Robust discrete time control

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April 2004,
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Outline

Part I

Introduction, examples and basic concepts

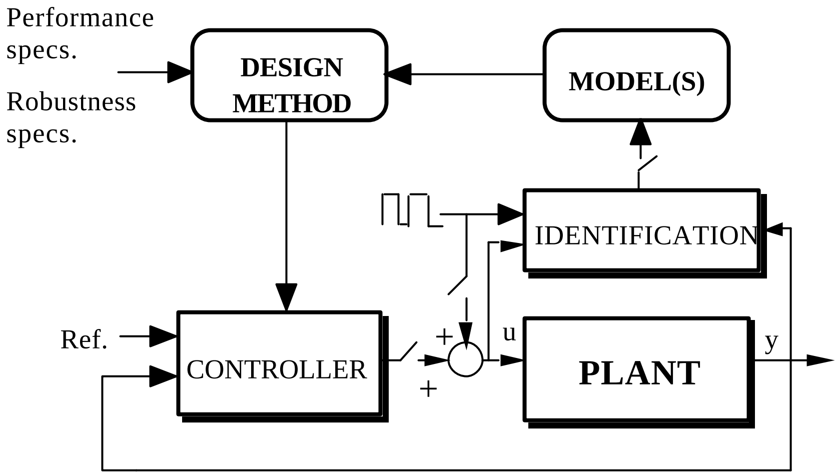
Part II

Design of robust discrete time controllers

Part III

Special topics

Controller Design and Validation



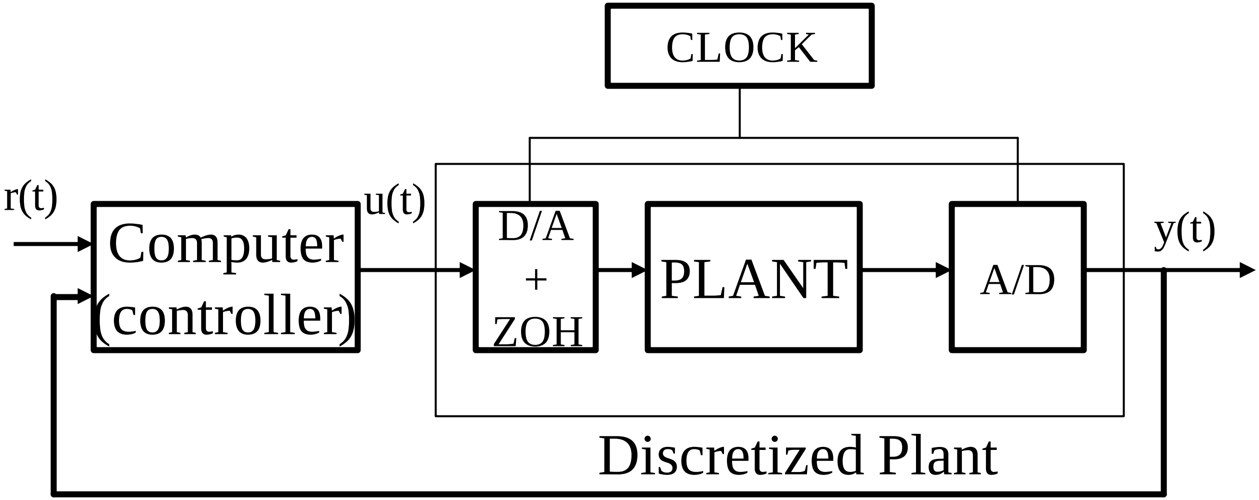
1) Identification of the dynamic model
2) **Performance and robustness specifications**
3) **Compatible robust controller design method**
4) Controller implementation
5) Real-time controller validation
   (and on site re-tuning)
6) Controller maintenance (same as 5)

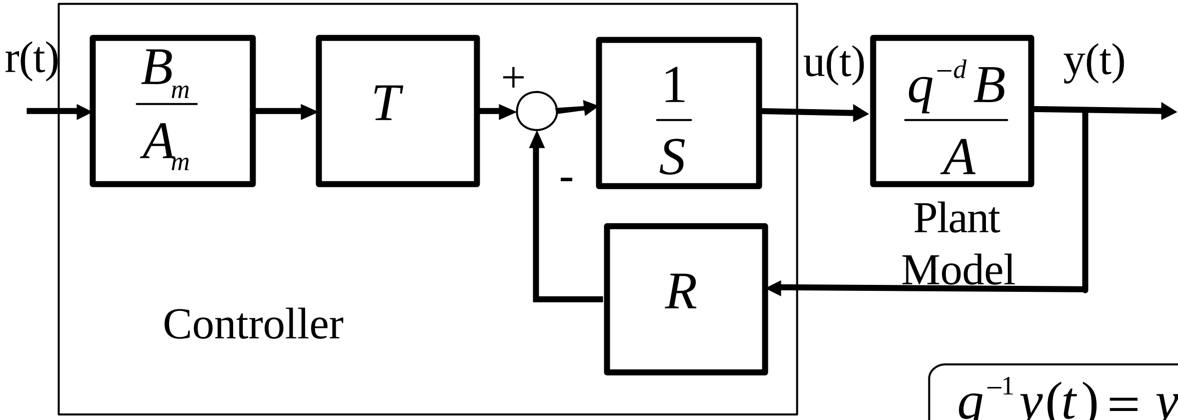
(5) and (6) require
identification in closed-loop

Part I
Outline

- The R-S-T digital controller
- Robust control. What is it ?
- Examples (4 applications)
- Sensitivity functions
- Stability of closed loop discrete time systems
- Robustness margins
- Robust stability
- Small gain theorem/ Passivity theorem / Circle criterion
- Description of uncertainties and robust stability
- Templates for the sensitivity functions

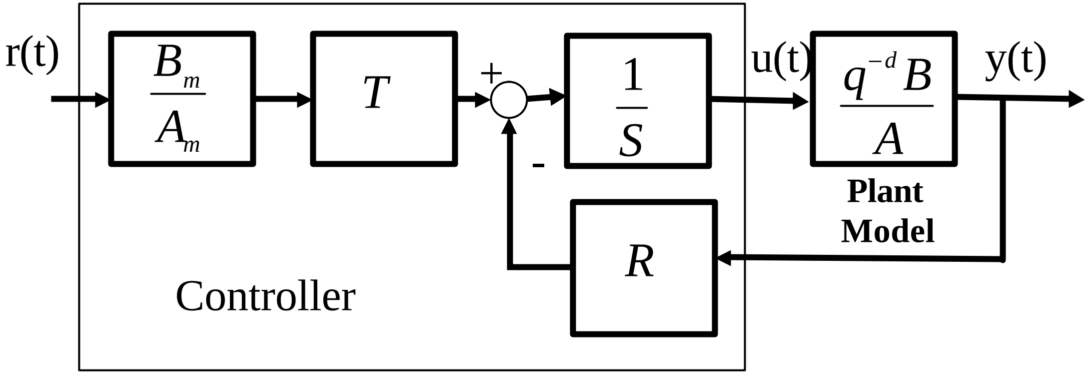
The R-S-T Digital Controller





$$q^{-1}y(t) = y(t-1)$$

The R-S-T Digital Controller



Plant Model:

$$G(q^{-1}) = H(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} = \frac{q^{-d-1}B^*(q^{-1})}{A(q^{-1})}$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_A} q^{-n_A} \quad B(q^{-1}) = b_1 q^{-1} + \ldots + b_{n_B} q^{-n_B} = q^{-1} B^*(q^{-1})$$

R-S-T Controller:

$$S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)$$

Characteristic polynomial (closed loop poles):

$$P(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1})$$

Robust control

Robustness of *what* ? with respect to *what* ?

Robustness of an *index of performance* with respect to *plant model uncertainties.*

Index of performance :

- closed loop stability
- time domain performance
- frequency domain performance

Plant model uncertainties:

- low quality design model
- uncertainties in a frequency range
- variations of the dynamic characteristics of the plant

Robust control

Robust controller : assures the desired index of performance of the closed loop for a set of given uncertainties

Robust closed loop : the index of performance is guaranteed for a set of given uncertainties

Robustness of a control system

A control system is said to be *robust* for a set of given uncertainties upon the nominal plant model if it guarantees stability and performance for all plant models in this set.

- To characterize the *robustness* of a closed loop system a frequency domain analysis is needed
- The *sensitivity functions* play a fundamental role in robustness studies
- The study of closed loop stability in the frequency domain gives valuable information for characterizing robustness

Robustness. Some questions

- **How to characterize "model uncertainty"?**
- **How to deal with "model uncertainty" ?**
- **How to characterize "controller robustness" ?**

Model uncertainty
- Structured (parametric uncertainty)
- Unstructured (specified in the frequency domain)

We will consider the "uncertainties" described in the frequency domain (i.e. effect of parameter uncertainty should be converted in the frequency domain).

Important concepts :

- **nominal model, uncertainty model, family of plant models**
- **nominal stability, robust stability**
- **nominal performance, robust performance**

Nominal model, uncertainty model, family of plant models

Nominal model : the model used for design

Uncertainty model : the description of the uncertainties in the frequency domain (magnitude, phase)

Family of plant models (P): characterized by the nominal plant model and the uncertainty model

The different true "plant models" belong to the family P

Nominal and robust stability and performance

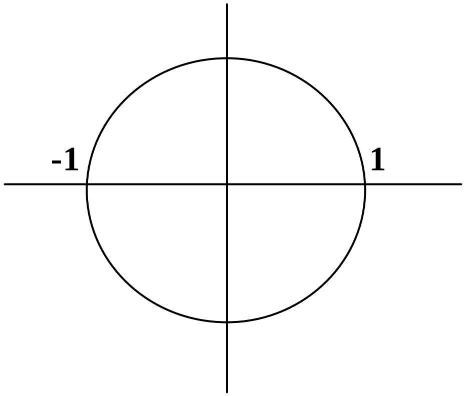
Nominal stability : closed loop as. stable for the "nominal model"

Robust stability: closed loop as. Stable for all the plant models belonging to the family (set) *P*

Nominal performance: performance for the "nominal model"

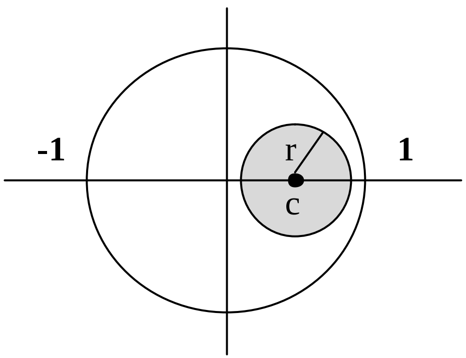
Robust performance: performance guaranteed for all plant models belonging to the family (set) *P*

Robust stability and robust performance



Robust stability:

The closed loop poles remain inside the unit circle (discrete time case) for all the models belonging to the family *P*



Robust performance:

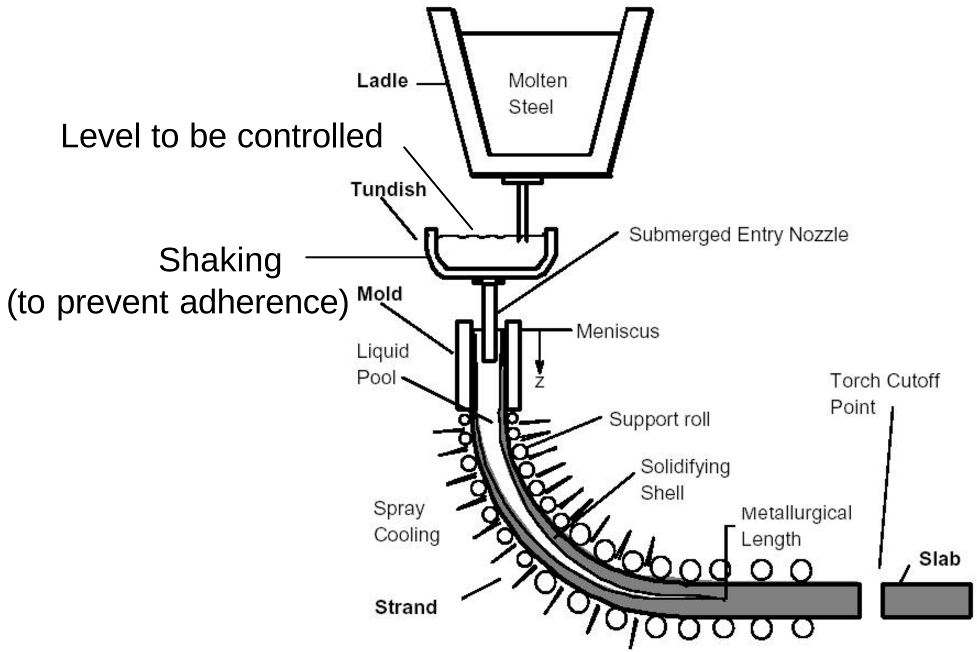
The closed loop poles remain inside the circle (c,r) for all the models belonging to the family *P*

