

## $(\delta, \varepsilon)$ -ball approximation of a shape: definition and complexity

Dominique Attali · Tuong-Bach  
Nguyen · Isabelle Sivignon

the date of receipt and acceptance should be inserted later

**Abstract** Given a set  $S$  in  $\mathbb{R}^n$ , a  $(\delta, \varepsilon)$ -ball approximation of  $S$  is defined as a collection of balls that covers the morphological erosion of  $S$  (by a ball of radius  $\varepsilon$ ) and remains inside the morphological dilation of  $S$  (by a ball of radius  $\delta$ ). We study the problem of computing a  $(\delta, \varepsilon)$ -ball approximation when  $S$  is itself defined as a finite union of balls. This problem relates to geometric set cover problems but is however different in nature. It offers a new framework for simplifying a collection of balls while controlling both the inner and outer distance to the shape. We prove that computing a  $(\delta, \varepsilon)$ -ball approximation of minimum cardinality is NP-complete for  $n = 2$ . Along the way, we study the boundary of unions of disks and their erosion, for which we derive a computational description.

**Keywords** shape approximation · union of balls · medial axis · covering · morphological erosion and dilation · complexity

**Mathematics Subject Classification (2000)** 68U05 · 52C99

---

Research partially supported by the French “Agence nationale pour la Recherche” under grants ANR-13-BS01-0008 TopData. Also partially supported by the French program “Grenoble Innovation Recherche” 2014 from Université Grenoble Alpes and Grenoble INP.

Dominique Attali · Tuong-Bach Nguyen · Isabelle Sivignon  
Univ. Grenoble Alpes, GIPSA-Lab, F-38000 Grenoble, France  
CNRS, GIPSA-Lab, F-38000 Grenoble, France  
Tel.: +33 4 76 57 43 74  
E-mail: dominique.attali, tuong-bach.nguyen, isabelle.sivignon@gipsa-lab.grenoble-inp.fr

## 1 Introduction

*Context.* In many applications, shapes are often represented as the union of a finite set of balls. Such a representation is used in biochemistry to model molecules [17, 10] or in computer graphics to detect collisions between objects [5]. Unions of balls have become ubiquitous representations of shapes, largely due to the existence of provably good conversion algorithms that allow to build them from other representations such as point clouds [2], polygonal meshes [15, 24, 27] or digital shapes [14]. Unfortunately, conversion algorithms generally output very large (and sometimes infinite) collections of balls. This may be the case when conversion algorithms aim at representing the shape as the union of its medial balls. A ball  $b$  is said to be *medial* for a given shape if every ball that contains  $b$  and is contained in the shape is equal to  $b$ . Assuming that balls are closed and the shape is compact, the union of medial balls provides an exact description of the shape, whose size however may be infinite. The centers of medial balls form the *medial axis*, whose stability and computation (either exact or approximated) have been widely studied; see [3] for a state-of-the-art report. For computationally demanding tasks such as the simulation of physical processes [19], shape interpolation or shape matching [30, 8], it is desirable that the collection of balls that describes a shape has a small size while its union still provides an accurate approximation of the shape. The quality of the approximation is usually measured by various quantities such as the Hausdorff distance or the difference in volume between the shape and its representation. In other situations such as collision detection, it may be important that the union of balls satisfies some *geometric constraints*. For instance, we may want the balls to cover the shape or at least a prescribed set of points.

*Ball approximations.* In this work, we introduce a novel way of constraining a collection of balls so that its union remains close to the shape. The idea is to force balls to cover a subset of the shape while remaining contained in a superset of the shape, hence allowing balls neither to cover exactly nor to be contained perfectly in the shape. To explain this more precisely, we need some notation and definitions. Unless stated otherwise, all balls considered in the paper are closed. We denote by  $b(x, r)$  the (closed) ball centered at  $x$  with radius  $r$ . Given a subset  $S \subseteq \mathbb{R}^n$  and a real number  $r \geq 0$ , we recall that the *dilation* of  $S$  by a ball of radius  $r$  is  $S^{\oplus r} = \cup_{x \in S} b(x, r)$  and the *erosion* of  $S$  by a ball of radius  $r$  is  $S^{\ominus r} = \{x \mid b(x, r) \subseteq S\}$ . Note that if  $S$  is closed, then both  $S^{\ominus r}$  and  $S^{\oplus r}$  are closed. We shall refer to  $S^{\ominus r}$  and  $S^{\oplus r}$  as the  $r$ -erosion and  $r$ -dilation.

**Definition 1** Let  $S \subseteq \mathbb{R}^n$ ,  $\varepsilon \geq 0$  and  $\delta \geq 0$ . A collection of balls  $\mathcal{B}$  is a  $(\delta, \varepsilon)$ -ball approximation of  $S$  if  $S^{\ominus \varepsilon} \subseteq \bigcup \mathcal{B} \subseteq S^{\oplus \delta}$ .

Let us make a few remarks. Consider a  $(\delta, \varepsilon)$ -ball approximation  $\mathcal{B}$  of  $S$ ; see Figure 1 for an example. Setting  $B = \bigcup \mathcal{B}$ , we thus have  $B \subseteq S^{\oplus \delta}$  and  $(S^{\ominus \varepsilon})^{\oplus \varepsilon} \subseteq B^{\oplus \varepsilon}$ . It follows that if  $S = (S^{\ominus \varepsilon})^{\oplus \varepsilon}$ , then  $B \subseteq S^{\oplus \delta}$  and  $S \subseteq B^{\oplus \varepsilon}$ ,

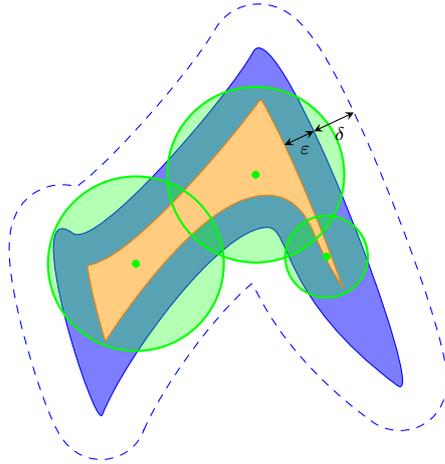


Fig. 1: A shape (in blue), its  $\varepsilon$ -erosion (in orange), its  $\delta$ -dilation (bounded by the dashed blue curve) and a  $(\delta, \varepsilon)$ -ball approximation (the three green balls).

or equivalently the Hausdorff distance between  $B$  and  $S$  is smaller than or equal to the maximum of  $\delta$  and  $\varepsilon$ , *i.e.*  $d_H(S, B) \leq \max\{\delta, \varepsilon\}$ . The condition  $S = (S^{\ominus\varepsilon})^{\oplus\varepsilon}$  can be seen as some regularity<sup>1</sup> assumption on  $S$ . Indeed, the equality  $S = (S^{\ominus\varepsilon})^{\oplus\varepsilon}$  is satisfied if every point of  $S$  can be covered by a ball of radius  $\varepsilon$ . In particular, the condition holds whenever medial balls have a radius larger than or equal to  $\varepsilon$ , which in turn holds whenever the reach of  $S$  is larger than or equal to  $\varepsilon$ . We recall that the reach of  $S$  was first introduced by Federer [18] and is the infimum of distances between points in the medial axis of  $S$  and points in the complement of  $S$ . Denoting the reach of  $S$  by  $\text{reach}(S)$ , we summarize our findings in the following remark:

*Remark 1* Let  $\mathcal{B}$  be a  $(\delta, \varepsilon)$ -ball approximation of  $S$ . If  $\text{reach}(S) \geq \max(\delta, \varepsilon)$  then  $d_H(\bigcup \mathcal{B}, S) \leq \max(\delta, \varepsilon)$ .

