Stability of switched linear hyperbolic systems by Lyapunov techniques

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***Abstract*—Switched linear hyperbolic partial differential equations are considered in this paper. They model infinite dimensional systems of conservation laws and balance laws, which are potentially affected by a distributed source or sink term. The dynamics and the boundary conditions are subject to abrupt changes given by a switching signal, modeled as a piecewise constant function and possibly a dwell time. By means of Lyapunov techniques some sufficient conditions are obtained for the exponential stability of the switching system, uniformly for all switching signals. Different cases are considered with or without a dwell time assumption on the switching signals, and on the number of positive characteristic velocities (which may also depend on the switching signal). Some numerical simulations are also given to illustrate some main results, and to motivate this study.**

I. Introduction

Lyapunov techniques are commonly used for the stability analysis of dynamical systems, such as those modeled by partial differential equations (PDEs). The present paper focuses on a class of one-dimensional hyperbolic equations that describe, for example, systems of conservation laws or balance laws (with a source term), see [5].

The exponential stabilizability of such systems is often proved by means of a Lyapunov function, as illustrated by the contributions from [9], [13] where different control problems are solved for particular hyperbolic equations. For more general nonlinear hyperbolic equations, the knowledge of Lyapunov functions can be useful for the stability analysis of a system of conservation laws (see [4]), or even for the design of exponentially stabilizing boundary controls (see [3]). Other control techniques may be useful, such as Linear Quadratic regulation [1] or semigroup theory [12, Chap. 6].

In this paper, the class of hyperbolic systems of balance laws is first considered without any switching rule and we state sufficient conditions to derive a Lyapunov function for this class of systems. It allows us to relax [5] where the Lyapunov stability for hyperbolic systems of balance laws has been first tackled (see also [4]). Then, switched systems are considered and sufficient conditions for the asymptotic stability of a class of linear hyperbolic systems with switched dynamics and switched boundary conditions are stated. Some stability conditions depend on the average dwell time of the switching signals (if such a positive dwell time does exist). The stability property depends on the classes of the switching rules applied to the dynamics (as in [11] for finite dimensional systems). The present paper is also related to [15] where unswitched time-varying hyperbolic systems are considered.

In [2], the condition of [10] is employed. It allows analyzing the stability of hyperbolic systems, assuming a stronger hypothesis on the boundary conditions. More precisely, our approach generalizes the condition of [4], which is known to be strictly weaker than the one of [10]. Therefore our stability conditions are strictly weaker than the ones of [2]. Moreover the technique in [2] is trajectory-based via the method of characteristics, while our approach is based on Lyapunov functions, allowing for numerically tractable conditions. Indeed, the obtained sufficient conditions are written in terms of matrix inequalities, which can be solved numerically. Furthermore the estimated speed of exponential convergence is provided and can be optimized. See [14] for the use of line search algorithms to numerically compute the variables in our stability conditions, and thus to compute Lyapunov functions. The main results and the computational aspects are illustrated on two examples of switched linear hyperbolic systems.

Due to space limitation, some proofs have been omitted and collected in [14].

Notation. The notation is standard. When G is invertible, then, $(G^{-1})^\top$ is denoted as $G^{-\top}$. Given some scalar values $(a_1,\dots,a_n)$, $\mathtt{diag}(a_1,\dots,a_n)$ is the matrix in $\mathbb{R}^{n\times n}$ with zero non-diagonal entries, and with $(a_1,\dots,a_n)$ on the diagonal. Moreover given two matrices A and B, $\mathtt{diag}[A,B]$ is the block diagonal matrix formed by A and B (and zero for the other entries). The notation $A\geq B$ means that $A-B$ is positive semidefinite. The usual Euclidian norm in $\mathbb{R}^n$ is denoted by $|\cdot|$ and the associated matrix norm is denoted $\|\cdot\|$, whereas the set of all functions $\phi:(0,1)\to\mathbb{R}^n$ such that $\int_0^1|\phi(x)|^2<\infty$ is denoted by $L^2((0,1);\mathbb{R}^n)$ that is equipped with the norm $\|\cdot\|_{L^2((0,1);\mathbb{R}^n)}$. Given a topological set $\mathcal{S}$, and an interval I in $\mathbb{R}_+$, the set $C^0(I,\mathcal{S})$ is the set of continuous functions $\phi:I\to\mathcal{S}$.

II. Linear hyperbolic systems

Let us first consider the following linear hyperbolic partial differential equation:

$$\partial_t y(t,x)+\Lambda\partial_x y(t,x)=Fy(t,x),\ x\in[0,1],\ t\in\mathbb{R}_+ \quad (1)$$

where $y:\mathbb{R}_+\times[0,1]\to\mathbb{R}^n$, F is a matrix in $\mathbb{R}^{n\times n}$, Λ is a diagonal matrix in $\mathbb{R}^{n\times n}$ such that $\Lambda=\mathtt{diag}(\lambda_1,\dots,\lambda_n)$, with $\lambda_k<0$ for $k\in\{1,\dots,m\}$ and $\lambda_k>0$ for $k\in\{m+1,\dots,n\}$. We use the notation $y=\begin{pmatrix}y^-\\y^+\end{pmatrix}$, where $y^-:\mathbb{R}_+\times$

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$[0,1] \to \mathbb{R}^m$ and $y^+ : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^{n-m}$. In addition, we consider the following boundary conditions:

$$\begin{pmatrix} y^-(t,1) \\ y^+(t,0) \end{pmatrix} = G \begin{pmatrix} y^-(t,0) \\ y^+(t,1) \end{pmatrix}, \quad t \in \mathbb{R}_+ \tag{2}$$

where G is a matrix in $\mathbb{R}^{n\times n}$. Let us introduce the matrices G_{--} in $\mathbb{R}^{m\times m}$, G_{-+} in $\mathbb{R}^{m\times(n-m)}$, G_{+-} in $\mathbb{R}^{(n-m)\times m}$ and G_{++} in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $G = \begin{pmatrix} G_{--} & G_{-+} \\ G_{+-} & G_{++} \end{pmatrix}$.

