Chapter I

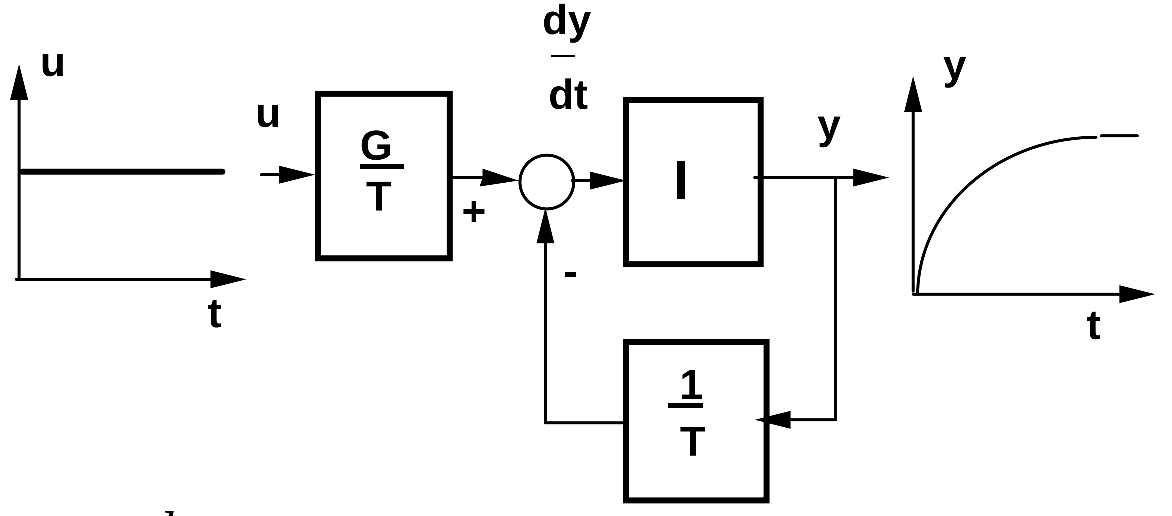
Continuous Control Systems : A Review

Chapter 1. Continuous Control Systems : A Review

Continuous Time Models

Time Domain

$$\frac{dy}{dt} = -\frac{1}{T}\,y(t) + \frac{G}{T}\,u(t) \quad (*)$$



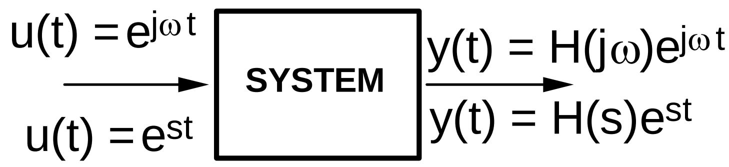
Obs.: $p = \frac{d}{dt}$

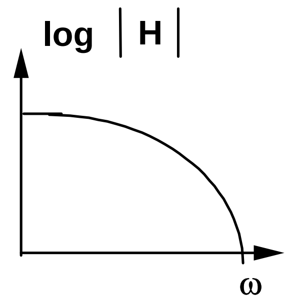
$$(p + \frac{1}{T})\,y(t) = \frac{G}{T}\,u(t)\,; \quad (*)$$

Continuous Time Models

Frequency Domain

u(t) = periodic input





$s = \sigma + j\omega$ *s = complex frequency*

$$y(t) = H(s)e^{st} \qquad \frac{dy(t)}{dt} = s\,H(s)\,e^{st}$$

$$(s + \frac{1}{T})H(s)e^{st} = \frac{G}{T}e^{st} \quad (*) \quad \rightarrow \quad \boxed{H(s) = \frac{G}{1 + sT}}$$

= transfer function

The transfer function can also be obtained by:

- replacing « p » with « s » in (*) (see slide #3)
- Laplace transform

Stability

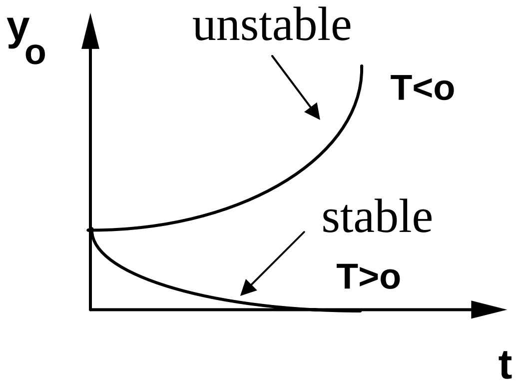
Ex.: 1st order system

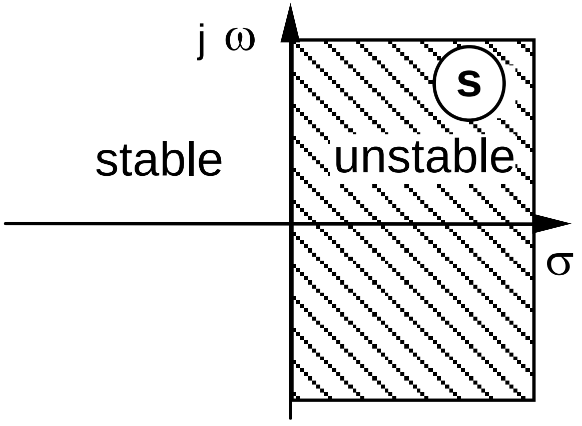
$$\frac{dy}{dt} = -\frac{1}{T}y(t) + \frac{G}{T}u(t) \quad \rightleftarrows \quad \text{T.F.}: \; H(s) = \frac{G}{1+sT}$$

Unforced response (u=0): $\frac{dy}{dt} + \frac{1}{T}y(t) = 0 \; ; \; y(0) = y_0$

Solution: $y(t) = Ke^{st}$? ? $\qquad \frac{dy}{dt} = s\,Ke^{st}$

$$Ke^{st}\left(s + \frac{1}{T}\right) = 0 \;\rightarrow\; \boxed{s = -\frac{1}{T} \; ; \; K = y_0} \;\rightarrow\; y(t) = y_0 e^{-t/T}$$