*Remark 2*  $S$  possesses  $(\delta, \varepsilon)$ -ball approximations of finite cardinality whenever  $S$  is compact and  $\delta + \varepsilon > 0$ .

Suppose  $S$  is compact and  $\delta + \varepsilon > 0$ . To build a finite  $(\delta, \varepsilon)$ -ball approximation of  $S$ , consider the collection of open balls with radius  $\delta + \varepsilon$  centered at points in  $S^{\ominus\varepsilon}$ . By construction, the collection covers  $S^{\ominus\varepsilon}$  which is compact because  $S$  is compact, implying that the collection has a finite subcover. By taking the closure of balls in that subcover, we get a collection of closed balls that are contained in  $S^{\oplus\delta}$  and thus form a finite  $(\delta, \varepsilon)$ -ball approximation of  $S$ .

<sup>1</sup> A set  $S$  is called *r-regular* if  $S = (S^{\ominus r})^{\oplus r} = (S^{\oplus r})^{\ominus r}$  [32]. In other words, the shape  $S$  is *r-regular* if it does not change neither under a morphological opening nor a morphological closing by a ball of radius  $r$ .

*Main result.* In this paper, we are interested in a discrete variant of the following problem:

**Problem 1 (Ball approximation)** Given a subset  $S \subseteq \mathbb{R}^n$ , two non-negative real numbers  $\delta$  and  $\varepsilon$ , find a  $(\delta, \varepsilon)$ -ball approximation of  $S$  of minimum cardinality.

Our main result is that a discrete decision version of this problem is NP-complete. To state the problem, we call any collection of balls whose centers have rational coordinates and whose radii are rational *rational*.

**Problem 2 (Rational ball approximation)** Given a subset  $S \subseteq \mathbb{R}^n$  which is the union of a rational finite collection of balls, two non-negative rational real numbers  $\varepsilon$  and  $\delta$  and an integer  $k > 0$ , answer the following question: does  $S$  have a rational  $(\delta, \varepsilon)$ -ball approximation with cardinality lower than or equal to  $k$ ?

The main result of this paper is that this problem is already difficult in dimension 2:

**Theorem 1** *The rational ball approximation problem is NP-complete in  $\mathbb{R}^2$ .*

*Related works.* As mentioned above, any shape  $S$  can be exactly described by the union of its medial balls  $\mathcal{S}$ . A  $(\delta, \varepsilon)$ -ball approximation of  $S$  of minimum cardinality is a collection of balls  $\mathcal{B}$  such that  $|\mathcal{B}| \leq |\mathcal{S}|$ . Indeed, note that, by Definition 1,  $\mathcal{S}$  itself is a  $(\delta, \varepsilon)$ -ball approximation of  $S$ . Therefore, solving Problem 1 offers a new framework for *simplifying unions of balls* with a control of the distance to the shape. Various algorithms have been proposed to compute simpler collections of balls. A family of algorithms was designed to deal with the sensitivity of the medial axis to small perturbations on the shape  $S$ : the general idea is to compute a geometric quantity for each medial ball and prune the balls whose quantity is below a given threshold (see for instance [12, 3, 11, 16] to cite only a few). A slightly different approach has been presented in [27] with the notion of scale axis, where balls of the medial axis are first inflated, the medial axis updated (resulting in a simplification) and its balls deflated. Last, hierarchical structures have also been proposed to obtain simpler collections of balls. In the context of collision detection, sphere-trees are built by iteratively merging and/or displacing balls while ensuring that the simpler set of balls includes the boundary of the input one [23, 5, 7].

The  $(\delta, \varepsilon)$ -ball approximation problem is also closely related to *geometric set cover* problems. Following [1] or [21], let  $\Sigma = (K, R)$  be a finite geometric range space where  $K$  is a *finite* set of points in  $\mathbb{R}^d$  and  $R$ , the range, is a *finite* family of simple shaped regions, like for instance rectangles, or disks. A subset  $C \subseteq R$  is a cover of  $K$  if any point of  $K$  belongs to at least one element of  $C$ . Computing a set cover of minimum cardinality is known as the set cover problem. The corresponding decision problem is well-known to be NP-complete, and various approximation algorithms have been proposed, from the simple greedy one [13, 25], to more convoluted algorithms using  $\varepsilon$ -nets

or multiplicative weights methods [6, 21, 1]. Several variations of the geometric set cover problem have also been introduced and investigated, notably when the range  $R$  is a family of balls. Instead of minimizing the number of balls used to cover the whole set  $K$ , a related problem is to cover the set  $K$  at best using a prescribed number of balls (see for instance a greedy algorithm in [10]). For more applicative purposes, especially in sensor networks, the balls of the range  $R$  can be of prescribed centers, but undefined radii, and the goal is to find a cover that minimizes a function of the radii of the balls [31].

The key difference between  $(\delta, \varepsilon)$ -ball approximation and set cover problems lies in the assumption of a finite range space for the latter one. If the shape  $K$  to cover and the range  $R$  are infinite, another family of results exists for which the range is the (infinite) set of congruent balls of a given radius. Results on that matter go back as far as [26], that establishes a relation between the (Lebesgue) measure of a bounded set of points  $K$  in  $\mathbb{R}^2$  and the number of congruent disks of radius  $r$  necessary to cover  $K$ . When  $K$  is a simple geometric shape (disk, square, equilateral triangle), results on the maximum size (radius, side length) of a shape  $K$  that can be covered by  $n$  unit disks are documented in [33].

*Paper overview.* We establish Theorem 1 by successively proving that:

- The rational ball approximation problem in  $\mathbb{R}^2$  is in NP, meaning that one can verify in polynomial time that a given rational collection of disks is indeed a solution of the problem. For this, we start by studying in Section 2 the boundary of the erosion of a union of disks, for which we give a full computational characterization. This result is then used in Section 3 to design a polynomial time algorithm to check that a given collection of disks covers the  $\varepsilon$ -erosion  $S^{\ominus\varepsilon}$  and is included in the  $\delta$ -dilation  $S^{\oplus\delta}$  using arrangements of circles.
- The rational ball approximation problem in  $\mathbb{R}^2$  is NP-hard. We prove this through a reduction from the well-known NP-complete vertex cover problem in Section 4.

## 2 Studying boundaries

Let  $\mathcal{S}$  be a finite collection of disks in  $\mathbb{R}^2$  and  $\varepsilon$  a non-negative real number. The purpose of this section is to provide a computational description of the boundary of the erosion  $(\bigcup \mathcal{S})^{\ominus\varepsilon}$ . For this, we introduce in Section 2.2 a superset of the boundary called the *wavefront* and characterize which part of the wavefront belongs to the boundary. In Section 2.3, we decompose the wavefront into elements that are easy to compute, based on the decomposition of  $\partial(\bigcup \mathcal{S})$  into vertices and edges described in Section 2.1.

## 2.1 Boundary of a finite union of disks

Given a finite set of disks  $\mathcal{S}$  in  $\mathbb{R}^2$ , we decompose the boundary of the union  $\bigcup \mathcal{S}$  into a finite set of elements that we call *vertices* and *edges*. For this, we associate to each point  $u \in \mathbb{R}^2$  the subsets of disks that have the property to represent  $\bigcup \mathcal{S}$  locally around  $u$  (see Definition 2) and rely on those subsets to distinguish between vertices or edge points of  $\partial(\bigcup \mathcal{S})$ .

