Robust discrete time control

Special topics

- Generalized stability margin and Vinnicombe distance
- Q – parametrization (Youla- Kucera)
- Predictor interpretation of R-S-T controllers
- Robustification of Linear GPC
- Complexity of robust controllers and controller reduction
- Some references (books)

Generalized stability margin. Preliminaries

Modulus margin:

$$\Delta M = \left(\left| S_{yp}(e^{-jw}) \right|_{\max_{w}} \right)^{-1} = \left\| S_{yp}(e^{-jw}) \right\|_{\infty}^{-1} \quad , \text{ for } w = 0 \text{ to } \pi f_S$$

Takes in account only one sensitivity function

- Stability of the closed loop system requires the stability of all the sensitivity functions
- Various sensitivity functions have to be considered for robust stability depending on the representation of the uncertainties.

Is it possible to characterize globally the robustness margin of the closed loop system taking in account simultaneously the four sensitivity functions ?

Generalized stability margin

Sensitivity matrix:

$$\mathrm{T}(K,G)=\mathbf{T}(e^{-jw})=\begin{vmatrix} S_{yr}(e^{-jw}) & S_{yv}(e^{-jw}) \\ -S_{up}(e^{-jw}) & S_{yp}(e^{-jw}) \end{vmatrix}$$

controller plant model

Generalized stability margin:

$$\boxed{\mathrm{b}(\mathrm{K},\mathrm{G})=\begin{cases} \left\|\mathrm{T}(\mathrm{K},\mathrm{G})\right\|_{\infty}^{-1} & \text{if } (\mathrm{K},\mathrm{G}) \text{ is stable} \\ 0 & \text{otherwise} \end{cases}}$$

Can be computed with *smarg.m*

Generalized stability margin. Computation

Singular value decomposition:

$$\mathbf{T}(jw) = \mathbf{U}\mathbf{S}(jw)\mathbf{V}^{*}$$

U,V : orthonormal matrices $\quad UU^{*} = U^{*}U = I,\, VV^{*} = V^{*}V = I$

$$\mathbf{S}(jw) = \begin{bmatrix} s_1(jw) & 0 & 0 & 0 \\ 0 & s_2(jw) & 0 & 0 \\ 0 & 0 & s_3(jw) & 0 \\ 0 & 0 & 0 & s_4(jw) \end{bmatrix}$$

$$|s_1(jw)| \geq |s_2(jw)| \geq |s_3(jw)| \geq |s_4(jw)| \quad \text{for} \quad w = 0 \text{ to } p\, f_S$$

Largest singular value of T: $\quad \overline{s}(jw) = s_1(jw)$

$$|T(jw)| = |\overline{s}(jw)| \qquad \text{(modulus of T)}$$

$$\boxed{\|\mathbf{T}(jw)\|_{\infty} = |\mathbf{T}(jw)|_{\max_{w}} = |\overline{s}(jw)|_{\max_{w}} \quad \text{, for } w = 0 \text{ to } pf_S}$$

Normalized distance between two transfer functions:

The winding number

n_{z_i} : number of unstable zeros of G $\qquad$ n_{p_i}: number of unstable poles of G

$$wno(G) = n_{z_i}(G) - n_{p_i}(G)$$

wno(G)>0 $\qquad$ wno(G)<0

Transfer functions G_1 *and* G_2 which can be compared:

$$wno(1 + G_2^* G_1) + n_{p_i}(G_1) - n_{p_i}(G_2) - n_{P_1}(G_2) = 0$$

number of poles on the unit circle

Normalized difference between two transfer functions:

$$\Psi[G_1(jw), G_2(jw)] = \frac{G_1(jw) - G_2(jw)}{\left(1 + |G_1(jw)|^2\right)^{1/2}\left(1 + |G_2(jw)|^2\right)^{1/2}}$$

Vinnicombe distance (ν-gap):

$$\boxed{d_n(G_1, G_2) = \left|\Psi[G_1(jw), G_2(jw)]\right|_{\max_w} = \left\|\Psi[G_1(jw), G_2(jw)]\right\|_\infty;\ 0 \le w \le p\ f_S}$$

$$0 \le d_n(G_1, G_2) < 1$$

Robust stability criterion (Vinnicombe)

The controller K which stabilizes the plant model G_1 will also stabilize model G_2 if:

$$\boxed{d_n(G_1, G_2) \leq b(K, G_1)}$$

A less restrictive condition:

$$\left| \Psi\left[G_1(jw), G_2(jw)\right] \right| \leq \left| \mathrm{T}(jw) \right|^{-1}; \quad 0 \leq w \leq p \; f_S$$

Vinnicombe G., (1993) : « Frequency domain uncertainty and the graph topology », *IEEE Trans. on Automatic Control*, vol. 38, no. 9, pp. 1371-1383.