- robust performance condition can be translated in a robust stability condition
- this is not the only way for solving robust performance design

Examples to illustrate robust control design

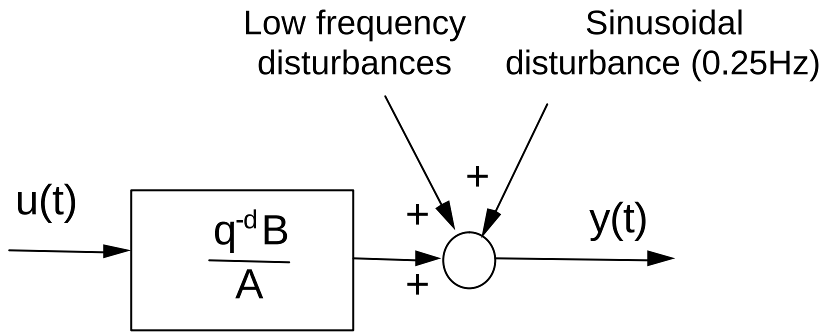
- Continuous steel casting
- Hot dip galvanizing
- Flexible transmission
- 360° flexible arm

Continuous steel casting



Continuos steel casting

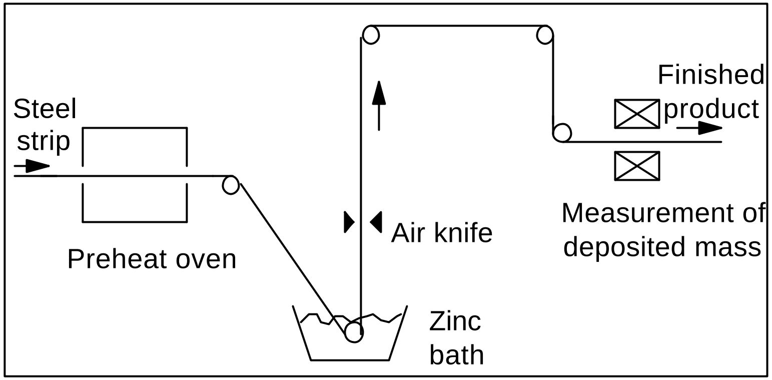
Plant (integrator): $A = 1 - q^{-1}$; $B = 0.5q^{-1}$; $d = 2$; $T_s = 1s$

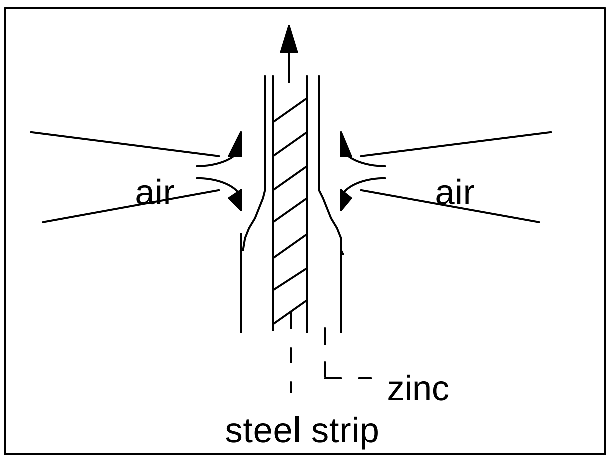


Specifications:

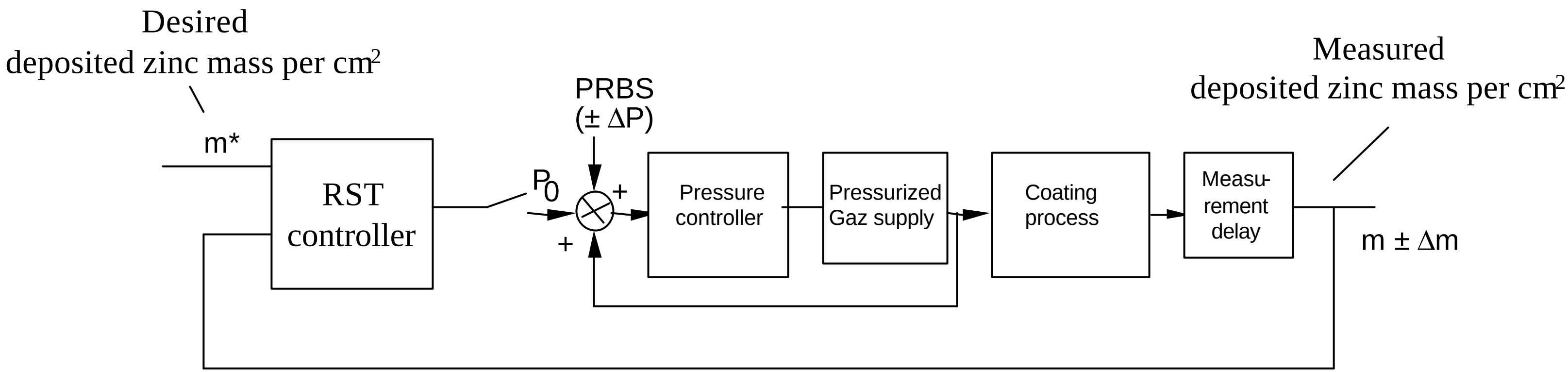
1. No attenuation of the sinusoidal disturbance (0.25 Hz)
2. Attenuation band in low frequencies: 0 à 0.03 Hz
3. Disturbance amplification at 0.07 Hz: < 3dB
4. Modulus margin > -6 dB and Delay margin > T_S
5. No integrator in the controller

Hot dip galvanizing. Control of the deposited zinc





Hot dip galvanizing. The control loops



- important time delay with respect to process dynamics
- time delay depends upon the steel strip speed
- sampling frequency tied to the steel strip speed
- constant integer delay in discrete time
- parameter variations of the process as a function of the type of product

Hot dip galvanizing. Model and specifications

Plant model for a type of product : $$\frac{q^{-7}\left(b_1 q^{-1}\right)}{1+a_1 q^{-1}}$$

Model : $T_S = 12\,\text{sec};\ b_1 = 0.3;\ a_1 = -0.2(-0.3))$

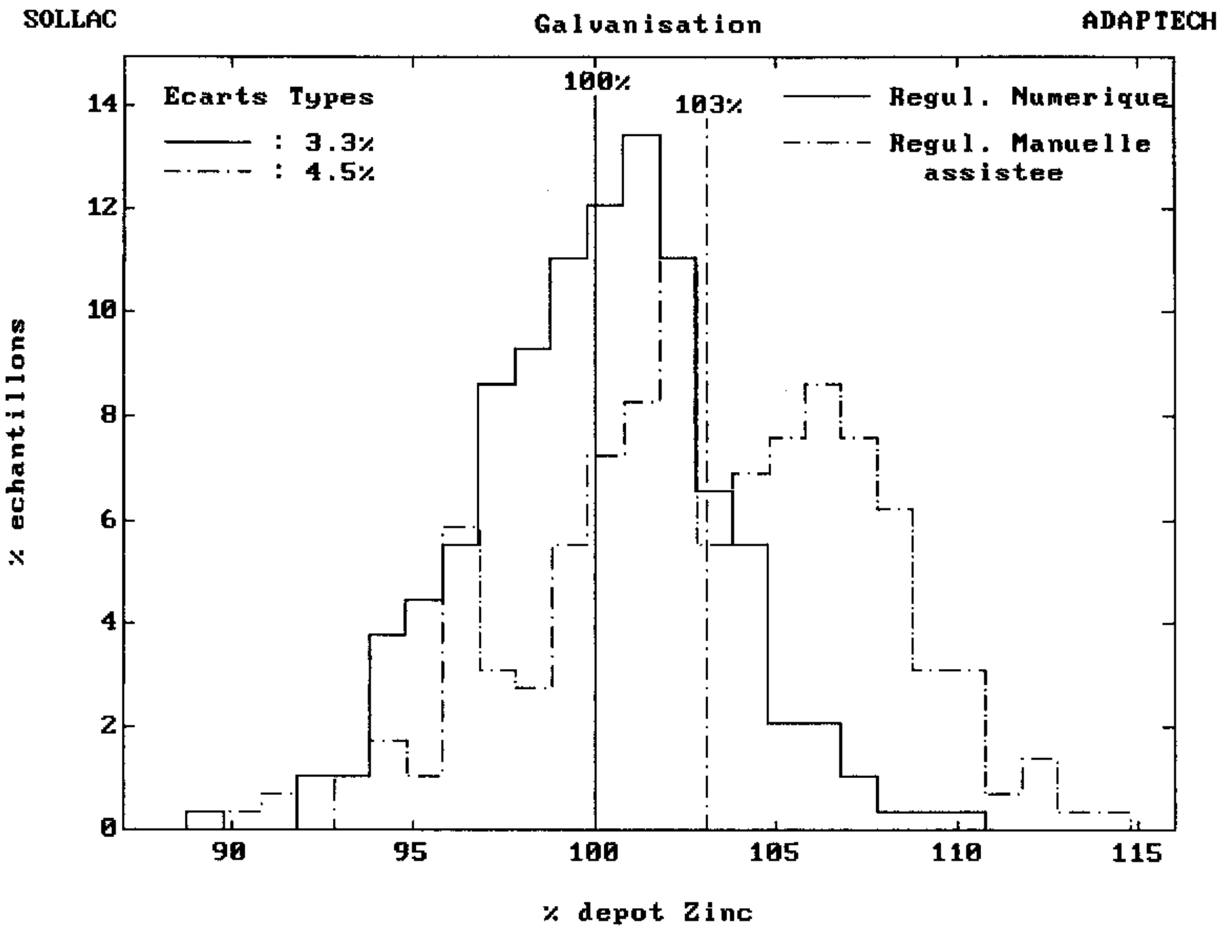
Specifications (performance) :