We shall consider an initial condition given by

$$y(0,x) = y^0(x), \quad x \in (0,1) \tag{3}$$

where $y^0 \in L^2((0,1);\mathbb{R}^n)$. Then, it can be shown (see e.g. [5]) that there exists a unique solution $y \in C^0(\mathbb{R}_+; L^2((0,1);\mathbb{R}^n))$ to the initial value problem (1)-(3). As these solutions may not be differentiable everywhere, the concept of weak solutions of partial differential equations has to be used (see again [5] for more details). The linear hyperbolic system (1)-(2) is said to be *globally exponentially stable* (GES) if there exist $\nu > 0$ and $C > 0$ such that, for every $y_0 \in L^2((0,1);\mathbb{R}^n)$; the solution to the initial value problem (1)-(3) satisfies

$$\|y(t,.)\|_{L^2((0,1);\mathbb{R}^n)} \le Ce^{-\nu t}\|y^0\|_{L^2((0,1);\mathbb{R}^n)}, \ \forall t \in \mathbb{R}_+. \tag{4}$$

Sufficient conditions for exponential stability of (1)-(3) have been obtained in [5] using a Lyapunov function. In this section, we present an extension of this result. This extension will be also useful for subsequent work on switched linear hyperbolic systems.

Let $\Lambda^+ = \mathtt{diag}(|\lambda_1|, \ldots, |\lambda_n|)$.

Proposition 2.1: *Let us assume that there exist $\nu > 0$, $\mu \in \mathbb{R}$ and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that, defining for each x in $[0,1]$, $\mathcal{Q}(x) = \mathtt{diag}[e^{2\mu x}Q^-, e^{-2\mu x}Q^+]$, $\mathcal{Q}(x)\Lambda = \Lambda\mathcal{Q}(x)$, the following matrix inequalities hold*

$$-2\mu\mathcal{Q}(x)\Lambda^+ + F^\top\mathcal{Q}(x) + \mathcal{Q}(x)F \le -2\nu\mathcal{Q}(x) \tag{5}$$

$$\begin{aligned} &\begin{pmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{pmatrix}^\top \mathcal{Q}(0)\Lambda \begin{pmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{pmatrix} \\ &\le \begin{pmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}^\top \mathcal{Q}(1)\Lambda \begin{pmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}. \end{aligned} \tag{6}$$

Then there exists C such that (4) holds and the linear hyperbolic system (1)-(2) is GES.

The complete proof of this proposition is detailed in [14] and is based on the inequality $\dot{V}(y(t,.)) \le -2\nu V(y(t,.))$ along the solutions of (1) with the boundary conditions (2), where the Lyapunov function V is defined by $V(y) = \int_0^1 y(x)^\top \mathcal{Q}(x) y(x)\, dx$.

If all the diagonal elements of Λ are different, the assumption that $\mathcal{Q}(x)\Lambda = \Lambda\mathcal{Q}(x)$ is equivalent to Q being diagonal positive definite[1]. The main contributions of the previous proposition with respect to the result presented in [5] is double: first, we do not restrict the values of parameter μ to be positive, this allows us to consider non-contractive boundary conditions (it will be the case for the numerical illustration considered in Example V-B); second, we provide an estimate of the exponential convergence rate (see [14] for more details).

[1] This equivalence follows from the computation of matrices $\mathcal{Q}(x)\Lambda$ and $\Lambda\mathcal{Q}(x)$, and from a comparison between each of their entries.

III. Switched linear hyperbolic systems

We now consider the case of switched linear hyperbolic partial differential equation of the form (see [2]) for all $x \in [0,1]$, and $t \in \mathbb{R}_+$,

$$\partial_t w(t,x) + L_{\sigma(t)}\partial_x w(t,x) = A_{\sigma(t)}w(t,x), \tag{7}$$

where $w : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^n$, $\sigma : \mathbb{R}_+ \to I$, I is a finite set (of *modes*), A_i and L_i are matrices in $\mathbb{R}^{n\times n}$, for $i \in I$. The partial differential equation associated with each mode is hyperbolic, meaning that for all $i \in I$, there exists an invertible matrix S_i in $\mathbb{R}^{n\times n}$ such that $L_i = S_i^{-1}\Lambda_i S_i$ where Λ_i is a diagonal matrix in $\mathbb{R}^{n\times n}$ satisfying $\Lambda_i = \mathtt{diag}(\lambda_{i,1}, \ldots, \lambda_{i,n})$, with $\lambda_{i,k} < 0$ for $k \in \{1, \ldots, m_i\}$ and $\lambda_{i,k} > 0$ for $k \in \{m_i+1, \ldots, n\}$. The matrices S_i can be written as

$$S_i = \begin{pmatrix} S_i^{-\top} & S_i^{+\top} \end{pmatrix}^\top \tag{8}$$

where S_i^- and S_i^+ are matrices in $\mathbb{R}^{m_i\times n}$ and $\mathbb{R}^{(n-m_i)\times n}$. We define the matrices $F_i = S_i A_i S_i^{-1}$ and $\Lambda_i^+ = \mathtt{diag}(|\lambda_{i,1}|, \ldots, |\lambda_{i,n}|)$ for $i \in I$. The boundary conditions are given by

$$B^0_{\sigma(t)}w(t,0) + B^1_{\sigma(t)}w(t,1) = 0, \quad t \ge 0 \tag{9}$$

where, for all $i \in I$, $B_i^0 = G_i^0 S_i$ and $B_i^1 = G_i^1 S_i$, G_i^0 and G_i^1 being matrices in $\mathbb{R}^{n\times n}$ that satisfy

$$G_i^0 = \begin{pmatrix} -G_{i--} & 0_{m_i,n-m_i} \\ -G_{i+-} & I_{n-m_i} \end{pmatrix}, \ G_i^1 = \begin{pmatrix} I_{m_i} & -G_{i-+} \\ 0_{m_i,n-m_i} & -G_{i++} \end{pmatrix}.$$

For $i \in I$, let us define the matrices in $\mathbb{R}^{n\times n}$, $G_i = \begin{pmatrix} G_{i--} & G_{i-+} \\ G_{i+-} & G_{i++} \end{pmatrix}$. We shall consider an initial condition given by

$$w(0,x) = w^0(x), \quad x \in (0,1) \tag{10}$$

where $w^0 \in L^2((0,1);\mathbb{R}^n)$.

