# Chapter IV

## Design of Digital Controllers in the Presence of Random Disturbances

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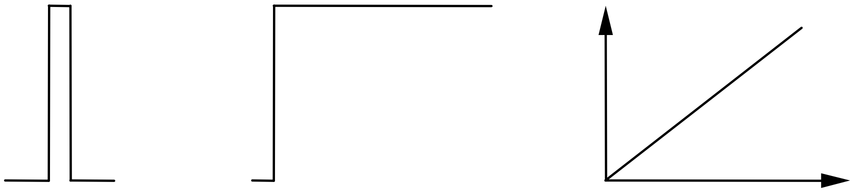
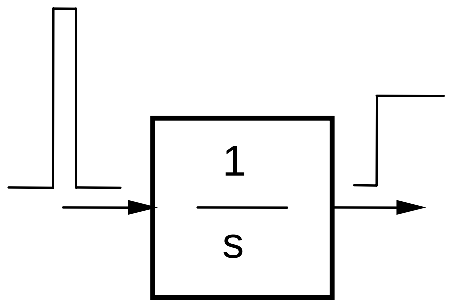
# Disturbance Representation

## Deterministic disturbances

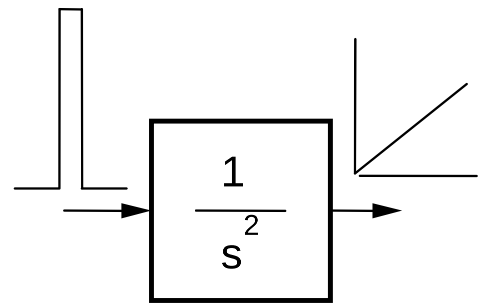


Dirac pulse    step    ramp

Can be described as a Dirac pulse passed through a filter




Disturbance model




Disturbance model

## Stochastic Disturbances

Can not be described in a deterministic way, since they are not reproducible.

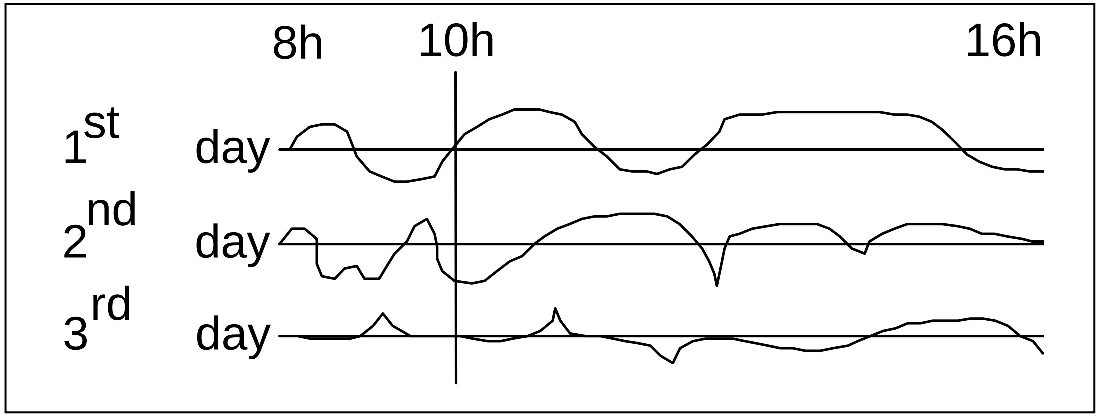
Most of the stochastic disturbances can be described as:

*A white noise passed through a filter*.

In a stochastic environment, the *white noise* play the role *of the Dirac Pulse*.

# Stochastic (random) Process

Example: record of a controlled variable in regulation (1 day)




• each evolution can be described by a different *f(t)* (*stochastic realization*)

• for a fixed time (ex.: 10h) for each *experiment* (day) one gets a different measured value (*random variable*)

• one can define a *statistitics* (mean value, variance) and *probabilities* of occurrence of the various values

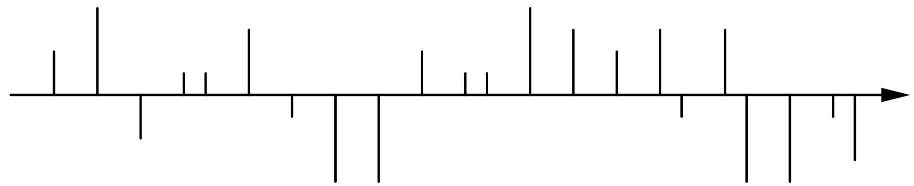
• if the *stochastic process* is *ergodic* the statistics over one *experiment* are significant

• if the *stochastic process* is *gaussian* the knowledge of the m.v. and variance allows to give the probability of occurrence of a certain value (Gauss bell – App.A)

# Discrete-time Gaussian White Noise

It is the fundamental *generator signal*

{*e(t)*}: Sequence of independent equally distributed Gaussian random variables with rero mean and variance $\sigma^2$ $(0,\sigma)$ ⟵ *standard deviation*



$$M.V. = E\{e(t)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e(t) = 0$$

$$var = E\{e^2(t)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e^2(t) = \sigma^2$$

*Independence* : The knowledge of *e(i)* does not allow to predict an approximation for *e(i+1), e(i+2)*….