**Definition 2** Given a subset  $X \subseteq \mathbb{R}^n$  and a point  $u \in \mathbb{R}^n$ , we say that a set of balls  $\mathcal{B}$  in  $\mathbb{R}^n$  *represents  $X$  locally around  $u$*  if:

- (i) for all  $b \in \mathcal{B}$ ,  $u \in b$ ;
- (ii) there exists  $r > 0$  such that  $X \cap b(u, r) = \bigcup \mathcal{B} \cap b(u, r)$ .

We first establish that such a set  $\mathcal{B}$  exists and is finite when  $X$  is the union of a finite collection of balls:

*Property 1* The set  $\mathcal{S}_u = \{b \in \mathcal{S} \mid u \in b\}$  represents  $\bigcup \mathcal{S}$  locally around  $u$ , for all  $u \in \mathbb{R}^n$ .

*Proof* For any disk  $b \in \mathcal{S}$ , either  $u \in b$  or the distance between  $u$  and  $b$  is positive. Because  $\mathcal{S}$  is finite,  $\mathcal{S} \setminus \mathcal{S}_u$  is also finite and we let  $r$  be the smallest distance between  $u$  and a disk in  $\mathcal{S} \setminus \mathcal{S}_u$ . We have  $r > 0$ , and by construction  $b(u, r) \cap \bigcup \mathcal{S} \subseteq \bigcup \mathcal{S} \setminus \bigcup (\mathcal{S} \setminus \mathcal{S}_u) \subseteq \bigcup \mathcal{S}_u \subseteq \bigcup \mathcal{S}$ . Thus  $\bigcup \mathcal{S} \cap b(u, r) = \bigcup \mathcal{S}_u \cap b(u, r)$ .  $\square$

Let  $S = \bigcup \mathcal{S}$  be the union of a finite collection of disks  $\mathcal{S}$  in  $\mathbb{R}^2$ . We are now ready to define vertices and edges of  $\partial S$ . We say that a point  $u \in \partial S$  is a *vertex* if all sets  $\mathcal{B}$  that represent  $S$  locally around  $u$  have cardinality greater than 2, that is,  $|\mathcal{B}| \geq 2$ . Letting  $\mathcal{V}$  be the collection of vertices of  $\partial S$ , we call any maximal connected component of  $\partial S \setminus \mathcal{V}$  an *edge* of  $\partial S$ . Then, for any point  $u$  on an edge, there exists a disk  $b \in \mathcal{S}$  such that  $\{b\}$  represents  $S$  locally around  $u$ . Notice that  $\{b\}$  is the unique set of cardinality 1 that represents  $S$  locally around  $u$ , because if two disks  $b_1$  and  $b_2$  share a common boundary point  $u$  and coincide in a neighborhood of  $u$ , then they are equal. We now show that this disk  $b$  is the same for all boundary points in a small neighborhood of  $u$ :

*Property 2* Consider  $u \in \partial S \setminus \mathcal{V}$  and let  $\{b\}$  that represents  $S$  locally around  $u$ . Then, there exists  $r > 0$  such that  $\{b\}$  is the unique cardinal-wise minimum element that represents  $S$  locally around  $v$  for all  $v \in \mathring{b}(u, r) \cap \partial S$ .

*Proof* Let  $r > 0$  such that  $S \cap b(u, r) = b \cap b(u, r)$ . Let  $\rho_v = r - \|u - v\|$  and observe that  $b(v, \rho_v) \subseteq b(u, r)$ . For all  $v \in \mathring{b}(u, r) \cap S$ , we thus have that  $S \cap b(v, \rho_v) = b \cap b(v, \rho_v)$  and it follows that for all points  $v \in \mathring{b}(u, r) \cap S$ ,  $\{b\}$  represents  $S$  locally around  $v$ . In particular, if  $v \in \mathring{b}(u, r) \cap \partial S$ , then  $v \in \partial S \setminus \mathcal{V}$ , and uniqueness and minimality stem from the definition of an edge.  $\square$

We deduce immediately that any edge is supported by the boundary of a unique disk in  $\mathcal{S}$ . Formally:

*Property 3* Let  $e$  be an edge of  $\partial S$ . Then there is a unique  $b \in \mathcal{S}$  such that (1)  $e \subseteq \partial b$ ; (2)  $\{b\}$  represents  $S$  locally around each point of  $e$ .

*Proof* Consider  $f : \partial S \setminus \mathcal{V} \rightarrow \mathcal{S}$  that associates to each point  $x \in \partial S \setminus \mathcal{V}$  the unique inclusion-minimum set that represents  $S$  locally around  $x$ . By Property 2,  $f$  is locally constant and therefore constant over each connected component of  $\partial S \setminus \mathcal{V}$ .  $\square$

## 2.2 Boundary of the erosion and wavefront

Recall that  $\mathcal{S}$  is a finite collection of disks in  $\mathbb{R}^2$  and let  $S = \bigcup \mathcal{S}$  be the union of those disks. In this section, we relate the boundary of the erosion  $\partial S^{\ominus \varepsilon}$  to a (possibly self-intersecting) curve that we call the *wavefront* of  $\partial S$  at time  $\varepsilon$ . Roughly speaking, the wavefront at time  $\varepsilon$  is obtained by moving with speed 1 some of the points on the boundary of  $S$  a given distance  $\varepsilon$  down the inward-pointing normals. Formally, the contribution of a point  $u \in \partial S$  to the wavefront at time  $\varepsilon$  is:

$$w_\varepsilon(u) = \{x \in \partial b(u, \varepsilon) \mid \exists r > 0, b(x, \varepsilon) \cap b(u, r) \subseteq S\}.$$

The wavefront of  $\partial S$  is obtained by taking the union of all contributions  $w_\varepsilon(\partial S) = \bigcup_{u \in \partial S} w_\varepsilon(u)$  (see Figure 2.b). It is not difficult to see that the wavefront contains the boundary of the erosion (see Figure 2.a). Indeed, if a point  $x$  belongs to the erosion  $\partial S^{\ominus \varepsilon}$ , then it is the center of a disk  $b(x, \varepsilon) \subseteq S$  whose boundary meets the boundary of  $S$  at some point  $u$ . It follows that  $x \in \partial b(u, \varepsilon)$  and thus belongs to the contribution  $w_\varepsilon(u)$  of point  $u$ . The goal of this section is to establish that the boundary of the erosion  $\partial S^{\ominus \varepsilon}$  is the set of points on the wavefront  $w_\varepsilon(\partial S)$  at distance  $\varepsilon$  or more to vertices of  $\partial S$  (see Figure 2.c):

**Theorem 2** Let  $\mathcal{V}$  be the vertex set of  $\partial S$ . For  $\varepsilon \geq 0$ , we have

$$\partial S^{\ominus \varepsilon} = w_\varepsilon(\partial S) \setminus (\mathcal{V}^{\oplus \varepsilon})^\circ.$$

*Proof* The inclusion  $\partial S^{\ominus \varepsilon} \subseteq w_\varepsilon(\partial S) \setminus (\mathcal{V}^{\oplus \varepsilon})^\circ$  is obvious. Let us prove the converse inclusion  $w_\varepsilon(\partial S) \setminus (\mathcal{V}^{\oplus \varepsilon})^\circ \subseteq \partial S^{\ominus \varepsilon}$ . Take a point  $x$  in  $w_\varepsilon(\partial S) \setminus (\mathcal{V}^{\oplus \varepsilon})^\circ$ . This implies that there is a point  $u \in \partial S \cap \partial b(x, \varepsilon)$  and  $r > 0$  such that  $b(x, \varepsilon) \cap b(u, r) \subseteq S$ . For  $t \geq 0$ , let  $z_t \in [ux)$  such that  $d(u, z_t) = t$ . Denote by  $b_t$  the disk centered at  $z_t$  and whose bounding circle goes through  $u$ , see Figure 3.a. By construction, the family of disks  $\{b_t\}$  is nested and their boundaries have a common intersection point  $u$ . Let us prove that  $b_\varepsilon \subseteq S$ , which entails  $x = z_\varepsilon \in \partial S^{\ominus \varepsilon}$ .