A *switching signal* is a piecewise constant function $\sigma : \mathbb{R}_+ \to I$, right-continuous, and with a finite number of discontinuities on every bounded interval of $\mathbb{R}_+$. This allows us to avoid the *Zeno behavior*, as described in [11]. The set of switching signals is denoted by $\mathcal{S}(\mathbb{R}_+, I)$. The discontinuities of σ are called *switching times*. The number of discontinuities of σ on the interval $(\tau, t]$ is denoted by $N_\sigma(\tau, t)$. Following [8], for $\tau_D > 0$, $N_0 \in \mathbb{N}$, we denote by $\mathcal{S}_{\tau_D,N_0}(\mathbb{R}_+, I)$ the set of switching signals verifying, for all $\tau < t$, $N_\sigma(\tau,t) \le N_0 + \frac{t-\tau}{\tau_D}$. The constant τ_D is called the *average dwell time* and N_0 the *chatter bound*.

We first provide an existence and uniqueness result for the solutions of (7)-(10):

Proposition 3.1: *For all $\sigma \in \mathcal{S}(\mathbb{R}_+, I)$, $w^0 \in L^2((0,1);\mathbb{R}^n)$, there exists a unique (weak) solution $w \in C^0(\mathbb{R}_+; L^2((0,1);\mathbb{R}^n))$ to the initial value problem (7)-(10).*

Proof: We build iteratively the solution between successive switching times. Let $(t_k)_{k\in K}$ denote the increasing switching times of σ, with $t_0 = 0$ and K be a (finite or infinite) subset of $\mathbb{N}$. Let us assume that we have been able to build a unique (weak) solution $w \in C^0([0,t_k]; L^2((0,1);\mathbb{R}^n))$ for some $k \ge 0$. Then, let i_k be the value of $\sigma(t)$ for $t \in [t_k, t_{k+1})$. Let us introduce the following notation, for all k in K and for all x in $[0,1]$,

$$y_k(t,x) = S_{i_k}w(t,x), \quad t \in [t_k, t_{k+1}]. \tag{11}$$

Note that closed time intervals are used on both sides due to technical reasons in this proof. Then, (7) gives that, for all k in K, y_k satisfies the following partial differential equation, for all $x \in [0,1]$, and $t \in [t_k, t_{k+1}]$,

$$\partial_t y_k(t,x) + \Lambda_{i_k} \partial_x y_k(t,x) = F_{i_k} y_k(t,x). \tag{12}$$

Also, we use the notations $y_k = \begin{pmatrix} y_k^- \\ y_k^+ \end{pmatrix}$, where $y_k^- : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^{m_{i_k}}$ and $y_k^+ : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^{n-m_{i_k}}$. The boundary conditions (9) give, for all k in K,

$$\begin{pmatrix} y_k^-(t,1) \\ y_k^+(t,0) \end{pmatrix} = G_{i_k} \begin{pmatrix} y_k^-(t,0) \\ y_k^+(t,1) \end{pmatrix}, \quad t \in [t_k, t_{k+1}] \ . \tag{13}$$

The initial condition ensuring the continuity of w at time t_k is the following:

$$y_k(t_k, x) = S_{i_k} w(t_k, x), \quad x \in (0,1) \ . \tag{14}$$

It follows from [5] that, for all k in K, there exists a unique (weak) solution $y_k \in C^0([t_k, t_{k+1}]; L^2((0,1);\mathbb{R}^n))$ to the initial value problem (12)-(14). Then, we can extend the (weak) solution to the initial value problem (7)-(10), from the initial time t_k, up to the switching time t_{k+1}; (14) ensures that $w \in C^0([0, t_{k+1}]; L^2((0,1);\mathbb{R}^n))$, and the uniqueness of y_k ensures that w is the unique solution. Finally, since there are only a finite number of switching times on every bounded intervals of $\mathbb{R}_+$, the solution can be defined for all times, resulting on a unique solution $w \in C^0(\mathbb{R}_+; L^2((0,1);\mathbb{R}^n))$. ■

IV. Stability of switched linear hyperbolic systems

Let $\mathcal{S} \subseteq \mathcal{S}(\mathbb{R}_+, I)$. The switched linear hyperbolic system (7)-(9) is said to be *globally uniformly exponentially stable* (GUES) with respect to the set of switching signals $\mathcal{S}$ if there exist $\nu > 0$ and $C > 0$ such that, for every $w_0 \in L^2((0,1);\mathbb{R}^n)$, for every $\sigma \in \mathcal{S}$, the solution to the initial value problem (7)-(10) satisfies $\|w(t,.)\|_{L^2((0,1);\mathbb{R}^n)} \leq Ce^{-\nu t}\|w^0\|_{L^2((0,1);\mathbb{R}^n)}$, $\forall t \in \mathbb{R}_+$. In this section, we provide sufficient conditions for the stability of switched linear hyperbolic systems.