•Modulus margin: $\Delta M \geq 0.5$

•Delay margin: $\Delta t \geq 2T_S$

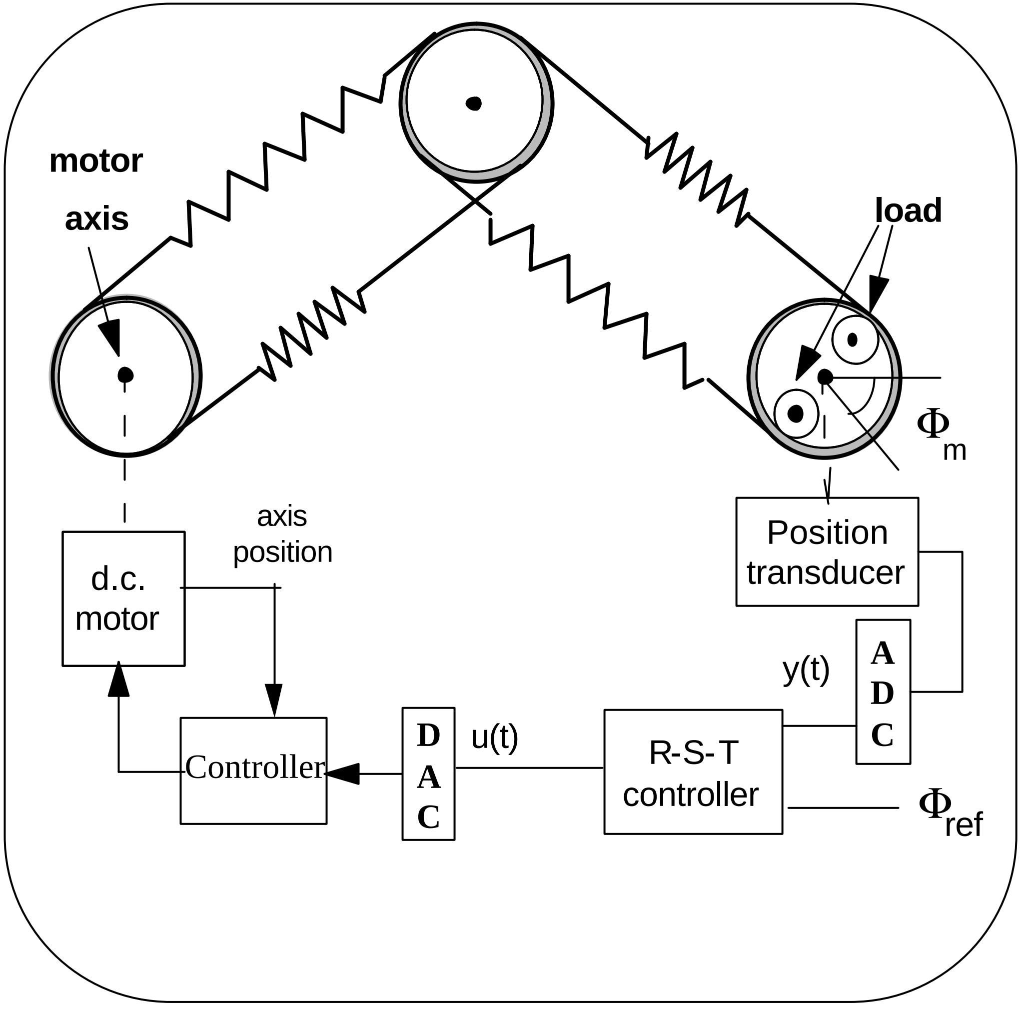
•Integrator

Hot dip galvanizing. Performance



Control of a Flexible Transmission

The flexible transmission



Control of a Flexible Transmission

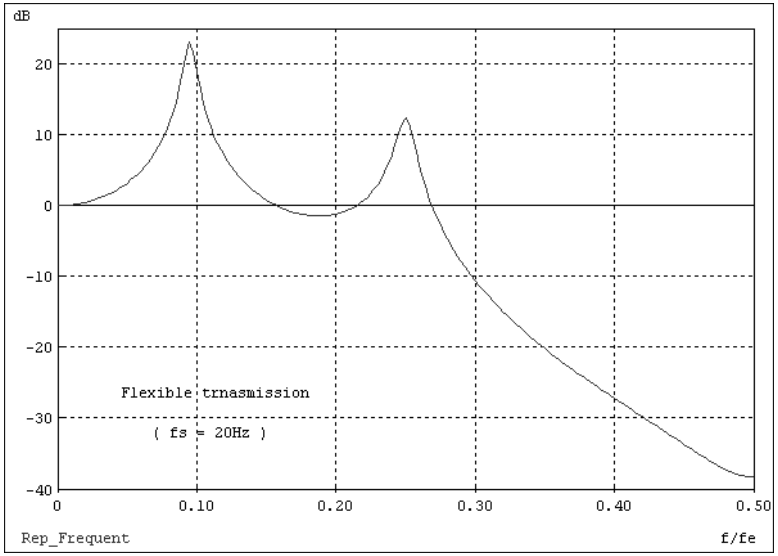
Sampling frequency : *20 Hz*

$$A(q^{-1}) = 1 - 1.609555q^{-1} + 1.87644q^{-2} - 1.49879q^{-3} + 0.88574q^{-4}$$
$$B(q^{-1}) = 0.3053q^{-1} + 0.3943q^{-2}$$
$$d = 2$$

Model

Vibration modes:

$$w_1 = 11.949\ rad/\sec, z_1 = 0.042;\ w_2 = 31.462\ rad/\sec, z_2 = 0.023$$



Specifications:

Tracking: $w_0 = 11.94\ rad/\sec, z = 0.9$

Dominant poles: $w_0 = 11.94\ rad/\sec, z = 0.8$

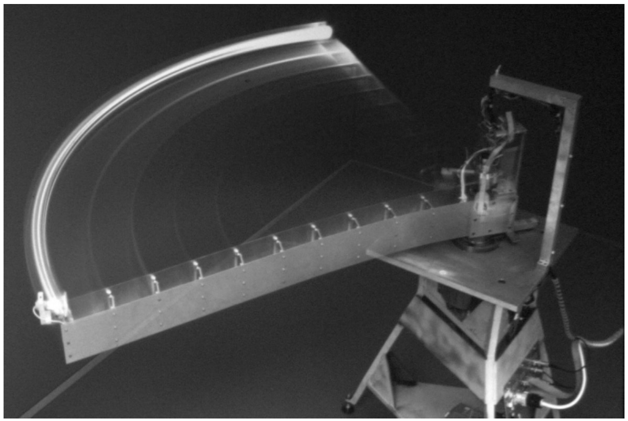
Robustness margins: $\Delta M \geq 0.5$ $\Delta t \geq 2T_S$

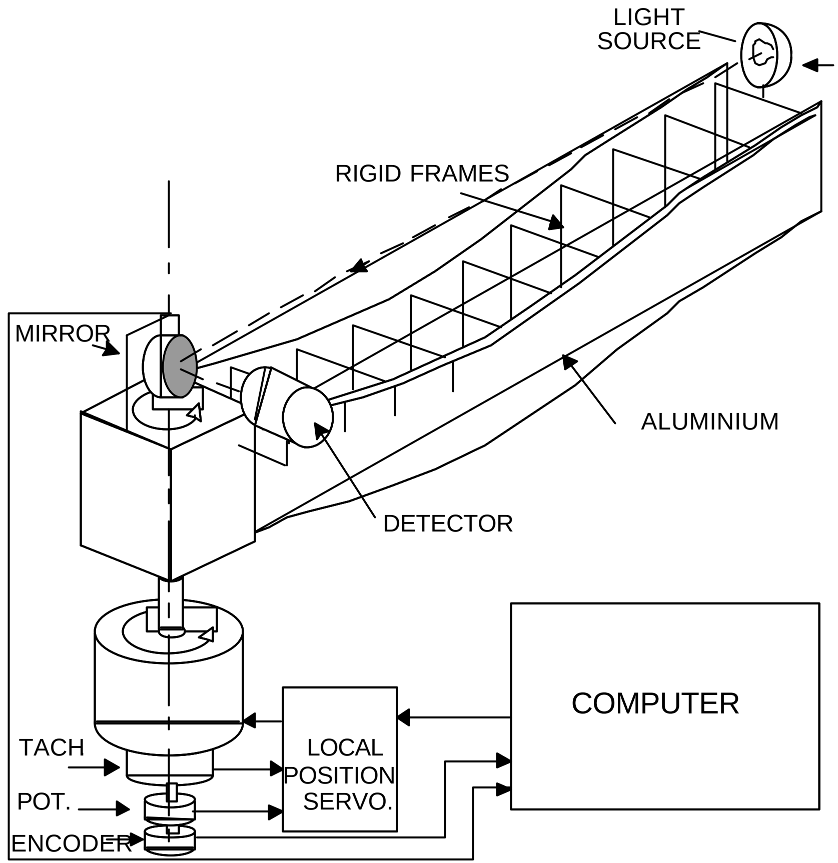
Zero steady state error (integrator)

Constraints on S_{up} :

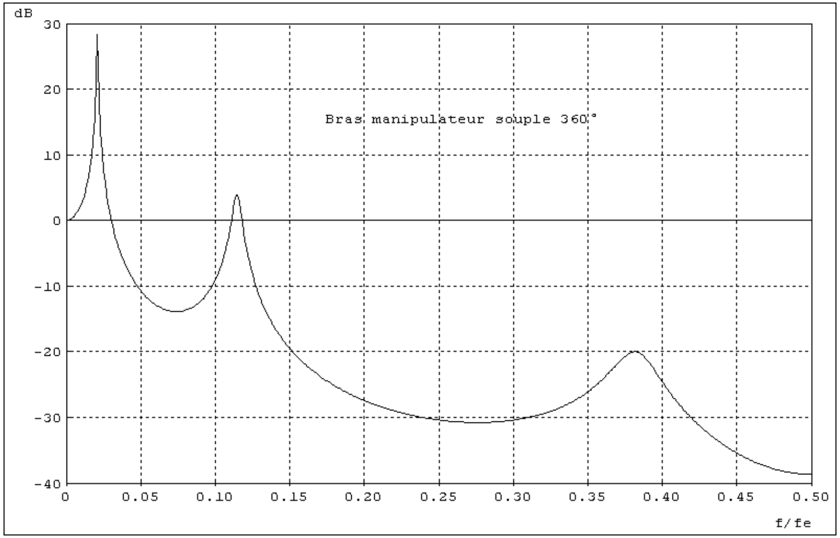
$$\left|S_{up}\right|_{\max} \leq 10\ dB\ \ for\ \ f \geq 0.35 f_S = 7\ Hz$$

360° Flexible Arm

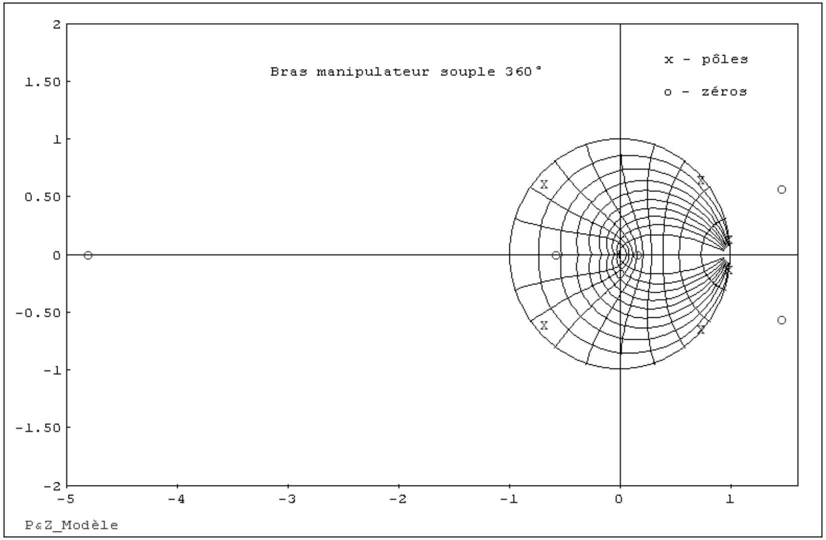




360° Flexible Arm



Frequency characteristics



Poles-Zeros

Unstable zeros !

(Identified Model)

360° Flexible Arm

Sampling frequency : *20 Hz*

Model –

$$A(q^{-1}) = 1 - 2.1049\, q^{-1} + 1.04851\, q^{-2} + 0.33836\, q^{-3} + 0.46\, q^{-4} - 1.5142\, q^{-5} + 0.7987\, q^{-6}$$

$$B(q^{-1}) = 0.0064\, q^{-1} + 0.0146\, q^{-2} - 0.0697\, q^{-3} + 0.044\, q^{-4} + 0.0382\, q^{-5} - 0.007\, q^{-6}$$

$$d = 0$$

Vibration modes:

$$w_1 = 2.617 rad/\sec, z_1 = 0.018;\ w_2 = 14.402 rad/\sec, z_2 = 0.025;\ w_3 = 48.117 rad/\sec, z_3 = 0.038$$

Specifications:

Tracking: $w_0 = 2.6173\, rad/\sec, z = 0.9$

Dominant poles: $w_0 = 2.6173\, rad/\sec, z = 0.8$

Robustness margins: $\Delta M \geq 0.5$ $\Delta t \geq 2T_S$

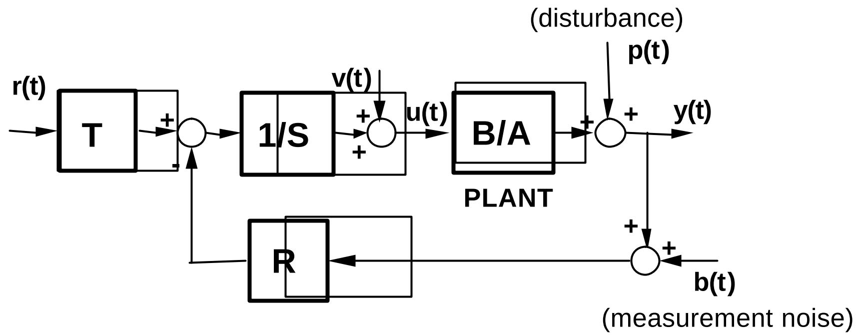
Zero steady state error (integrator)

Constraints on S_{up} : $|S_{up}| \leq 15\, dB$ *for* $f < 4\, Hz$; $|S_{up}| \leq 0\, dB$ *for* $4.5 \leq f < 6.5\, Hz$;

$|S_{up}| < 15\, dB$ *for* $6.5 \leq f < 8\, Hz$; $|S_{up}| < 10\, dB$ *for* $8 \leq f \leq 10\, Hz$

Robust Control. Basic concepts

Digital control in the presence of disturbances and noise



Output sensitivity function
(p —— y)

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

Input sensitivity function
(p —— u)

$$S_{up}(z^{-1}) = \frac{-A(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

Noise-output sensitivity function
(b —— y)

$$S_{yb}(z^{-1}) = \frac{-B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

Input disturbance-output sensitivity function
(v —— y)

$$S_{yv}(z^{-1}) = \frac{B(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

All four sensitivity functions should be stable ! (see book pg.102 - 103)

All four sensitivity functions must be stable !