First, observe that for  $t > 0$  small enough, we have  $b_t \subseteq b(x, \varepsilon) \cap b(u, r) \subseteq S$  and  $\partial b_t \cap \partial S = \{u\}$ . Hence, as we increase continuously  $t$  from 0 to  $+\infty$ , the

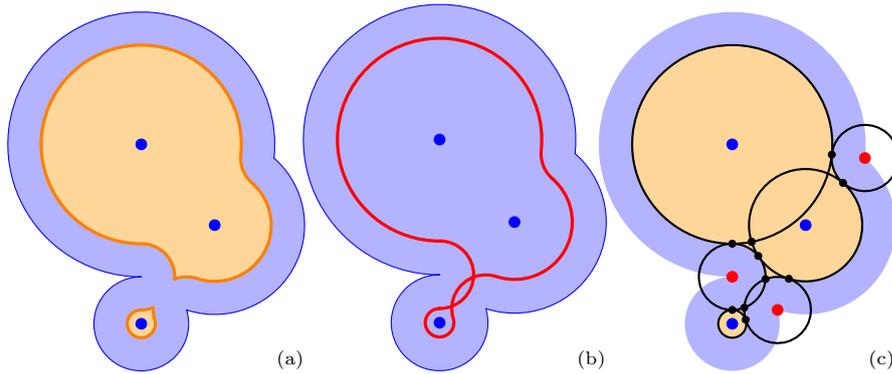


Fig. 2: (a) A shape defined as the union of three (blue) disks and its erosion (in orange); (b) wavefront of the boundary (red curve); (c) arrangement of circles used to decompose the erosion into cells (see Section 3.1). Three of the circles are centered on the (red) vertices of the boundary. The vertices of the arrangement are the black dots.

center  $z_t$  moves on the half-line  $[ux)$  and the ball  $b_t$  which is contained in  $S$  for small values of  $t$  goes eventually outside  $S$  because  $S$  is bounded and  $b_\infty$  is not. Let  $T$  the largest  $t$  for which  $b_t \subseteq S$ . For all  $t < T$ , we have  $\partial b_t \cap \partial S = \{u\}$ . We have  $T > 0$  and suppose for a contradiction that  $T < \varepsilon$ . We claim that  $b_T$  meets the boundary of  $S$  in another point  $y \neq u$  as in Figure 3.b. Indeed, for  $t > T$ , let  $y_t \in b_t \cap S^c$ . The sequence  $(y_t)$  admits a limit point  $y \in \partial b_T$ . If we had  $y = u$ , then for  $t$  close enough to  $T$ , we would have  $b_t \subseteq b_\varepsilon$  and  $y_t \in b_\varepsilon \cap b(u, r) \subseteq S$ , which is impossible. Thus  $y \neq u$ . Since by hypothesis  $b_\varepsilon = b(x, \varepsilon)$  does not contain any vertex of  $\partial S$  in its interior, neither does  $b_T \subseteq b_\varepsilon$  and  $y$  belongs to some edge  $e$  of  $\partial S$ . This edge is supported by the boundary of a ball  $b \in \mathcal{S}$  that represents  $S$  locally around  $y$ . It follows that  $b$  and  $b_T$  are tangent at  $y$  with  $b_T \subseteq b$ . Since  $b \subseteq S$  cannot contain  $u \in \partial S$  in its interior, we get  $b = b_T \subseteq b_\varepsilon$  and the edge  $e$  is either the whole circle  $\partial b_T$  or the punctured circle  $\partial b_T \setminus \{u\}$ . In both cases,  $\partial b_T \subseteq \partial S$ . Since  $b_T$  is contained in  $b_\varepsilon$  which is contained in  $S$  locally around  $u$ , the only possibility is that  $b_T = b_\varepsilon$ , yielding  $T = \varepsilon$  and therefore a contradiction.  $\square$

### 2.3 Computational description of the wavefront

The goal of this section is to give a computational description of the wavefront. In particular, we shall see that the wavefront is composed of circular arcs; convex circular arcs are contributed by edges of  $\partial S$  while concave circular arcs are contributed by vertices of  $\partial S$ . To describe those arcs, let us start by studying the contribution of a point  $u \in \partial S$  to the wavefront at time  $\varepsilon$ . We show that this contribution can be expressed as the intersection of  $\partial b(u, \varepsilon)$  and an affine conical hull with apex  $u$ . Consider a finite collection of points

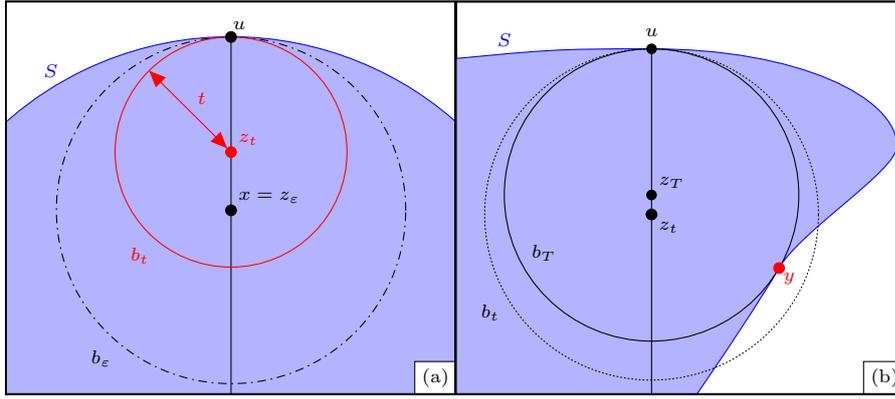


Fig. 3: Notation for the proof of Theorem 2. (a) Construction of  $z_t$  and  $b_t$ ; (b) second boundary point  $y$ .

$C = \{c_1, c_2, \dots, c_k\}$  in  $\mathbb{R}^2$ . We recall that the affine conical hull with support  $C$  and apex  $u \in \mathbb{R}^2$  is

$$\text{coni}(u, C) = \left\{ u + \sum_{i=1}^k \lambda_i (c_i - u) \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \right\}.$$

For technical reasons, we shall need to exclude some of the points on the boundary of affine conical hulls. This motivates the following definition:

$$\text{coni}^*(u, C) = \text{coni}(u, C) \cup \left( \bigcup_{c \in C} [uc] \right).$$

**Theorem 3** Let  $S = \bigcup \mathcal{S}$  be a finite union of disks and consider  $u \in \partial S$ . Let  $\mathcal{B} \subseteq \mathcal{S}$  that represents  $S$  locally around  $u$ , and  $C$  the centers of disks in  $\mathcal{B}$ . We have

$$w_\varepsilon(u) = \text{coni}^*(u, C) \cap \partial b(u, \varepsilon).$$

An illustration is available in Figure 4. Before proving the theorem, we describe for each element on the boundary of the union  $\bigcup \mathcal{S}$  its contribution to the wavefront:

*Contribution of vertices.* Assume that  $v$  is a vertex of  $\partial S$ . Then any set  $\mathcal{B}$  that represents  $S$  locally around  $v$  is such that  $|\mathcal{B}| \geq 2$ , and  $w_\varepsilon(v)$  is a concave circular arc in the general case. In the special case when  $\mathcal{S}$  has two disks  $b_1$  and  $b_2$  meeting tangentially at  $v$  and whose pair  $\mathcal{B} = \{b_1, b_2\}$  represents locally  $S$  around  $v$ , then  $w_\varepsilon(v) = [c_1 c_2] \cap \partial b(v, \varepsilon)$ , where  $c_i$  is the center of  $b_i$  for  $i \in \{1, 2\}$ . It follows that  $w_\varepsilon(v)$  is composed of zero, one or two degenerate circular arcs (each reduced to a point), depending on whether zero, one or two disks among  $b_1$  and  $b_2$  have radii larger than or equal to  $\varepsilon$ .