A. Mode independent sign structure of characteristics

Assume first that the number of negative and positive characteristics of the linear partial differential equations associated with each mode is constant, that is for all $i \in I$, $m_i = m$.

We provide a first result giving sufficient conditions such that stability holds for all switching signals. See [14] for a proof where a common Lyapunov function equivalent to the L^2-norm is used. An alternative proof can be obtained by checking some semigroup properties and by using [7] (where the equivalence is shown between the existence of a common Lyapunov function commensurable with the squared norm and the global uniform exponential stability).

Theorem 1: *Let us assume that, for all $i \in I$, $m_i = m$ and that there exist $\nu > 0$, $\mu \in \mathbb{R}$ and diagonal positive definite matrices Q_i in $\mathbb{R}^{n \times n}$, $i \in I$ such that the following matrix inequalities hold, for all $i \in I$ and for all x in $[0,1]$,*

$$-2\mu \mathcal{Q}_i(x)\Lambda_i^+ + F_i^\top \mathcal{Q}_i(x) + \mathcal{Q}_i(x) F_i \leq -2\nu \mathcal{Q}_i(x), \tag{15}$$

$$\begin{pmatrix} I_m & 0_{m,n-m} \\ G_{i+-} & G_{i++} \end{pmatrix}^\top \mathcal{Q}_i(0)\Lambda_i \begin{pmatrix} I_m & 0_{m,n-m} \\ G_{i+-} & G_{i++} \end{pmatrix} \leq \begin{pmatrix} G_{i--} & G_{i-+} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}^\top \mathcal{Q}_i(1)\Lambda_i \begin{pmatrix} G_{i--} & G_{i-+} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}, \tag{16}$$

where $\mathcal{Q}_i(x) = \mathtt{diag}[e^{2\mu x}Q_i^-, e^{-2\mu x}Q_i^+]$, $Q_i = \begin{pmatrix} Q_i^- & 0 \\ 0 & Q_i^+ \end{pmatrix}$, Q_i^- and Q_i^+ are diagonal positive matrices in $\mathbb{R}^{m_i \times m_i}$ and $\mathbb{R}^{(n-m_i)\times(n-m_i)}$, together with the following matrix equalities, for all $i, j \in I$,

$$(S_i^+)^\top Q_i^+ S_i^+ = (S_j^+)^\top Q_j^+ S_j^+, \quad (S_i^-)^\top Q_i^- S_i^- = (S_j^-)^\top Q_j^- S_j^-. \tag{17}$$

Then, the switched linear hyperbolic system (7)-(9) is GUES with respect to the set of switching signals $\mathcal{S}(\mathbb{R}_+, I)$.

The numerical computation of the unknown variables, satisfying the sufficient conditions of Theorem 1, is explained in [14].

For systems that do not satisfy the assumptions of the previous theorem, but whose dynamics in each mode satisfy independently the assumptions of Proposition 2.1 (i.e. the dynamics in each mode is stable), it is possible to show that the system is stable provided that the switching is slow enough:

Theorem 2: *Let us assume that, for all $i \in I$, $m_i = m$ and that there exist $\nu > 0$, $\gamma \geq 1$, $\mu_i \in \mathbb{R}$, diagonal positive definite matrices Q_i in $\mathbb{R}^{n\times n}$, such that the following matrix inequalities hold, for all x in $[0,1]$,*

$$-2\mu_i \mathcal{Q}_i(x)\Lambda_i^+ + F_i^\top \mathcal{Q}_i(x) + \mathcal{Q}_i(x) F_i \leq -2\nu \mathcal{Q}_i(x), \tag{18}$$

$$\begin{pmatrix} I_m & 0_{m,n-m} \\ G_{i+-} & G_{i++} \end{pmatrix}^\top \mathcal{Q}_i(0)\Lambda_i \begin{pmatrix} I_m & 0_{m,n-m} \\ G_{i+-} & G_{i++} \end{pmatrix} \leq \begin{pmatrix} G_{i--} & G_{i-+} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}^\top \mathcal{Q}_i(1)\Lambda_i \begin{pmatrix} G_{i--} & G_{i-+} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}, \tag{19}$$

where $\mathcal{Q}_i(x) = \mathtt{diag}[e^{2\mu_i x}Q_i^-, e^{-2\mu_i x}Q_i^+]$, $Q_i = \begin{pmatrix} Q_i^- & 0 \\ 0 & Q_i^+ \end{pmatrix}$, Q_i^- and Q_i^+ are diagonal positive matrices in $\mathbb{R}^{m_i \times m_i}$ and $\mathbb{R}^{(n-m_i)\times(n-m_i)}$, together with the following matrix inequalities, for all $i, j \in I$,

$$(S_i^+)^\top Q_i^+ S_i^+ \leq \gamma (S_j^+)^\top Q_j^+ S_j^+, \tag{20}$$

$$(S_i^-)^\top Q_i^- S_i^- \leq \gamma (S_j^-)^\top Q_j^- S_j^-. \tag{21}$$

Let $\Delta_\mu = \max(\mu_1, \ldots, \mu_n) - \min(\mu_1, \ldots, \mu_n)$, then, for all $N_0 \in \mathbb{N}$, for all $\tau_D > \frac{\ln(\gamma)}{2\nu} + \frac{\Delta_\mu}{\nu}$, the switched linear hyperbolic system (7)-(9) is GUES with respect to the set of switching signals $\mathcal{S}_{\tau_D, N_0}(\mathbb{R}_+, I)$.