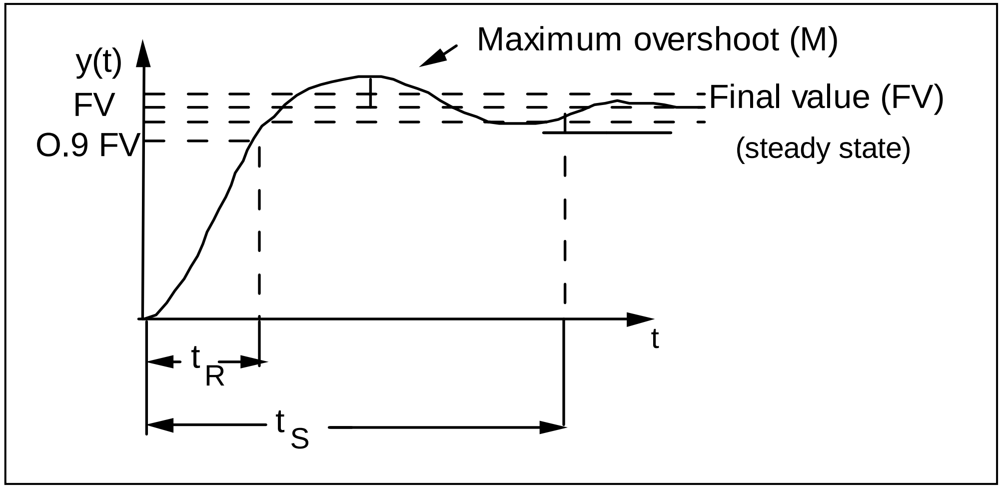
Stability

The stability (or instability) of a system depends on the roots of the transfer function denominator:

***Stability (asymptotically)*: for all roots (s_i) it holds $Re\ s_i < 0$**
***Instability*: there is at least one root s for which $Re\ s > 0$**
$Re\ s = 0$: boundary of stability

Stability criteria are available for proving the existence of unstable roots without any explicit calculation of roots.
(ex: Routh – Hurwitz criteria)

Time Responses

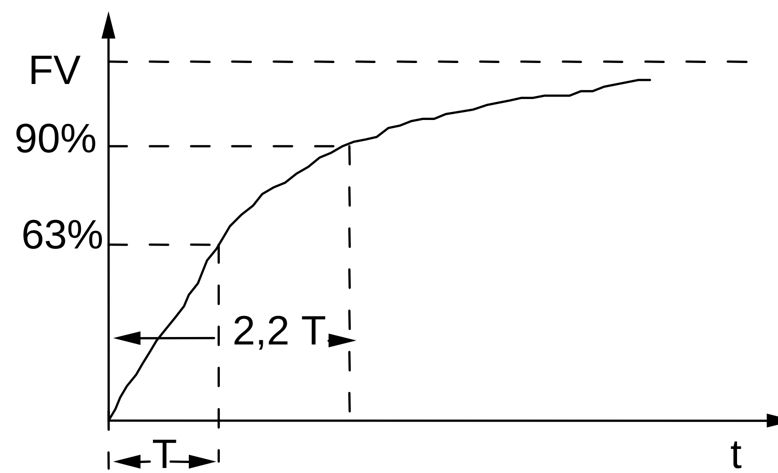


Input : *unit step*

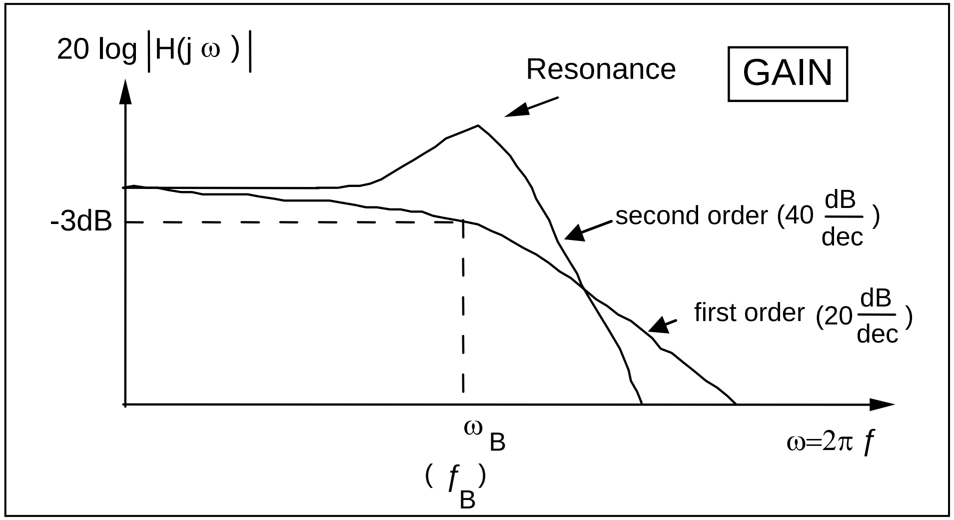
t_R – rising time
t_S – settling time (+/- tolerance)
FV –finale value
M –max. overshoot (% FV)

Ex : 1st order $H(s) = \dfrac{G}{1 + sT}$

FV = G (*static gain*);
$t_R = 2.2T$;
$t_S = 2.2T$ (+/-10%);
M = 0



Frequency Responses



$(|H(j\omega)|\ dB = 20 \log |H(j\omega)|$

$\omega\ (rad/s) = 2\pi f$ (Hz)

Slope : It depends on the number of poles and zeros and on their frequency distribution

Asymptotic Slope (high freq.)

$$\frac{\Delta G}{\Delta \omega} = -(n - m) \times 20\, dB / dec$$

n = # poles; m= # zeros

– $f_{BP}(\omega_{BP})$ (*bandwidth*) : the frequency (radian frequency) from which the zero-frequency (steady-state) gain $G(0)$ is attenuated more than 3 dB.

$$G(\omega_{BP}) = G(0) - 3dB; \qquad (G(\omega_{BP}) = 0.707\, G(0))$$

– $f_C(\omega_C)$ (*cut-off frequency*) : the frequency (rad/s) from which the attenuation introduced with respect to the zero frequency is greater than N dB.