Zhu K., (1998) : *Essentials of robust control*, Prentice Hall, N.J., U.S.A.
Landau I.D.(2002) *Commande des sytèmes- conception, identification et mise en oeuvre,* Hermes, Paris

Q – parametrization (Youla –Kucera)

Q - parametrization

Central controller: $[R_0(q^{-1}), S_0(q^{-1})]$.

CL poles: $P(q^{-1}) = A(q^{-1})S_0(q^{-1}) + q^{-d}B(q^{-1})R_0(q^{-1})$.

Control: $S_0(q^{-1})\, u(t) = -R_0\,(q^{-1})\; y(t)$

Q-parametrization : $$Q(q^{-1}) = \frac{b(q^{-1})}{a(q^{-1})} \in RH_{\infty}$$

(set of rational,stable and proper transfer functions)

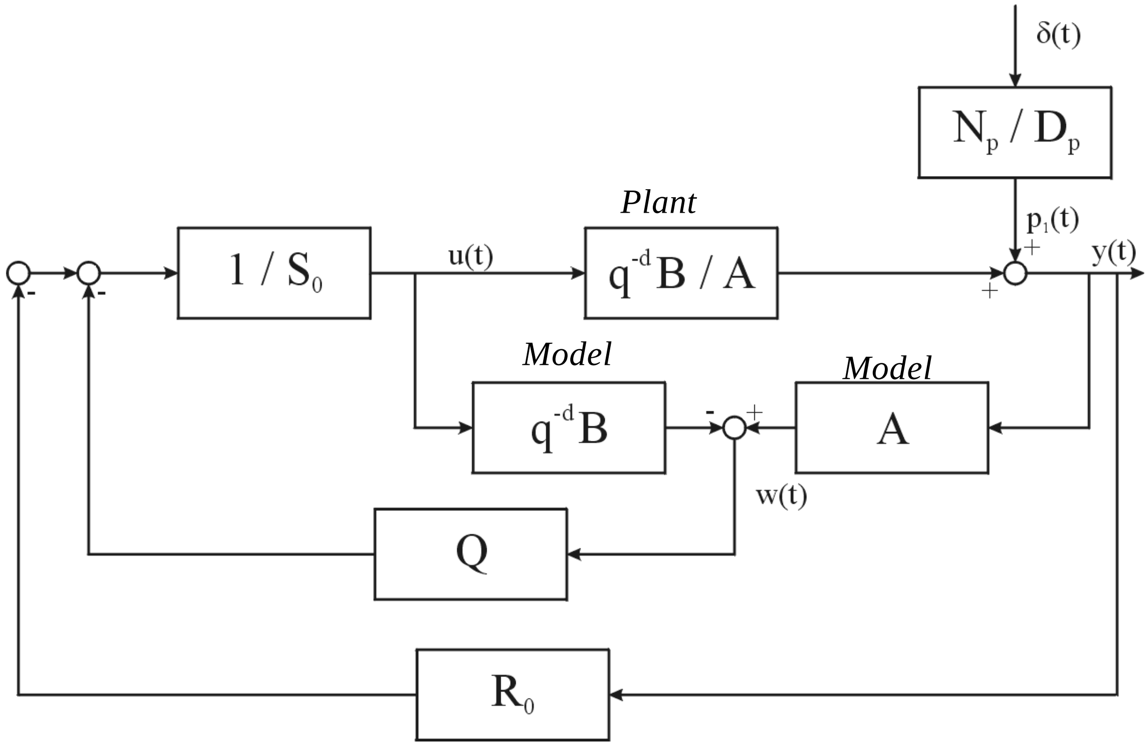
$R(z^{1}) = R_0(q^{-1}) + A(q^{-1})Q(q^{-1})$;
$S(q^{-1}) = S_0(z^{-1}) - q^{-d}B(q^{-1})Q(q^{-1})$.