Proof: Let $(t_k)_{k\in K}$ denote the increasing switching times of σ, with $t_0 = 0$ and K is a (finite or infinite) subset of $\mathbb{N}$. For $k \in K$, let i_k be the value of $\sigma(t)$ for $t \in [t_k, t_{k+1})$, and let y_k be given by (11). It satisfies the boundary conditions (13). Given the diagonal matrices Q_i satisfying the assumptions of Theorem 2, let $M_i^- = (S_i^-)^\top Q_i^- S_i^-$ and $M_i^+ = (S_i^+)^\top Q_i^+ S_i^+$. The proof is based on the use of multiple Lyapunov functions. More precisely, denoting $\mathcal{M}_{i_k}(x) = e^{2\mu_{i_k} x} M_{i_k}^- + e^{-2\mu_{i_k} x} M_{i_k}^+$, let us define, for all w in $C^0([0,\infty); L^2((0,1);\mathbb{R}^n))$, for all t in $\mathbb{R}_+$,

$$V(w(t,.)) = \int_0^1 w(t,x)^\top \mathcal{M}_{i_k}(x) w(t,x)\, dx, \tag{22}$$

if k is such that $t \in [t_k, t_{k+1})$, which may be rewritten as $V(w(t,.)) = \int_0^1 y_k(t,x)^\top \mathcal{Q}_{i_k}(x) y_k(t,x)\, dx$, if $t \in [t_k, t_{k+1})$.

Note that $\mathcal{Q}_{i_k}(x)$ commute with Λ_{i_k} since these matrices are diagonal. Using (18) and (19), and following the proof of Proposition 1, we get that, along the solutions of (7)-(9), for all $k \in K$ and $t \in [t_k, t_{k+1})$,

$$V(w(t,.)) \leq V(w(t_k,.)) e^{-2\nu(t-t_k)}. \tag{23}$$

The function V may be not continuous at the switching times any more. Nevertheless, by (20) and (21), we have that, for all k in K,

$$\begin{aligned}
&V(w(t_{k+1},.))\\
&= \int_0^1 \Big(w(t_{k+1},x)^\top M^-_{i_{k+1}} w(t_{k+1},x) e^{2\mu_{i_{k+1}} x}\\
&\quad + w(t_{k+1},x)^\top M^+_{i_{k+1}} w(t_{k+1},x) e^{-2\mu_{i_{k+1}} x}\Big)\, dx\\
&\leq \gamma \int_0^1 \Big(w(t_{k+1},x)^\top M^-_{i_k} w(t_{k+1},x) e^{2\mu_{i_{k+1}} x}\\
&\quad + w(t_{k+1},x)^\top M^+_{i_k} w(t_{k+1},x) e^{-2\mu_{i_{k+1}} x}\Big)\, dx\\
&\leq \gamma e^{2\Delta_\mu} \int_0^1 \big(w(t_{k+1},x)^\top M^-_{i_k} w(t_{k+1},x) e^{2\mu_{i_k} x}\\
&\quad + w(t_{k+1},x)^\top M^+_{i_k} w(t_{k+1},x) e^{-2\mu_{i_k} x}\big)\, dx\\
&\leq \gamma e^{2\Delta_\mu} \lim_{t \to t^-_{k+1}} V(w(t,.))
\end{aligned}$$

where the continuity of w is used in the last inequality (it follows from Proposition 3.1). Then, it follows from (23) that, for all k in K, $V(w(t_{k+1},.)) \leq \gamma e^{2\Delta_\mu} V(w(t_k,.)) e^{-2\nu(t_{k+1}-t_k)}$, and it allows us to prove recursively that, for all $t \in \mathbb{R}_+$, $V(w(t,.)) \leq \left(\gamma e^{2\Delta_\mu}\right)^{N_\sigma(0,t)} V(w^0) e^{-2\nu t} \leq \left(\gamma e^{2\Delta_\mu}\right)^{(N_0 + \frac{t}{\tau_D})} V(w^0) e^{-2\nu t}$. Let $\bar{\nu} = \nu - \frac{\Delta_\mu}{\tau_D} - \frac{\ln(\gamma)}{2\tau_D}$, the assumption on the average dwell time gives that $\bar{\nu} > 0$ and the previous inequality yields $\forall t \in \mathbb{R}_+,\ V(w(t,.)) \leq \left(\gamma e^{2\Delta_\mu}\right)^{N_0} V(w^0) e^{-2\bar{\nu} t}$ which allows us to conclude that the switched linear hyperbolic system is GUES with respect to the set of switching signals $\mathcal{S}_{\tau_D, N_0}(\mathbb{R}_+, I)$. This concludes the proof of Theorem 2. ∎

Remark 4.1: Setting $\gamma = 1$ and $\mu_i = \mu$ for all $i \in I$, we recover the assumptions of Theorem 1. In that case we have $\Delta_\mu = 0$: there is no positive lower bound imposed on the average dwell time, which is consistent with Theorem 1. ◦

Remark 4.2: Note that the existence of $\gamma \geq 1$ such that (20) holds is equivalent to $\mathtt{Ker}(S_i^+) = \mathtt{Ker}(S_j^+)$, for all $i, j \in I$. Therefore the existence of $\gamma \geq 1$ such that (20) and (21) are satisfied is equivalent to $\mathtt{Ker}(S_i^+) = \mathtt{Ker}(S_j^+)$ and $\mathtt{Ker}(S_i^-) = \mathtt{Ker}(S_j^-)$, for all $i, j \in I$ (and also, by recalling $L_i = S_i^{-1} \Lambda_i S_i$, the subspace associated with all positive (resp. negative) eigenvalues of L_i does not depend on i). If this condition does not hold, stability can still be analyzed using other stability results presented in the following section. ◦

B. Mode dependent sign structure of characteristics

We now relax the assumption on the number of negative and positive characteristics. As in the previous section, we provide a first result giving sufficient conditions such that stability holds for all switching signals (see [14] for the proof):

Theorem 3: *Let us assume that there exist $\nu > 0$ and diagonal positive definite matrices Q_i in $\mathbb{R}^{n \times n}$, $i \in I$ such that, for all $i \in I$,*

$$F_i^\top Q_i + Q_i F_i \leq -2\nu Q_i, \tag{24}$$

$$G_i^T Q_i \Lambda_i^+ G_i \leq Q_i \Lambda_i^+, \tag{25}$$

and such that, for all $i, j \in I$,

$$S_i^\top Q_i S_i = S_j^\top Q_j S_j. \tag{26}$$

Then, the switched linear hyperbolic system (7)-(9) is GUES with respect to the set of switching signals $\mathcal{S}(\mathbb{R}_+, I)$.