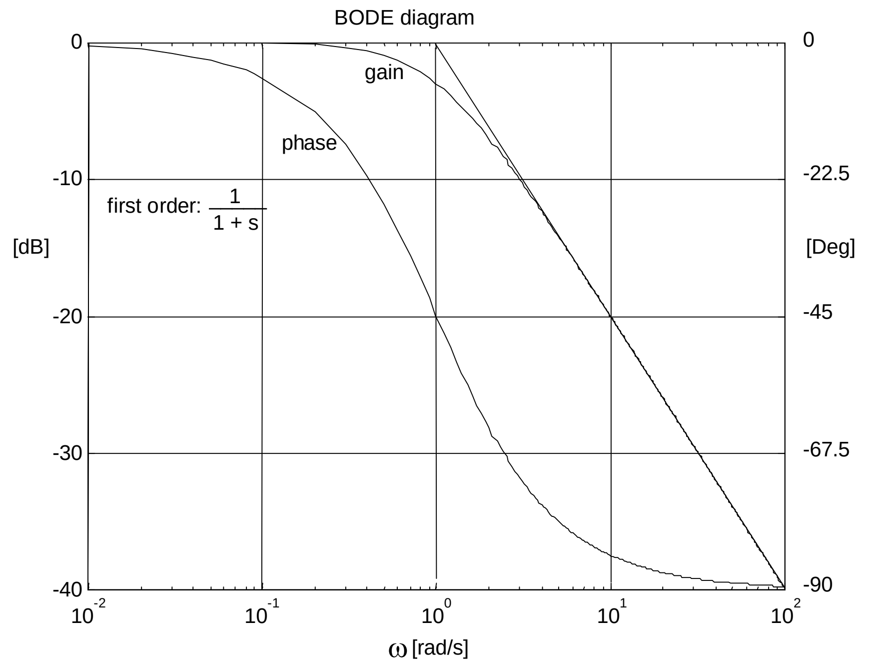
$$G(j\omega_C) = G(0) - N\, dB$$

– Q (*resonance factor*) : the ratio between the gain corresponding to the maximum of the frequency response curve and the value $G(0)$.

1st Order System Frequency Response

$$H(j\omega)=\frac{G}{1+j\omega T}=\left|H(j\omega)\right|e^{j\phi(\omega)}=\left|H(j\omega)\right|\angle\phi(\omega)$$

$$G(\omega)=\left|H(j\omega)\right|=\frac{G}{\sqrt{(1+\omega^2T^2)}}\ ;\ \angle\phi(\omega)=\tan^{-1}\left[\frac{\operatorname{Im}G(j\omega)}{Re\,G(j\omega)}\right]=\tan^{-1}\left[-\omega T\right]$$



Second Order System Analysis

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_0 \frac{dy(t)}{dt} + \omega_0^2 y(t) = \omega_0^2 u(t)$$

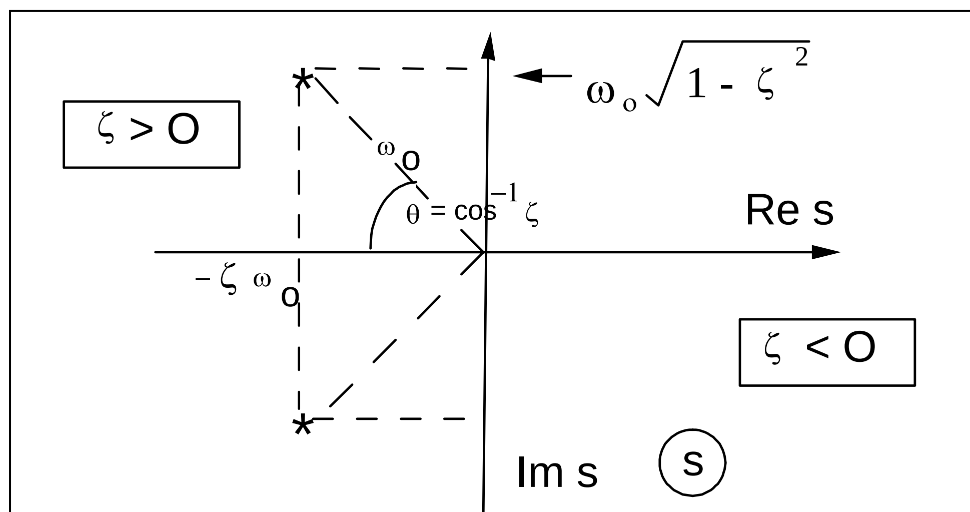
T.F.: $$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\,\omega_0 s + \omega_0^2}$$

ω_0 : natural frequency ($\omega_0 = 2\,\pi f_0$)
ζ : damping factor

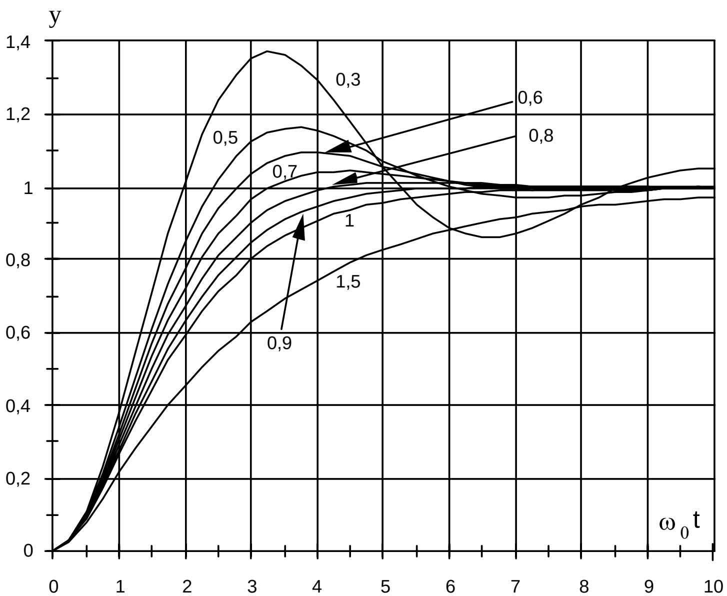
$|\zeta| < 1$: complex poles (oscillatory response). $$s_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$$

$|\zeta|\;\; 1$: real poles (aperiodic response). $$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$\zeta > 0$: asymptotically stable system
$\zeta < 0$: unstable system