Control: $$S\,(q^{-1})u(t) = -R\,(q^{-1})y(t)$$

$$S_0(q^{-1})u(t) = -R_0(q^{-1})y(t) - Q(q^{-1})[A(q^{-1})y(t) - q^{-d}B(q^{-1})u(t)]$$
$$= -R_0(q^{-1})y(t) - Q(q^{-1})w(t)$$

where: $$w(t) = A(q^{-1})y(t) - q^{-d}B(q^{-1})u(t)$$

Q – parametrization (equivalent scheme)



Q – parametrization (Q = FIR)

$$Q(q^{-1}) = b(q^{-1}) = b_0 + b_1 q^{-1} + \ldots; a(q^{-1}) = 1$$

Central controller: $[R_0(q^{-1}),S_0(q^{-1})]$.

CL poles: $P(q^{-1})=A(q^{-1})S_0(q^{-1})+q^{-d}B(q^{-1})R_0(q^{-1})$.

Control: $S_0(q^{-1})\,u(t) = -R_0\,(q^{-1})\,y(t)$

Q-parametrization :
$R(z^1)=R_0(q^{-1})+A(q^{-1})Q(q^{-1})$;
$S(q^{-1})=S_0(z^{-1})-q^{-d}B(q^{-1})Q(q^{-1})$.

Closed Loop Poles :

$$P(q^{-1}) = A(q^{-1})\left[S_0(q^{-1}) - q^{-d}B(q^{-1})Q(q^{-1})\right]$$
$$+ q^{-d}B(q^{-1})\left[R_0(q^{-1}) + A(q^{-1})Q(q^{-1})\right] = A(q^{-1})S_0(q^{-1}) + q^{-d}B(q^{-1})R_0(q^{-1})$$

- ***Closed Loop Poles remain unchanged !***
- ***Sensitivity Functions will be different***

Q – parametrization. Sensitivity functions

Q-parametrization :

$R(z^{1})=R_0(q^{-1})+A(q^{-1})Q(q^{-1})$;
$S(q^{-1})=S_0(z^{-1})-q^{-d}B(q^{-1})Q(q^{-1})$.

Sensitivity functions:

$$S_{yp}=\frac{AS}{P}=\frac{AS_0}{P}-\frac{q^{-d}B}{P}Q=S_{yp0}-\frac{q^{-d}B}{P}Q$$

$$S_{up}=-\frac{AR}{P}=-\frac{AR_0}{P}-\frac{A}{P}Q=S_{up0}-\frac{A}{P}Q$$

$$S_{yb}=-\frac{q^{-d}BR}{P}=-\frac{q^{-d}BR_0}{P}-\frac{A}{P}Q=S_{yb0}-\frac{A}{P}Q$$

$$S_{yv}=\frac{q^{-d}BS}{P}=\frac{q^{-d}BS_0}{P}-\frac{q^{-d}B}{P}Q=S_{yv0}-\frac{q^{-d}B}{P}Q$$

General form:

$$\boxed{S_{xy}=S_{xy0}-\frac{q^{-d}B(or\,A)}{P}Q}$$

Afine in Q

Q – parametrization (Q = FIR)

- Q can be used to "tune" the robustness of the closed loop
- Q can be used to introduce an equivalent "internal model"
- Q can be used to introduce an"equivalent pre-specified" filter in the controller
- Q as a FIR does not allow to introduce "additional poles" (useful for robustness)

Example : internal model

$$S = S' H_s = S_0 - q^{-d} BQ$$

Solve Bezout equation:

$$\underbrace{S'}_{?} H_S + q^{-d} B \underbrace{Q}_{?} = S_0$$

$$n_{S_0} \le n_{H_S} + n_B + d - 1$$

$$n_Q = n_{H_s} - 1$$

$$n_{S'} = n_B + d - 1$$

Q – parametrization (general case)

$$Q(q^{-1}) = \frac{b(q^{-1})}{a(q^{-1})} \in RH_{\infty}$$

$$\frac{R}{S} = \frac{R_0 a + Ab}{S_0 a - q^{-d} Bb}$$

$$P_a = A(S_0 a - q^{-d} Bb) + q^{-d} B(R_0 a + Ab) = Pa$$

α introduces additional poles (auxiliary poles)

$$S_{xy} = S_{xy0} - \frac{q^{-d} B(or\,A)}{P} \frac{b}{a} = S_{xy0} - T_{xy} \frac{b}{a}$$

Afine in β/α but not in β and α

Q – parametrization (general case)

Rantzer & Megretski (IEEE-TAC, 1994 result:

