Chapter I

Continuous Control Systems : A Review
Chapter 1. Continuous Control Systems : A Review

1.1 Continuous-time Models
  1.1.1 Time Domain
  1.1.2 Frequency Domain
  1.1.3 Stability
  1.1.4 Time Response
  1.1.5 Frequency Response
  1.1.6 Study of the Second-order System
  1.1.7 Systems with Time Delay
  1.1.8 Non-minimum Phase Systems

1.2 Closed-loop Systems
  1.2.1 Cascaded Systems
  1.2.2 Transfer Function of Closed-loop Systems
  1.2.3 Steady-state Error
  1.2.4 Rejection of Disturbances
  1.2.5 Analysis of Closed-loop Systems in the Frequency Domain: Nyquist Plot and Stability Criterion

1.3 PI and PID Controllers
  1.3.1 PI Controller
  1.3.2 PID Controller

1.4 Concluding Remarks

1.5 Notes and References
Continuous Time Models

Time Domain

\[ \frac{dy}{dt} = -\frac{1}{T} y(t) + \frac{G}{T} u(t) \quad (*) \]

Obs.: \[ p = \frac{d}{dt} \]

\[ (p + \frac{1}{T}) y(t) = \frac{G}{T} u(t); \quad (*) \]

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
**Continuous Time Models**

**Frequency Domain**

\[ u(t) = \text{periodic input} \]

\[ u(t) = e^{j\omega t} \]

\[ y(t) = H(j\omega)e^{j\omega t} \]

\[ y(t) = H(s)e^{st} \]

\[ s = \sigma + j\omega \]

\[ s = \text{complex frequency} \]

\[ y(t) = H(s)e^{st} \]

\[ \frac{dy(t)}{dt} = sH(s)e^{st} \]

\[ (s + \frac{1}{T})H(s)e^{st} = \frac{G}{T}e^{st} \quad (\ast) \]

\[ H(s) = \frac{G}{1 + sT} \]

The transfer function can also be obtained by:
- replacing « \( p \) » with « \( s \) » in (\ast) (see slide #3)
- Laplace transform

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
Stability

Ex.: 1st order system

\[ \frac{dy}{dt} = -\frac{1}{T} y(t) + \frac{G}{T} u(t) \quad \longleftrightarrow \quad \text{T.F. : } H(s) = \frac{G}{1 + sT} \]

Unforced response (u=0):

\[ \frac{dy}{dt} + \frac{1}{T} y(t) = 0 ; y(0) = y_0 \]

Solution:

\[ y(t) = Ke^{st} \quad \frac{dy}{dt} = sKe^{st} \]

\[ Ke^{st} \left( s + \frac{1}{T} \right) = 0 \quad \rightarrow \quad s = -\frac{1}{T} ; K = y_0 \quad \rightarrow \quad y(t) = y_0 e^{-t/T} \]

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
The stability (or instability) of a system depends on the roots of the transfer function denominator:

**Stability (asymptotically):** for all roots \( s_i \) it holds \( \text{Re} \ s_i < 0 \)

**Instability:** there is at least one root \( s \) for which \( \text{Re} \ s > 0 \)

\( \text{Re} \ s = 0 \) : boundary of stability

Stability criteria are available for proving the existence of unstable roots without any explicit calculation of roots. (ex: Routh – Hurwitz criteria)
**Time Responses**

Input: *unit step*

- $t_R$ – rising time
- $t_S$ – settling time (+/- tolerance)
- FV – final value
- M – max. overshoot (% FV)

Ex: 1st order

$$H(s) = \frac{G}{1 + s T}$$

FV = $G$ (*static gain*);

- $t_R = 2.2T$;
- $t_S = 2.2T$ (+/-10%);
- $M = 0$
Frequency Responses

\[ |H(j\omega)| \text{dB} = 20 \log |H(j\omega)| \]
\[ \omega (\text{rad/s}) = 2\pi f(\text{Hz}) \]

**Slop e**: It depends on the number of poles and zeros and on their frequency distribution.

\[ \frac{\Delta G}{\Delta \omega} = -(n - m) \times 20 \text{dB/dec} \]
\[ n = \# \text{poles}; m = \# \text{zeros} \]

\(-f_{BP}(\omega_{BP}) \) (**bandwidth**): the frequency (radian frequency) from which the zero-frequency (steady-state) gain \( G(0) \) is attenuated more than 3 dB.