Second Order System – Normalized Time Responses



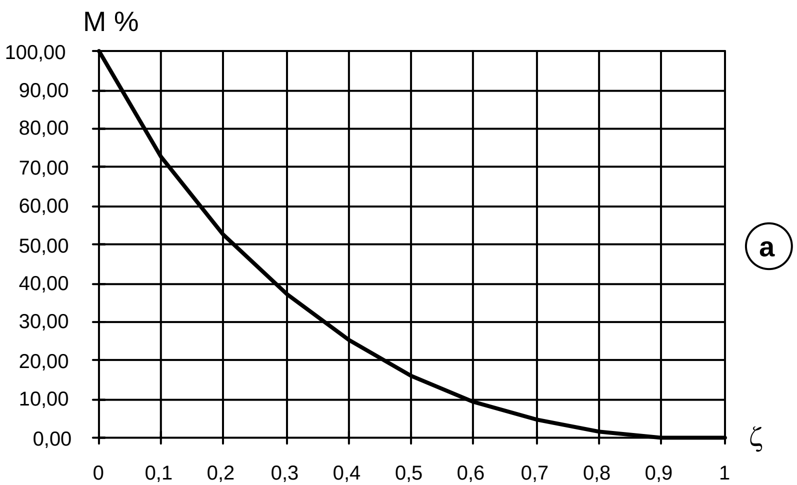
$\omega_0 t_M$ =normalized time response

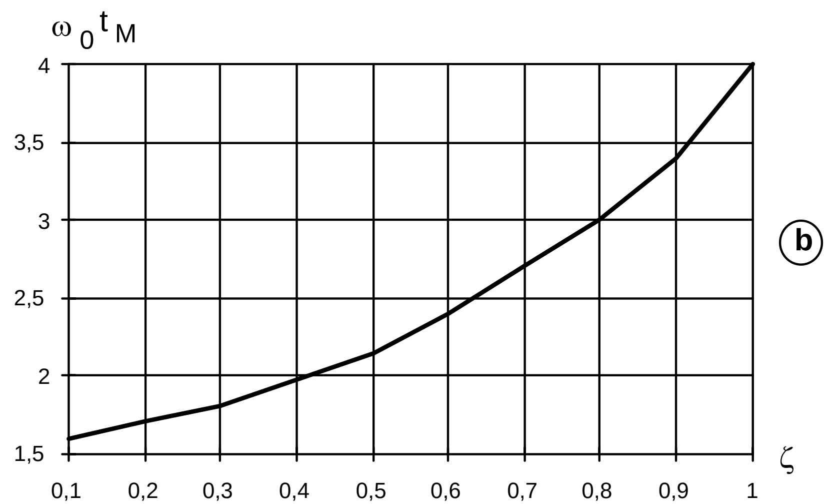
What choice for ω_0 and ζ ?

$M_{desired} \to \zeta\,(diagr.\ a) \to \omega_0 t_M\,(diagr.\ b)$

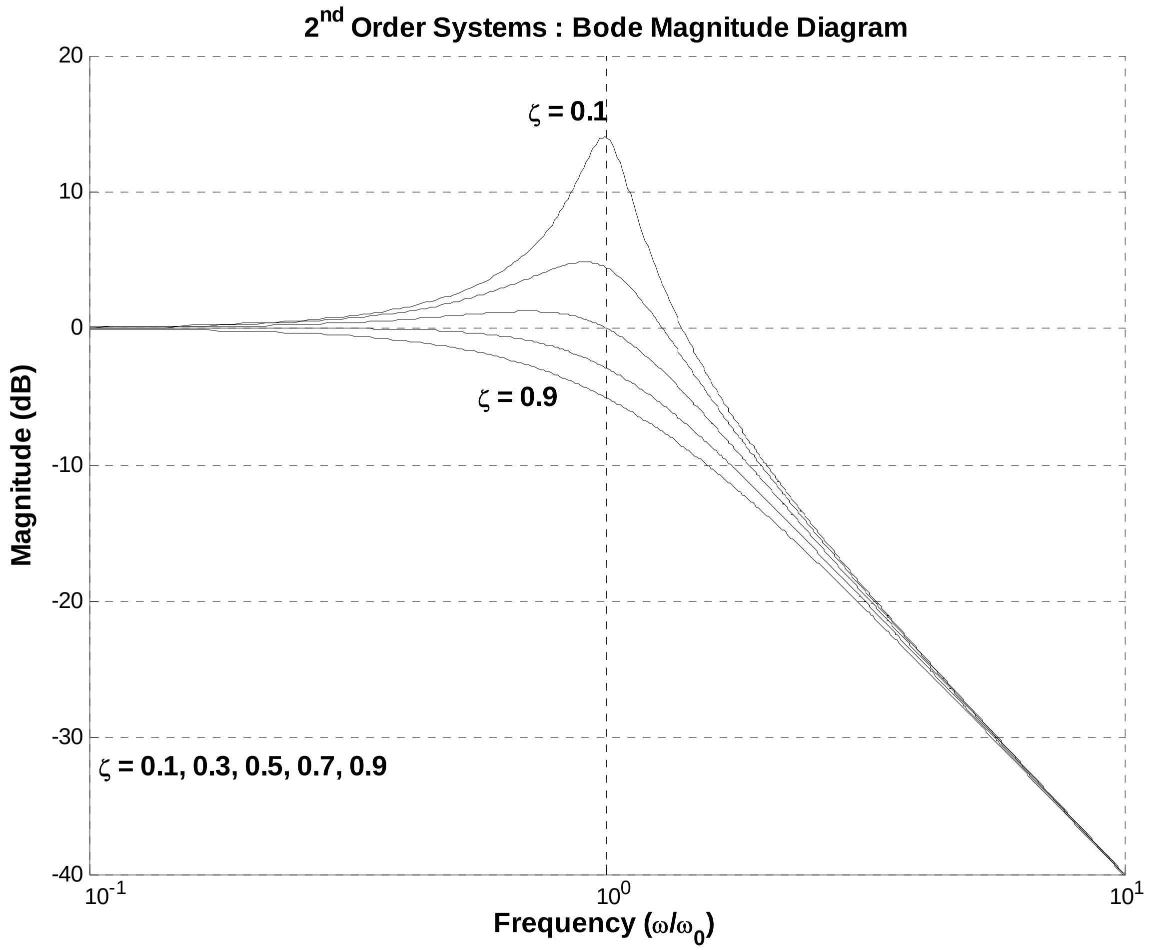
$\omega_0 = (\omega_0 t_M)/(t_M)_{desired}$