If it exist a' and b' such that:

$$\left\| \bar{T}_1 + \bar{T}_2 \frac{b'}{a'} \right\|_\infty < 1 \qquad (*)$$

This implies the existence of a and b such that :

$$\left| \bar{T}_1 a + \bar{T}_2 b \right| < \mathrm{Re}\{a\} \quad \forall |z| = 1 \qquad (**)$$

$$\bar{T}$$

(convex condition on a and b – see Langer's thesis for a proof)

But (*) is nothing else that the condition for satisfying the upper template on a sensitivity function:

$$\left| S_{xy0} + T_{xy} \frac{b'}{a'} \right| < \left| W_{xy}^{-1} \right| \Rightarrow \left\| W_{xy} S_{xy0} + W_{xy} T_{xy} \frac{b'}{a'} \right\|_\infty < 1$$

And one can uses (**) with: $\bar{T}_1 = W_{xy} S_{xy0}; \bar{T}_2 = W_{xy} T_{xy}$ for finding α and β

Then convex optimization techniques can be used to solve "automatically" the problem (Langer,Automatica, June 99 and OPTREG(Matlab Toolbox))

Parametrization of α and β for convex optimization

Ritz method

$$\alpha(x_a) = 1 + \sum_1^N x_{ak}\alpha_k; \quad \beta(x_a) = x_{b0} + \sum_1^N x_{bk}\beta_k$$

$$\alpha_k = \beta_k = \left(\frac{z_0 - z^{-1}}{1 + z_0 z^{-1}}\right)$$

$$x^T = \left[x_a^T \; x_a^T\right]$$ Parameter vector to be optimized

The order of the resulting controller R/S is augmented by N

An exercise

How to introduce a band stop filter H_S/P_F ?

$$P_F = a$$

$$S' H_S + q^{-d} B b = S_0 a$$

? ?

$$n_{S_0} \leq n_{H_S} + n_B + d - 1$$

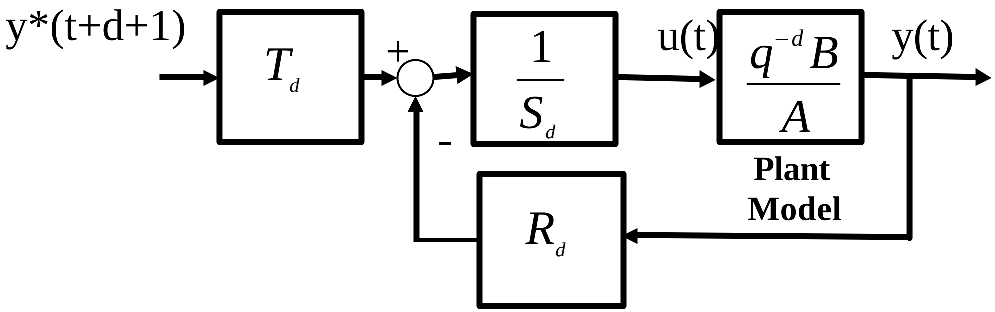
$$n_b = n_a + n_{H_s} - 1$$

$$n_{S'} = n_a + n_B + d - 1$$

Predictor interpretation of R-S-T controllers
The case of systems with delay

See also:
I.D. Landau (1995) : Robust digital control of systems with time delay (the Smith predictor revisited), Int. J. of Control, v.62,no.2 pp 325-347

The R-S-T Digital Controller



Plant Model: $$G(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} = \frac{q^{-d-1}B^*(q^{-1})}{A(q^{-1})}$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_A} q^{-n_A} \quad B(q^{-1}) = b_1 q^{-1} + \ldots + b_{n_B} q^{-n_B} = q^{-1}B^*(q^{-1})$$

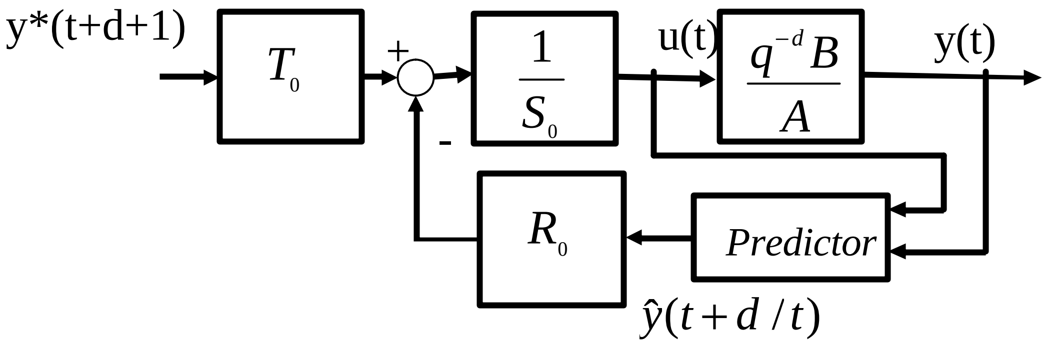
R-S-T Controller: $$S_d(q^{-1})u(t) = T_d(q^{-1})y^*(t+d+1) - R_d(q^{-1})y(t)$$