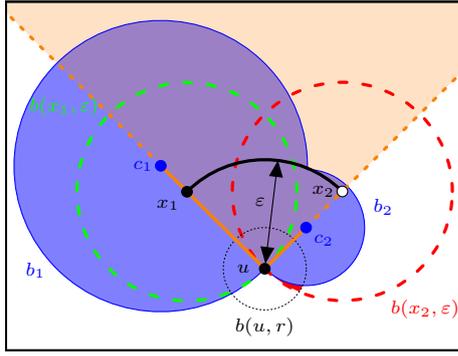


Fig. 4: A shape represented locally around  $u$  by two (blue) disks  $b_1$  and  $b_2$ . The contribution of  $u$  to the wavefront is the (black) circular arc, closed at  $x_1$  and open at  $x_2$ . By Theorem 3, it is the intersection of the (orange) affine conical hull and  $\partial b(u, \varepsilon)$ . The endpoint  $x_1$  belongs to the wavefront (unlike the endpoint  $x_2$ ) because the disk  $b(x_1, \varepsilon)$  is locally included in the shape around  $u$  (unlike the disk  $b(x_2, \varepsilon)$ ).

*Contribution of edges.* Let  $e$  be an edge of  $\partial S$ . By Proposition 3, there is a disk  $b \in \mathcal{S}$  whose boundary supports  $e$  and which represents  $S$  locally around each point  $u \in e$ . Letting  $c$  be the center of  $b$  and plugging in Theorem 3 the equality  $C = \{c\}$  yields  $w_\varepsilon(u) = [uc] \cap \partial b(u, \varepsilon) = [uc] \cap \partial b^{\ominus \varepsilon}$  for all  $u \in e$ . It follows that the contribution  $w_\varepsilon(e) = \bigcup_{u \in e} w_\varepsilon(u)$  of an edge  $e$  is either empty if  $b$  has a radius smaller than  $\varepsilon$  or is the convex circular arc obtained by projecting  $e$  orthogonally onto  $b^{\ominus \varepsilon}$  otherwise.

*Boundary of the erosion.* The boundary of the erosion  $\partial S^{\ominus \varepsilon}$  being a subset of the wavefront, it is also composed of circular arcs supported by the set of circles (see Figure 2.c):

$$\mathcal{C}_\varepsilon(\mathcal{S}) = \{\partial b(v, \varepsilon) \mid v \in \mathcal{V}\} \cup \{\partial b^{\ominus \varepsilon} \mid b \in \mathcal{S} \text{ and } \text{radius}(b) \geq \varepsilon\}.$$

The remainder of this section is dedicated to the proof of Theorem 3. The proof relies on two technical lemma that we now state. Consider a family of disks that all share a common boundary point  $u$ . The first technical lemma says that if the center of one of the disk  $b_0$  is contained in the open affine conical hull formed by the centers of other disks, then  $b_0$  is contained in the union of the other disks locally around  $u$ . The second technical lemma gives a slightly weaker version of the converse.

**Lemma 1** *Let  $u \in \mathbb{R}^2$  and consider  $k$  points  $c_1, c_2, \dots, c_k \in \mathbb{R}^2$ . Let  $c_0 \in \text{coni}(u, \{c_i, 1 \leq i \leq k\})$ . For  $0 \leq i \leq k$ , let  $b_i$  be the disk centered at  $c_i$  whose bounding circle goes through  $u$ . Then, there exists  $r > 0$  such that  $b_0 \cap b(u, r) \subseteq \bigcup_{i=1}^k b_i$ .*

*Proof* Consider the closed half-space  $H_0$  through  $u$  containing  $b_0$ . We prove the theorem by establishing that  $H_0 \cap b(u, r) \subseteq \bigcup_{i=1}^k b_i$ .

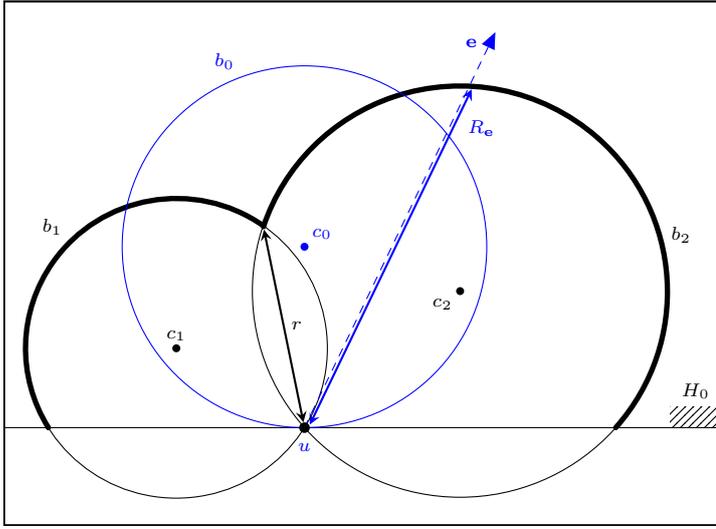


Fig. 5: Notation for the proof of Lemma 1. The bold curve represents the set of points at distance  $R_{\mathbf{e}}$  from the origin in directions  $\mathbf{e}$  such that  $\langle \mathbf{e}, c_0 - u \rangle \geq 0$ .

Notice that for all unit vectors  $\mathbf{e}$  and real numbers  $\rho$ , the point  $u + \rho\mathbf{e}$  belongs to  $b_i$  if and only if  $0 \leq \rho \leq 2\langle \mathbf{e}, c_i - u \rangle$ . It follows that  $u + \rho\mathbf{e}$  belongs to the union  $\bigcup_{i=1}^k b_i$  if and only if  $0 \leq \rho \leq R_{\mathbf{e}}$  where  $R_{\mathbf{e}} = 2 \max_{1 \leq i \leq k} \langle \mathbf{e}, c_i - u \rangle$ ; see Figure 5.

Let  $r = \inf\{R_{\mathbf{e}} \mid \langle \mathbf{e}, c_0 - u \rangle \geq 0, \|\mathbf{e}\| = 1\}$ . We next prove that  $r > 0$ . Consider a unit vector  $\mathbf{e}$  with  $\langle \mathbf{e}, c_0 - u \rangle \geq 0$ . Because  $c_0$  lies in the interior of  $\text{coni}(u, \{c_i, 1 \leq i \leq k\})$ , there exist  $k$  real positive numbers  $\lambda_1, \lambda_2, \dots, \lambda_k$  such that  $c_0 = u + \sum_{i=1}^k \lambda_i(c_i - u)$ ; see Appendix B. By linearity,

$$0 \leq \sum_{i=1}^k \lambda_i \langle \mathbf{e}, c_i - u \rangle \leq \left( \sum_{i=1}^k \lambda_i \right) \max_{1 \leq i \leq k} \langle \mathbf{e}, c_i - u \rangle.$$

Assume for a contradiction that  $\max_{1 \leq i \leq k} \langle \mathbf{e}, c_i - u \rangle \leq 0$ . Since  $\lambda_i > 0$  for all  $1 \leq i \leq k$ , this would imply that  $\langle \mathbf{e}, c_i - u \rangle = 0$  for all  $1 \leq i \leq k$ , showing that all vectors  $c_i - u$  are collinear. Hence, the affine conical hull  $\text{coni}(u, \{c_i, 1 \leq i \leq k\})$  would be either a line or a half-line, contradicting our hypothesis that its interior  $\text{coni}(u, \{c_i, 1 \leq i \leq k\})$  contains  $c_0$  and is non-empty.