The assumptions of the previous theorem are quite strong. To assure the asymptotic stability for switching signals with a sufficiently large dwell time, weaker assumptions are needed. More precisely, considering the assumptions of Theorem 2, the last main result of this paper can be stated:

Theorem 4: *Let us assume that there exist $\nu > 0$, $\gamma \geq 1$, $\mu_i \in \mathbb{R}$, and diagonal positive definite matrices Q_i in $\mathbb{R}^{n \times n}$, $i \in I$ such that the matrix inequalities (18), (19) hold (where the same notation for $\mathcal{Q}_i(x)$ is used) together with the following matrix inequalities, for all $i, j \in I$,*

$$S_i^\top Q_i S_i \leq \gamma S_j^\top Q_j S_j. \tag{27}$$

Let $\bar{\Delta}_\mu = 2|\mu_i|$ if I is a singleton and $\bar{\Delta}_\mu = 2\max_{i \neq j \in I}(|\mu_i| + |\mu_j|)$ else. Then, for all $N_0 \in \mathbb{N}$, for all $\tau_D > \frac{\ln(\gamma)}{2\nu} + \frac{\bar{\Delta}_\mu}{\nu}$, the switched linear hyperbolic system (7)-(9) is GUES with respect to the set of switching signals $\mathcal{S}_{\tau_D, N_0}(\mathbb{R}_+, I)$.

Proof: We use the same notations as in Theorem 2, and we consider the candidate Lyapunov function (22). Using (18) and (19), Equation (23) still holds along the solutions of (7)-(9). Moreover, for all k in K,

$$\begin{aligned}
&V(w(t_{k+1},.))\\
&\leq e^{2|\mu_{i_{k+1}}|} \int_0^1 y_{k+1}(t_{k+1},x)^\top Q_{i_{k+1}} y_{k+1}(t_{k+1},x) dx\\
&\leq \gamma e^{2|\mu_{i_{k+1}}|} \int_0^1 y_{k+1}(t_{k+1},x)^\top Q_{i_k} y_{k+1}(t_{k+1},x) dx\\
&\leq \gamma e^{2|\mu_{i_{k+1}}| + 2|\mu_{i_k}|} \int_0^1 y_k(t_{k+1},x)^\top \mathcal{Q}_{i_k}(x) y_k(t_{k+1},x) dx\\
&\leq \gamma e^{2\bar{\Delta}_\mu} \lim_{t \to t^-_{k+1}} V(w(t,.)).
\end{aligned}$$

The end of the proof follows the same lines as that of Theorem 2. ∎

Let us note that Theorem 3 can be deduced from Theorem 4 by selecting $\gamma = 1$ and $\mu_i = 0$ for all $i \in I$.

V. Examples

A. Mode independent sign structure of characteristics

Consider the wave equation: $\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0$, where $x \in [0,1]$, $t \in \mathbb{R}_+$, and $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$. The solutions of the previous equations can be written as $u(t,x) = w_1(t,x) + w_2(t,x)$ with $w = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$ verifying

$$\partial_t w(t,x) + L\partial_x w(t,x) = 0, \quad x \in [0,1],\ t \in \mathbb{R}_+ \tag{28}$$

where $L = \mathtt{diag}(-1,1)$. We consider for this hyperbolic system the following switching boundary conditions:

$$\begin{aligned} w_1(t,1) &= \begin{cases} -1.2\, w_2(t,1) & \text{if } i(t) = 1 \\ -0.6\, w_2(t,1) & \text{if } i(t) = 2 \end{cases}, \\ w_2(t,0) &= \begin{cases} 0.6\, w_1(t,0) & \text{if } i(t) = 1 \\ 1.2\, w_1(t,0) & \text{if } i(t) = 2 \end{cases}. \end{aligned} \tag{29}$$

This is a switched linear hyperbolic system of the form (7)-(9) with $L_1 = L_2 = L$, $A_1 = A_2 = 0_2$, $S_1 = S_2 = I_2$, $G_1 = \left(\begin{smallmatrix} 0 & -1.2 \\ 0.6 & 0 \end{smallmatrix}\right)$ and $G_2 = \left(\begin{smallmatrix} 0 & -0.6 \\ 1.2 & 0 \end{smallmatrix}\right)$. With the notations defined in the previous sections, we also have $\Lambda_1^+ = \Lambda_2^+ = I_2$ and $F_1 = F_2 = 0_2$. In this case (15) becomes

$$-\mu\Lambda_i^+ \leq -\nu I_n. \tag{30}$$

We were not able to apply Theorem 1 as we could not find $\nu > 0$, $\mu \in \mathbb{R}$, and diagonal positive definite matrices Q_1 and Q_2 such that the set of matrix inequalities (16), (17) and (30) hold.

Actually, this could be explained by the fact that it is possible to find a switching signal that destabilizes the system as shown on the left part of Figure 1 (where a periodic switching signal is used with a period equal to 1).