Characteristic polynomial (closed loop poles):

$$P(q^{-1}) = P_0(q^{-1})P_{pr}(q^{-1}) = A(q^{-1})S_d(q^{-1}) + q^{-d}B(q^{-1})R_d(q^{-1})$$

Closed loop poles for system with d = 0 Predictor poles

R-S-T controller –equivalent representation



(R_0, S_0, T_0) Controller for the sytem without delay (d=0)

Equivalent controller equations:

$S_0 u(t) = T_0 y^*(t+d+1) - R_0 \hat{y}(t+d/t)$ (*) Controller for system with d = 0

$P_{pr} \hat{y}(t+d/t) = Fy(t) + BEu(t)$ (**) d-steps ahead predictor

R_0 and S_0 are solutions of: $AS_0 + BR_0 = P_0$

E and F are solutions of: $AE + q^{-d}F = P_{pr}$

R-S-T controller –equivalent representation

Combining (*) and (**) gives

$$(P_{pr}S_0 + BR_0E)u(t) = T_0P_{pr}y^*(t+d+1) - R_0Fy(t)$$

which yelds the closed loop poles :

$$(P_{pr}S_0 + BR_0E)A + q^{-d}BR_0F = P_{pr}S_0A + BR_0(AE + q^{-d}F) = P_0P_{pr}$$

Remark:
Various interpretation possible of the imposed dynamics for the closed loop without delay and the predictor dynamics
(Which are the *dominant poles* ? Which are the *auxiliary poles* ?)

Internal model control – another equivalent representation

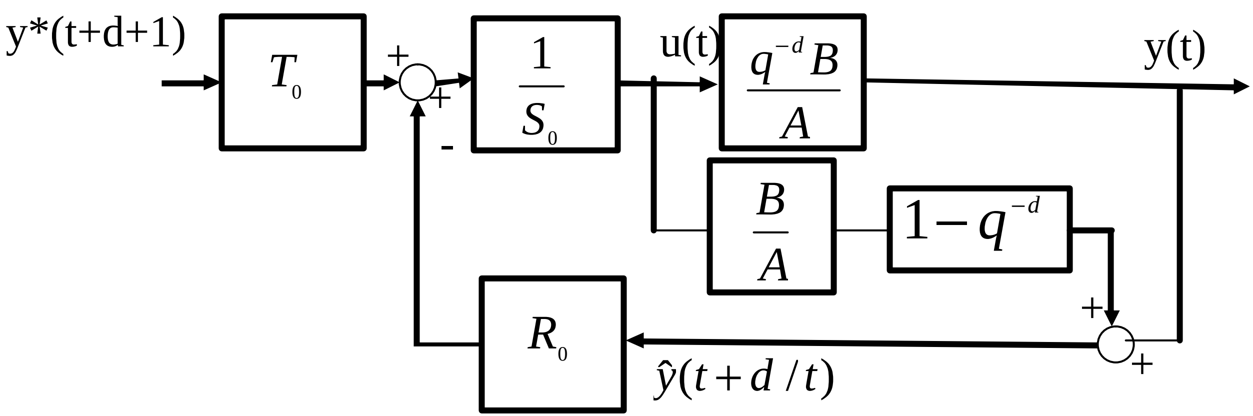
Particular case : $P_{pr}=A$ (IMC)

$$S_0 u(t) = T_0 y^*(t+d+1) - R_0 \hat{y}(t+d+1)$$ controller

$$P_{pr}\hat{y}(t+d/t) = Fy(t) + BEu(t)$$ predictor

$$AE + q^{-d}F = A \Rightarrow F = A;\ E = 1-q^{-d}$$ predictor parameters for IMC

$$\hat{y}(t+d) = \frac{(1-q^{-d})B}{A}u(t) + y(t)$$ Predictor (IMC)