Plant model: $\dfrac{z^{-d}B(z^{-1})}{A(z^{-1})}$ $\qquad A(z^{-1})$ is unstable

Supose : $R(z^{-1}) = A(z^{-1})$ (poles compensation by the controller zeros)

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})A(z^{-1})} = \frac{S(z^{-1})}{S(z^{-1}) + B(z^{-1})}$$

$$S_{up}(z^{-1}) = -\frac{A(z^{-1})A(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})A(z^{-1})} = -\frac{A(z^{-1})}{S(z^{-1}) + B(z^{-1})}$$

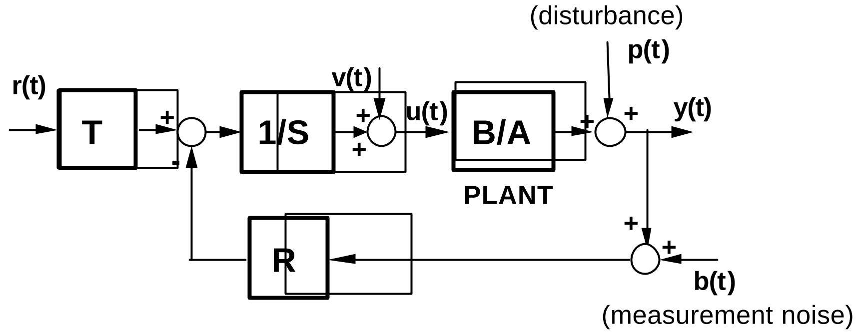
$$S_{yb}(z^{-1}) = -\frac{B(z^{-1})A(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})A(z^{-1})} = -\frac{B(z^{-1})}{S(z^{-1}) + B(z^{-1})}$$

$$S_{yv}(z^{-1}) = \frac{B(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})A(z^{-1})} = \frac{B(z^{-1})S(z^{-1})}{A(z^{-1})[S(z^{-1}) + B(z^{-1})]}$$

S_{yv} is *unstable* while S_{yp}, S_{up}, and S_{yb} may be stable if :

$$S(z^{-1}) + B(z^{-1}) = 0 \Rightarrow |z| < 1$$

Complementary sensitivity function



For $T = R$ one has:

$$S_{yr}(z^{-1}) = \frac{B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})} = -S_{yb}(z^{-1})$$

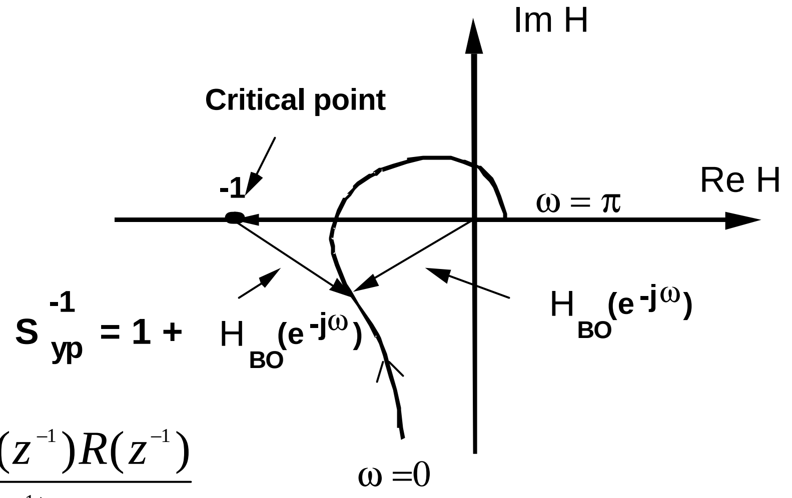
$$S_{yp}(z^{-1}) - S_{yb}(z^{-1}) = S_{yp}(z^{-1}) + S_{yr}(z^{-1}) = 1$$

Stability of closed loop discrete time systems

The Nyquist is used like in continuous time

(can be displayed with WinReg ou *Nyquist_OL.sci(.m))*

$$H_{OL}(e^{-jw}) = \frac{B(e^{-jw})R(e^{-jw})}{A(e^{-jw})S(e^{-jw})}$$



$$S_{yp}^{-1}(z^{-1}) = 1 + H_{OL}(z^{-1}) = \frac{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}$$

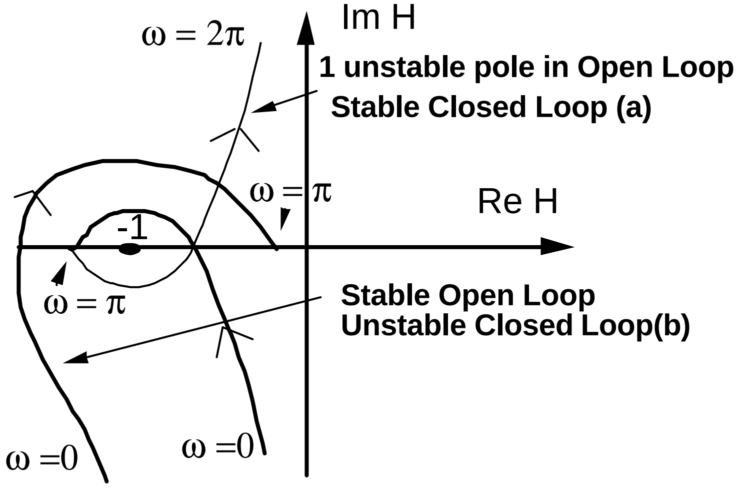
Nyquist criterion (discrete time –O.L. is stable)

The Nyquist plot of the open loop transfer fct. $H_{OL}(e^{-jw})$ traversed in the sense of growing frequencies (from 0 to $0.5f_S$) leaves the critical point[-1, j0] on the left

Stability of closed loop discrete time systems

Nyquist criterion (discrete time –O.L. is unstable)

The Nyquist plot of the open loop transfer fct. $H_{OL}(e^{-jw})$ traversed in the sense of growing frequencies (from 0 et f_S) leaves the critical point[-1, j0] on the left and the number of encirclements of the critical pointcounter clockwise should be equal to the number of unstable poles in open loop.



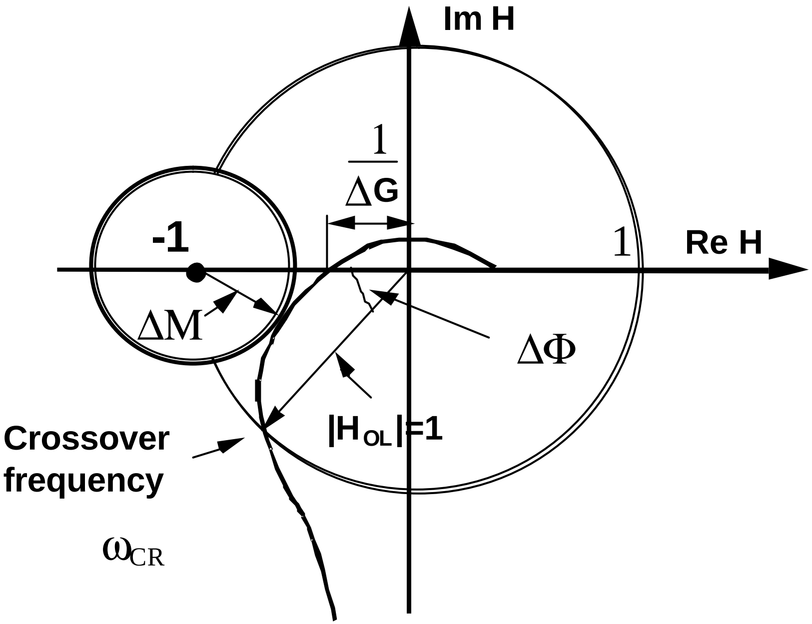
Remarks:

-The controller poles may become unstable if high performances are required without using an appropriate design method

-The Nyquist plot from $0.5f_S$ to f_S is the symmetric with respect to the real axis of the Nyquist plot from 0 to $0.5f_S$

Marges de robustesse

The minimal distance with respect to the critical point characterizes the robustness of the CL with respect to uncertainties on the plant model parameters(or their variations)



-Gain margin DG

-Phase margin $D\phi$

-Delay margin $D\tau$

-Modulus margin DM

Robustness margins

Gain margin

$$\Delta G = \frac{1}{\left| H_{BO}(jw_{180}) \right|} \qquad \text{pour} \qquad \angle f(w_{180}) = -180^{\circ}$$

Phase margin

$$\Delta f = 180^{0} - \angle f(w_{cr}) \quad \textit{pour} \quad \left| H_{BO}(jw_{cr}) \right| = 1$$

$$\Delta f = \min_{i} \Delta f_{i}$$

If there several intersections with the unit circle

Delay margin

$$\Delta t = \frac{\Delta f}{w_{cr}}$$

Several intersections points: $\Delta t = \min_{i} \frac{\Delta f_{i}}{w_{cr}^{i}}$

Modulus margin

$$\Delta M = \left| 1 + H_{BO}(jw) \right|_{\min} = \left| S_{yp}^{-1}(jw) \right|_{\min} = \left(\left| S_{yp}(jw) \right|_{\max} \right)^{-1}$$

Robustness margins – typical values

Gain margin : $\Delta G \geq 2$ (*6 dB*) [*min : 1,6* (*4 dB*)]

Phase margin : $30° \leq \Delta\phi \leq 60°$

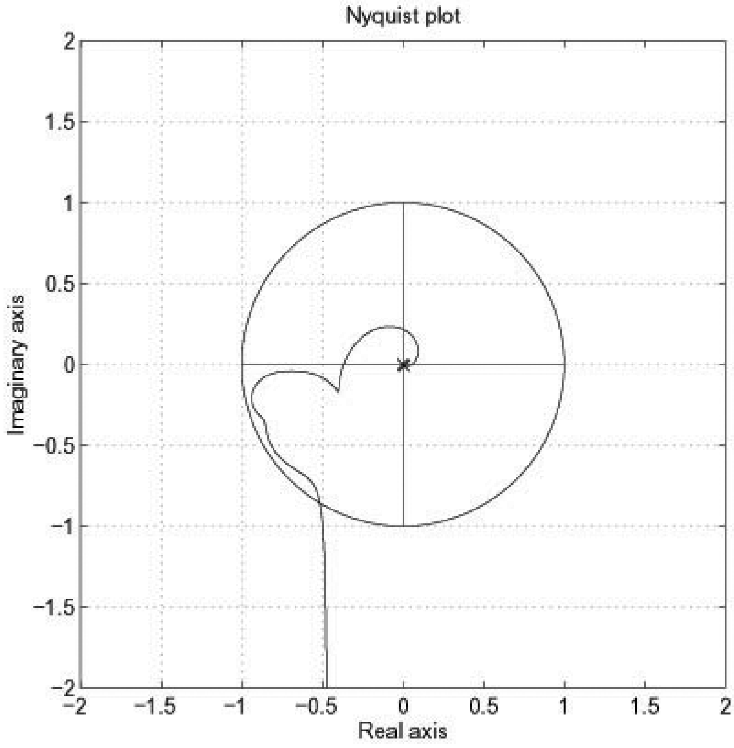
Delay margin : fraction of system delay (10%) or
of time response (10%) (often $1.T_S$)

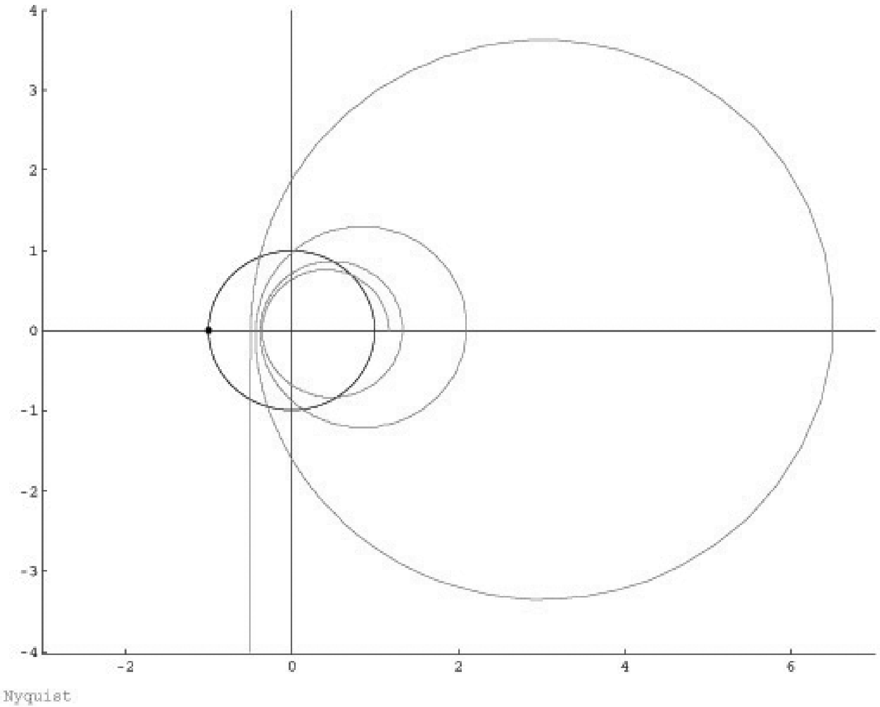
Modulus margin : $\Delta M \geq 0.5$ (- *6 dB*) [*min : 0,4* (*-8 dB*)]

A modulus margin $\Delta M \geq 0.5$ implies $\Delta G \geq 2$ *et* $\Delta\phi > 29°$
Attention ! The converse is not generally true