We can prove the exponential stability for a set of switching signals with an assumption on the average dwell time using Theorem 2. Let us remark that since $F_1 = F_2 = 0$, (18) is equivalent to $\mu_i \geq \nu$ for $i \in \{1,2\}$. One can verify that Equations (19), (20) and (21) hold as well for the choices $Q_1 = \left(\begin{smallmatrix} 0.75 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $Q_2 = \left(\begin{smallmatrix} 1.5 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma = 2$. Then, Theorem 2 guarantees the stability of the switched linear hyperbolic system for switching signals with average dwell time greater than $\frac{\ln(\gamma)}{2\nu} = 2.3105$. The right part of Figure 1 shows the stable behavior of the switched linear hyperbolic system for a periodic switching signal with a period equal to 2.4. To illustrate Theorem 1, we add a diagonal damping term to (28) defined by, for all $x \in [0,1]$ and $t \in \mathbb{R}_+$,

$$\partial_t w(t,x) + L\partial_x w(t,x) = Aw(t,x), \tag{31}$$

where $A = \mathtt{diag}(-0.3,-0.3)$. The boundary conditions are given by (29). Now, $A_1 = A_2 = F_1 = F_2 = A$ and the other matrices of the system remain unchanged. In the present case (30) is equivalent to $\nu \leq \mu + 0.3$. One can verify that Theorem 1 applies with $\mu = -0.2$, $\nu = 0.1$ and $Q_1 = Q_2 = \left(\begin{smallmatrix} 1.5 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then, Theorem 1 guarantees the stability of the switched linear hyperbolic system for all switching signals. Figure 2 shows the stable behavior of the switched linear hyperbolic system for a periodic switching signal of period 1.

B. *Mode dependent sign structure of characteristics*

To illustrate the results of Section IV-B, we consider the following switched linear hyperbolic system, for all $x \in [0,1]$ and $t \in \mathbb{R}_+$,

$$\partial_t w(t,x) + L_{i(t)}\partial_x w(t,x) = Fw(t,x), \tag{32}$$

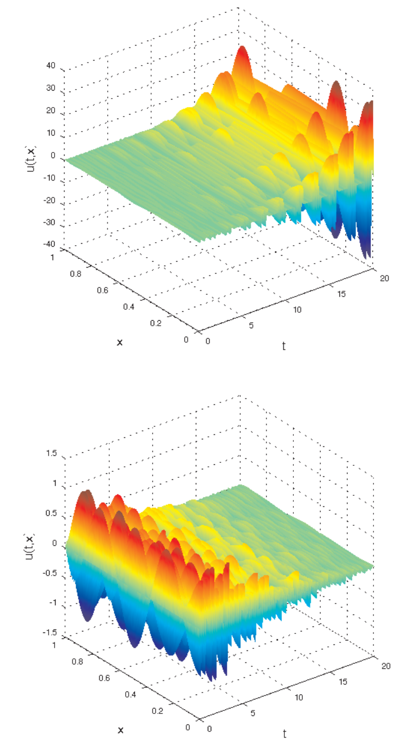

Fig. 1. Time evolution of $u = w_1 + w_2$, solution of (28)-(29) for periodic switching signals of period 1 (up) and 2.4 (down).

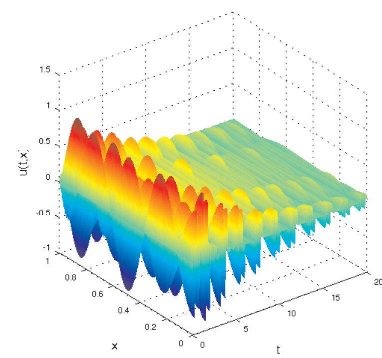

Fig. 2. Time evolution of $u = w_1 + w_2$, solution of (29)-(31) for a periodic switching signal of period 1.

where $w : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$, $i(t) \in I = \{1,2\}$, $L_1 = 1$ and $L_2 = -1$ and $F \in \mathbb{R}$. The boundary conditions are given by

$$\begin{aligned} w(t,0) &= Gw(t,1) \quad \text{if } i(t) = 1 \\ w(t,1) &= Gw(t,0) \quad \text{if } i(t) = 2 \end{aligned} \tag{33}$$

and $G > 0$. This is a switched linear hyperbolic system of the form (7)-(9) with $A_1 = A_2 = F$, $S_1 = S_2 = 1$, and $G_1 = G_2 = G$. With the notation defined in the previous sections, we also have $\Lambda_1^+ = \Lambda_2^+ = 1$ and $F_1 = F_2 = F$.

We assume that $F < -\ln(G)$; if this does not hold, then it can be shown that the linear hyperbolic systems in each mode are both not asymptotically stable. If $F < 0$ and $G \leq 1$, then Theorem 3 applies with $Q_1 = Q_2 = 1$ and $\nu = -F$. Hence, in that case Theorem 3 guarantees the stability of the switched linear hyperbolic system for all switching signals. If $G > 1$ (resp. $F > 0$) then the condition (25) (resp. (24)) of Theorem 3 does not hold and thus Theorem 3 does not apply.

If $G > 1$, let $F < \mu < -\ln(G)$, then Theorem 4 holds with $\mu_1 = \mu_2 = \mu$, $\nu = \mu - F$, $\gamma = 1$ and $Q_1 = Q_2 = 1$. Then, Theorem 4 guarantees the stability of the switched linear hyperbolic system for switching signals with average dwell time τ_D greater than $\frac{\bar{\Delta}_\mu}{\nu} = \frac{-2\mu}{\mu - F}$ for any $\mu \in (F, -\ln(G))$. The minimal value of $\frac{-2\mu}{\mu-F}$ in this interval is $\frac{-2\ln(G)}{\ln(G)+F}$; therefore the stability of the switched linear hyperbolic system is guaranteed for switching signals with τ_D greater than $\frac{-2\ln(G)}{\ln(G)+F}$. For $G = 2$ and $F = -1$, in that case the minimal required τ_D is 4.5178. Figure 3 shows unstable and stable behaviors for these values of G and F and for periods equal to 1.2 and 4.6.

