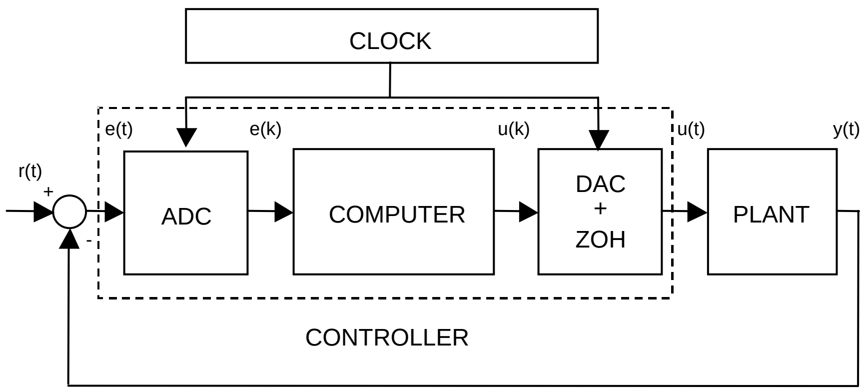
Chapter II

Computer Control Systems

Chapter 2. Computer Control Systems

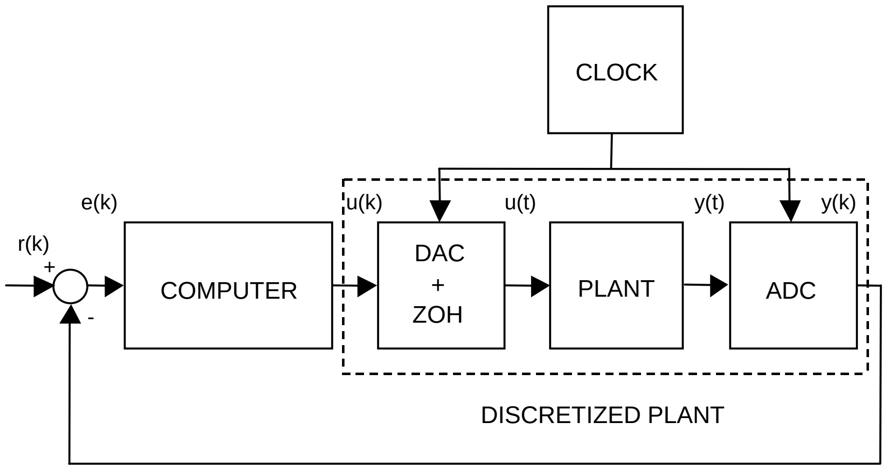
Computer control system

Digital implementation of an analog controller



- Continuous time emulation by fast sampling
- Strong constraint on computer resources

Digital control system



- Sampling time depends on the system bandwidth
- Efficient use of computer resources

Choice of the sampling period for control purposes

$$f_{BW}^{CL} : closed\ loop\ bandwidth$$

$$f_s = (6\ up\ to\ 25)\ f_{BW}^{CL}$$

1st order : $H(s) = \frac{1}{1 + sT_0}$ $\qquad f_{BW} = f_0 = \frac{1}{2\pi T_0}$

$T_s = 1/f_s$: sampling period

$$\frac{T_0}{4} < T_s < T_0$$

2nd order : $H(s) = \frac{\omega_0^2}{\omega_0^2 + 2\zeta\omega_0 s + s^2}$

$$\zeta = 0.7 \Rightarrow f_{BW} = \frac{\omega_0}{2\pi}$$

$$\zeta = 1 \Rightarrow f_{BW} = \frac{0.6\,\omega_0}{2\pi}$$

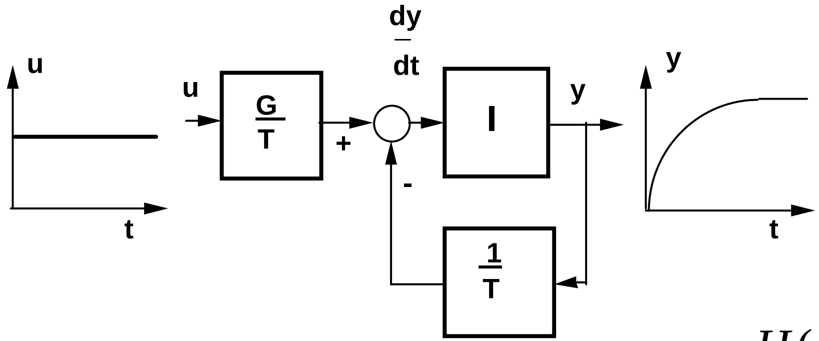
$0.25 \le \omega_0 T_s \le 1 ; \zeta = 0.7$

$0.4 \le \omega_0 T_s \le 1.75 ; \zeta = 1$

$$0.25 \le \omega_0 T_s \le 1.5 ; 0.7 \le \zeta \le 1$$

Discrete-time models

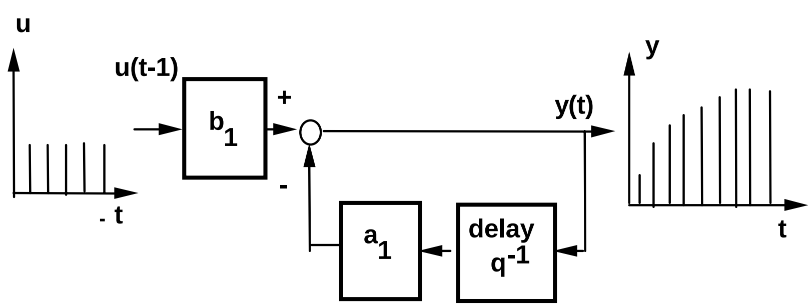
Remind : Continuous-time model



$$\frac{dy}{dt} = -\frac{1}{T}y(t) + \frac{G}{T}u(t)$$

$$H(s) = \frac{G}{1 + sT}$$

Discrete-time model



t = normalized discrete time (t/T_s)

$$y(t) = -a_1 y(t-1) + b_1 u(t-1)$$

$$H(z^{-1}) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}$$

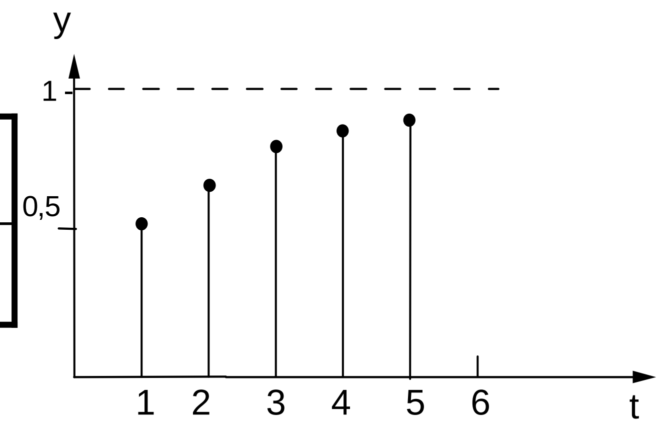
Discrete-time models: *Time domain*

t = normalized discrete time (t/Ts)   t = 1, 2, 3, 4, ……

$$y(t) = -a_1 y(t-1) + b_1 u(t-1) \qquad u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \qquad y(0) = 0$$