To conclude, consider a point  $x \in H_0 \cap b(u, r)$ . Let  $\rho = \|x - u\|$  and  $\mathbf{e}$  the unit vector such that  $x = u + \rho\mathbf{e}$ . Because  $x \in H_0$ ,  $\langle \mathbf{e}, c_0 - u \rangle \geq 0$ . Because  $x \in b(u, r)$ ,  $0 \leq \rho \leq r \leq R_{\mathbf{e}}$ . Hence,  $x \in \bigcup_{i=1}^k b_i$  and  $H_0 \cap b(u, r) \subseteq \bigcup_{i=1}^k b_i$ .  $\square$

**Lemma 2** *Let  $u \in \mathbb{R}^2$  and  $k+1$  points  $c_0, c_1, \dots, c_k \in \mathbb{R}^2 \setminus \{u\}$ . For  $0 \leq i \leq k$ , let  $b_i$  be the disk centered at  $c_i$  whose bounding circle goes through  $u$ . If there exists  $r > 0$  such that  $b_0 \cap b(u, r) \subseteq \bigcup_{i=1}^k b_i$ , then  $c_0 \in \text{coni}(u, \{c_i, 1 \leq i \leq k\})$ .*

*Proof* Let  $H_i^+$  be the closed half-space through  $u$  containing  $b_i$  and  $H_i^-$  the closed half-space through  $u$  that avoids the interior of  $b_i$ . The inclusion  $b_0 \cap b(u, r) \subseteq \bigcup_{i=1}^k b_i$  implies that  $(H_0^+)^{\circ} \subseteq \bigcup_{i=1}^k (H_i^+)^{\circ}$  and  $\bigcap_{i=1}^k H_i^- \subseteq H_0^-$ . We consider two cases:

- (a) The open half-space  $(H_0^+)^{\circ}$  is equal to one of the open half-spaces  $(H_i^+)^{\circ}$ , say  $(H_1^+)^{\circ}$ . Clearly,  $c_0 \in \text{coni}(u, \{c_1\})$ .
- (b) The open half-space  $(H_0^+)^{\circ}$  is distinct from all other half-spaces  $(H_i^+)^{\circ}$  and contained in the union of two of them, say  $(H_1^+)^{\circ}$  and  $(H_2^+)^{\circ}$ . In that case,  $u$  and the two centers  $c_1$  and  $c_2$  are not aligned and the vectors  $c_1 - u$  and  $c_2 - u$  span  $\mathbb{R}^2$ . It follows that there exist two real numbers  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$  such that  $c_0 - u = \lambda_1(c_1 - u) + \lambda_2(c_2 - u)$ . Let us prove that  $\lambda_1 > 0$  and  $\lambda_2 > 0$ . Consider a vector  $t_1$  orthogonal to  $c_1 - u$  and choose it so that  $u + t_1 \in H_2^-$ . We have  $\langle t_1, c_0 - u \rangle = \lambda_2 \langle t_1, c_2 - u \rangle$ . By choice of  $t_1$ , we have  $u + t_1 \in H_1^- \cap H_2^- \subseteq H_0^-$  and therefore  $\langle t_1, c_0 - u \rangle \leq 0$  and  $\langle t_1, c_2 - u \rangle \leq 0$ , showing that  $\lambda_2 > 0$ . Hence,  $c_0 \in \text{coni}(u, \{c_1, c_2\})$ .  $\square$

We are now ready to give the proof of Theorem 3.

*Proof of Theorem 3* Let us establish the inclusion  $\text{coni}(u, C) \cap \partial b(u, \varepsilon) \subseteq w_\varepsilon(u)$ . First, consider  $x \in \text{coni}(u, C) \cap \partial b(u, \varepsilon)$ . By Lemma 1, there exists  $r > 0$  such that  $b(x, \varepsilon) \cap b(u, r) \subseteq \bigcup \mathcal{B} \subseteq S$ . Hence,  $x \in w_\varepsilon(u)$ . Second, suppose that there exists  $c \in C$  such that  $\{x\} = [uc] \cap \partial b(u, \varepsilon)$ . Clearly,  $b(x, \varepsilon) \subseteq b(c, \|c - u\|) \subseteq S$ . Once again,  $x \in w_\varepsilon(u)$ .

Conversely, let us establish the inclusion  $w_\varepsilon(u) \subseteq \text{coni}(u, C) \cap \partial b(u, \varepsilon)$ . Consider  $x \in w_\varepsilon(u)$ . By definition,  $x \in \partial b(u, \varepsilon)$  and there exists  $r > 0$  such that  $b(x, \varepsilon) \cap b(u, r) \subseteq S$ . Since  $\mathcal{B}$  represents  $S$  locally around  $u$ , we also have  $b(x, \varepsilon) \cap b(u, r) \subseteq \bigcup \mathcal{B}$  for  $r$  small enough. It follows by Lemma 2 that  $x \in \text{coni}(u, C) \cap \partial b(u, \varepsilon)$ . If  $x \in \text{coni}(u, C)$ , we are done. Otherwise,  $x$  lies on the boundary of  $\text{coni}(u, C)$ . This boundary consists of half-lines originating from  $u$  and going through some point of  $C$ . Let  $c \in C$  such that  $c$  supports one such half-line and  $x \in [uc]$ . Without loss of generality, we may assume that the half-line  $[uc]$  only meets  $C$  at  $c$  itself. Let us prove that  $x \in [uc]$ . Consider  $b$  the disk of  $\mathcal{B}$  centered at  $c$ , and notice that, since  $[uc] \subseteq \partial \text{coni}(u, C)$ ,  $b$  supports an edge of  $\partial S$  in the neighbourhood of  $u$ . Assume by contradiction that  $x \in [uc] \setminus [uc]$ . Then  $b \subseteq b(x, \varepsilon)$ . In particular  $b(x, \varepsilon)$  contains in its interior and in a neighborhood of  $u$  part of the edge supported by  $b$ , implying that  $x \notin w_\varepsilon(u)$ .  $\square$

### 3 Rational $(\delta, \varepsilon)$ -ball approximation is in NP

To check whether a finite collection of disks  $\mathcal{B}$  is a  $(\delta, \varepsilon)$ -ball approximation of a finite union of disks  $\bigcup \mathcal{S}$ , we have to test the two inclusions  $(\bigcup \mathcal{S})^{\ominus \varepsilon} \subseteq \bigcup \mathcal{B} \subseteq (\bigcup \mathcal{S})^{\oplus \delta}$  in polynomial time. To do so, we build on results of Section 3 and describe a method that given the triplet  $(\mathcal{S}, \varepsilon, \mathcal{B})$  returns true if and

only if the first inclusion  $(\bigcup \mathcal{S})^{\ominus \varepsilon} \subseteq \bigcup \mathcal{B}$  holds. To test the second inclusion, it suffices to apply the same method to the triplet  $(\mathcal{B}, 0, \{s^{\oplus \delta} \mid s \in \mathcal{S}\})$ .

Before describing our method in Section 3.2, we explain in Section 3.1 how to encode the erosion (or union) of a finite union of disks using cells of an arrangement of circles.