\( G(\omega_{BP}) = G(0) - 3dB; \quad (G(\omega_{BP}) = 0.707 G(0)) \)

\(-f_{C}(\omega_{C}) \) (**cut-off frequency**): the frequency (rad/s) from which the attenuation introduced with respect to the zero frequency is greater than \( N \text{dB} \).

\( G(j\omega_{C}) = G(0) - N \text{dB} \)

\(-Q \) (**resonance factor**): the ratio between the gain corresponding to the maximum of the frequency response curve and the value \( G(0) \).

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
1st Order System Frequency Response

\[ H(j\omega) = \frac{G}{1 + j\omega T} = |H(j\omega)|e^{j\phi(\omega)} = |H(j\omega)|\angle\phi(\omega) \]

\[ G(\omega) = |H(j\omega)| = \frac{G}{\sqrt{1 + (\omega T)^2}} ; \quad \angle\phi(\omega) = \tan^{-1}\left[ \frac{\text{Im} G(j\omega)}{\text{Re} G(j\omega)} \right] = \tan^{-1}[-\omega T] \]
Second Order System Analysis

\[ \frac{d^2 y(t)}{dt^2} + 2\zeta\omega_0 \frac{dy(t)}{dt} + \omega_0^2 y(t) = \omega_0^2 u(t) \]

T.F.: \[ H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \]

\( \omega_0 \) : natural frequency (\( \omega_0 = 2\pi f_0 \))
\( \zeta \) : damping factor

\(|\zeta| < 1\) : complex poles (oscillatory response). \[ s_{1,2} = -\zeta\omega_0 \pm j\omega_0 \sqrt{1 - \zeta^2} \]

\(|\zeta| > 1\) : real poles (aperiodic response). \[ s_{1,2} = -\zeta\omega_0 \pm \omega_0 \sqrt{\zeta^2 - 1} \]

\(\zeta > 0\) : asymptotically stable system
\(\zeta < 0\) : unstable system

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
Second Order System – Normalized Time Responses

\[ \omega_0 t_M = \text{normalized time response} \]

**What choice for \( \omega_0 \) and \( \zeta \)?**

\[ M_{\text{desired}} \rightarrow \zeta(\text{diagr } a) \rightarrow \omega_0 t_M (\text{diagr } b) \]

\[ \omega_0 = \frac{(\omega_0 t_M)}{(t_M)_{\text{desired}}} \]

- Use of functions \texttt{omega_dmp.sci(m)}

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
Second Order System – Normalized Frequency Responses

2nd Order Systems : Bode Magnitude Diagram

- Magnitude (dB)
- Frequency ($\omega / \omega_0$)

$\zeta = 0.1$, $\zeta = 0.9$

$\zeta = 0.1, 0.3, 0.5, 0.7, 0.9$
The time delay does not modify the system gain but it introduces a phase shift proportional to the frequency

\[ H_{delay}(j \omega) = e^{-j\omega\tau} = 1 \left\langle \phi(\omega) \right\rangle \quad \text{with} \quad \left\langle \phi(\omega) \right\rangle = -\omega \tau \text{ (rad)} \]
Non-minimum Phase Systems

For Continuous-time Systems (only) → one or more \textit{unstable zeros}

\[ H(s) = \frac{1 - sa}{(1 + s)(1 + 0.5s)} \]
**Closed Loop Systems**

\[ \begin{align*}
&\text{Controller} \\
&\text{Plant} \\
&\text{Controller} + \text{Plant} \\
&u(t) \quad r(t) \quad y(t) \\
&\text{Controller} - \text{Plant} \\
\end{align*} \]

**Cascaded Systems**

\[ y_1(t) = H_1(s) e^{st} = u_2 \]

\[ u_1(t) = e^{st} \]

\[ y_2(t) = H_2(s)u_2(t) = H_2(s)H_1(s)u_1(t) = H(s)e^{st} \]

\[ H(s) = H_n(s)H_{n-1}(s)\ldots H_2(s)H_1(s) \]