# Independence Test

Autocorrelation (covariance) function:

$$R(i) = E\left\{e(t)e(t-i)\right\} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} e(t)e(t-i)$$

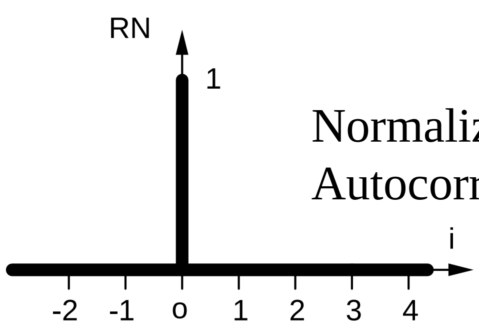
*Rem.:* $R(0) = var = \sigma^2$

Normalized autocorrelation (covariance) function :

$$RN(i) = \frac{R(i)}{R(0)} \qquad (RN(0) = 1)$$

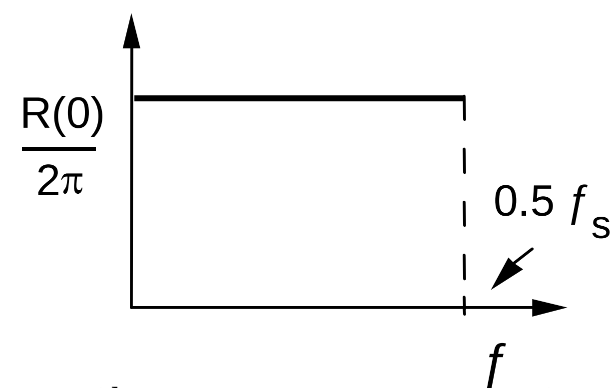
Whiteness (independence test) : $R(i) = RN(i) = 0 \quad i = 1, 2, 3..-1, -2\ldots$

White Noise

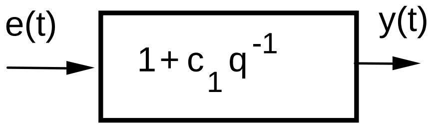



Normalized Autocorrelations

Spectral Density



# Moving Average Process – MA



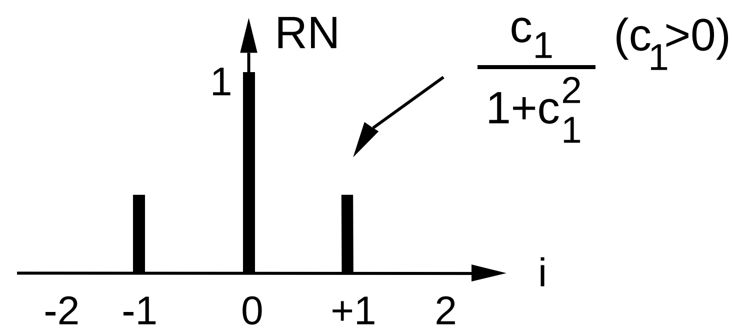

$$y(t) = e(t) + c_1 e(t-1) = (1 + c_1 q^{-1})e(t)$$

$$V.M. = E\{y(t)\} = \frac{1}{N}\sum_{t=1}^{N} y(t) = \frac{1}{N}\sum_{t=1}^{N} e(t) + c_1\frac{1}{N}\sum_{t=1}^{N} e(t-1) = 0$$

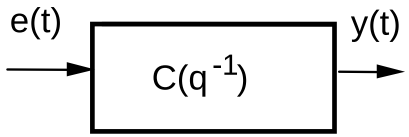
$$R_y(0) = E\{y^2(t)\} = \frac{1}{N}\sum_{t=1}^{N} y^2(t) = (1 + c_1^2)\sigma^2$$

$$R_y(1) = E\{y(t)y(t-1)\} = \frac{1}{N}\sum_{t=1}^{N} y(t)y(t-1) = \frac{1}{N}c_1\sum_{t=1}^{N} e^2(t) = c_1^2\sigma^2$$

$$R_y(2) = R_y(3) = .. = 0$$



# Moving Average Process – MA




$$y(t) = e(t) + \sum_{i=1}^{n_c} c_i e(t-i) = C(q^{-1})e(t)$$

$$C(q^{-1}) = 1 + \sum_{i=1}^{n_c} c_i q^{-i} = 1 + q^{-1}C^*(q^{-1})$$

$$R(i) = 0 \quad i \geq n_C + 1 \quad i \geq -(n_C + 1)$$

Spectral density:

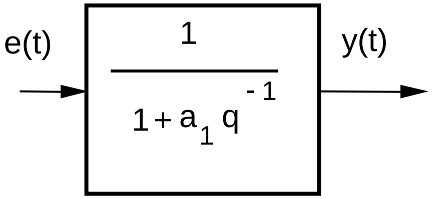
$$\phi_e$$

$$\phi_y(\omega) = C(e^{j\omega})C(e^{-j\omega})\frac{\sigma^2}{2\pi} = \left|C(e^{j\omega})\right|^2 \frac{\sigma^2}{2\pi}$$

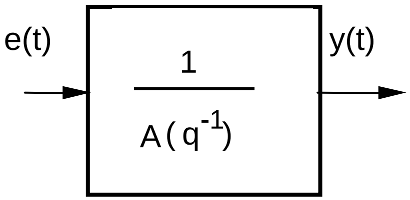
Relationship spectral density/transfer function:

$$\phi_y(z) = C(z)C(z^{-1})\phi_e(z) \, ; \, z = e^{j\omega}$$

# Auto-regressive Process – AR




$$y(t) = -a_1 y(t-1) + e(t) = \frac{e(t)}{1 + a_1 q^{-1}} \qquad |a_1| < 1$$




$$y(t) = -\sum_{i=1}^{n_A} a_i y(t-1) + e(t) \longrightarrow A(q^{-1})y(t) = e(t)$$

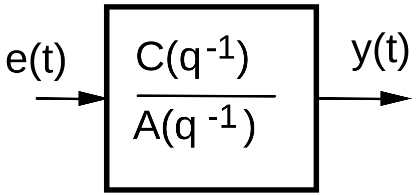
$$A(q^{-1}) = 1 + \sum_{i=1}^{n_A} a_i q^{-i} = 1 + q^{-1} A^*(q^{-1})$$

Asymptotically stable

Spectral density:

$$\phi_y(z) = \frac{1}{A(z)} \frac{1}{A(z^{-1})} \phi_e(z) \qquad \phi_y(\omega) = \phi_y(z)\Big|_{z=e^{j\omega}}$$

# Auto-regressive Moving Average Process - ARMA




$$y(t) = -\sum_{i=1}^{n_A} a_i y(t-i) + \sum_{i=1}^{n_C} c_i e(t-i) + e(t)$$

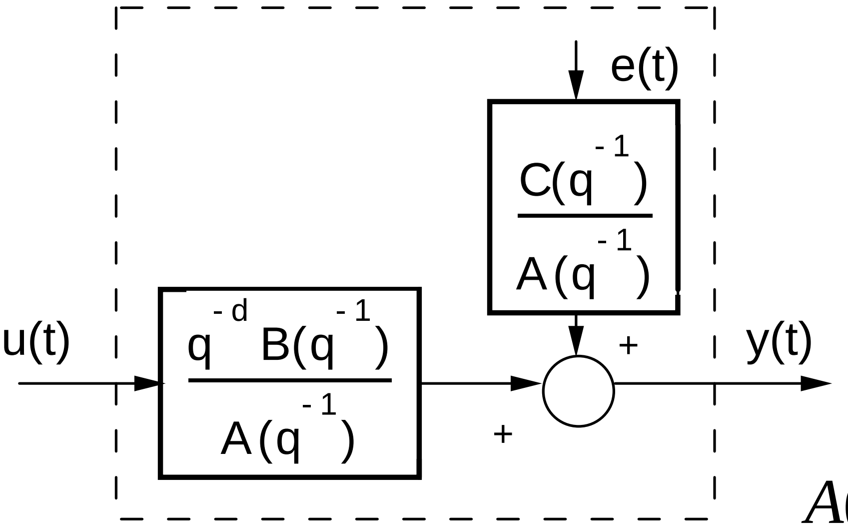
$$A(q^{-1})y(t) = C(q^{-1})e(t)$$

Asymptotically stable

Spectral density:

$$\phi_y(z) = \left(\frac{C(z)}{A(z)}\right)\left(\frac{C(z^{-1})}{A(z^{-1})}\right)\phi_e(z)$$

# ARMAX Process (*A.R.M.A. with « exogenous » input*)




$$y(t) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(t) + \boxed{\frac{C(q^{-1})}{A(q^{-1})}e(t)}$$

Disturbance

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})e(t)$$

$$y(t+1) = -\sum_{i=1}^{n_A} a_i y(t+1-i) + \sum_{i=1}^{n_B} b_i u(t+1-d-i) + \sum_{i=1}^{n_C} c_i e(t+1-i) + e(t+1)$$