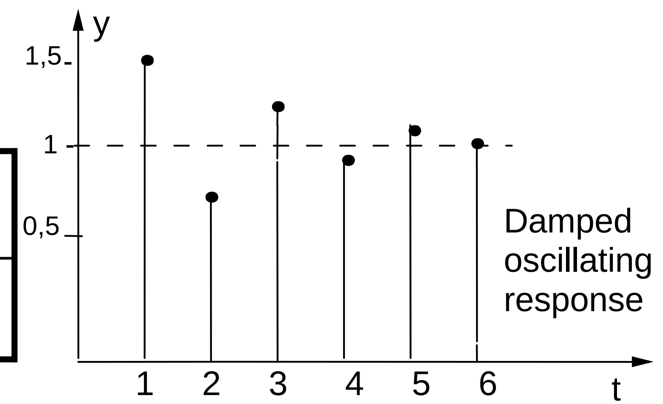
Case 1 : $a_1 = -0.5 \quad b_1 = 0.5$

t	0	1	2	3	4	5
y(t)	0	0.5	0.75	0.875	0.937	0.969



Case 2 : $a_1 = 0.5 \quad b_1 = 1.5$

t	0	1	2	3	4	5
y(t)	0	1.5	0.75	1.125	0.937	1.062



Backward shift operator (q^{-1})

$$q^{-1}y(t) = y(t-1)\ ; \quad q^{-d}y(t) = y(t-d)$$

$$y(t) = -a_1 y(t-1) + b_1 u(t-1) \longrightarrow (1 + a_1 q^{-1})y(t) = b_1 q^{-1}u(t)$$

Pulse transfer operator

$$(1 + a_1 q^{-1})y(t) = b_1 q^{-1}u(t)$$

$$\downarrow$$

$$y(t) = \frac{b_1 q^{-1}}{1 + a_1 q^{-1}} u(t) = H(q^{-1})u(t)$$

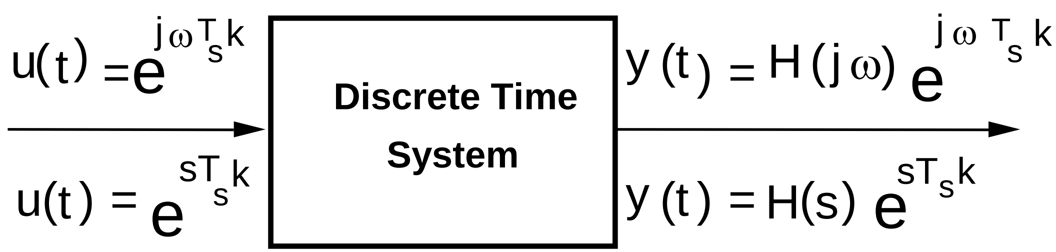
$H(q^{-1})$: Pulse transfer operator

Discrete time model : *Frequency domain*

Exponential complex function (continue time) $\begin{cases} e^{j\omega t} = \cos\omega t + j\sin\omega t \\ e^{st} \quad (s = \sigma + j\omega) \end{cases}$

Exponential complex function (discrete time, T_s: sampling period)

$t = kT_s$ $\qquad$ $\boxed{e^{j\omega T_s k} \;;\quad e^{sT_s k}}$ $\qquad$ $k = 1, 2, 3, ..$



$$y(t) = H(s)e^{sT_s k} \qquad y(t-1) = H(s)e^{sT_s(k-1)} = e^{-sT_e}H(s)e^{sT_s k} = e^{-sT_s}y(t)$$

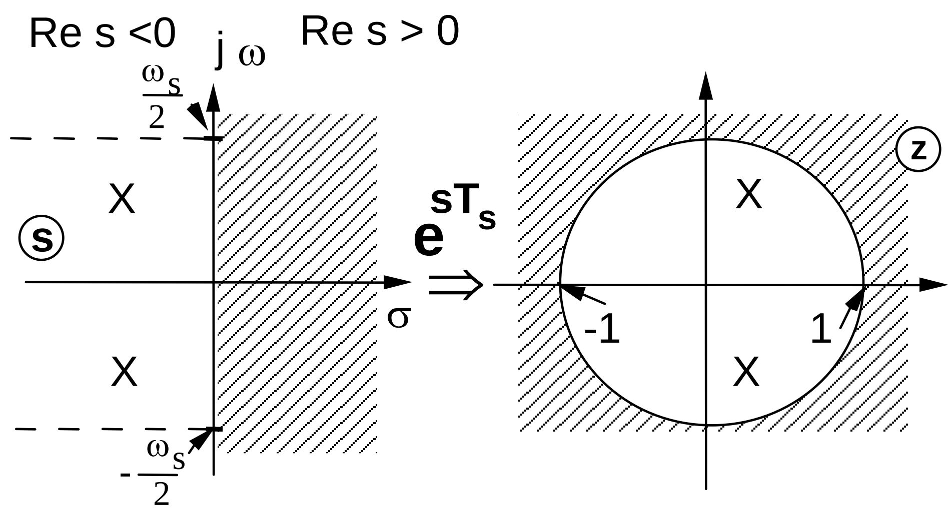
Discrete time model : *Frequency domain*

Exemple:

$$y(t) = -a_1 y(t-1) + b_1 u(t-1) \qquad u(t) = e^{sT_e k}$$

$$(1 + a_1 e^{-sT_s}) H(s) e^{sT_s k} = b_1 e^{-sT_s} e^{sT_s k} \longrightarrow H(s) = \frac{b_1 e^{-sT_s}}{1 + a_1 e^{-sT_s}}$$

$$z = e^{sT_s}$$



Transfer function: $$H(z^{-1}) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}$$

Discrete time model – *General form*

$$(*)\quad y(t) = -\sum_{i=1}^{n_A} a_i\, y(t-i) + \sum_{i=1}^{n_B} b_i\, u(t-d-i)$$

d –delay (integer multiple of the sampling period)

$$1 + \sum_{i=1}^{n_A} a_i q^{-i} = A(q^{-1}) = 1 + q^{-1} A^*(q^{-1}) \; ; \quad A^*(q^{-1}) = a_1 + a_2 q^{-1} + \dots + a_{n_A} q^{-n_A+1}$$

$$\sum_{i=1}^{n_B} b_i q^{-i} = B(q^{-1}) = q^{-1} B^*(q^{-1}) \; ; \quad B^*(q^{-1}) = b_1 + b_2 q^{-1} + \dots + b_{n_B} q^{-n_B+1}$$

$$(*)\; A(q^{-1}) y(t) = q^{-d} B(q^{-1}) u(t)$$

$$(*)\; A(q^{-1}) y(t+d) = B(q^{-1}) u(t)$$ (Predictive form)

$$(*)\; y(t) = H(q^{-1}) u(t) \; ; \quad H(q^{-1}) = \frac{q^{-d} B(q^{-1})}{A(q^{-1})}$$ - pulse transfer operator

$$q^{-1} \to z^{-1} \qquad H(z^{-1}) = \frac{q^{-z} B(z^{-1})}{A(z^{-1})}$$ - transfer function