Remark : "Smith predictor" like structure

Robustification of linear generalized predictive control

Robustification of linear generalized predictive control

Minimize

$$J(t,h_p,h_c,h_i) = E\left\{ \sum_{j=h_i}^{h_p} \left[P(q^{-1})y(t+j) - P(q^{-1})y^*(t+j)\right]^2 + \lambda \left[\bar{Q}(q^{-1})u(t+j-h_i)\right]^2 \right\}$$

in a receding horizon sense, subject to the constraint

$$\bar{Q}(q^{-1})u(t+i) = 0; \quad h_c \leq i \leq h_p$$

Solution : an R-S-T controller with well defined closed loop poles

Two approaches for robustification:

- Selection of P and Q plus the poles of the predictor
- Use of Q parametrization on top of the computed controller

Selection of P ,Q and the poles of the predictor

Integratror

$$\bar{Q} = \frac{H_S}{H_R}; \quad \bar{Q} = \frac{1 - q^{-1}}{1 + q^{-1}}$$

Opening the loop at $0.5f_S$

The *P* polynomial and the dynamics of the predictor appear in the closed loop poles in addition of the poles introduced by the criterion minimization.They can be Selected to improve the robustness

Use of Q parametrization on top of the computed controller

Use the computed controller as the “central controller” and add the Q – parametrization

To optimize Q parameters use:

- Convex optimization (as for pole placement)
- Use linear programming on a fixed size Q (see Rodriguez, ECC03)

Complexity of robust controllers and Controller reduction

CONTROLLER REDUCTION. Why ?

- Complex Models
- Robust Control Design

→ High Order Controllers

Example : The Flexible Transmission
(Robust control benchmark, EJC no. 2/1995 and no.2-4/1999)

Model complexity : $G(q^{-1}) = \dfrac{q^{-d}B(q^{-1})}{A(q^{-1})}$ $n_A = 4;\ n_B = 2;\ d = 2$

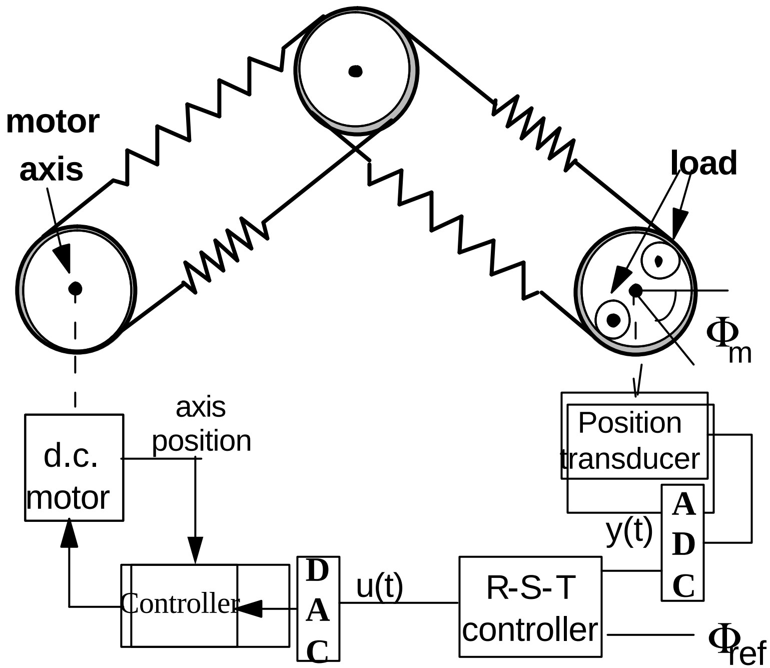
Fixed controller part : Integrator

Pole placement design : $K(q^{-1}) = \dfrac{R(q^{-1})}{S(q^{-1})}$ $n_R = 4;\ n_S = 4$

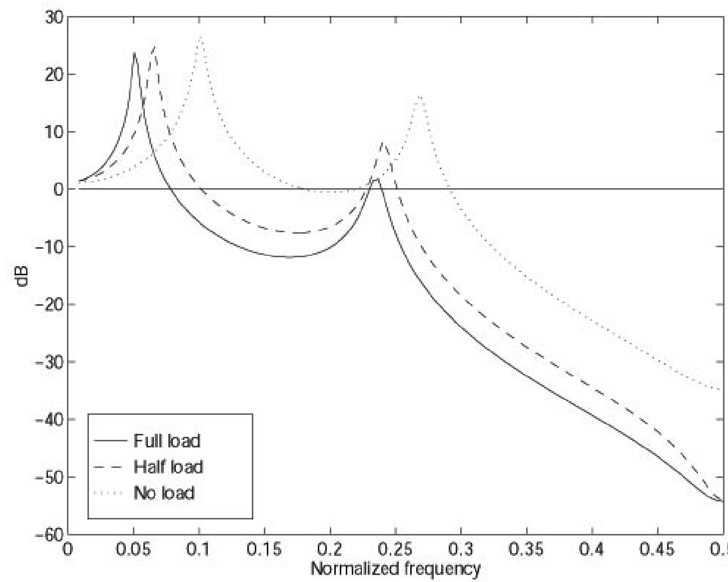
Complexity of controllers achieving 100 % of specifications:

Max : $n_R = 9;\ n_S = 9$ (Nordin) **Min** : $n_R = 7;\ n_S = 7$ (Langer)

Robust control of a flexible transmission



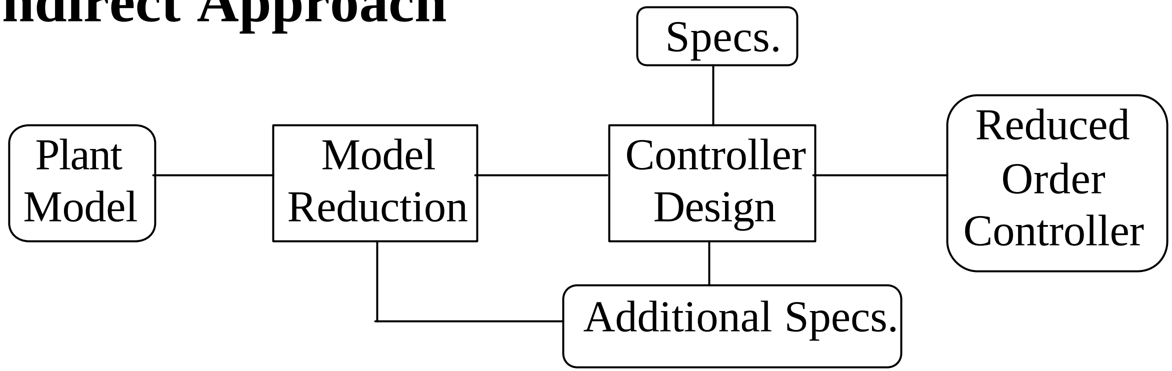
Frequency characteristics for various load



The main vibration mode varies by 100%

Approaches to Controller Reduction

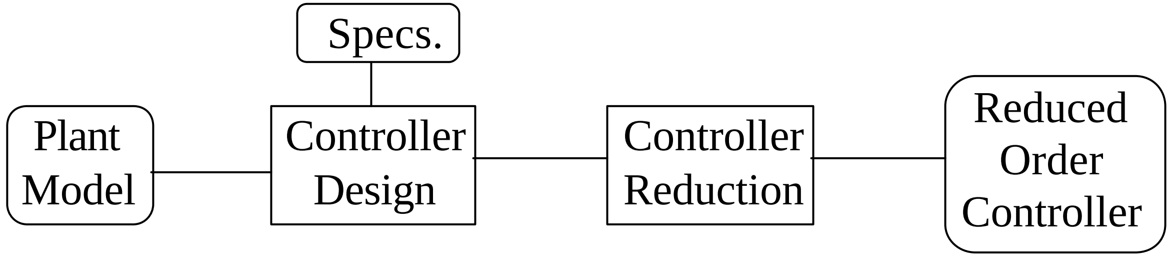
Indirect Approach



-Does not guarantee resulting controllers of desired order

- Propagation of model errors

Direct Approach



- Approximation carried in the final step
- Further controller reduction for "indirect approach"

Controller Reduction

Basic rule :

Controller reduction should be done with the aim to preserve as much as possible the closed loop properties.

Reminder :

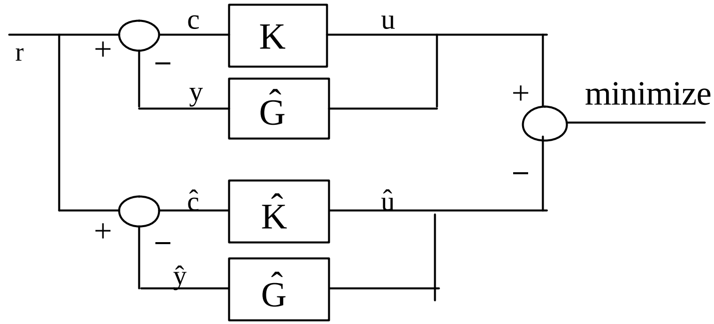
Controller reduction without taking into account the closed loop properties can be a disaster !