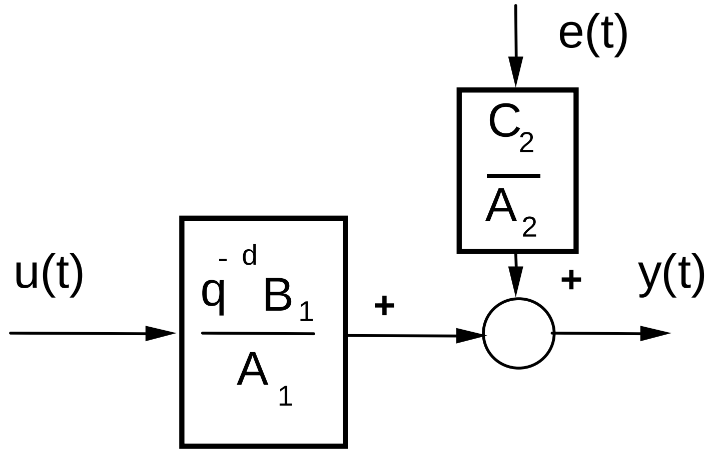
$$y(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C(q^{-1})e(t+1)$$

*Remark*: in general $n_C = n_A$

Example: $n_A = 1;\, n_B = 1;\, n_C = 1;\, d = 0$

$$y(t+1) = -a_1 y(t) + b_1 u(t) + c_1 e(t) + e(t+1)$$

# Generality of the ARMAX Process




$$y(t) = \frac{q^{-d} B_1(q^{-1})}{A_1(q^{-1})} u(t) + \frac{C_2(q^{-1})}{A_2(q^{-1})} e(t)$$

$$\Downarrow$$

$$y(t) = \frac{q^{-d} B_1 A_2}{A_1 A_2} u(t) + \frac{C_2 A_1}{A_1 A_2} e(t) = \frac{q^{-d} B}{A} u(t) + \frac{C}{A} e(t)$$

$$A = A_1 A_2 \; ; B = B_1 A_2 \; ; C = C_2 A_1$$

# Optimal Prediction

$\hat{y}(t+1/t) =$ Prediction of *y(t+1)* based on the measures of *u* and *y* available up to *t*

Prediction error: $\varepsilon(t+1) = y(t+1) - \hat{y}(t+1)$

Objective: $\hat{y}(t+1/t) = \hat{y}(t+1) = f(y(t), y(t-1), \ldots, u(t), u(t-1), \ldots)$

such that : $E\left\{[y(t+1) - \hat{y}(t+1)]^2\right\} = \min$

Example : $y(t+1) = -a_1 y(t) + b_1 u(t) + c_1 e(t) + e(t+1)$

$$\varepsilon(t+1) = y(t+1) - \hat{y}(t+1) = [-a_1 y(t) + b_1 u(t) + c_1 e(t) - \hat{y}(t+1)] + e(t+1)$$

$$E\left\{[y(t+1) - \hat{y}(t+1)]^2\right\} = E\left\{[-a_1 y(t) + b_1 u(t) + c_1 e(t) - \hat{y}(t+1)]^2\right\} + E\left\{e^2(t+1)\right\}$$
$$+ 2E\{\underbrace{e(t+1)[-}_{=0}a_1 y(t) + b_1 u(t) + c_1 e(t)]\}$$

**Optimality condition:** $E\left\{[-a_1 y(t) + b_1 u(t) + c_1 e(t) - \hat{y}(t+1)]^2\right\} = 0$

$$\hat{y}(t+1)\big|_{opt} = -a_1 y(t) + b_1 u(t) + c_1 e(t) \longrightarrow \varepsilon(t+1)\big|_{opt} = y(t+1) - \hat{y}(t+1)\big|_{opt} = e(t+1)$$

$$\varepsilon(t) = e(t) \longrightarrow \hat{y}(t+1)\big|_{opt} = -a_1 y(t) + b_1 u(t) + c_1 \varepsilon(t)$$

# Optimal prediction

ARMAX:

$$y(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C(q^{-1})e(t+1)$$

Optimal predictor (theoretical):

$$\hat{y}(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C^*(q^{-1})e(t)$$

Prediction error:

$$\varepsilon(t+1)\big|_{opt} = y(t+1) - \hat{y}(t+1) = e(t+1)$$

Optimal predictor (implementation):

$$\hat{y}(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C^*(q^{-1})\varepsilon(t)$$