### 3.1 Capturing the erosion through an arrangement

Recall that an arrangement of circles is the decomposition of  $\mathbb{R}^2$  into open cells induced by the circles. Cells in the arrangement are either vertices, edges or faces. Such arrangements have a size quadratic in the number of circles and can be computed in polynomial time [29]. In this section, we let  $\mathcal{A}_\varepsilon(\mathcal{S})$  be the arrangement formed by circles in  $\mathcal{C}_\varepsilon(\mathcal{S})$  and show that the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$  can be captured by cells in  $\mathcal{A}_\varepsilon(\mathcal{S})$ . This means that we can label cells in  $\mathcal{A}_\varepsilon(\mathcal{S})$  as IN or OUT in such a way that cells with label IN partition the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$ . We shall see that labeling the arrangement can be done in polynomial time. Furthermore, the arrangement can be computed in an exact manner [35] because all circles involved in the construction are rational (center coordinates and radii are rational numbers).

We adopt the convention that each time a circle in  $\mathcal{C}_\varepsilon(\mathcal{S})$  has radius zero, it counts as a vertex in the arrangement  $\mathcal{A}_\varepsilon(\mathcal{S})$ . We already know from Section 2 that the boundary of the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$  is supported by circles in the collection  $\mathcal{C}_\varepsilon(\mathcal{S})$ . Hence, the edges and vertices of  $\mathcal{A}_\varepsilon(\mathcal{S})$  capture the boundary of the erosion. It follows that every cell of  $\mathcal{A}_\varepsilon(\mathcal{S})$  is either inside or outside of the erosion. We establish the following result:

**Lemma 3** *We can label in polynomial time  $\mathcal{A}_\varepsilon(\mathcal{S})$  in such a way that cells with label IN partition the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$ .*

To label cells in the arrangement, we need the following technical lemma:

**Lemma 4** *Let  $e$  be an edge of  $\mathcal{A}_\varepsilon(\mathcal{S})$  that supports the boundary of the erosion. Then  $e$  is adjacent to a face in the erosion and a face in its complement.*

*Proof* Let  $S = \bigcup \mathcal{S}$  and  $x \in e$ . By Theorem 2, there is  $u \in \partial S$  such that  $x \in w_\varepsilon(u) \setminus (\mathcal{V}^{\oplus \varepsilon})^\circ$ . Let  $z_t$  be the point on half-line  $[ux)$  such that  $d(z_t, u) = t$  (see Figure 3.a). To establish the lemma, we proceed in three steps:

(1) First, we prove that  $z_t \notin (\mathcal{V}^{\oplus t})^\circ$  for all  $t$  close enough to  $\varepsilon$ . Recall that  $x \notin (\mathcal{V}^{\oplus \varepsilon})^\circ$  or equivalently  $d(x, \mathcal{V}) \geq \varepsilon$ . We claim that if we exclude  $u$  from the vertex set  $\mathcal{V}$ , then the distance from  $x$  to the remaining vertices of  $\mathcal{V}$  can only be larger, that is,  $d(x, \mathcal{V} \setminus \{u\}) > \varepsilon$ . Indeed, if this is not the case, then  $x$  would be equidistant to  $u$  and a vertex  $v \in \mathcal{V} \setminus \{u\}$ . This would imply that  $x$  lies on at least two circles of the arrangement (the one supporting  $e$  and another bounding  $b(v, \varepsilon)$ ). Hence,  $x$  would be a vertex of the arrangement, contradicting our assumption that  $x$  lies on edge  $e$ . It follows that we can

find  $T > \varepsilon$  such that for all  $0 \leq t \leq T$ , we have  $d(z_t, \mathcal{V} \setminus \{u\}) \geq d(z_T, \mathcal{V} \setminus \{u\}) - d(z_t, z_T) > T - (T - t) = t$ . Hence,  $d(z_t, \mathcal{V}) \geq t$  for all  $0 \leq t \leq T$  or equivalently  $z_t \notin (\mathcal{V}^{\oplus t})^\circ$  for all  $0 \leq t \leq T$ .

(2) Second, we prove that  $z_t \in w_t(u)$  for all  $t$  close enough to  $\varepsilon$ . Let  $F$  a collection of disks that represents  $S$  locally around  $u$  and  $C$  the collection of their centers. Recall that by Theorem 3,  $w_t(u) = \text{coni}(u, C) \cap \partial b(u, t)$ , hence  $x \in \text{coni}(u, C)$ . We want to prove that we also have  $z_t \in \text{coni}(u, C)$ , for  $t$  in an open neighbourhood of  $\varepsilon$ . If  $x \in \text{coni}(u, C)$ , then for all  $t > 0$ ,  $z_t \in \text{coni}(u, C)$ . If  $x \notin \text{coni}(u, C)$ , then there is  $c \in C$  such that  $x \in [uc]$ . Without loss of generality we can assume that  $\text{coni}(u, C) \cap [uc] = [uc]$ . We claim that  $x \neq c$ . Indeed let  $b = b(c, r)$  be the disk in  $\mathcal{S}$  centered at  $c$ . Assume for a contradiction that  $x = c$ . Then,  $r = \varepsilon$  and  $b^{\ominus \varepsilon} = \{c\}$ , showing that  $c = x$  is a vertex of the arrangement and contradicting our assumption  $x \in e$ . Thus,  $x \neq c$  and  $r > \varepsilon$ . Also for all  $0 \leq t \leq r$ , we have  $z_t \in [uc] \subseteq \text{coni}(u, C)$ , hence  $z_t \in w_t(u)$ .

(3) Steps (1) and (2) imply that for  $t$  close enough to  $\varepsilon$ , we have  $z_t \in w_t(u) \setminus (\mathcal{V}^{\oplus t})^\circ$ . By Theorem 2, this implies that  $z_t \in \partial(S^{\ominus t})$  and thus that  $d(z_t, \partial S) = t$ . In particular, for  $t < \varepsilon$ ,  $d(z_t, \partial S) < \varepsilon$  and  $z_t \in (S^{\ominus \varepsilon})^c$  while for  $t > \varepsilon$ ,  $d(z_t, \partial S) > \varepsilon$  and  $z_t \in S^{\ominus \varepsilon}$ , which suffices to conclude.  $\square$

We now prove Lemma 3.

*Proof of Lemma 3* We label cells in  $\mathcal{A}_\varepsilon(\mathcal{S})$  as follows:

(a) We label as BOUNDARY all cells on the boundary of the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$ , using results in Section 2. More precisely, we compute first vertices and edges on the boundary of  $\bigcup \mathcal{S}$  (building for instance the arrangement  $\mathcal{A}_0(\mathcal{S})$ ). Then, for each element on the boundary of  $\bigcup \mathcal{S}$ , we compute its contribution to the wavefront. Each contribution is composed of cells and we label the cells belonging to the boundary of the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$  as BOUNDARY.

(b) We label as IN each face covered by a disk  $b^{\ominus \varepsilon}$  for  $b \in \mathcal{S}$  and label as OUT each face covered by a disk  $b(v, \varepsilon)$  for  $v \in \mathcal{V}$ . If no faces have received a label so far, this means that  $S^{\ominus \varepsilon}$  is either empty or a finite collection of singletons. In this case, each singleton is labeled IN, all other cells are labeled OUT and we are done. Otherwise, we continue.

(c) Starting from a face either with label IN or label OUT, we do a traversal of the arrangement  $\mathcal{A}_\varepsilon(\mathcal{S})$ , stepping from one face to an adjacent face. If we cross an edge labeled BOUNDARY, the face currently visited receives a label different from the face on the other side of the crossed edge. Otherwise, the face currently visited receives the same label as the face on the other side. Lemma 4 guarantees that after this step, every face is properly labeled.