- Use of functions ***omega_dmp.sci(.m)***

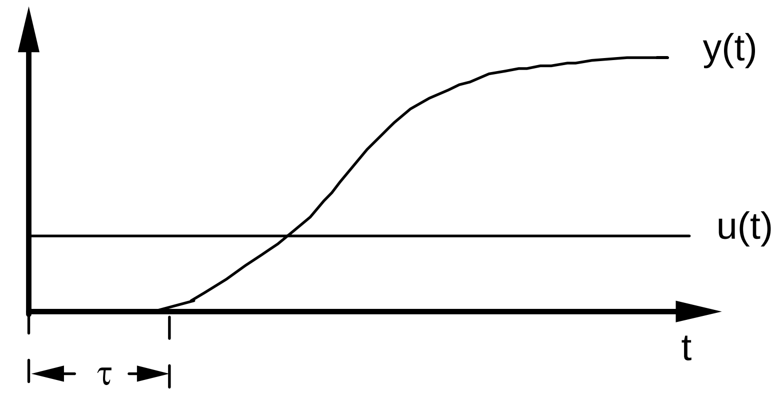




Second Order System – Normalized Frequency Responses



System with Time Delay



$$\frac{dy}{dt} = -\frac{1}{T}y(t) + \frac{G}{T}u(t-\tau) \rightleftarrows \quad \text{T.F.:} \quad H(s) = \frac{Ge^{-s\tau}}{1+sT}$$

Rem.(frequency domain):

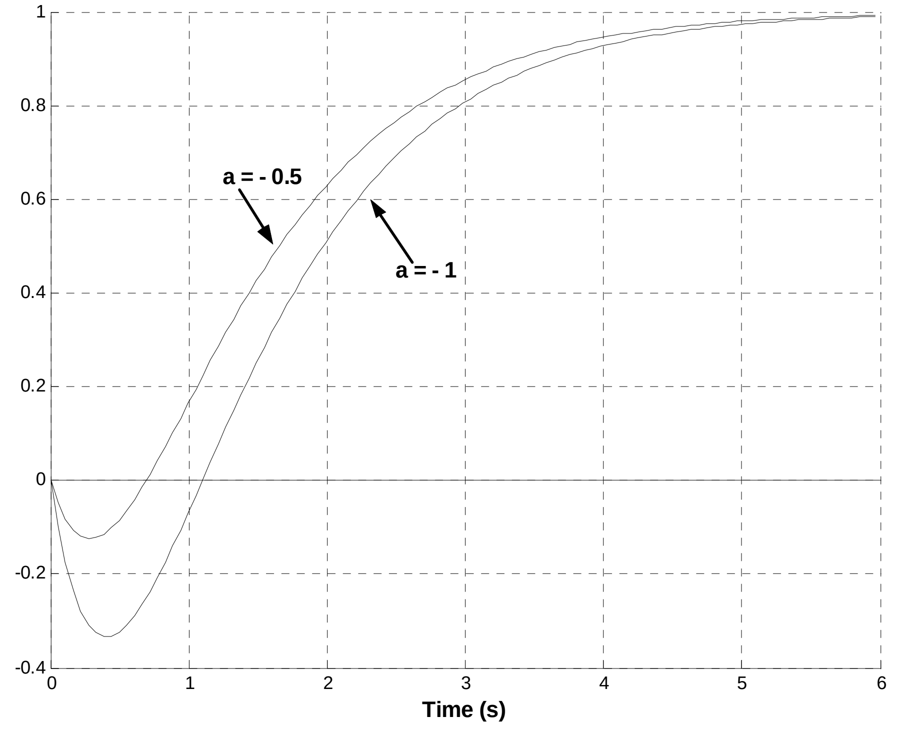
The time delay does not modify the system gain but it introduces a phase shift proportional to the frequency

$$H_{delay}(j\omega) = e^{-j\omega\tau} = |\,1\,|\angle\,\phi(\omega) \qquad \text{with} \quad \angle\,\phi(\omega) = -\,\omega\,\tau\,(rad)$$

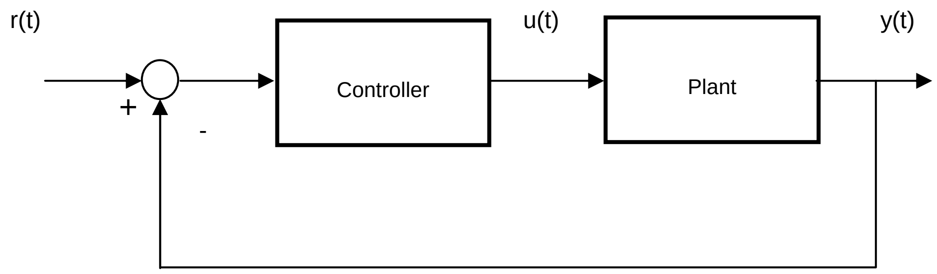
Non-minimum Phase Systems

For Continuous-time Systems (only) → one or more *unstable zeros*

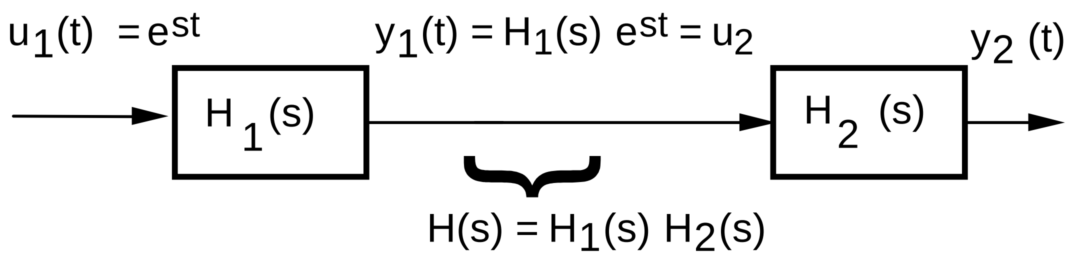
$$H(s) = \frac{1 - sa}{(1+s)(1+0.5s)} \longrightarrow$$