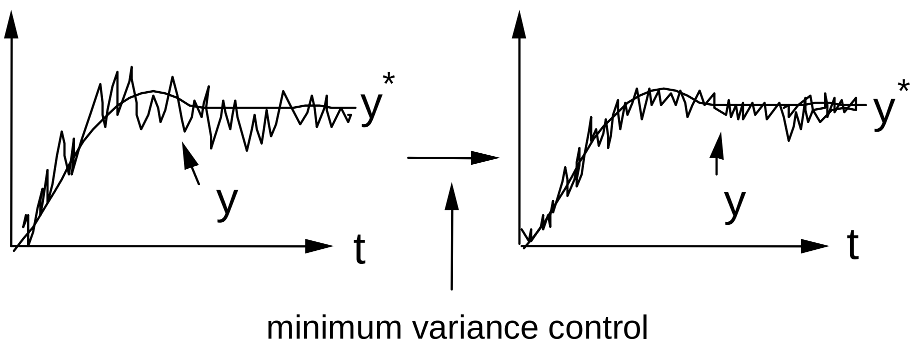
*One replaces the unknown white noise by the prediction error*

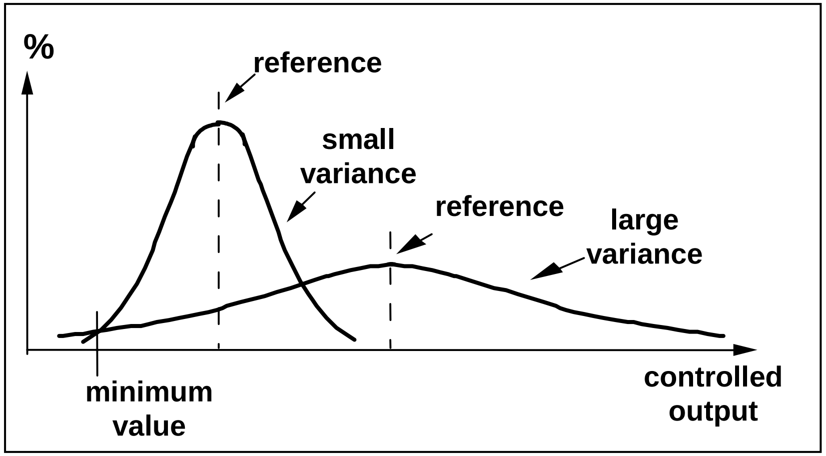
# Minimum Variance Tracking and Regulation

- random disturbances
- the discrete time plant model has stable zeros

Objective: *minimization of the output variance (standard deviation)*

$$J(u(t)) = E\left\{\left[y(t) - y^*(t)\right]^2\right\} \approx \frac{1}{N}\sum_{t=1}^{N}\left[y(t) - y^*(t)\right]^2 = \min$$




- A model for the disturbance has to be considered
- Plant + disturbance: ARMAX model

# Minimum Variance Tracking and Regulation

Plant + disturbance: $y(t+1) = -a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) + e(t+1)$

Reference trajectory: $y^*(t+1)$

Criterion computation:

$$E\{[y(t+1) - y^*(t+1)]^2\} = E\{[-a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) - y^*(t+1)]^2\}$$
$$+E\{e^2(t+1)\} + 2E\{e(t+1)[-a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) - y^*(t+1)]\}$$
$$\underbrace{\qquad}_{=0}$$

Optimality condition: $E\{[-a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) - y^*(t+1)]^2\} = 0$

Control law (theoretical):

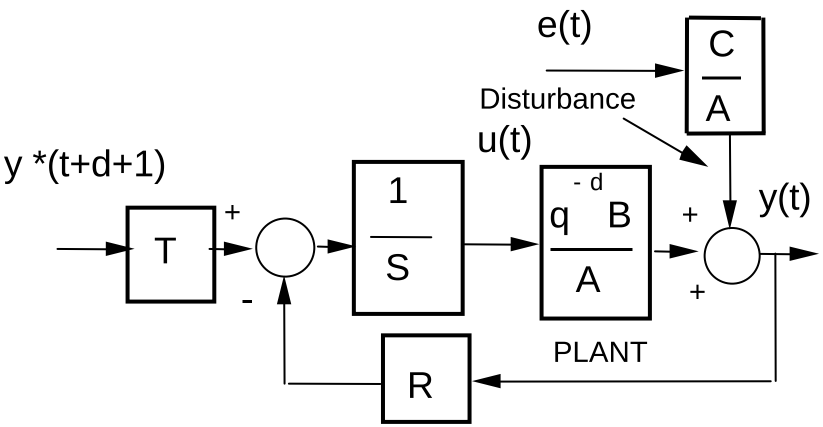
$$u(t) = \frac{y^*(t+1) - c_1 e(t) + a_1 y(t)}{b_1 + b_2 q^{-1}} \rightarrow y(t+1) - y^*(t+1) = e(t+1) \longrightarrow y(t) - y^*(t) = e(t)$$