Stability of discrete time models

Time domain

$$y(t) = -a_1 y(t-1)\,; \quad y(0) = y_0$$

$$y(1) = -a_1 y_0\,;\; y(2) = (-a_1)^2 y_0\,;\; y(t) = (-a_1)^t y_0$$

$$\lim_{t\to\infty} y(t) = 0 \quad \Rightarrow \quad |a_1| < 1$$

Pulse transfer operator (function)

$$(1 + a_1 q^{-1}) y(t) = 0$$

$1 + a_1 q^{-1}$ Is the denominator of the pulse transfer operator

$1 + a_1 z^{-1}$ Is the denominator of the transfer function

Stability condition: $1 + a_1 q^{-1} (or\ z^{-1}) = 0 \Rightarrow |q(or\ z)| < 1$

Stability condition: general case

$$1 + a_1 z^{-1} + a_2 z^{-2} ..... + a_n z^{-n} = 0 \Rightarrow |z_i| < 1 \ \ for \ \ i = 1, 2,..\, n$$

Steady state (static gain)

$$\textit{Continuous time}: \quad s = 0 \Rightarrow \textit{Discrete time}: z = e^{sT_s} = 1$$

$$G(0) = \left. \frac{b_1 z^{-1}}{1 + a_1 z^{-1}} \right|_{z=1} = \frac{b_1}{1 + a_1}$$

General case:

$$G(0) = H(1) = H(z^{-1})\Big|_{z=1} = \left. \frac{z^{-d} B(z^{-1})}{A(z^{-1})} \right|_{z=1} = \frac{\sum_{i=1}^{n_B} b_i}{1 + \sum_{i=1}^{n_A} a_i}$$

First order systems with delay

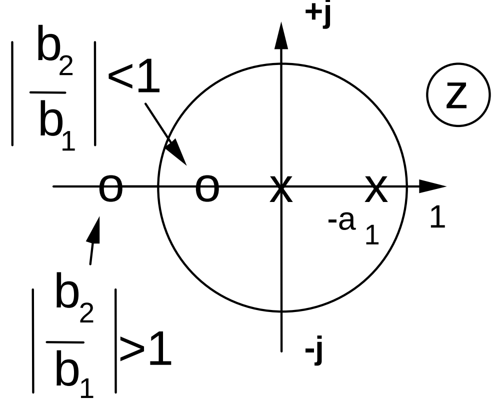
Fractional delay

Continuous time model $H(s) = \frac{G e^{-s\tau}}{1+T_s}$ $\tau = d.T_s + L\ ;\ 0 < L < T_s$

Discrete time model $H(z^{-1}) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2})}{1 + a_1 z^{-1}} = \frac{z^{-d-1}(b_1 + b_2 z^{-1})}{1 + a_1 z^{-1}}$

$a_1 = -e^{-\frac{T_s}{T}}$ $b_1 = G(1 - e^{\frac{L-T_s}{T}})$ $b_2 = Ge^{-\frac{T_s}{T}}(e^{\frac{L}{T}} - 1)$

Remark: For $L > 0.5T_s \Rightarrow b_2 > b_1 \Rightarrow unstable\ \ zero\ (\left|-\frac{b_2}{b_1}\right| > 1)$