Closed Loop Systems



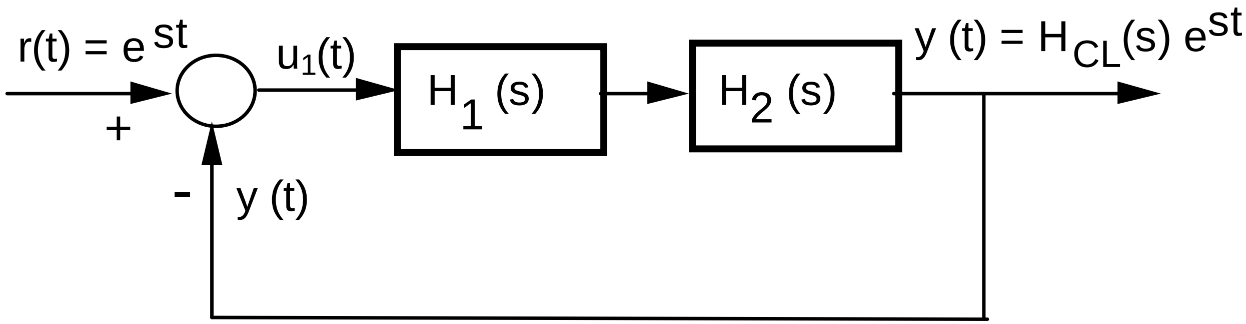
Cascaded Systems



$$y_2(t) = H_2(s)u_2(t) = H_2(s)H_1(s)u_1(t) = H(s)e^{st}$$

$$H(s) = H_n(s)....H_2(s)H_1(s)$$

Closed Loop Systems

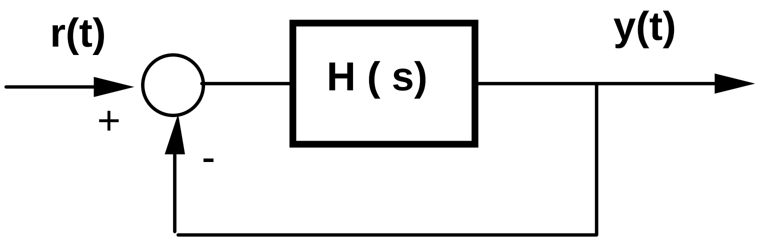


$$y(t) = H_{CL}(s)e^{st} = H_2(s)H_1(s)u(t); \;\; u(t) = r(t) - y(t)$$

$$y(t)\left[1 + H_2(s)H_1(s)\right] = H_2(s)H_1(s)r(t)$$

$$H_{CL}(s) = \frac{H_2(s)\,H_1(s)}{1 + H_2(s)\,H_1(s)}$$

Steady State Error



$$H(s) = \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n} = \frac{B(s)}{A(s)}$$

$$H_{CL}(s) = \frac{H(s)}{1 + H(s)} = \frac{B(s)}{A(s) + B(s)}$$

Steady State (static): r(t) = const. $s = 0$ ⟶

$$y = H_{CL}(0)\, r = \frac{B(0)}{A(0) + B(0)}\, r = \frac{b_0}{a_0 + b_0}\, r$$

Null Steady State Error $(y = r)$: $H_{CL}(0) = 1$ ⟶ $\frac{b_0}{a_0 + b_0} = 1 \Rightarrow a_0 = 0$

$a_0 = 0$ ⟶ $A(s) = s(a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}) = s.A'(s)$

$$H(s) = \boxed{\frac{1}{s}} \cdot \frac{B(s)}{A'(s)}$$

Internal Model Principle

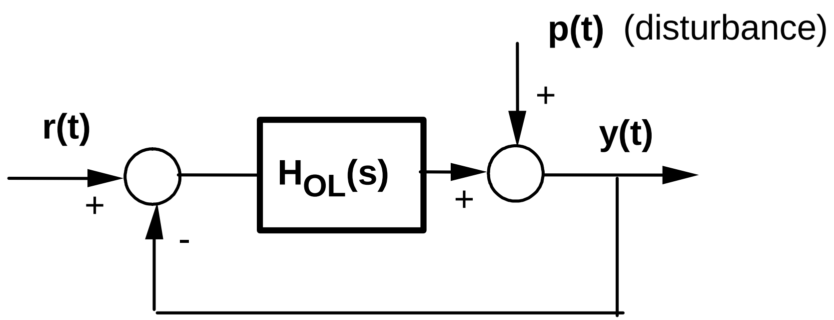
Null steady state error for a constant reference:
The T.F. of the direct channel must contain an ***integrator***

Rem.: step = output of an *integrator* for a Dirac pulse as input
integrator = internal model of the step

Internal Model Principle *:* In order to obtain a null steady state error, *H(s)* must contain the *internal model* of the reference *r(t)*

Internal Model of a signal $x(t)$ = T.F. of the filter that generates the signal $x(t)$ for a Dirac pulse as input

Rejection of Disturbances



T.F. disturbance/output:
(*sensitivity function*)

$$S_{yp}(s) = \frac{1}{1+H_{OL}(s)} = \frac{A(s)}{A(s)+B(s)}$$

Objective : attenuation of the effect of disturbances on the output at certain frequency regions

Typical case : cancellation of the effect of constant disturbances (step) in steady state $(t \to \infty, s \to 0)$

$$y = S_{yp}(0)\, p = \frac{A(0)}{A(0)+B(0)}\, p = \frac{a_0}{a_0+b_0}\, p \qquad y = 0 \longrightarrow a_0 = 0$$

An *integrator* is required in the direct channel

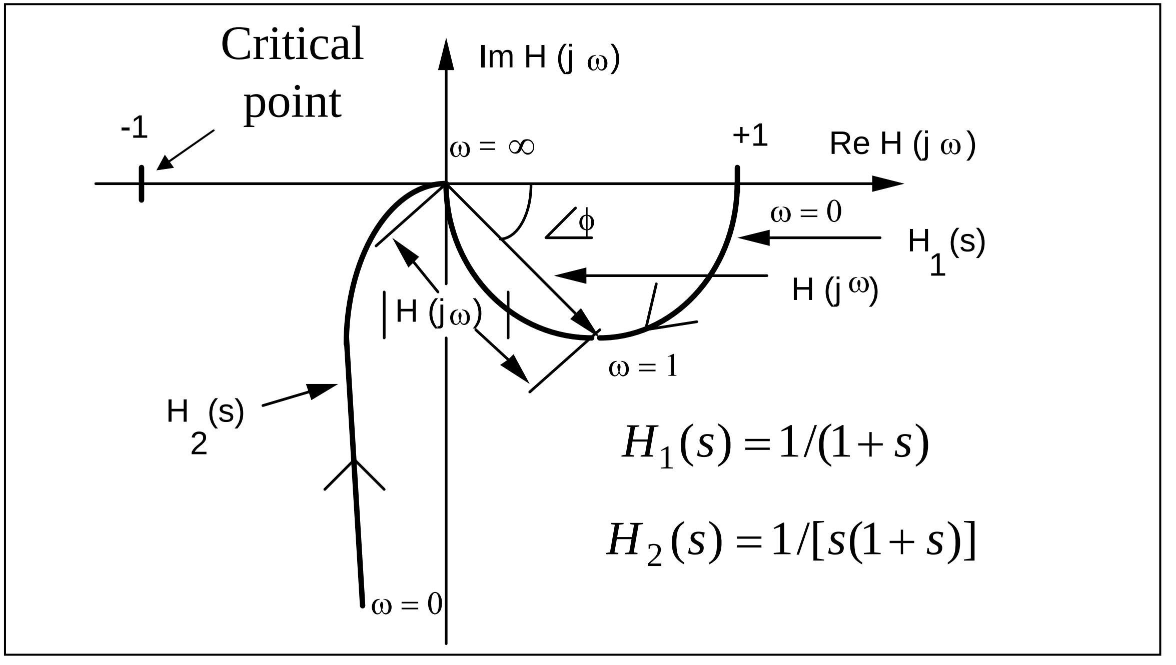
For a perfect rejection of a disturbance in steady state the direct channel must contain *the internal model of the disturbance*