Control law (implementation):

$$u(t) = \frac{(1 + c_1 q^{-1}) y^*(t+1) - (c_1 - a_1) y(t)}{b_1 + b_2 q^{-1}} = \frac{T(q^{-1}) y^*(t+1) - R(q^{-1}) y(t)}{S(q^{-1})}$$

**Same control law as for « Tracking and regulation with independent objectives » by taking $P(q^{-1}) = C(q^{-1})$**

# Closed Loop Poles




$$u(t) = \frac{T(q^{-1})y^*(t+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$H_{BF}(q^{-1}) = \frac{T(q^{-1})q^{-(d+1)}B^*(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-1}B^*(q^{-1})R(q^{-1})}$$

$A(q\text{-}1) = 1 + a_1 q^{-1}$ ; $B(q^{-1}) = q^{-1} B^*(q^{-1})$ ; $B^*(q^{-1}) = b_1 + b_2 q^{-1}$ ; $d = 0$

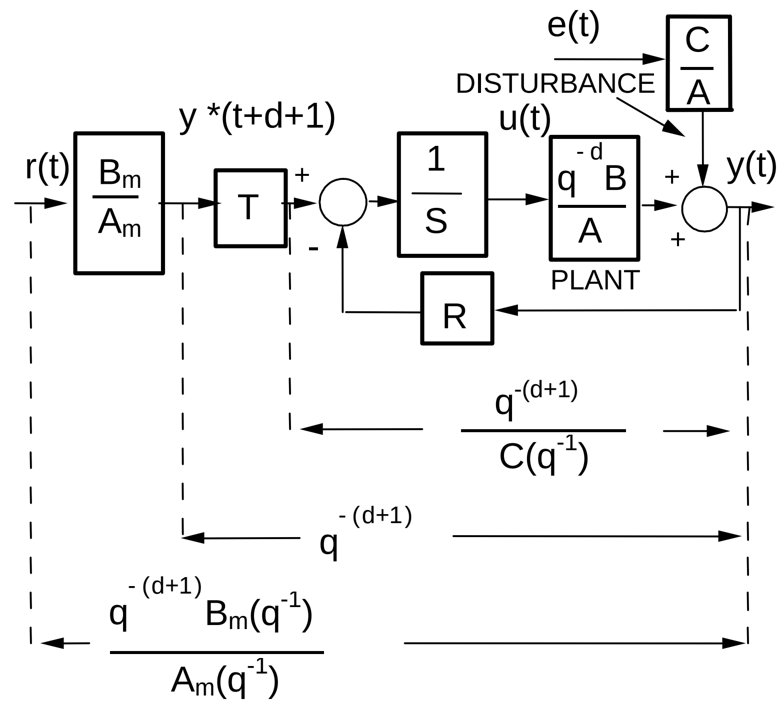
$T(q^{-1}) = C(q^{-1}) = 1 + c_1 q^{-1}$; $S(q^{-1}) = B^*(q^{-1}) = b_1 + b_2 q^{-1}$ ; $R(q^{-1}) = r_0 = c_1 - a_1$

$$H_{BF}(q^{-1}) = \frac{T(q^{-1})q^{-1}}{A(q^{-1}) + q^{-1}R(q^{-1})} = \frac{T(q^{-1})q^{-1}}{C(q^{-1})} = q^{-1}$$

Closed loop poles

*The disturbance model ($C(q^{-1})$) defines the closed loop poles and therefore the regulation performance*

# Minimum Variance Tracking and Regulation – general case




Same computations as for « tracking and regulation with independent objectives » by taking $P(q^{-1}) = C(q^{-1})$ (see Chapter 3)

# Minimum Variance Tracking and Regulation – General Case

$$u(t) = \frac{T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$T(q^{-1}) = C(q^{-1})\,;\, S(q^{-1}) = B^*(q^{-1})S'(q^{-1})$$

$$A(q^{-1})S'(q^{-1}) + q^{-(d+1)}B^*(q^{-1})R(q^{-1}) = C(q^{-1})$$

Solving with *predisol.sci(.m) or with WinReg (Adaptech)*

Prediction error : $$y(t+d+1) - y^*(t+d+1) = \underbrace{S'(q^{-1})e(t+d+1)}_{\text{MA of order } d}$$

*Optimality test:*

$$R(i) = \frac{1}{N}\sum_{t=1}^{N}\left[y(t) - y^*(t)\right]\cdot\left[y(t-i) - y^*(t-i)\right] \qquad i = 0,1,2,\ldots$$

$$\boxed{RN(i) \approx 0 \quad i \geq d+1}$$

theoretical

$$\boxed{\left|RN(i)\right| \leq 02.17\sqrt{N} \quad i \geq d+1}$$

pratical

# Minimum Variance Tracking and Regulation. Example

Plant:

- $d = 0$
- $B(q\text{-}1) = 0.2\ q\text{-}1 + 0.1\ q\text{-}2$
- $A(q\text{-}1) = 1 - 1.3\ q\text{-}1 + 0.42\ q\text{-}2$

Tracking dynamics → $Ts = 1s,\ \omega 0 = 0.5\ rad/s,\ \zeta = 0.9$

- $Bm = +0.0927 + 0.0687\ q\text{-}1$
- $Am = 1 - 1.2451\ q\text{-}1 + 0.4066\ q\text{-}2$

*Disturbance polynomial* → $C(q\text{-}1) = 1 - 1.34\ q\text{-}1 + 0.49\ q\text{-}2$

*Pre-specifications: Integrator*

****** CONTROL LAW ******

$S(q\text{-}1)\ u(t) + R(q\text{-}1)\ y(t) = T(q\text{-}1)\ y^*(t+d+1)$

$y^*(t+d+1) = [(Bmq\text{-}1)/Am(q\text{-}1)]\ .\ ref(t)$

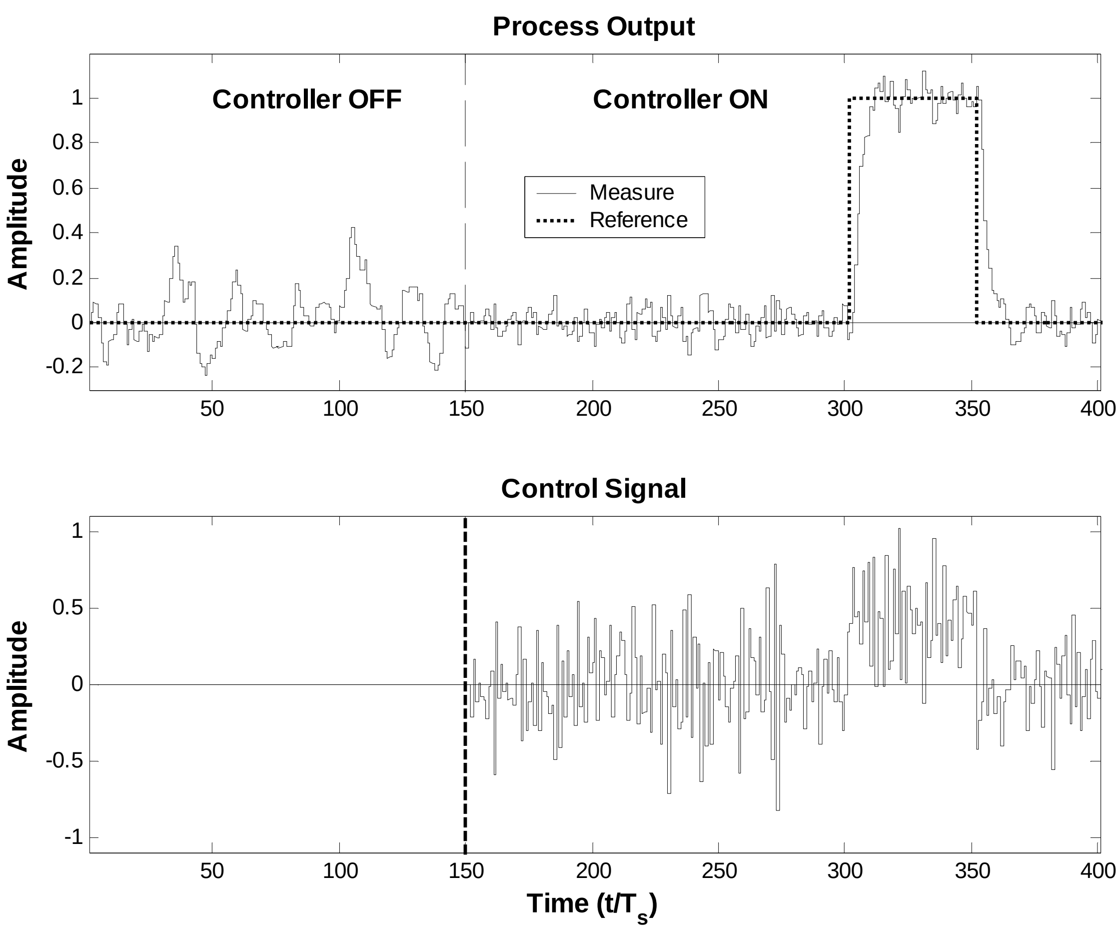
*Controller:*

- $R(q\text{-}1) = 0.96 - 1.23\ q\text{-}1 + 0.42\ q\text{-}2$
- $S(q\text{-}1) = 0.2 - 0.1\ q\text{-}1 - 0.1\ q\text{-}2$
- $T(q\text{-}1) = C(q\text{-}1)$