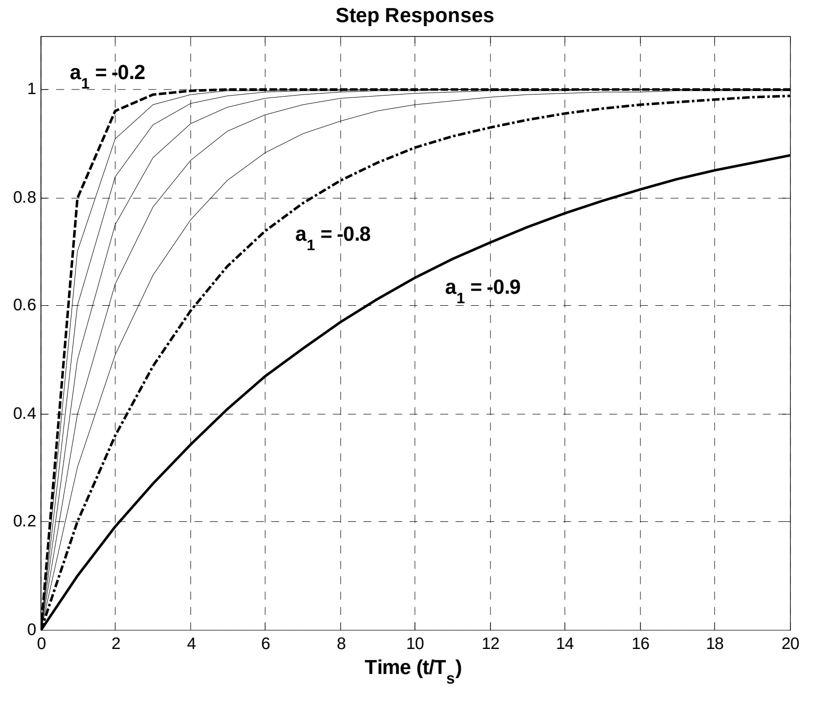
o - zero
x - pole

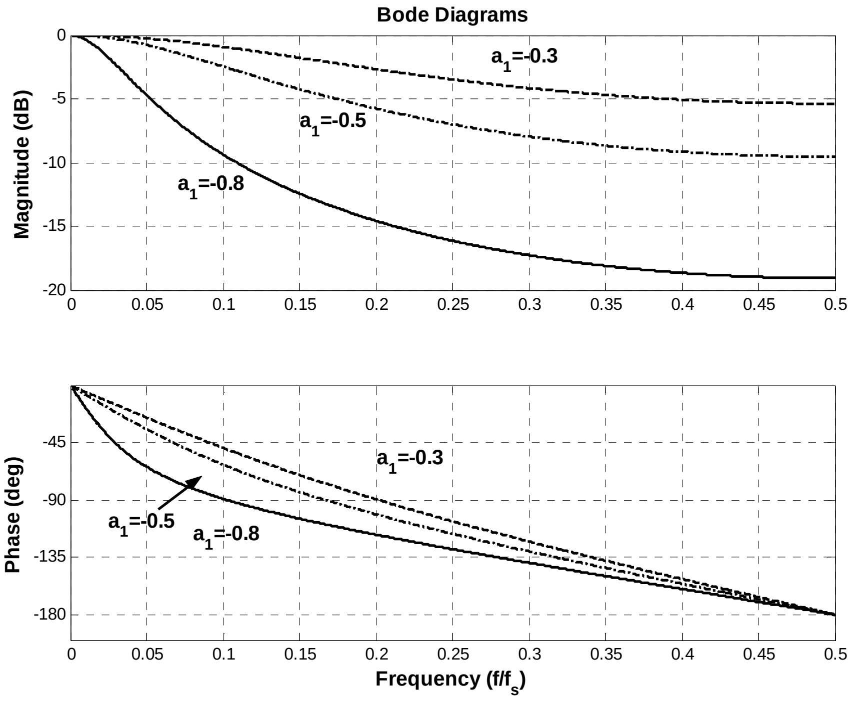
First order discrete time systems

$$H(z^{-1}) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}$$

Time responses

Frequency responses



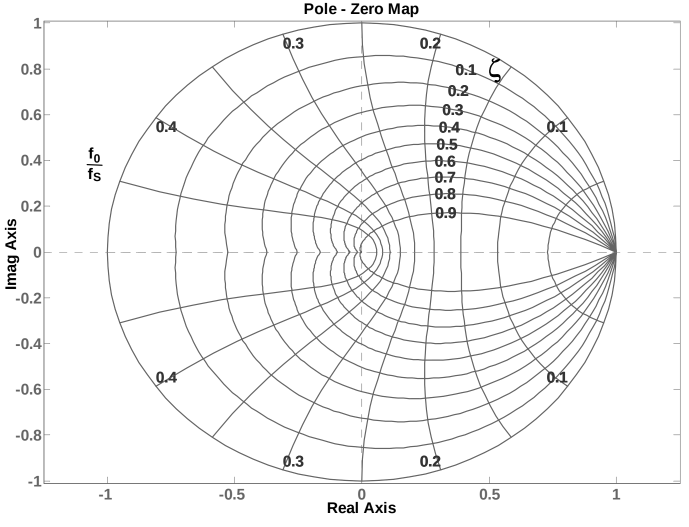


Second order discrete time systems

Discretization of 2nd order continuous time system with time delay

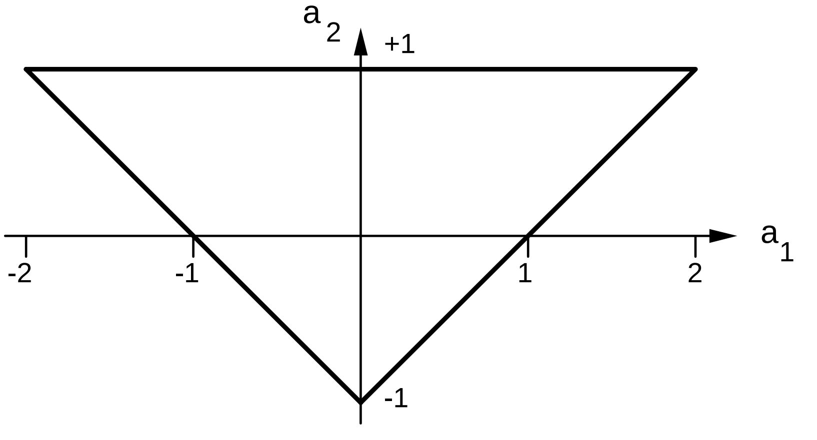
$$H(z^{-1}) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2})}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Poles: $z_{1,2} = e^{-\zeta \omega_0 T_s \pm j\omega_0 \sqrt{1-\zeta^2} T_s}$ $(continuous\ poles\ obtained\ by: e^{sT_s})$



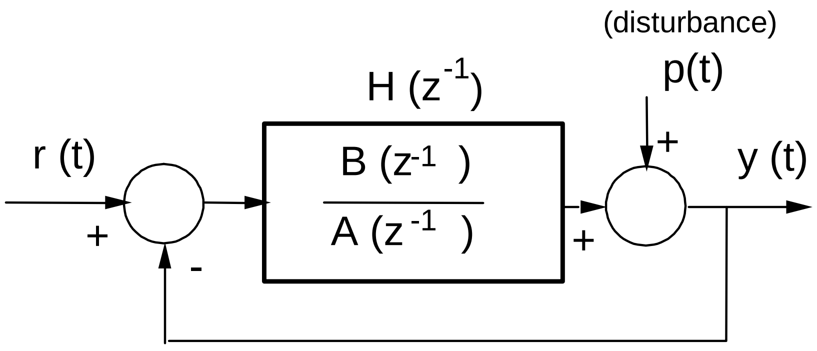
The curves for ζ= const and $\omega_0 T_s / 2\pi = f_0 / f_s = constant$

in the z-plane for a second-order discrete-time system



Stability domain for a_1 and a_2

Disturbances rejection



T.F. $p \longrightarrow y : S_{yp}$

S_{yp}: *Output Sensitivity function*

$$S_{yp}(z^{-1}) = \frac{1}{1 + H(z^{-1})} = \frac{A(z^{-1})}{A(z^{-1}) + B(z^{-1})}$$

Steady state ($z = 1$):

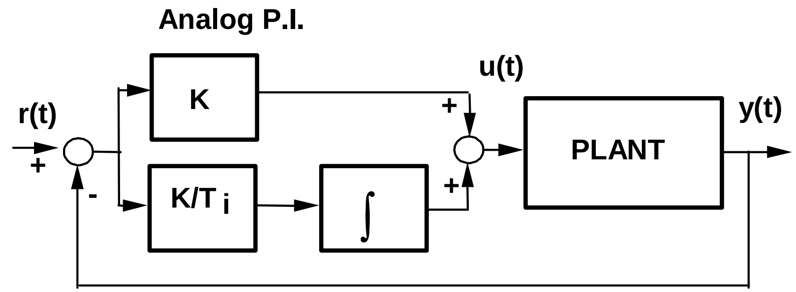
$$y = S_{yp}(1)\, p = \frac{A(1)}{A(1) + B(1)}\, p$$

Perfect rejection of constant disturbances : A(1) = 0

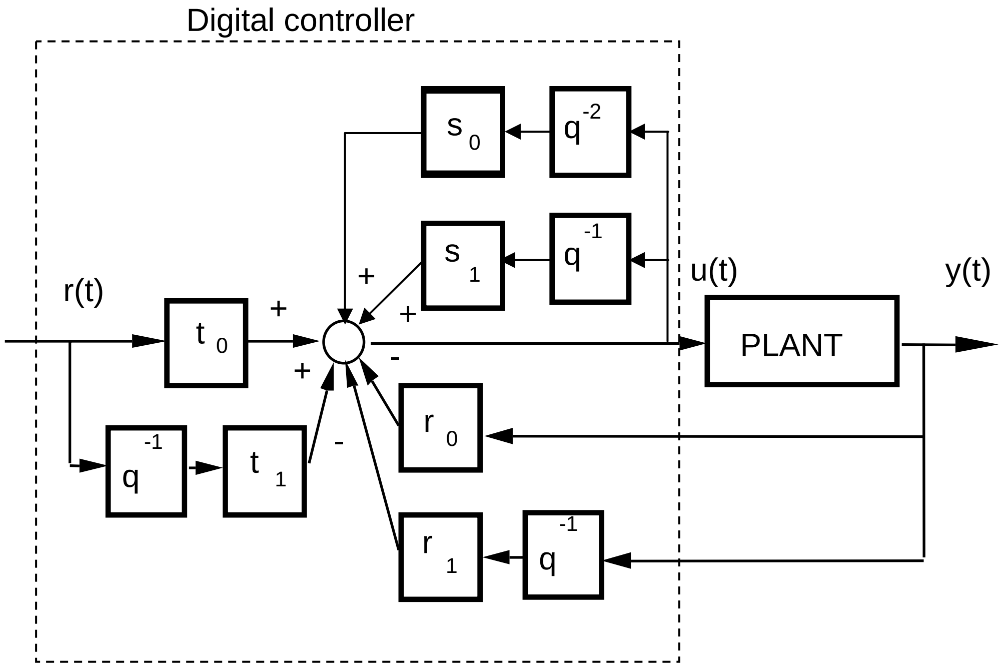
An integrator in the feedforward channel is required

Digital controller structure

Analog (PI) controller

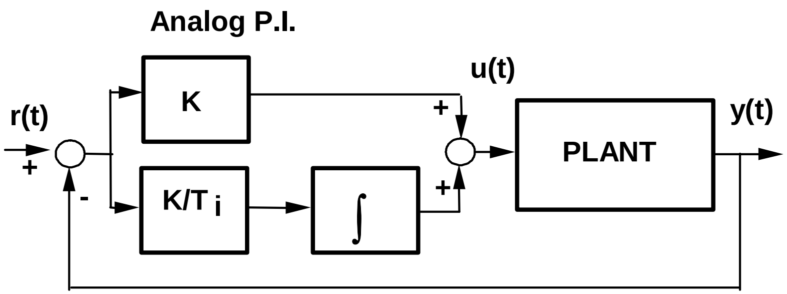


Digital controller



The control is a weighted average of the measured output at instants *t*, *t-1*,...., $t\text{-}n_A$ .., of the previous control values at instants *t-1*, *t-1*…, $t\text{-}n_B$… and of the reference signal at instants *t*, *t-1*,…, the weights being the coefficients of the controller.