Nyquit Plot and Stability Criterion

Objective:

Analysis of stability and robustness of closed loop systems

$$H_{CL}(j\omega) = \mathrm{Re}\, H_{CL}(j\omega) + j\, \mathrm{Im}\, H_{CL}(j\omega) = \left| H_{CL}(j\omega) \right| . \angle \phi(\omega)$$

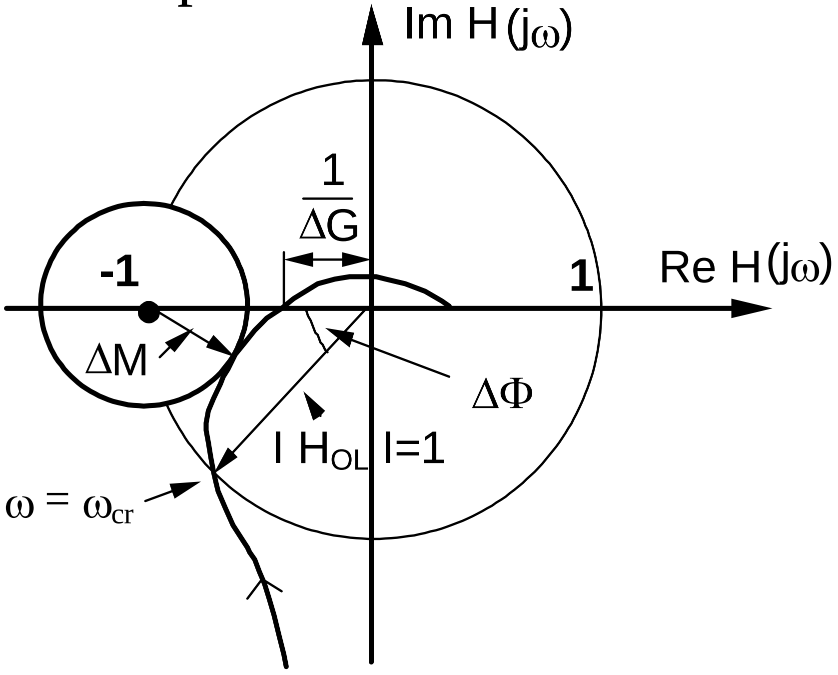


Stability Criterion:

The plot of $H_{OL}(j\omega)$ traversed in the sense of growing frequencies must leave the critical point on the left.

Robustness Margins

The *robustness* of the closed loop with respect to system parameters variations (or model uncertainties) is related to the minimal distance between the Nyquist plot for the nominal plant model and the "critical point"



- Gain Margin ΔG
- Phase Margin $\Delta \phi$
- **Delay Margin $\Delta \tau$**
- **Modulus Margin ΔM**

Robustness Margins

Gain Margin

$$\Delta G = \frac{1}{\left|H_{OL}(j\omega_{180})\right|} \qquad \text{for} \qquad \angle\phi(\omega_{180}) = -180^\circ$$

Phase Margin

$$\Delta\phi = 180^0 - \angle\phi(\omega_{cr}) \qquad \textit{for} \quad \left|H_{OL}(j\omega_{cr})\right| = 1$$

$$\Delta\phi = \min_i \Delta\phi_i$$

For several crossings with the unit circle

Delay Margin

$$\Delta\tau = \frac{\Delta\phi}{\omega_{cr}}$$

For several crossings:

$$\Delta\tau = \min_i \frac{\Delta\phi_i}{\omega_{cr}^i}$$

Modulus Margin

$$\Delta M = \left|1 + H_{OL}(j\omega)\right|_{\min} = \left|S_{yp}^{-1}(j\omega)\right|_{\min} = \left(\left|S_{yp}(j\omega)\right|_{\max}\right)^{-1}$$

Robustness Margins –typical values

Gain Margin : $\Delta G \geq 2$ (6 *dB*) [*min : 1,6* (4 *dB*)]

Phase Margin : $30° \leq \Delta\phi \leq 60°$

Delay Margin : fraction of the system delay (10%) or
of the rising time (10%)

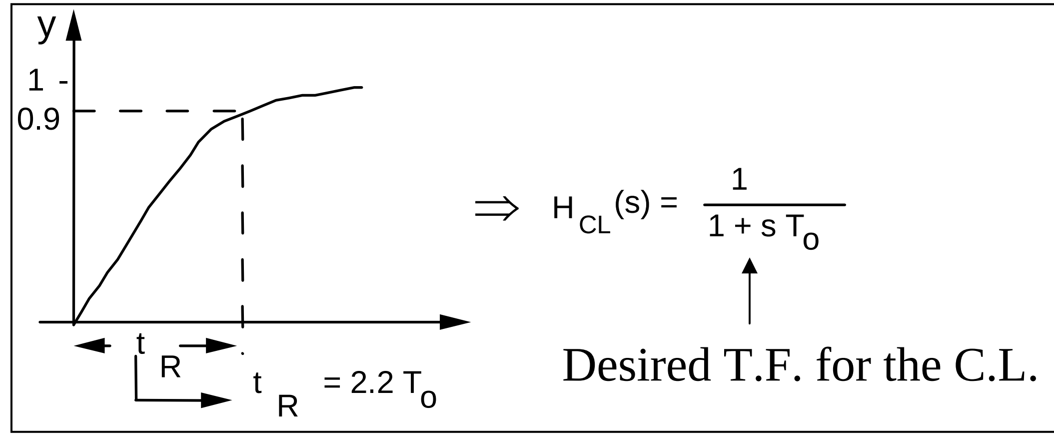
Modulus Margin : $\Delta M \geq 0.5$ (- 6 *dB*) [*min : 0,4* (-8 *dB*)]

A Modulus Margin $\Delta M \geq 0.5$ implique $\Delta G \geq 2$ *and* $\Delta\phi > 29°$
Attention: the converse is not true !