*Gain margin: 2.084* *Phase margin: 61.8 deg*

*Modulus margin: 0.520 (- 5.68 dB)* *Delay margin: 1.3 s*

# Minimum Variance Tracking and Regulation. Example




*Attention*: For robustness and actuator stress one may be obliged to add auxiliary poles (see book pg. 190)

# Minimum Variance Tracking and Regulation
# The case of unstable zeros

In this case minimum variance control can not be applied

Solutions:

- Use of pole placement with a special choice of the closed loop poles
- Generalized minimum variance tracking and regulation (modified criterion)

# Use of pole placement

$$B^*(q^{-1}) = B^+(q^{-1})B^-(q^{-1})$$

Unstable factor

$B^-{}'(q^{-1})$ Reciprocical polynomial (stable) of $B^-(q^{-1})$

(one reverses the order of the coefficients)

Closed loop poles

$$P(q^{-1}) = B^+(q^{-1})B^-{}'(q^{-1})C(q^{-1})$$

$$= A(q^{-1})S(q^{-1}) + q^{-(d+1)}B^*(q^{-1})R(q^{-1})$$

For details and examples, see book pg.192-195

# Generalized Minimum Variance Tracking and Regulation

Criterion:

$$E\left\{\left[y(t+d+1)-y^{*}(t+d+1)+\frac{Q(q^{-1})}{C(q^{-1})}u(t)\right]^{2}\right\}=\min$$

$$Q(q^{-1})=\frac{\lambda(1-q^{-1})}{1+\alpha q^{-1}}$$

Particular case : $\alpha = 0$

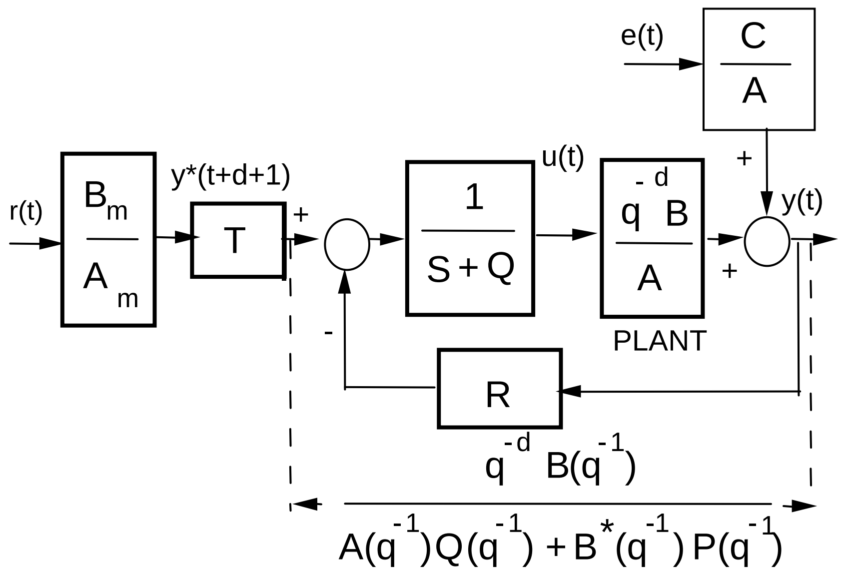
Weighting the control variations

$$E\left\{\left[y(t+d+1)+\frac{\lambda}{C(q^{-1})}[u(t)-u(t-1)]-y^{*}(t+d+1)\right]^{2}\right\}=\min$$

Controllere: $$u(t)=\frac{C(q^{-1})y^{*}(t+d+1)-R(q^{-1})y(t)}{S(q^{-1})+Q(q^{-1})}$$

Allows to stabilize the controller and the system (but not always!)

# Generalized minimum variance tracking and regulation




Design:

- One computes a minimum variance tracking/regulation controller without taking in account the unstable nature of *B. (Q(q*$^{-1}$*)=0)*
- One introduces $Q(q^{-1})$ and search for $\lambda > 0$ with stabilizes the controller and the closed loop

*A solution does not alwyas exists in particular when there are several unstable zeros*

*For details, see book pg.195-197*

## Some Concluding Remarks

- For a good control in a stochastic environment one need a model for the disturbances.
- Many stochastic disturbances can be modeled as a white noise passed through a filter.
- The knowledge of this filter (*disturbance model*) is enough.
- The ARMAX model for (plant + disturbance) is often used.
- Minimum variance control can be used only for discrete time plant models with stable zeros.
- This control strategy is the *dual* of tracking and regulation with independent objectives used in the deterministic case (with $P = C$).
- This technique allows to better understand the choice of CL poles
- For discrete time models with unstable zeros approximated solutions has to be used.
- Never use a minimum variance controller without a robustness analysis.