The *modulus margin* defines also the tolerance with respect to nonlinearities

Robustness margins





Good gain and phase margin
Bad modulus margin

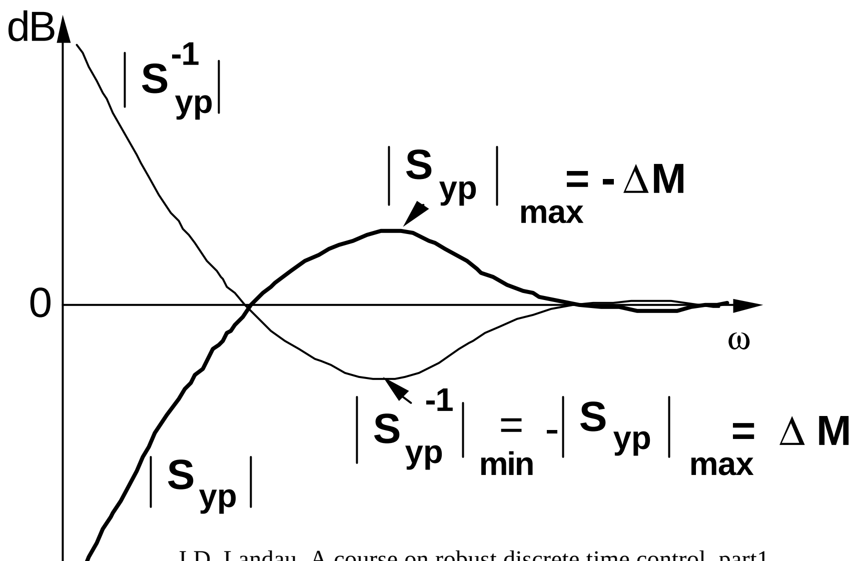
Good gain and phase margin
Bad delay margin

Modulus margin and sensitivity function

$$\Delta M = \left|1 + H_{OL}(z^{-1})\right|_{\min} = \left|S_{yp}^{-1}(z^{-1})\right|_{\min} = \left(\left|S_{yp}(z^{-1})\right|_{\max}\right)^{-1} =$$

$$\left(\left|\frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}\right|_{\max}\right)^{-1} \quad pour \quad z^{-1} = e^{-j2pf}$$

$$\left|S_{yp}(e^{-jw})\right|_{\max} dB = \Delta M^{-1} dB = -\Delta M \; dB$$



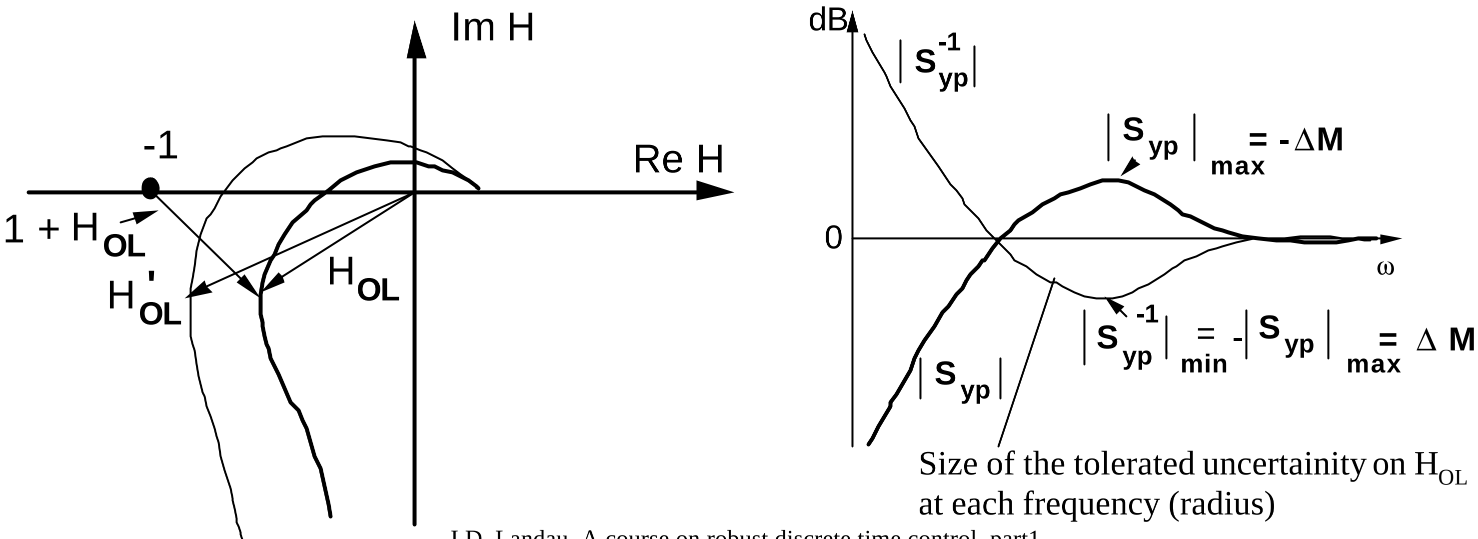
Robust stability

To assure stability in the presence of uncertainties (or variations) on the dynamic chatacteristics of the plant model

H_{OL} – nominal F.T.; H'_{OL} –Different from H_{OL} (perturbed)

Robust stability condition (sufficient cond.):

$$\left|H'_{OL}(z^{-1}) - H_{OL}(z^{-1})\right| < \left|1 + H_{OL}(z^{-1})\right| = \left|S_{yp}^{-1}(z^{-1})\right| = \left|\frac{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}\right| = \left|\frac{P(z^{-1})}{A(z^{-1})S(z^{-1})}\right| \; ; \quad z^{-1} = e^{-jw} \qquad (*)$$



Tolerance to plant additive uncertainty

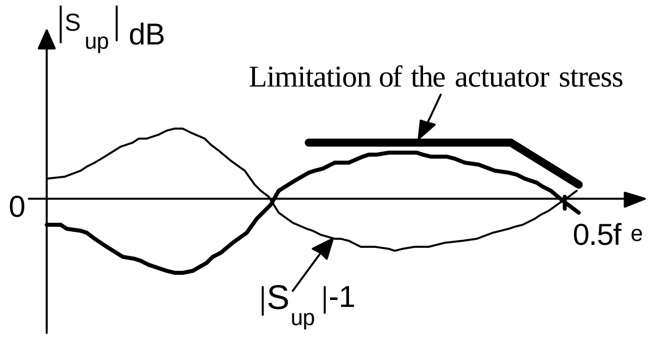
From previous slide :

$$\left|\frac{B'(z^{-1})R(z^{-1})}{A'(z^{-1})S(z^{-1})} - \frac{B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}\right| = \left|\frac{R(z^{-1})}{S(z^{-1})}\right| \cdot \left|\frac{B'(z^{-1})}{A'(z^{-1})} - \frac{B(z^{-1})}{A(z^{-1})}\right| < \left|\frac{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}\right| \quad (*)$$

H'$_{OL}$ H$_{OL}$ H' H

$$\left|\frac{B'(z^{-1})}{A'(z^{-1})} - \frac{B(z^{-1})}{A(z^{-1})}\right| < \left|\frac{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}{A(z^{-1})R(z^{-1})}\right| = \left|\frac{P(z^{-1})}{A(z^{-1})R(z^{-1})}\right| = \left|S_{up}^{-1}(z^{-1})\right| \quad (**)$$

H' H



(Size of tolerated uncertainties)

Tolerance to plant normalized uncertainty (multiplicative uncertainty)

From (**), previous slide:

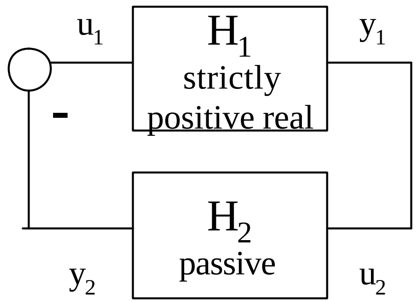
$$\frac{\left|\frac{B'(z^{-1})}{A'(z^{-1})}-\frac{B(z^{-1})}{A(z^{-1})}\right|}{\left|\frac{B(z^{-1})}{A(z^{-1})}\right|}<\left|\frac{A(z^{-1})S(z^{-1})+B(z^{-1})R(z^{-1})}{B(z^{-1})R(z^{-1})}\right|=\left|\frac{P(z^{-1})}{B(z^{-1})R(z^{-1})}\right|=\left|S_{yb}^{-1}(z^{-1})\right|$$

The inverse of the modulus of the "complementary sensitivity function" gives at each frequency the tolerance with respect to "normalized (multiplicative) uncertainty"

Relation between additive and multiplicative uncertainty:

$$H' = H + (H' - H) = H\left(1 + \frac{H' - H}{H}\right)$$

Passivity (Hyperstability) Theorem



H_1: Strictly positive real transfer function (state x)

H_2: linear or nonlinear, time invariant or time-varying

$$h_2(0,t_1) \geq \sum_{t=0}^{t_1} y_2^T(t)u_2(t) \geq -g_2^2 \;;\; g_2^2 < \infty \;;\; \nabla t_1 \geq 0$$

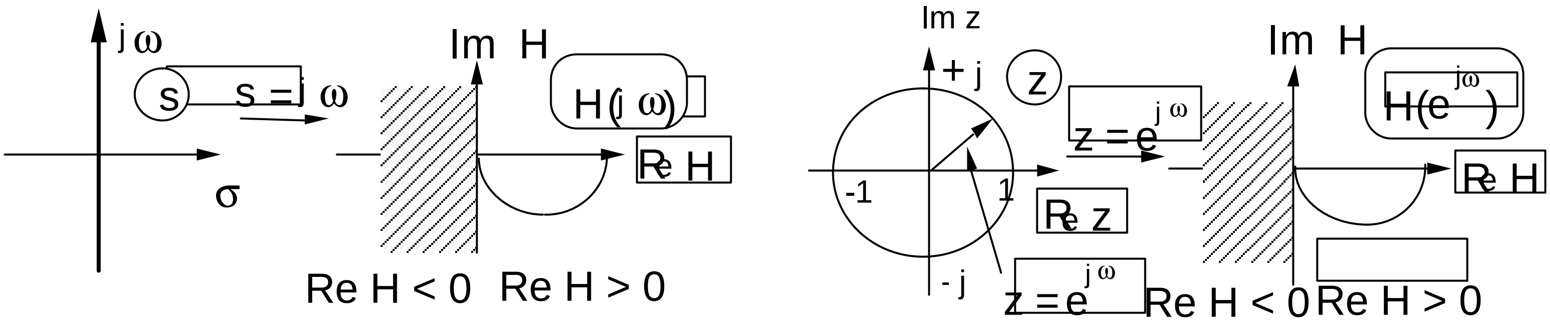
Then :

$$\lim_{t\to\infty} x(t) = 0 \;;\; \lim_{t\to\infty} u_1(t) = 0 \;;\; \lim_{t\to\infty} y_1(t) = 0$$

Strictly positive real transfer function (SPR)

Frequency domain

continuous **discrete**

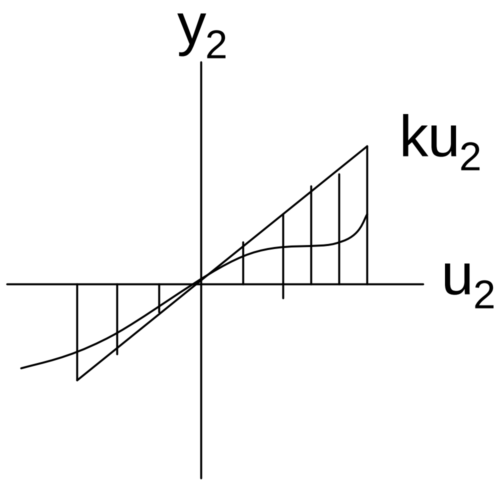


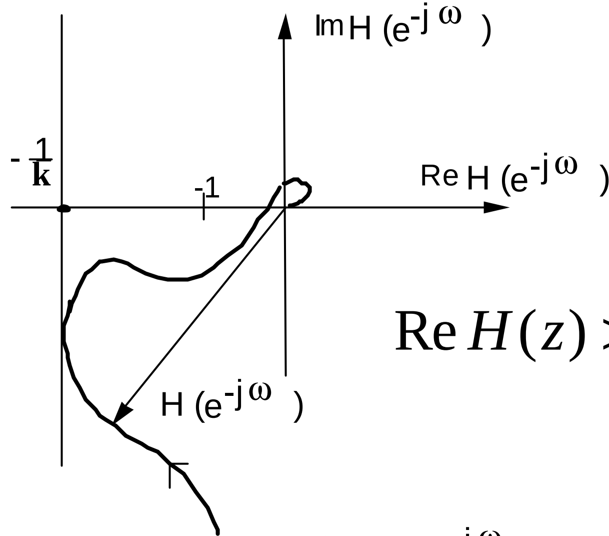
- asimptotycally stable
- $\mathrm{Re}\, H(e^{jw}) > 0$ *for all* $\left|e^{jw}\right| = 1, \ (0 < w < p)$ *(discrete time case)*

Input – output property (time domain)