The Modulus Margin also defines the tolerance with respect to non-linearities (see pp.73-75)

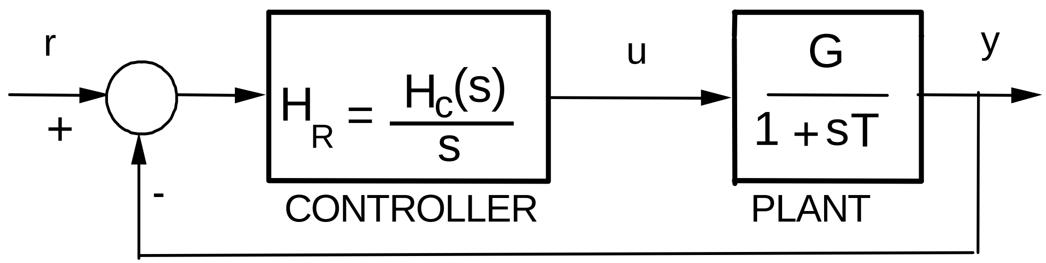
PI Controller

Plant : $G/(1+sT)$



Objectives :

1) Null steady state error
2) Rising time t_R



$$H_{CL}(s) = \frac{H_c(s)G}{G\,H_c(s) + s + s^2 T} = \frac{1}{1 + s\,T_0} = \frac{H_c(s)G}{H_c(s)G\left(1 + s\,T_0\right)}$$

$H_c(s)\ G\ s\ T_0 = s^2\ T + s$ ➞ $$H_c(s) = \frac{1}{G\,T_0}(1 + s\,T)$$

PI Controller

$$H_R(s) = \frac{\frac{1}{GT_0}(1+sT)}{s} = \frac{T}{GT_0}\left[1+\frac{1}{Ts}\right] = K\left[1+\frac{1}{T_i s}\right]$$

Proportional Gain (K) **Integral Action** (T_i)

Remark:

The controller parameters depend on the desired prformances (T_0) and on the plant transfer function parameters (G,T)

PID Controller

Several structures of the PID controller are possible.

For example, consider the structure :

proportional gain → K; integral action → $\frac{1}{T_i s}$; deravative action → $T_d s$; filtering on the deravative action → N

$$H_{PID}(s) = K\left(1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s}\right) \qquad (*)$$

Plant :

$$H(s) = \frac{G}{(1 + sT_1)(1 + sT_2)} = \frac{b_0}{1 + a_1 s + a_2 s^2}$$

Objectives:

1) t_R, M

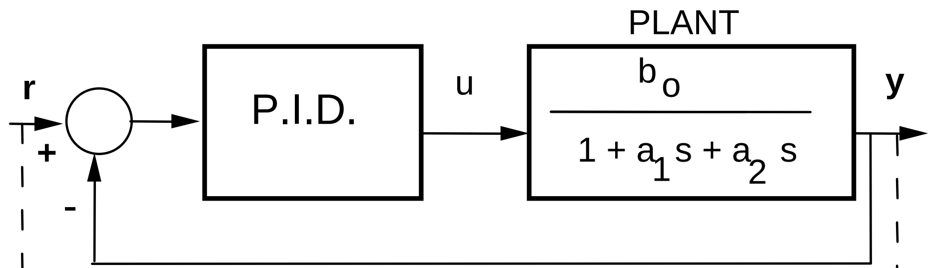
2) Null steady state error

See slide #18 →

$$H_{CL}(s) = \frac{\omega_0^2}{\omega_0^2 + 2\zeta\,\omega_0 s + s^2}$$

↑ Desired Closed Loop transfer function

PID Controller



$$H_{CL}(s) = \frac{\omega_o^2}{\omega_o^2 + 2\zeta\omega_o s + s^2}$$

$$H_{PID}(s) = \frac{K\left[1 + s\left(T_i + \frac{T_d}{N}\right) + s^2\left(T_i T_d + \frac{T_i T_d}{N}\right)\right]}{T_i s\left(1 + \frac{T_d}{N} s\right)} \qquad (*)$$

PID T.F. Numerator = Plant T.F. Denominator

PID Controller

$$H_{OL}(s) = H(s)\cdot H_{PID}(s) = \frac{Kb_0}{T_i s\left(1+\frac{T_d}{N}s\right)}$$

$$a_1 = T_i + \frac{T_d}{N};$$

$$a_2 = T_i T_d\left(1+\frac{1}{N}\right).$$

$$H_{CL}(s) = \frac{Kb_0}{Kb_0 + T_i s + \frac{T_i T_d}{N}s^2} = \frac{\frac{Kb_0 N}{T_i T_d}}{\frac{Kb_0 N}{T_i T_d} + \frac{N}{T_d}s + s^2} = \frac{\omega_0^2}{\omega_0^2 + 2\zeta\omega_0 s + s^2}$$

$$T_i = a_1 - \frac{T_d}{N} = a_1 - \frac{1}{2\zeta\omega_0} \qquad T_d = \frac{a_2}{T_i} - \frac{T_d}{N} = \frac{a_2}{T_i} - \frac{1}{2\zeta\,\omega_0} \qquad K = \frac{\omega_0 T_i}{2\zeta\, b_0} \qquad \frac{T_d}{N} = \frac{1}{2\zeta\omega_0}$$

The controller parameters depend on the desired prformances (ω_0 ,ζ) and on the plant transfer function parameters (a_1, a_2, b_0)

Concluding Rermarks

- The dynamics of a plant running around a specific operative point can be often described by a *linear dynamic model*.
-The linear dynamic systems are described by *linear differential equations* in the time domain and by *transfer functions* in the frequency domain.
- The control systems are closed loop systems containing: a controller, the plant (which contains the actuator and the sensor) and the *feedback loop*.
-The desired closed loop performances can be expressed by the desired (frequency) characteristics of the closed loop system.
- The Nyquist plot (frequency domain) plays a fundamental role for the closed system stability analysis and its robustness with respect to plant parameters variations.