Digital PI controller 1/2



$$u(t) = K\left[1 + \frac{1}{pT_i}\right][r(t) - y(t)]$$

Discretization: $\frac{dx}{dt} = px \approx \frac{x(t) - x(t-1)}{T_s} = \frac{1 - q^{-1}}{T_s} x(t)$; $\int x dt = \frac{1}{p} x \approx \left[\frac{T_s}{1 - q^{-1}}\right] x(t)$

The p operator is replaced by $(1 - q^{-1})/T_s$

Digital PI : $$u(t) = \frac{K(1 - q^{-1}) + \frac{KT_s}{T_i}}{1 - q^{-1}}[r(t) - y(t)]$$

After multiplying by $(1 - q^{-1})$: $$S(q^{-1})u(t) = T(q^{-1})r(t) - R(q^{-1})y(t)$$

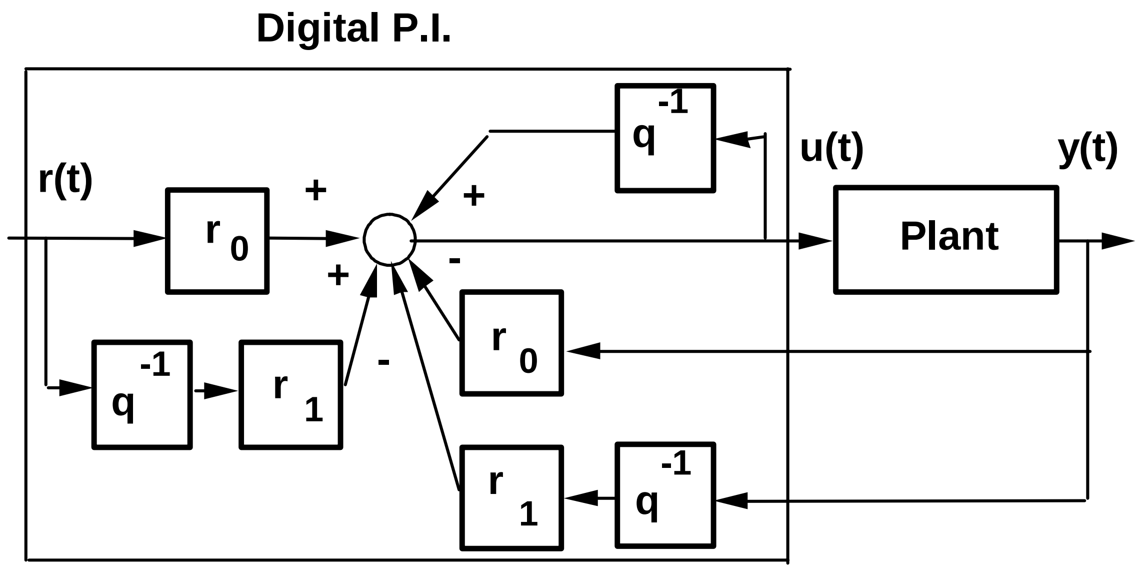
$$S(q^{-1}) = 1 - q^{-1} = 1 + s_1 q^{-1} ; \quad (s_1 = -1)$$

$$T(q^{-1}) = K(1 + T_s/T_i) - Kq^{-1} = r_0 + r_1 q^{-1}$$

Digital PI controller 2/2

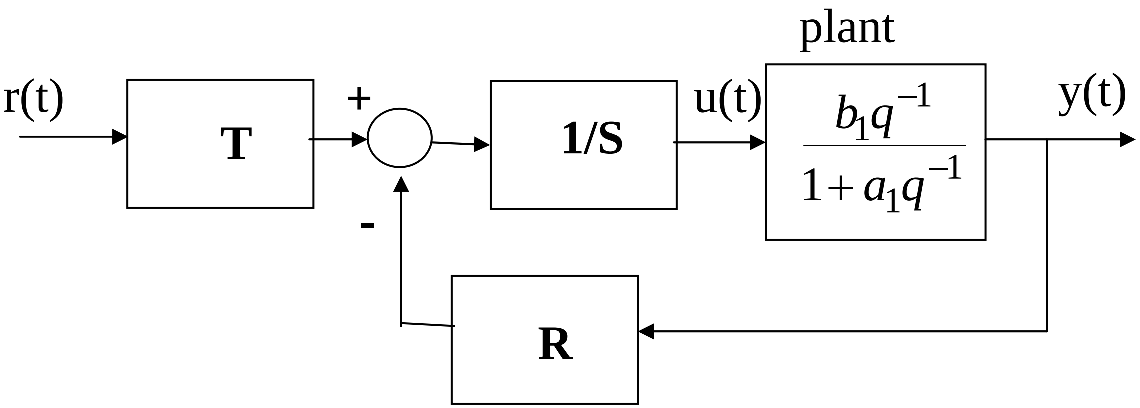
$$S(q^{-1})u(t) = T(q^{-1})r(t) - R(q^{-1})y(t)$$

$$S(q^{-1}) = 1 - q^{-1} = 1 + s_1 q^{-1}\ ; \quad (s_1 = -1) \qquad T(q^{-1}) = K(1 + T_e/T_i) - Kq^{-1} = r_0 + r_1 q^{-1}$$



$$u(t) = u(t-1) - R(q^{-1})y(t) + T(q^{-1})r(t)$$

Closed loop with a digital PI controller 1/2



$$R(q^{-1}) = T(q^{-1}) = r_0 + r_1 q^{-1} \quad ; \quad S(q^{-1}) = 1 - q^{-1} \quad ; \qquad \boxed{r_0, r_1 = ?}$$

$$H_{CL}(q^{-1}) = \frac{B(q^{-1})T(q^{-1})}{A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1})} = \frac{B(q^{-1})T(q^{-1})}{P(q^{-1})}$$

Desired poles (discretization of a 2nd order filter (ω_0,ζ)

$$(1 + a_1 q^{-1})(1 - q^{-1}) + b_1 q^{-1}(r_0 + r_1 q^{-1}) = 1 + p_1 q^{-1} + p_2 q^{-2}$$

$$1 + (a_1 - 1 + r_0 b_1) q^{-1} + (b_1 r_1 - a_1) q^{-2} = 1 + p_1 q^{-1} + p_2 q^{-2}$$