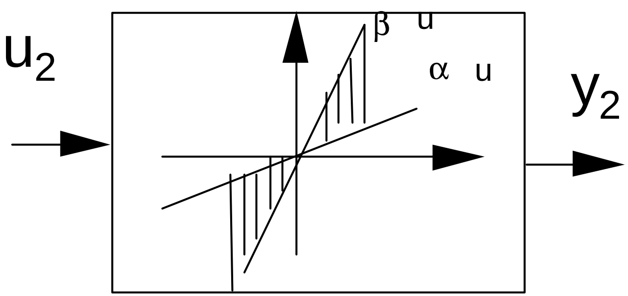
$$h_1(0,t_1) \geq \sum_{t=0}^{t_1} y_1^T(t)u_1(t) \geq -g_1^2 + k\left\|u_1\right\|_{2T}^2 \ ; g_1^2 < \infty \, ; k > 0 \, ; \nabla t_1 \geq 0$$

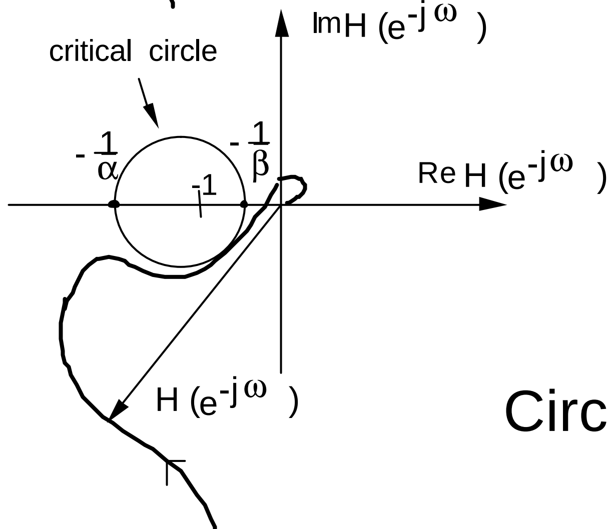
Stability criteria for nonlinear time/varying feedback system





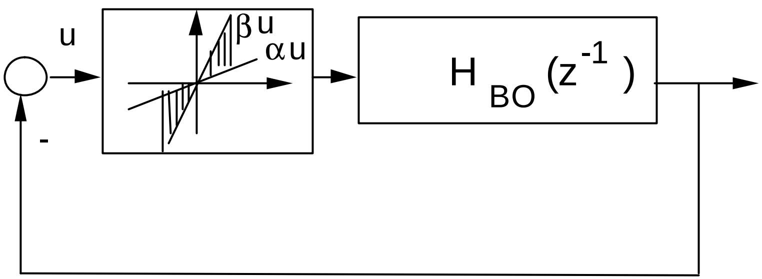
$$\operatorname{Re} H(z) > -1/k \; ; \; \forall |z| = 1$$

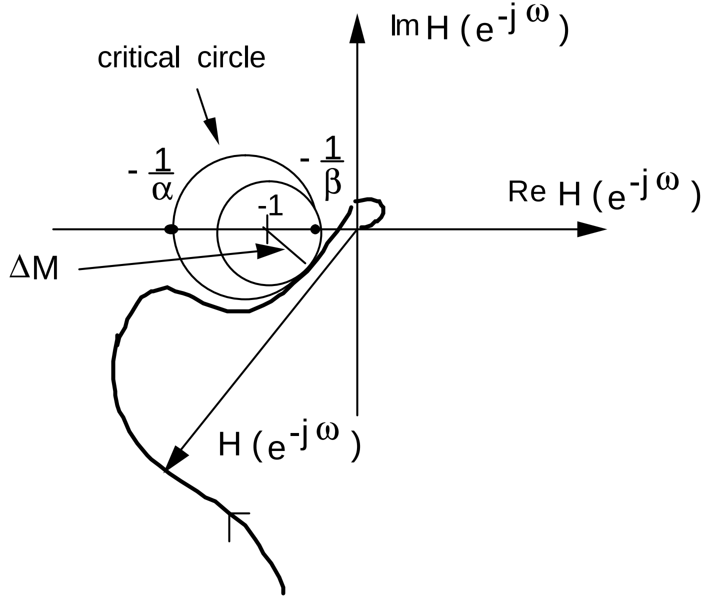




Circle criterion

Stability in the presence of nonlinearities (tolerance of nonlinearities)



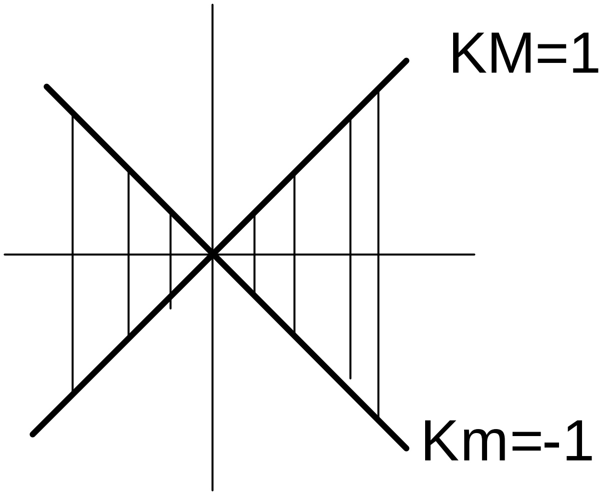


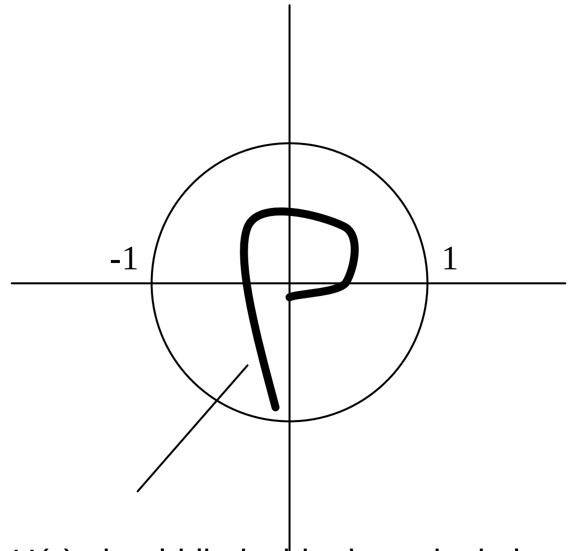
The modulus margin defines the tolerance with respect to nonlinear and/or time varying elements.

The tolerance sector is defined by :

Min gain: $1/(1+\Delta M)$ Max gain: $1/(1-\Delta M)$

The circle criterion : a particular case





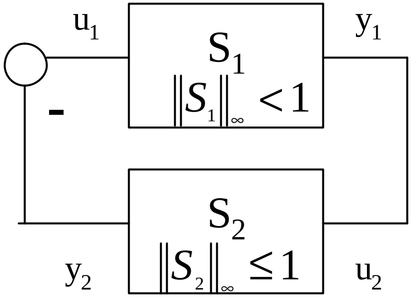
H(z) should lie inside the unit circle

The modulus of the H(z) should be smaller than 1 at all frequencies i.e.:

$$\|H(z)\|_{\infty} < 1$$

This is the “small gain theorem”

Small gain theorem



S_1: linear time invariant (state *x*)

$$\|S_1\|_\infty < 1$$

S_2: $\|S_2\|_\infty \le 1$

Then:

$$\lim_{t\to\infty} x(t) = 0\,;\ \lim_{t\to\infty} u_1(t) = 0\,;\ \lim_{t\to\infty} y_1(t) = 0$$

It will be used to characterize "robust stability"

Relationship between *passivity theorem* and *small gain theorem*

If H is *passive*, $\|S\|_\infty = \left\| \dfrac{H-1}{H+1} \right\|_\infty \leq 1$

If $\|S\|_\infty \leq 1 \Rightarrow H = \dfrac{1+S}{1-S}$ is *passive*

If H is *strictly passive* , $\|S\|_\infty = \left\| \dfrac{H-1}{H+1} \right\|_\infty < 1$

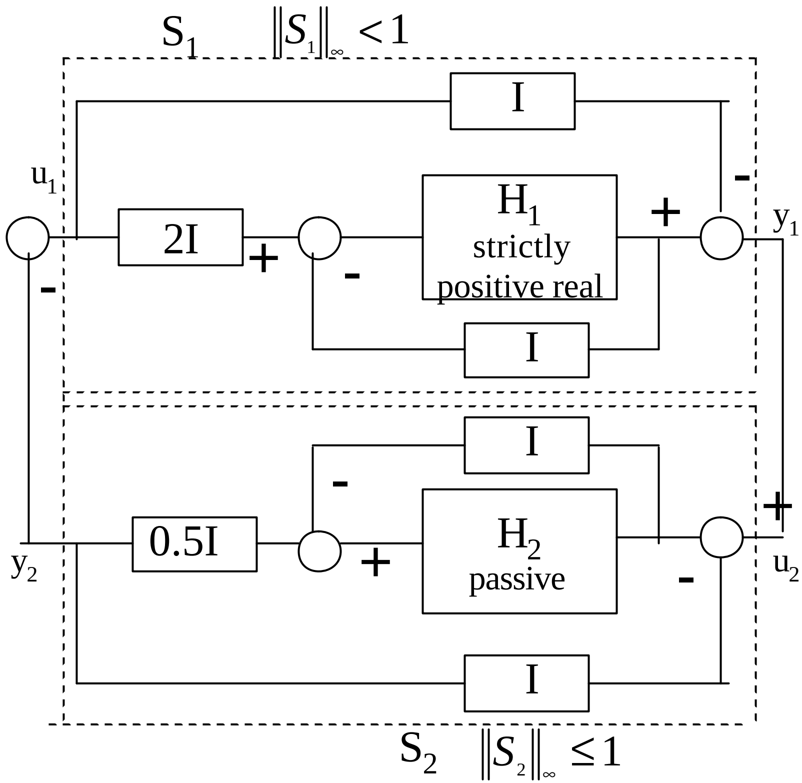
If $\|S\|_\infty < 1 \Rightarrow H = \dfrac{1+S}{1-S}$ is *strictly passive*

Hint for a proof:

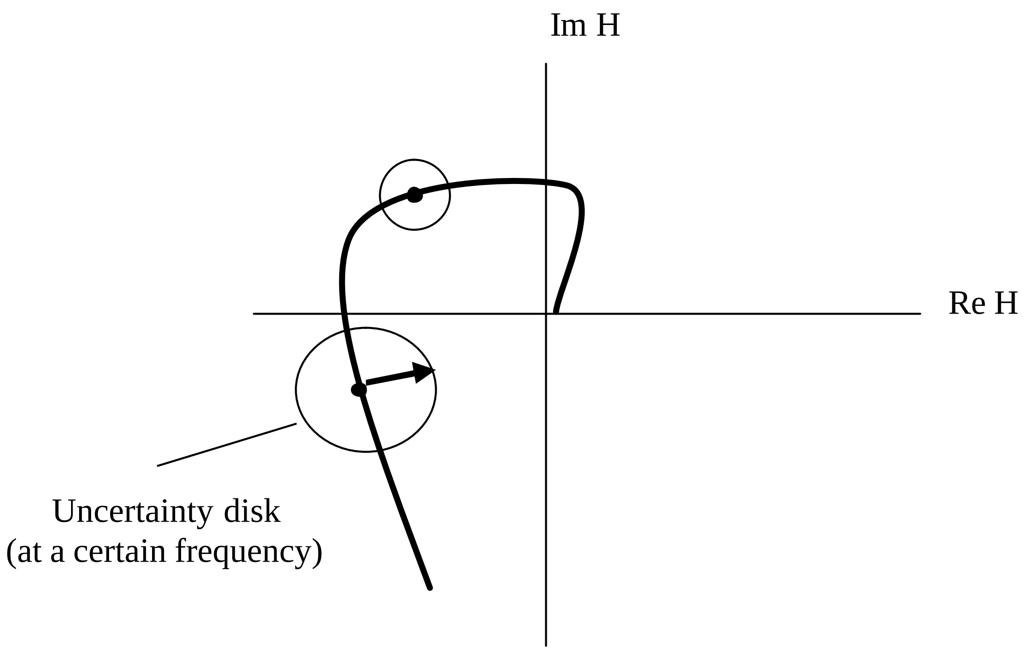
$$|S(e^{-jw})|^2 = \frac{|H(e^{-jw})-1|^2}{|H(e^{-jw})+1|^2} = \frac{|H|^2 - 2\,\mathrm{Re}\,H + 1}{|H|^2 + 2\,\mathrm{Re}\,H + 1} = 1 - \frac{4\,\mathrm{Re}\,H}{|H(e^{-jw})+1|^2}$$

Remark:
Correct writing in the general operator case $\|S\|_\infty = \|(H-1)(H+1)^{-1}\|_\infty \leq 1$