If $F > 0$, let G and μ such that $F < \mu < -\ln(G)$, then Theorem 4 holds with $\mu_1 = \mu_2 = \mu$, $\nu = \mu - F$, $\gamma = 1$ and $Q_1 = Q_2 = 1$. Then, Theorem 4 guarantees the stability of the switched linear hyperbolic system for switching signals with τ_D greater than $\frac{\bar{\Delta}_\mu}{\nu} = \frac{2\mu}{\mu-F}$ for any $\mu \in (F, -\ln(G))$. The minimal value of $\frac{2\mu}{\mu-F}$ in this interval is $\frac{2\ln(G)}{\ln(G)+F}$; therefore stability of the switched linear hyperbolic system is guaranteed for switching signals with $\tau_D > \frac{2\ln(G)}{\ln(G)+F}$. For $G = 0.5$ and $F = -0.1$, the minimal required τ_D is 2.3372. Figure 4 shows unstable and stable behaviors for these values of G and F and for periods equal 0.9 and 2.4.

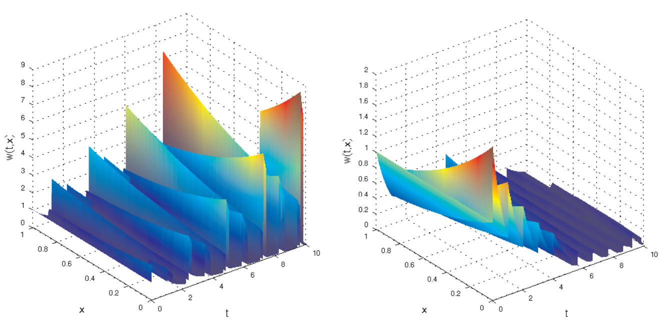

Fig. 3. Time evolution of w, solution of (32)-(33) with $F = -1$ and $G = 2$, for periodic switching signals of period 1.2 (left) and 4.6 (right).

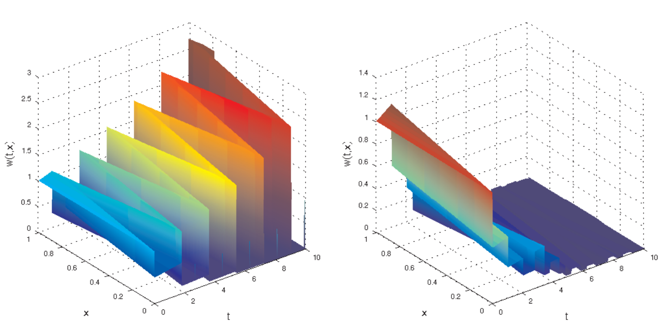

Fig. 4. Time evolution of w, solution of (32)-(33) with $F = 0.1$ and $G = 0.5$, for periodic switching signals of period 0.9 (left) and 2.4 (right).

VI. Conclusion

In this paper, some sufficient conditions have been derived for the exponential stability of hyperbolic PDE with switching signals defining the dynamics and the boundary conditions. This stability analysis has been done with Lyapunov functions and exploiting the dwell time assumption, if it holds, of the switching signals. The sufficient stability conditions are written in terms of matrix inequalities which lead to numerically tractable problems.

This work lets many questions open. In particular, exploiting the sufficient conditions for the derivation of switching stabilizing boundary controls (as for the physical application considered in [6]) seems to be a natural extension. The generalization of the results to linear hyperbolic systems with space-varying entries may also be studied.

References

[1] I. Aksikas, A. Fuxman, J.F. Forbes, and J.J. Winkin. LQ control design of a class of hyperbolic PDE systems: Application to fixed-bed reactor. *Automatica*, 45(6):1542–1548, 2009.

[2] S. Amin, F.M. Hante, and A.M. Bayen. Exponential stability of switched linear hyperbolic initial-boundary value problems. *IEEE Transactions on Automatic Control*, 57(2):291–301, 2012.

[3] J.-M. Coron, G. Bastin, and B. d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.

[4] J.-M. Coron, B. d'Andréa Novel, and G. Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, 2007.

[5] A. Diagne, G. Bastin, and J.-M. Coron. Lyapunov exponential stability of linear hyperbolic systems of balance laws. *Automatica*, 48(1):109–114, 2012.

[6] V. Dos Santos and C. Prieur. Boundary control of open channels with numerical and experimental validations. *IEEE Transactions on Control Systems Technology*, 16(6):1252–1264, 2008.

[7] F.M. Hante and M. Sigalotti. Converse Lyapunov theorems for switched systems in Banach and Hilbert spaces. *SIAM Journal on Control and Optimization*, 49(2):752–770, 2011.

[8] J.P. Hespanha and A.S. Morse. Stability of switched systems with average dwell-time. In *38th IEEE Conference on Decision and Control (CDC)*, volume 3, pages 2655–2660, Phoenix, AZ, USA, 1999.

[9] M. Krstic and A. Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57:750–758, 2008.

[10] T.-T. Li. *Global classical solutions for quasilinear hyperbolic systems*, volume 32 of *RAM: Research in Applied Mathematics*. Masson, Paris, 1994.

[11] D. Liberzon. *Switching in systems and control*. Springer, 2003.

[12] Z.-H. Luo, B.-Z. Guo, and O. Morgul. *Stability and stabilization of infinite dimensional systems and applications*. Communications and Control Engineering. Springer-Verlag, New York, 1999.

[13] C. Prieur and J. de Halleux. Stabilization of a 1-D tank containing a fluid modeled by the shallow water equations. *Systems & Control Letters*, 52(3-4):167–178, 2004.

[14] C. Prieur, A. Girard, and E. Witrant. Stability of switched linear hyperbolic systems by Lyapunov techniques (full version). Technical report, arXiv:1307.4973, http://fr.arXiv.org/abs/1307.4973, 2013.

[15] C. Prieur and F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems*, 24(1):111–134, 2012.