Some basic references :

- Anderson &Liu : IEEE-TAC, August 1989
- Anderson : IEEE Control Magazine, August 1993

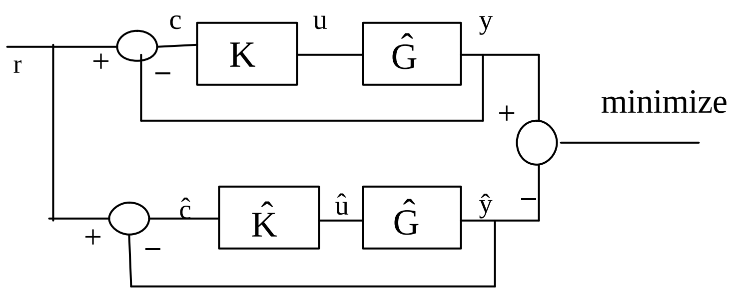
Controller Reduction - Objectives

Input matching



$$\hat{K}^* = \arg\min_{\hat{K}} \left\| \hat{S}_{up} - \hat{\hat{S}}_{up} \right\| = \arg\min_{\hat{K}} \left\| \hat{S}_{yp} (K - \hat{K}) \hat{\hat{S}}_{yp} \right\|$$

Ouput matching

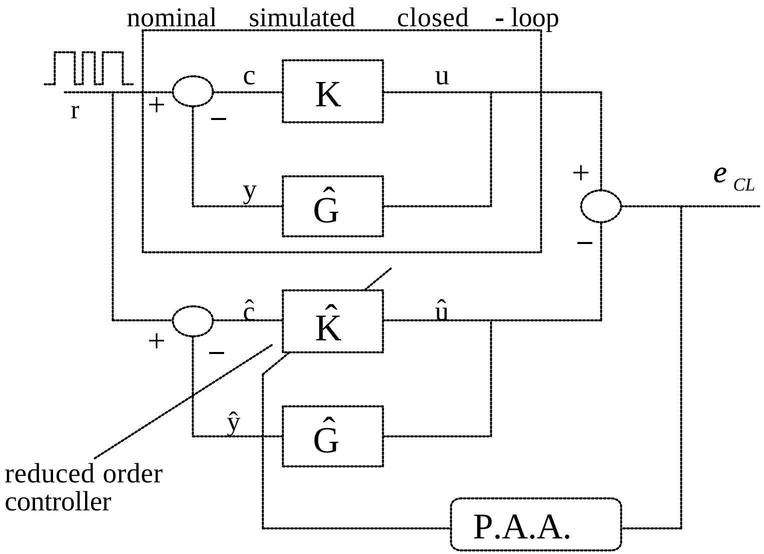


$$\hat{K}^* = \arg\min_{\hat{K}} \left\| \hat{S}_{yr} - \hat{\hat{S}}_{yr} \right\| = \arg\min_{\hat{K}} \left\| \hat{S}_{yp} - \hat{\hat{S}}_{yp} \right\| = \arg\min_{\hat{K}} \left\| \hat{S}_{yp} (K - \hat{K}) \hat{\hat{S}}_{yv} \right\|$$

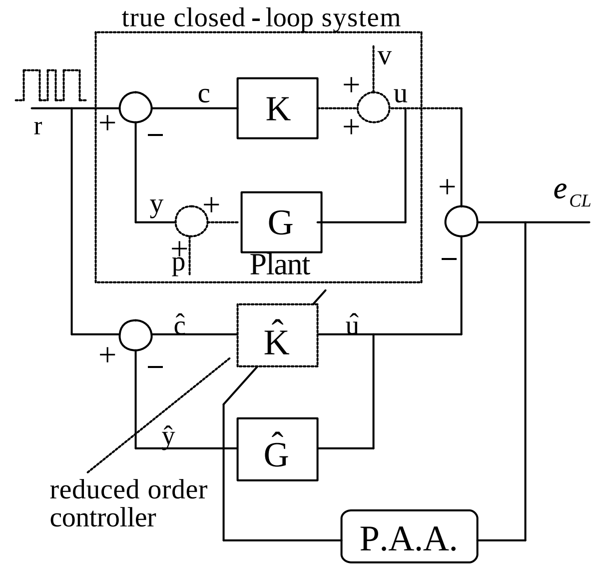
Identification of Reduced Order Controllers

Input Matching (CLIM)

Use of simulated data



Use of real data



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