---

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
Closed Loop Systems

\[ r(t) = e^{st} \]

\[ u(t) \]

\[ H_1(s) \]

\[ H_2(s) \]

\[ y(t) = H_{CL}(s) e^{st} \]

\[ y(t) = H_2(s) H_1(s) u(t); \quad u(t) = r(t) - y(t) \]

\[ y(t)[1 + H_2(s) H_1(s)] = H_2(s) H_1(s) r(t) \]

\[ H_{CL}(s) = \frac{H_2(s) H_1(s)}{1 + H_2(s) H_1(s)} \]
Steady State Error

Steady State (static): \( r(t) = \text{const.} \quad s = 0 \quad \rightarrow \quad y = H_{CL}(0) r = \frac{B(0)}{A(0) + B(0)} r = \frac{b_0}{a_0 + b_0} r \)

Null Steady State Error \((y = r) : H_{CL}(0) = \mathbf{1} \quad \rightarrow \quad \frac{b_0}{a_0 + b_0} = 1 \quad \Rightarrow \quad a_0 = 0 \)

\[ a_0 = 0 \quad \rightarrow \quad A(s) = s(a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}) = s. A'(s) \]

\[ H(s) = \frac{1}{s} \cdot \frac{B(s)}{A'(s)} \]
Internal Model Principle

Null steady state error for a constant reference:
*The T.F. of the direct channel must contain an integrator*

Rem.: step = output of an integrator for a Dirac pulse as input
integrator = internal model of the step

*Internal Model Principle*: In order to obtain a null steady state error, $H(s)$ must contain the internal model of the reference $r(t)$

*Internal Model* of a signal $x(t) = \text{T.F. of the filter that generates the signal } x(t) \text{ for a Dirac pulse as input}
Rejection of Disturbances

T.F. disturbance/output: 
(sensitivity function)

\[ S_{yp}(s) = \frac{1}{1 + H_{OL}(s)} = \frac{A(s)}{A(s) + B(s)} \]

**Objective**: attenuation of the effect of disturbances on the output at certain frequency regions

**Typical case**: cancellation of the effect of constant disturbances (step) in steady state \((t \to \infty, s \to 0)\)

\[ y = S_{yp}(0)p = \frac{A(0)}{A(0) + B(0)} p = \frac{a_0}{a_0 + b_0} p \quad y = 0 \quad \Rightarrow \quad a_0 = 0 \]

An integrator is required in the direct channel

For a perfect rejection of a disturbance in steady state the direct channel must contain *the internal model of the disturbance*
Nyquist Plot and Stability Criterion

Objective:
Analysis of stability and robustness of closed loop systems

\[ H_{CL}(j\omega) = \text{Re} \, H_{CL}(j\omega) + j \, \text{Im} \, H_{CL}(j\omega) = |H_{CL}(j\omega)| \angle \phi(\omega) \]

Stability Criterion:
The plot of \( H_{OL}(j\omega) \) traversed in the sense of growing frequencies must leave the critical point on the left.

\[ H_1(s) = \frac{1}{1+s} \]
\[ H_2(s) = \frac{1}{s(1+s)} \]
The robustness of the closed loop with respect to system parameters variations (or model uncertainties) is related to the minimal distance between the Nyquist plot for the nominal plant model and the “critical point”

- Gain Margin $\Delta G$
- Phase Margin $\Delta \phi$
- Delay Margin $\Delta \tau$
- Modulus Margin $\Delta M$
Robustness Margins

Gain Margin
\[ \Delta G = \frac{1}{|H_{OL}(j\omega_{180})|} \text{ for } \angle \phi(\omega_{180}) = -180^\circ \]

Phase Margin
\[ \Delta \phi = 180^\circ - \angle \phi(\omega_{cr}) \text{ for } |H_{OL}(j\omega_{cr})| = 1 \]
\[ \Delta \phi = \min_{i} \Delta \phi_i \text{ For several crossings with the unit circle} \]