Closed loop with a digital PI controller 2/2

$$\begin{cases} a_1 - 1 + r_0 b_1 = p_1 \\ b_1 r_1 - a_1 = p_2 \end{cases}$$

Digital PI controller parameters :

$$r_1 = \frac{p_2 + a_1}{b_1} \qquad r_0 = \frac{p_1 - a_1 + 1}{b_1}$$

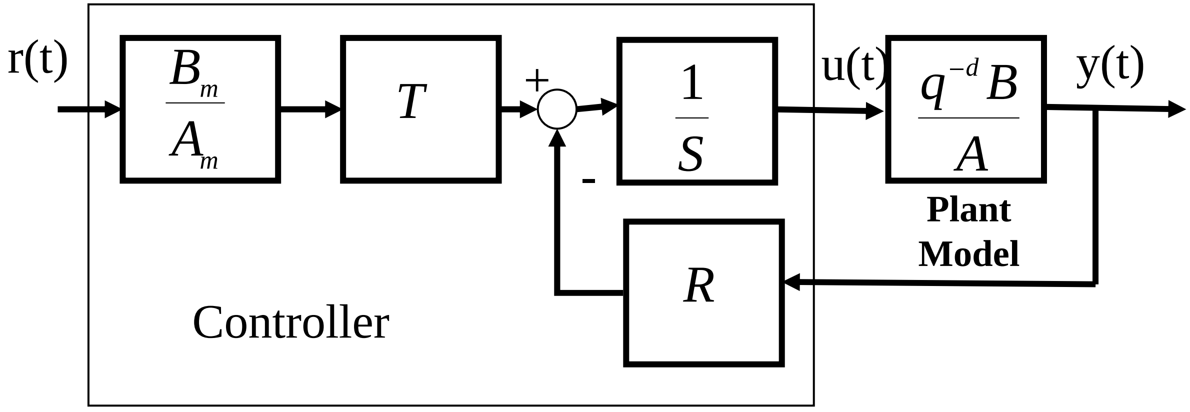
They depend on model parameters (a_1, b_1) and desired performances (p_1, p_2)

In this particuliar case the parameters of an equivalent continuous time PI controller can be computed

$$T(q^{-1}) = K(1 + T_s/T_i) - Kq^{-1} = r_0 + r_1 q^{-1}$$

$$K = -r_1 \qquad T_i = -\frac{T_e r_1}{r_1 + r_0}$$

The R-S-T Digital Controller



Plant Model:

$$G(q^{-1}) = H(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} = \frac{q^{-d-1}B^*(q^{-1})}{A(q^{-1})}$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_A} q^{-n_A} \quad B(q^{-1}) = b_1 q^{-1} + \ldots + b_{n_B} q^{-n_B} = q^{-1}B^*(q^{-1})$$

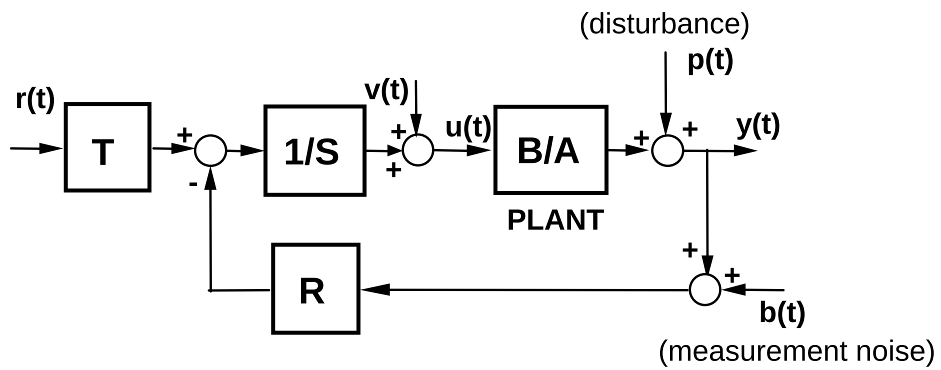
R-S-T Controller:

$$S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)$$

Characteristic polynomial (closed loop poles):

$$P(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1})$$

Digital control in the presence of disturbances and noise



Output sensitivity function
(p ⟶ y)

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

Input sensitivity function
(p ⟶ u)

$$S_{up}(z^{-1}) = \frac{-A(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

Noise-output sensitivity function
(b ⟶ y)

$$S_{yb}(z^{-1}) = \frac{-B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

Input disturbance-output sensitivity function
(v ⟶ y)

$$S_{yv}(z^{-1}) = \frac{B(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}$$

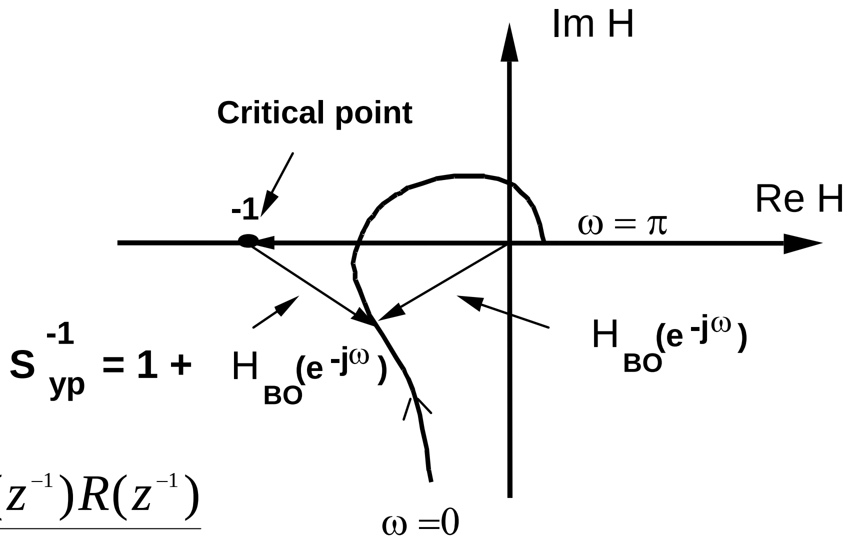
All four sensitivity functions should be stable !

Stability of closed loop discrete time systems

The Nyquist is used like in continuous time

(can be displayed with WinReg ou *Nyquist_OL.sci(.m))*

$$H_{OL}(e^{-j\omega}) = \frac{B(e^{-j\omega})R(e^{-j\omega})}{A(e^{-j\omega})S(e^{-j\omega})}$$



$$S_{yp}^{-1}(z^{-1}) = 1 + H_{OL}(z^{-1}) = \frac{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}$$

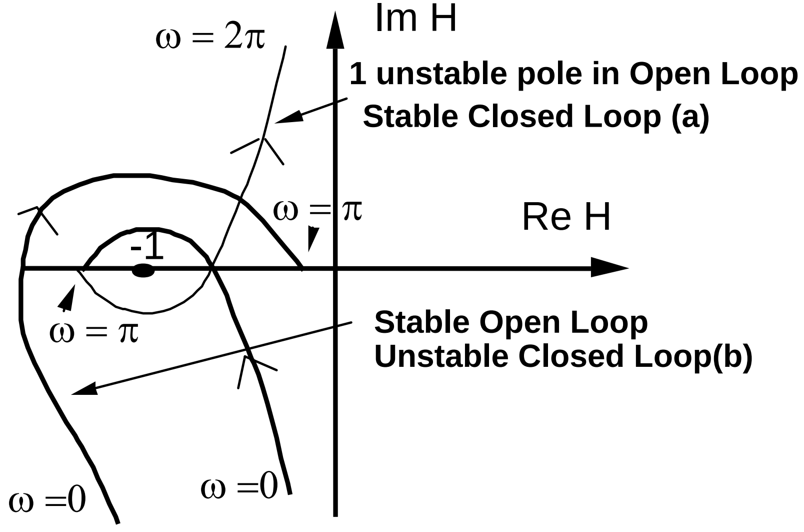
Nyquist criterion (discrete time –O.L. is stable)

