = q_{j+1}(0, t) \quad j = 1, n - 1
\]
Network in characteristic form

Using the change of coordinates

\[ h_i = \phi_{i1} + \phi_{i2}, \quad q_i = (B_i \sqrt{g H_i^*} + \frac{Q_0}{H_i^*}) \phi_{i1} - (B_i \sqrt{g H_i^*} - \frac{Q_0}{H_i^*}) \phi_{i2} \]

- Network in new coordinates

\[
\begin{align*}
\partial_t \phi_{i1}(x, t) + \lambda_{i1} \partial_x \phi_{i1}(x, t) &= 0 \\
\partial_t \phi_{i2}(x, t) - \lambda_{i2} \partial_x \phi_{i2}(x, t) &= 0 \\
y_i(t) &= \phi_{i1}(L, t) + \phi_{i2}(L, t) + H_i^*,
\end{align*}
\]

where \( \lambda_{i1} = \sqrt{g H_i^*} + \frac{Q_0}{B_i H_i^*} > 0 \), \( \lambda_{i2} = \sqrt{g H_i^*} - \frac{Q_0}{B_i H_i^*} > 0 \).

- Boundary conditions at junctions

\[
\begin{align*}
\phi_{i2}(L, t) &= R_{i2} \phi_{i1}(L, t) + u_i(t) \\
\phi_{i1}(0, t) &= R_{i1} \phi_{i2}(0, t) + \alpha_i \phi_{(i-1)1}(L, t) + \delta_i \phi_{(i-1)2}(L, t),
\end{align*}
\]

Here \( R_{i1}, R_{i2}, \alpha_i, \delta_j \) are constants.
Application of PI control design

PI controller design

\[ u_i(t) = K_{iP} (y_i(t) - H_i^*) + K_{il} \int_0^t (y_i(s) - H_i^*) ds \]

where

\[ K_{iP} = \frac{-2Q_0(B_i \sqrt{gH_i^*} + \frac{Q_0}{H_i^*}) + \alpha_i}{2Q_0(B_i \sqrt{gH_i^*} - \frac{Q_0}{H_i^*}) + \alpha_i} \]

\[ K_{il} = -\mu \left(1 + e^{\mu L} \right) \frac{\sqrt{gH_i^*} + \frac{Q_0}{B_i H_i^*}}{\sqrt{gH_i^*} - \frac{Q_0}{B_i H_i^*}} \frac{4Q_0(B_i \sqrt{gH_i^*})}{2Q_0(B_i \sqrt{gH_i^*} - \frac{Q_0}{H_i^*}) + \alpha_i} \]

\( \mu \) is tuning parameter chosen small enough.
Numerical simulations

- Numerical application for 3 channels \((n=3)\),
  - Length \(L = 100\ m\), base width \(B = 4\ m\).
  - Set-points \(H_1^* = 10\ m\), \(H_2^* = 8\ m\), \(H_3^* = 6.5\ m\), constant discharge \(Q_0 = 7\ m^3/s\).
  - Output disturbances \(w_{1o} = 0.1\), \(w_{2o} = 0.2\), \(w_{20} = 0.15\); and control disturbances \(w_{1c} = 0.02\), \(w_{2c} = 0.03\), \(w_{2c} = 0.01\).

- Simulations for the output regulation

![Output measurements graph](image.png)

**Figure:** Output measurements \(y_i(t)\)
Numerical simulations

- Simulations for the stability

**Figure:** $H_1(x, t)$

**Figure:** $H_2(x, t)$

**Figure:** $H_3(x, t)$

**Figure:** $Q_1(x, t)$

**Figure:** $Q_2(x, t)$

**Figure:** $Q_3(x, t)$
Plan

1. Introduction
2. Statement of the problem and main result
3. Lyapunov techniques and the proof of the main result
4. Application for Saint Venant model
5. Conclusions
Conclusions

Obtained results

- Study a network class of $n$ linear $2 \times 2$ hyperbolic systems.
- Design $n$ boundary PI controllers at each junction.
- Prove the stability of the closed-loop system in $L^2$ norm and output regulation based on Lyapunov direct method.
- Apply the control design for a practical network of $n$ fluid flow Saint Venant systems.

Perspectives

- Extend the PI control design for networks of $2 \times 2$ nonlinear hyperbolic PDE systems.
- Study the problem of optimal PI controllers (eg. the optimal value of $\mu$).

Submitted to Automatica
## References


THANK YOU FOR YOUR ATTENTION!