Relationship between *passivity theorem* and *small gain theorem*



Description of uncertainties in the frequency domain



1) It needs a description by a transfer function which may have any phase but a modulus < 1
2) The size of the radius will vary with the frequency and is characterized by a transfer function

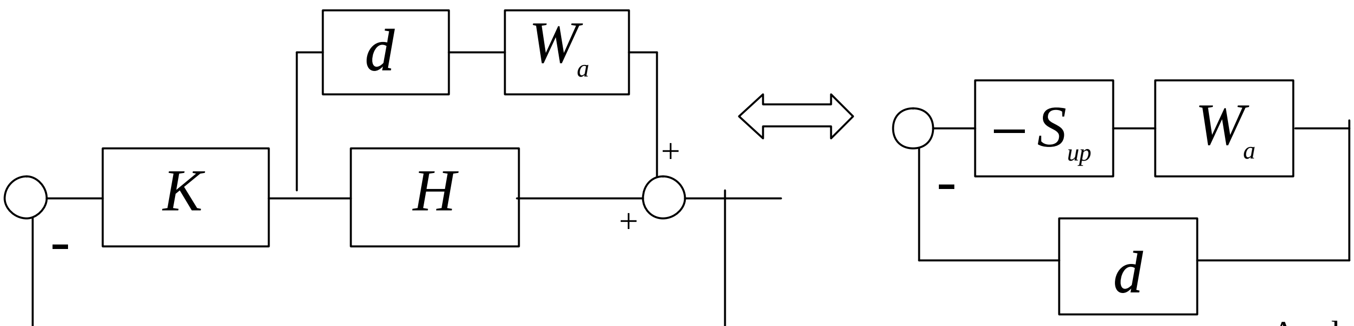
Additive uncertainty

$$H'(z^{-1}) = H(z^{-1}) + d(z^{-1})W_a(z^{-1})$$

$d(z^{-1})$ any stable transfer function with $\|d(z^{-1})\|_\infty \leq 1$

$W_a(z^{-1})$ a stable transfer function

$$|H'(z^{-1}) - H(z^{-1})|_{\max} = \|H'(z^{-1}) - H(z^{-1})\|_\infty = \|W_a(z^{-1})\|_\infty$$



Apply small gain theorem

$K = R/S;\ H = z^{-d}B/A$

Robust stability condition: $\|S_{up}(z^{-1})W_a(z^{-1})\|_\infty < 1$

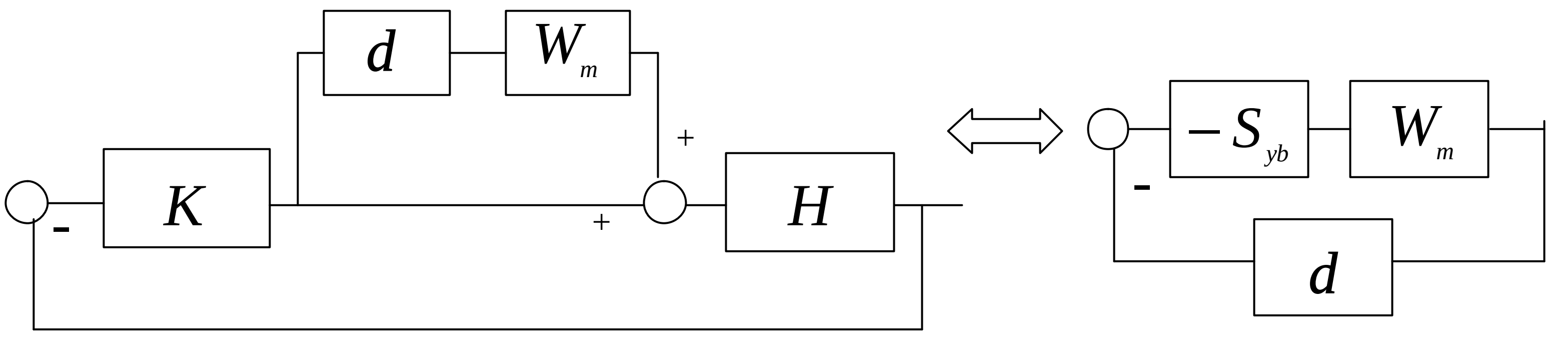
Multiplicative uncertainties

$$H'(z^{-1}) = H(z^{-1})\left[1 + d(z^{-1})W_m(z^{-1})\right]$$

$d(z^{-1})$ any stable transfer function with $\|d(z^{-1})\|_\infty \le 1$

$W_m(z^{-1})$ a stable transfer function

$$W_a(z^{-1}) = H(z^{-1})W_m(z^{-1})$$



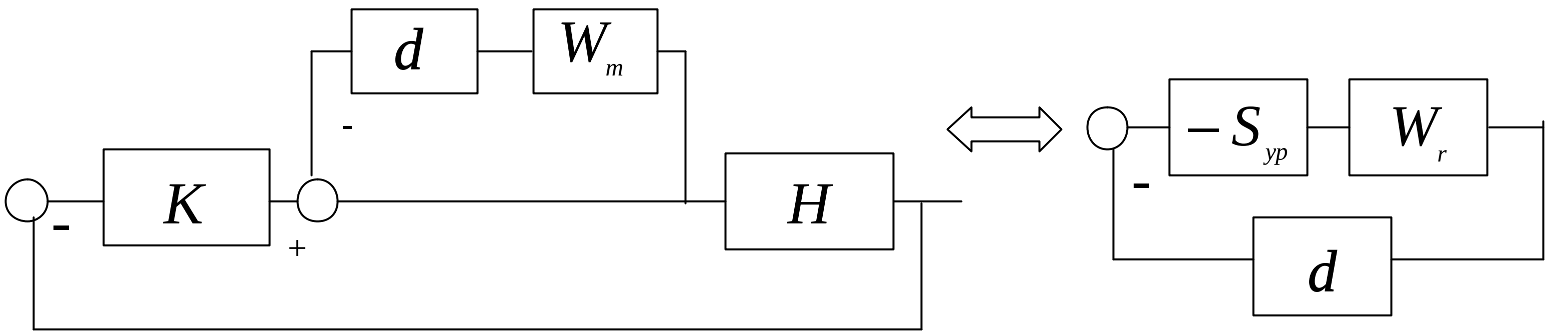
Robust stability condition: $\|S_{yb}(z^{-1})W_m(z^{-1})\|_\infty < 1$

Feedback uncertainties on the input

$$H'(z^{-1}) = \frac{H(z^{-1})}{\left[1 + d(z^{-1})W_r(z^{-1})\right]}$$

$d(z^{-1})$ any stable transfer function with $\left\| d(z^{-1}) \right\|_\infty \le 1$

$W_r(z^{-1})$ a stable transfer function



Robust stability condition: $\left\| S_{yp}(z^{-1})W_r(z^{-1}) \right\|_\infty < 1$

Robust stability conditions

$H, H' \in \mathrm{P}(W, d)$ —— Family (set) of plant models

Robust stability :

The feedback system is asymptotically stable for all the plant models belonging to the family $\mathrm{P}(W, d)$

- Additive uncertainties

$$\left\| S_{up}(z^{-1}) W_a(z^{-1}) \right\|_\infty < 1 \quad \Longleftrightarrow \quad \left| S_{up}(e^{-jw}) \right| < \left| W_a(e^{-jw}) \right|^{-1} \quad 0 \le w \le p$$

- Multiplicative uncertainties

$$\left\| S_{yb}(z^{-1}) W_m(z^{-1}) \right\|_\infty < 1 \quad \Longleftrightarrow \quad \left| S_{yb}(e^{-jw}) \right| < \left| W_m(e^{-jw}) \right|^{-1} \quad 0 \le w \le p$$

- Feedback uncertainties on the input (or output)

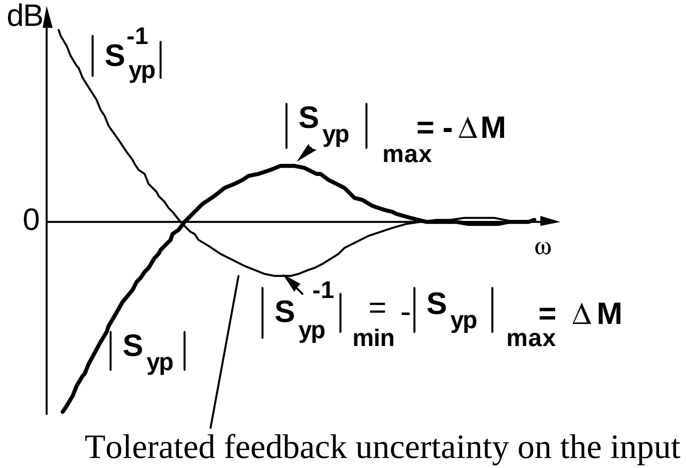
$$\left\| S_{yp}(z^{-1}) W_r(z^{-1}) \right\|_\infty < 1 \quad \Longleftrightarrow \quad \left| S_{yp}(e^{-jw}) \right| < \left| W_r(e^{-jw}) \right|^{-1} \quad 0 \le w \le p$$

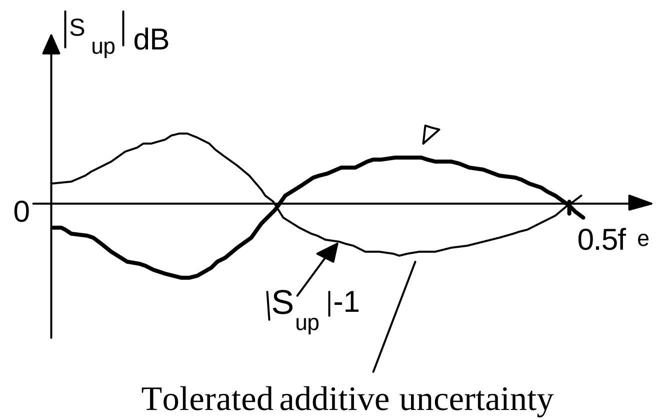
Robust stability and templates for the sensitivity functions

Robust stability condition:

$$\left|S_{xy}(e^{-jw})\right| < \left|W_z(e^{-jw})\right|^{-1} \quad 0 \le w \le p$$

- *The functions* $\left|W(z^{-1})\right|^{-1}$ *(the inverse of the size of the uncertainties) define an "upper" template for the sensitivity functions*
- *Conversely the frequency profile of* $\left|S_{xy}(e^{-jw})\right|$ *can be interpreted in terms of tolerated uncertainties*





Modulus margin and robust stability

Modulus margin: $\left|S_{yp}(e^{-jw})\right| < \Delta M$

Robust stability cond.: $\left|S_{yp}(e^{-jw})\right| < \left|W_r(e^{-jw})\right|^{-1} \quad 0 \le w \le p$

Possible uncertainties characterized by:

$$W_r^{-1}(z^{-1}) = \Delta M$$

$$d(z^{-1}) = l \; f(z^{-1})\,; -1 \le l \le 1$$

$$f(z^{-1}) = 1, z^{-1}, z^{-2}, \ldots, \frac{z^{-1} + z^{-2}}{2}$$

Examples of families of plant models for which robust stability is guaranteed:

$$H'(z^{-1}) = H(z^{-1}) \frac{1}{1 - l \Delta M}$$

$$H'(z^{-1}) = H(z^{-1}) \frac{1}{1 - l \Delta M z^{-1}}$$

Delay margin and robust stability

$$\Delta t = 1.T_s$$

$$H(z^{-1}) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})} \qquad H'(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}$$

$$\frac{H'(z^{-1}) - H(z^{-1})}{H(z^{-1})} = z^{-1} - 1$$

Can be interpreted as a multiplicative uncertainty

$$H'(z^{-1}) = H(z^{-1})\left[1 + d(z^{-1})W_m(z^{-1})\right] = H(z^{-1})\left[1 + (z^{-1} - 1)\right]$$

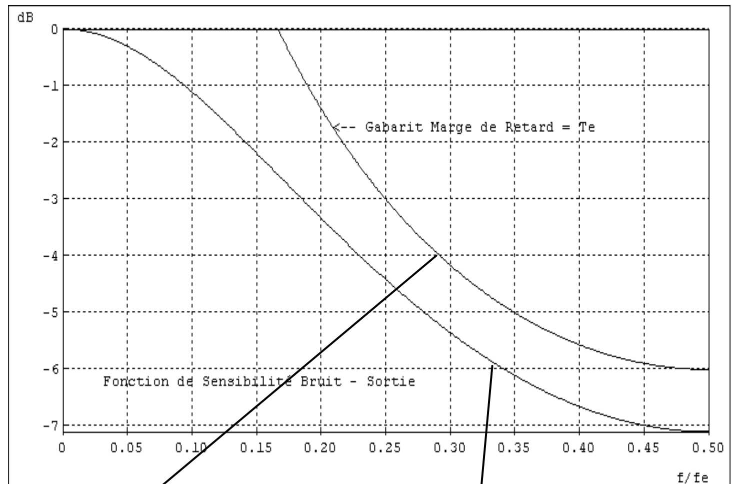
$$d(z^{-1}) = 1 ; W_m(z^{-1}) = (z^{-1} - 1)$$