The Nyquist plot of the open loop transfer fct. $H_{OL}(e^{-j\omega})$ traversed in the sense of growing frequencies (from 0 to $0.5f_S$) leaves the critical point[-1, j0] on the left

Stability of closed loop discrete time systems

Nyquist criterion (discrete time –O.L. is unstable)

The Nyquist plot of the open loop transfer fct. $H_{OL}(e^{-j\omega})$ traversed in the sense of growing frequencies (from 0 et f_S) leaves the critical point[-1, j0] on the left and the number of encirclements of the critical pointcounter clockwise should be equal to the number of unstable poles in open loop.



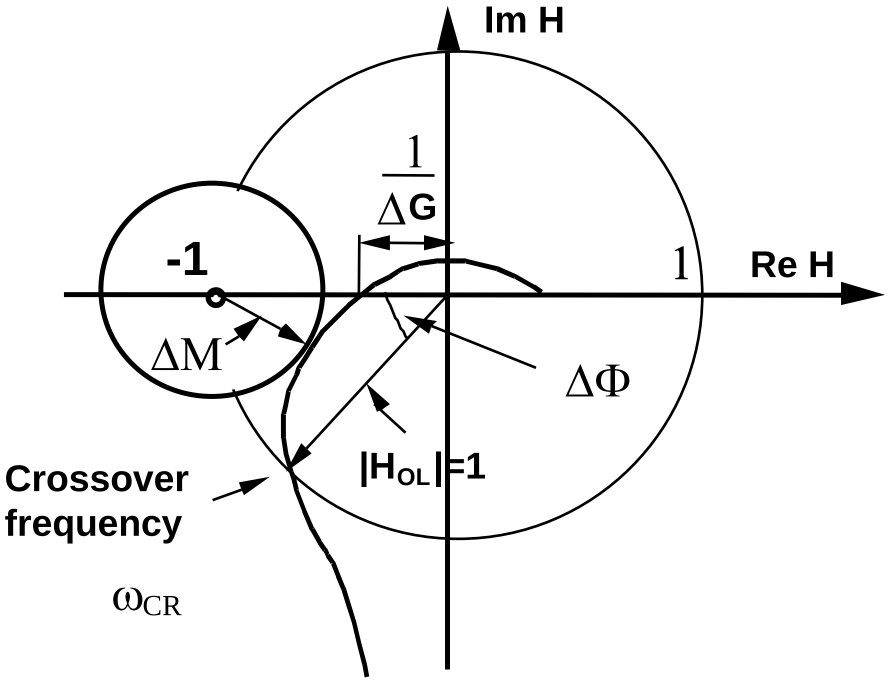
Remarks:

-*The controller poles may become unstable if high performances are required without using an appropriate design method*

-*The Nyquist plot from $0.5f_S$ to f_S is the symmetric with respect to the real axis of the Nyquist plot from 0 to $0.5f_S$*

Robustness margins

The minimal distance with respect to the critical point characterizes the robustness of the CL with respect to uncertainties on the plant model parameters(or their variations)



-Gain margin ΔG

-Phase margin $\Delta\phi$

-Delay margin $\Delta\tau$

-Modulus margin ΔM

Robustness margins – typical values

Gain margin : $\Delta G \geq 2$ (6 dB) [*min : 1,6* (4 dB)]

Phase margin : $30° \leq \Delta\phi \leq 60°$

Delay margin : fraction of system delay (10%) or
of time response (10%) (often $1.T_S$)

Modulus margin : $\Delta M \geq 0.5$ (- 6 dB) [*min : 0,4* (-8 dB)]

A modulus margin $\Delta M \geq 0.5$ implies $\Delta G \geq 2$ *et* $\Delta\phi > 29°$
Attention ! The converse is not generally true

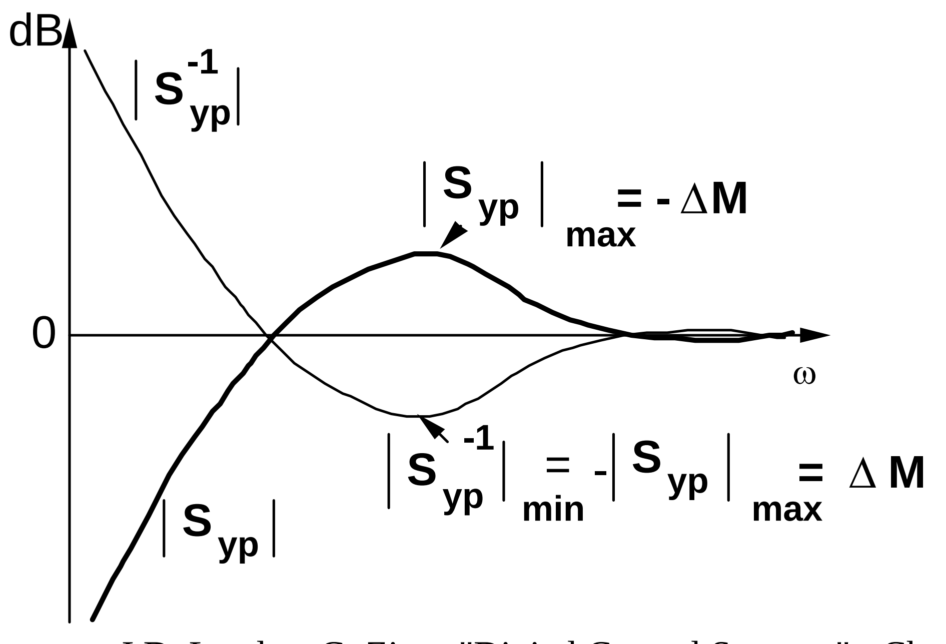
The *modulus margin* defines also the tolerance with respect to nonlinearities

Modulus margin and sensitivity function

$$\Delta M = \left|1 + H_{OL}(z^{-1})\right|_{\min} = \left|S_{yp}^{-1}(z^{-1})\right|_{\min} = \left(\left|S_{yp}(z^{-1})\right|_{\max}\right)^{-1} =$$

$$\left(\left|\frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}\right|_{\max}\right)^{-1} \quad pour \quad z^{-1} = e^{-j2\pi f}$$

$$\left|S_{yp}(e^{-j\omega})\right|_{\max} dB = \Delta M^{-1} dB = -\Delta M\ dB$$