Delay Margin
\[ \Delta \tau = \frac{\Delta \phi}{\omega_{cr}} \text{ For several crossings: } \Delta \tau = \min_{i} \frac{\Delta \phi_i}{\omega_{cr}} \]

Modulus Margin
\[ \Delta M = |1 + H_{OL}(j\omega)|_{\text{min}} = |S_{yp}^{-1}(j\omega)|_{\text{min}} = \left( |S_{yp}(j\omega)|_{\text{max}} \right)^{-1} \]
Robustness Margins – typical values

Gain Margin: \( \Delta G \geq 2 \) (6\,dB) \[ \text{min : 1,6 (4\,dB)} \]

Phase Margin: \( 30^\circ \leq \Delta \phi \leq 60^\circ \)

Delay Margin: fraction of the system delay (10%) or of the rising time (10%)

Modulus Margin: \( \Delta M \geq 0.5 \) (-6\,dB) \[ \text{min : 0,4 (-8\,dB)} \]

A Modulus Margin \( \Delta M \geq 0.5 \) implique \( \Delta G \geq 2 \) and \( \Delta \phi > 29^\circ \)
Attention: the converse is not true!

The Modulus Margin also defines the tolerance with respect to non-linearities (see pp.73-75)
PI Controller

Plant: \[ G/(1+sT) \]

Objectives:
1) Null steady state error
2) Rising time \( t_R \)

\[ H_{CL}(s) = \frac{1}{1+sT_0} \]

Desired T.F. for the C.L.

\[ H_{CL}(s) = \frac{H_c(s)G}{G + H_c(s)s + s^2T} = \frac{1}{1+sT_0} = \frac{H_c(s)G}{H_c(s)G(1+sT_0)} = \frac{H_c(s)G}{s^2T + s} \]

\[ H_c(s)G s T_0 = s^2T + s \quad \Rightarrow H_c(s) = \frac{1}{GT_0}(1+sT) \]
**PI Controller**

\[
H_R(s) = \frac{1}{s} \left(1 + sT\right) = \frac{T}{GT_0} \left[1 + \frac{1}{Ts}\right] = K \left[1 + \frac{1}{Ti s}\right]
\]

**Remark:**
The controller parameters depend on the desired performances \(T_0\) and on the plant transfer function parameters \((G, T)\)
Several structures of the PID controller are possible. For example, consider the structure:

\[
H_{PID}(s) = K \left( 1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s} \right)
\]  

(\*)

Plant:

\[
H(s) = \frac{G}{(1 + s T_1)(1 + s T_2)} = \frac{b_0}{1 + a_1 s + a_2 s^2}
\]

Objectives:

1) \(t_R, M\) \(\longrightarrow\) See slide #18 \(\rightarrow\) \(H_{CL}(s) = \frac{\omega_0^2}{\omega_0^2 + 2\zeta \omega_0 s + s^2}\)

2) Null steady state error

Desired Closed Loop transfer function
PID Controller

\[ H_{CL}(s) = \frac{\omega_o^2}{\omega_o^2 + 2\zeta \omega_o s + s^2} \]

\[ H_{PID}(s) = K \left[ 1 + s \left( T_i + \frac{T_d}{N} \right) + s^2 \left( T_i T_d + \frac{T_i T_d}{N} \right) \right] \]

PID T.F. Numerator = Plant T.F. Denominator

I.D. Landau, G. Zito - "Digital Control Systems" - Chapter 1
The controller parameters depend on the desired performances $(\omega_0, \zeta)$ and on the plant transfer function parameters $(a_1, a_2, b_0)$
Concluding Remarks

- The dynamics of a plant running around a specific operative point can be often described by a *linear dynamic model*.
- The linear dynamic systems are described by *linear differential equations* in the time domain and by *transfer functions* in the frequency domain.
- The control systems are closed loop systems containing: a controller, the plant (which contains the actuator and the sensor) and the *feedback loop*.
- The desired closed loop performances can be expressed by the desired (frequency) characteristics of the closed loop system.
- The Nyquist plot (frequency domain) plays a fundamental role for the closed system stability analysis and its robustness with respect to plant parameters variations.