Robust stability condition:

$$\left|S_{yb}(e^{-jw})\right| < \left|W_m(e^{-jw})\right|^{-1} \quad 0 \le w \le p$$

$$\left|S_{yb}(z^{-1})\right| < \frac{1}{\left|z^{-1} - 1\right|} ; z = e^{-jw}, 0 \le w \le p$$

$$\left|S_{yb}^{-1}(z^{-1})\right| dB < -20 \log\left|1 - z^{-1}\right| ; z = e^{-jw}$$



Define a template on S_{yb} $\qquad$ S_{yb}

Delay margin template on S_{yp}

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})} \qquad S_{yb}(z^{-1}) = \frac{-B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

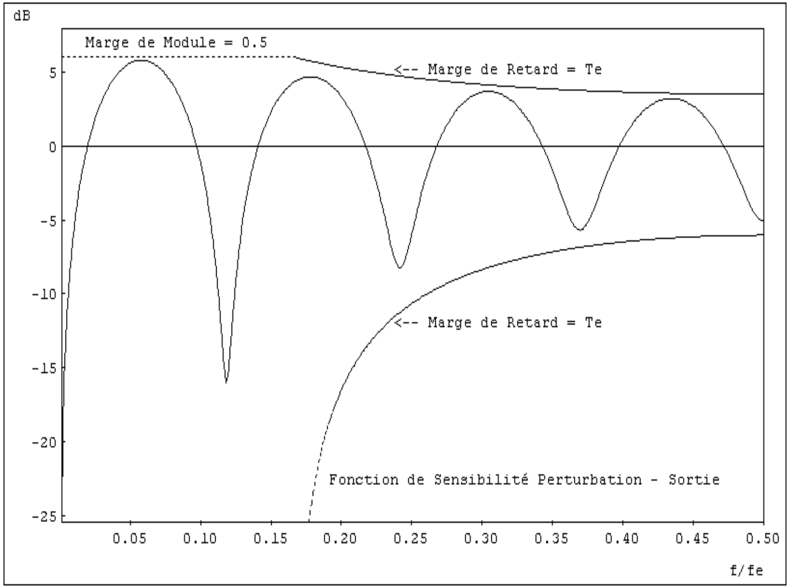
$$S_{yp}(z^{-1}) = 1 + S_{yb}(z^{-1})$$

$$1 - \left|S_{yb}(z^{-1})\right| \leq \left|S_{yp}(z^{-1})\right| \leq 1 + \left|S_{yb}(z^{-1})\right|$$

$$\left|S_{yb}(z^{-1})\right| < \frac{1}{\left|z^{-1} - 1\right|}; z = e^{-jw}, 0 \leq w \leq p$$

$$1 - \left|1 - z^{-1}\right|^{-1} \leq \left|S_{yp}(z^{-1})\right| \leq 1 + \left|1 - z^{-1}\right|^{-1}$$

Approximate condition



Defintion of “templates” for the sensitivity functions

Nominal performance requirements and *robust stability* conditions lead to the definition of desired templates on the sensitivity functions

The union of various templates $W_i^{-1}(e^{-jw})$ will define an *upper* and a *lower* template

Upper template:

$$\left|W^{-1}(e^{-jw})\right|_{\sup} = \min_i \left[\left|W_{S_1}^{-1}(e^{-jw})\right|, \ldots \left|W_{S_n}^{-1}(e^{-jw})\right| \right]$$

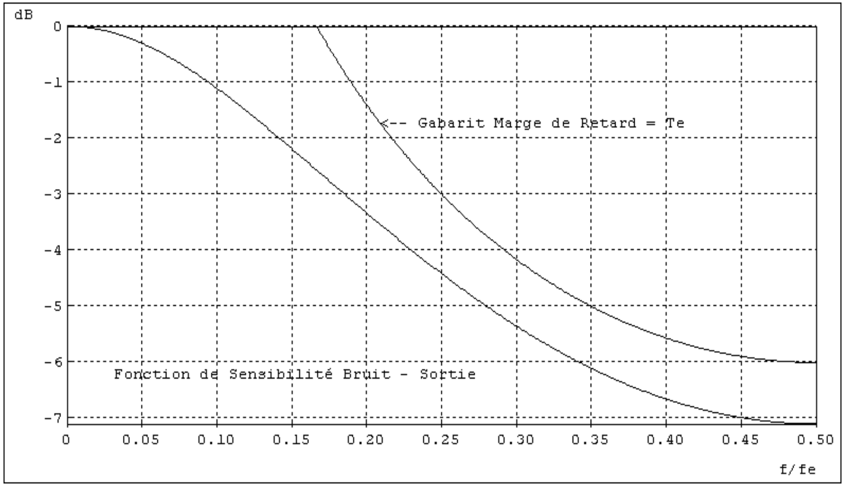
Lower template:

$$\left|W^{-1}(e^{-jw})\right|_{\inf} = \max_i \left[\left|W_{I_1}^{-1}(e^{-jw})\right|, \ldots \left|W_{I_n}^{-1}(e^{-jw})\right| \right]$$

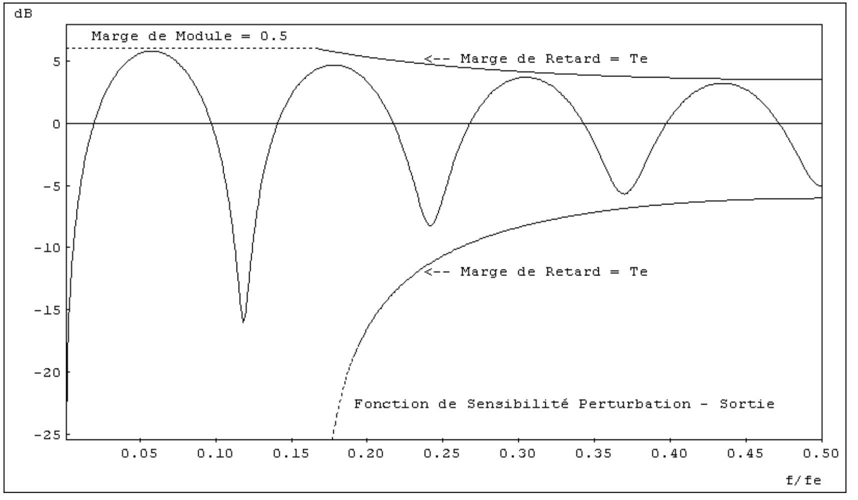
Frequency templates on the sur sensitivity functions

The robust stability conditions allow to define frequency templates on the sensitivity functions which guarantee the delay margin and the modulus margin;

The templates are essential for the design a good controller

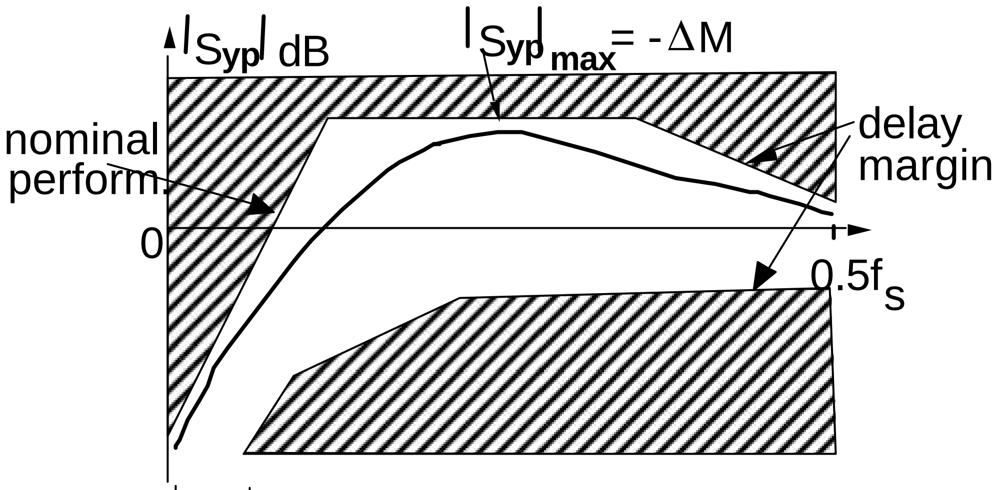


Frequency template on the noise-output sensitivity function S_{yb} for $Dt = T_S$

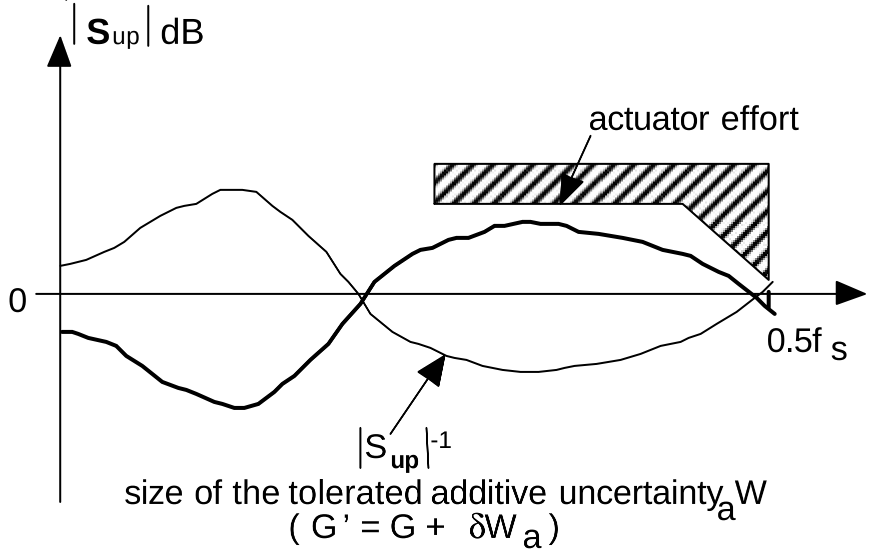


Frequency template on the output sensitivity function S_{yp} for $Dt = T_S$ and $DM = 0.5$

Templates for the Sensitivity Functions

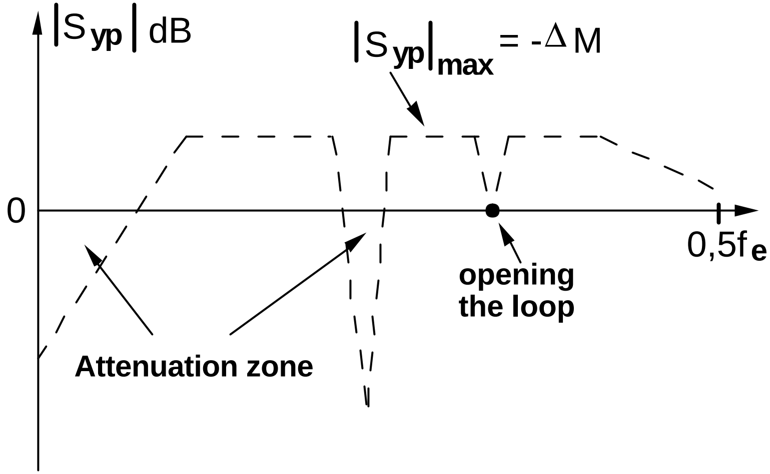


Output Sensitivity Function

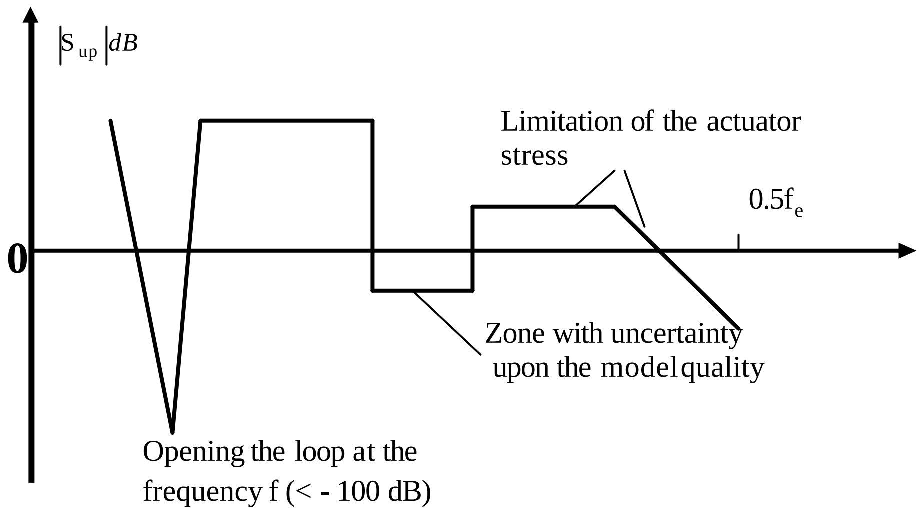


Input Sensitivity Function

Templates for the Sensitivity Functions



Output Sensitivity Function



Input Sensitivity Function