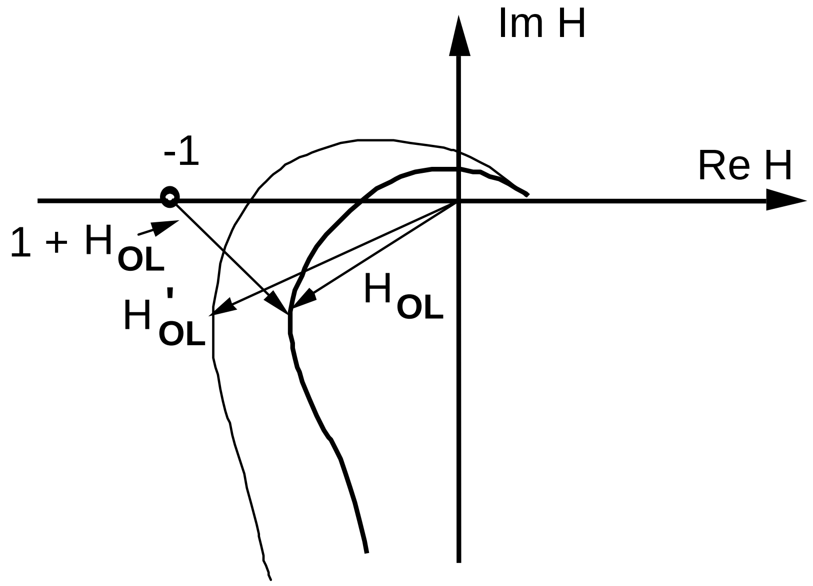
Robust stability

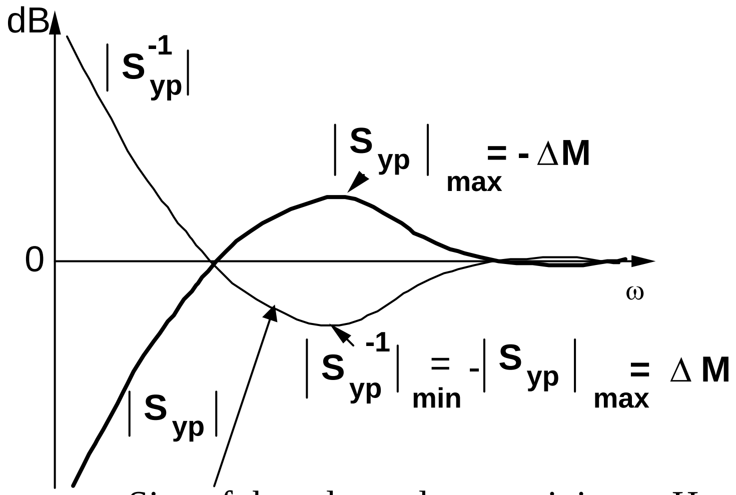
To assure stability in the presence of uncertainties (or variations) on the dynamic chatacteristics of the plant model

H_{OL} – nominal F.T.; H'_{OL} –Different from H_{OL} (perturbed)

Robust stability condition (sufficient cond.):

$$\left|H'_{OL}(z^{-1}) - H_{OL}(z^{-1})\right| < \left|1 + H_{OL}(z^{-1})\right| = \left|S_{yp}^{-1}(z^{-1})\right| = \left|\frac{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}\right| = \left|\frac{P(z^{-1})}{A(z^{-1})S(z^{-1})}\right| \quad ; \quad z^{-1} = e^{-j\omega} \qquad (*)$$



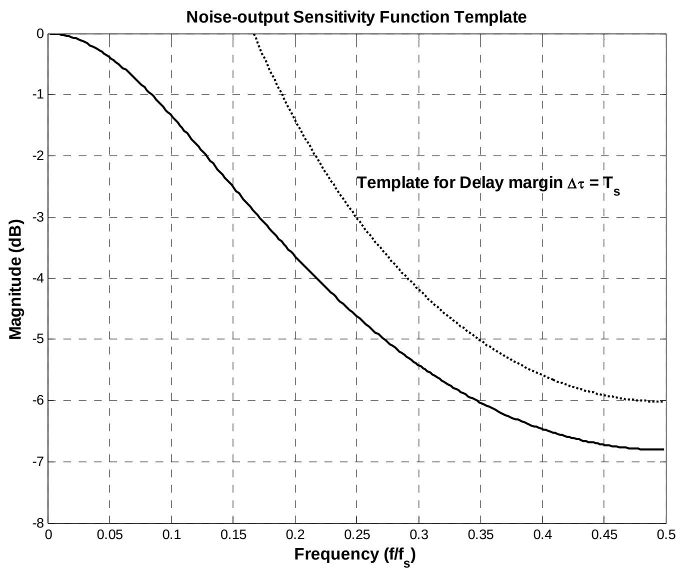


Size of the tolerated uncertainity on H_{OL} at each frequency (radius)

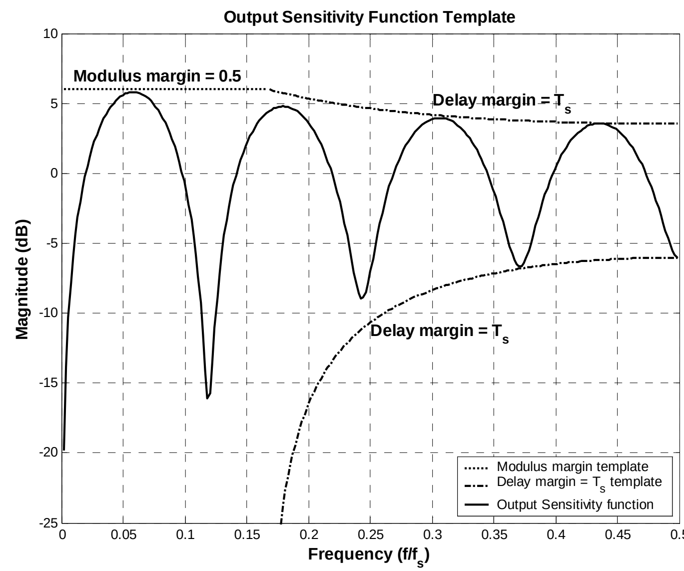
Frequency templates on the sensitivity functions

The robust stability conditions allow to define frequency templates on the sensitivity functions which guarantee the delay margin and the modulus margin;

The templates are essential for designing a good controller



Frequency template on the noise-output sensitivity function S_{yb} for $\Delta\tau = T_S$



Frequency template on the output sensitivity function S_{yp} for $\Delta\tau = T_S$ and $\Delta M = 0.5$

Concluding remarks

-Discrete time dynamic models are described by recursive equations:

$$y(t) = -\sum_{i=1}^{n_A} a_i y(t-i) + \sum_{i=1}^{n_B} b_i u(t-d-i)$$

u(t) = input ; y(t) = output ; d= integer delay (number of sampling periods)

-The q^{-1} operator is a convenient tool for coding recursive equations.

-The input/output behavior is describef by the transfer operator:

$$H(q^{-1}) = \frac{q^{-d} B(q^{-1})}{A(q^{-1})} \qquad A(q^{-1}) = 1 + \sum_{i=1}^{n_A} a_i q^{-i} \qquad B(q^{-1}) = 1 + \sum_{i=1}^{n_B} b_i q^{-i}$$

-Digital controllers have a canonical form (R-S-T). They are two degrees of freedom controllers:

$$S(q^{-1})u(t) = -R(q^{-1})y(t) + T(q^{-1})r(t)$$

r(t) = reference

-The Nyquist plot (frequency domain) and the sensitivity functions are essential for studyng closed loop stability and robustness with respect to plant parameters uncertainties.