(d) To label remaining vertices and edges, we use the fact that the erosion is a closed set. We thus give label IN to every vertex or edge bounding a face labeled IN. Other vertices and edges receive the label OUT.

Clearly, all steps can be done in time polynomial in the size of the arrangement.  $\square$

### 3.2 Checking set inclusion

We explain in this section how to check the inclusion  $(\bigcup \mathcal{S})^{\ominus \varepsilon} \subseteq \bigcup \mathcal{B}$  in polynomial time. We compute the arrangement  $\mathcal{A}_\varepsilon(\mathcal{S})$  and label its cells as IN or OUT depending on whether they belong to the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$  or not as explained in the previous section. Likewise, we compute a second arrangement  $\mathcal{A}_0(\mathcal{B})$  and label its cells as IN or OUT depending on whether they belong to the union  $\bigcup \mathcal{B}$  or not. We then overlay the two arrangements. This operation consists in computing the arrangement of the joint collection of circles, and it is possible to do so while inheriting the labels from both input arrangements [4, 34]. By inspecting the labels of the overlay, we can thus determine if every cell in the erosion is also in the union and conclude on the inclusion test. All operations can be done in polynomial time. In Appendix A, we provide an alternative method that does not use any overlay.

## 4 Rational $(\delta, \varepsilon)$ -ball approximation is NP-hard

In this section, we establish that the rational  $(\delta, \varepsilon)$ -ball approximation problem in  $\mathbb{R}^n$  is already NP-hard for  $n = 2$ . We prove NP-hardness through a reduction from a variant of the vertex cover problem. Recall that for a graph  $G = (V, E)$ , a subset  $C \subseteq V$  is a vertex cover of  $G$  if every edge of  $E$  is incident to a vertex of  $C$ . Finding a minimum vertex cover is NP-hard, even when restricted to cubic planar graphs, that is, planar graphs of degree at most 3 [20]. We shall perform the reduction from this particular variant. Given a cubic planar graph  $G = (V, E)$  and real non-negative numbers  $\delta$  and  $\varepsilon$  with  $\delta + \varepsilon > 0$ , we show how to build a finite collection  $\mathcal{S}(G, \delta, \varepsilon)$  of disks in the plane such that  $G$  has a vertex cover of cardinality  $k$  if and only if  $\bigcup \mathcal{S}(G, \delta, \varepsilon)$  admits a  $(\delta, \varepsilon)$ -ball approximation of cardinality  $k + L$ , where  $L$  is an integer depending only on  $\mathcal{S}(G, \delta, \varepsilon)$ .

### 4.1 Reduction from vertex cover

*Gadgets.* We build the set of disks  $\mathcal{S}(G, \delta, \varepsilon)$  from the pair  $(G, \delta, \varepsilon)$  by converting each vertex of  $G$  into a *vertex gadget* and each edge of  $G$  into an *edge gadget*. Given a vertex  $v \in V$ , the vertex gadget associated to  $v$  is a singleton containing only one disk of radius  $\lambda\varepsilon$ : any value of lambda in  $[0, 1[$  works for our construction, meaning in particular that these disks are not mandatory, but including them make the explanations below clearer. In Figures 6, 7, 9 and 10 we choose  $\lambda = 0.8$ . The center of that disk is called a *ghost* and is denoted  $g_v$ . The reason for this is that ghosts won't belong to the  $\varepsilon$ -erosion of  $\bigcup \mathcal{S}(G, \delta, \varepsilon)$ . Given an edge  $\{x, y\} \in E$ , the edge gadget associated to  $\{x, y\}$  is a finite sequence of disks of odd length at least three; see Figure 6.b. The disks in the sequence all share the same radius  $\varepsilon$ . The centers of those disks are called *rotulae*. We distinguish two types of rotulae: *Extreme* rotulae are

the centers of the two extreme disks in the sequence and are denoted as  $c_{xy}$  and  $c_{yx}$ . *Regular* rotulae are the centers of other disks. The rotulae of two consecutive disks in the sequence are said to be neighbors of one another. Note that extreme rotulae have only one neighbor and regular rotulae have two neighbors. We say that an extreme rotula  $c_{xy}$  and a ghost  $g_v$  are *linked* to one another whenever  $x = v$ . Hence, each extreme rotula is linked to a unique ghost, whereas a ghost may be linked to up to three extreme rotulae depending on the degree of the vertex that generated the ghost. Finally, we define  $\mathcal{S}(G, \delta, \varepsilon)$  as the collection of disks resulting from the conversion of vertices and edges into vertex gadgets and edge gadgets respectively.

At this point, we haven't yet specified the centers of disks, nor the number of rotulae per edge that we need (only that it should be an odd number). We postpone to Section 4.2 the description of how disks composing gadgets are chosen. This will be done from an orthogonal grid drawing of  $G$ ; see Figure 6 for an example. At this stage it suffices to know that we can always choose disks composing gadgets so that they fulfill the properties (i)-(viii) listed below. To simplify notations, we shall refer to  $\mathcal{S}(G, \delta, \varepsilon)$  and  $S(G, \delta, \varepsilon) = \bigcup \mathcal{S}(G, \delta, \varepsilon)$  simply as  $\mathcal{S}$  and  $S$ .

- (i) The  $\varepsilon$ -erosion  $S^{\oplus \varepsilon}$  is exactly the collection of rotulae.
- (ii) Any disk  $b \subseteq S^{\oplus \delta}$  contains at most 2 regular rotulae.
- (iii) If a disk  $b \subseteq S^{\oplus \delta}$  contains 2 regular rotulae, then they are neighbors of each other and  $b$  does not contain any other rotula, neither regular nor extreme.
- (iv) Let  $c$  be a regular rotula,  $c_1$  and  $c_2$  its two neighbors. There exist two disks  $b_1, b_2 \subseteq S^{\oplus \delta}$  such that  $\{c_1, c\} \subseteq b_1$  and  $\{c_2, c\} \subseteq b_2$ .
- (v) Any disk  $b \subseteq S^{\oplus \delta}$  contains at most 3 extreme rotulae.
- (vi) If a disk  $b \subseteq S^{\oplus \delta}$  contains 2 or 3 extreme rotulae, then these extreme rotulae are linked to the same ghost and  $b$  only contains extreme rotulae linked to that ghost.
- (vii) Let  $g$  be a ghost. There exists a disk  $b \subseteq S^{\oplus \delta}$  that contains all extreme rotulae linked to  $g$ .
- (viii) If a disk  $b \subseteq S^{\oplus \delta}$  contains both a regular rotula and an extreme rotula, then these are neighbors and  $b$  does not contain any other rotula, neither regular nor extreme.

*Canonical coverings.* Recall that each edge of  $G$  is associated to an odd number of rotulae (which are the centers of disks in the associated edge gadget). Given an edge  $e \in E$ , we ask for the smallest number of disks contained in  $S^{\oplus \delta}$  that suffices to cover the rotulae of  $e$ . Let  $2\ell(e) + 1$  be the number of rotulae in  $e$ , where  $\ell(e) \geq 1$  is an integer. Since  $e$  has two extreme rotulae, the number of regular rotulae is  $2\ell(e) - 1$ . From property (ii) we need at least  $\lceil (2\ell(e) - 1)/2 \rceil = \ell(e)$  disks in  $S^{\oplus \delta}$  in order to cover these regular rotulae. By (iv) there always exists a collection of  $\ell(e)$  disks in  $S^{\oplus \delta}$  which covers these rotulae plus one of the two extreme rotulae of  $e$ , and that extreme rotula can be chosen arbitrarily. Indeed, it suffices to cover pairs of neighbouring rotulae

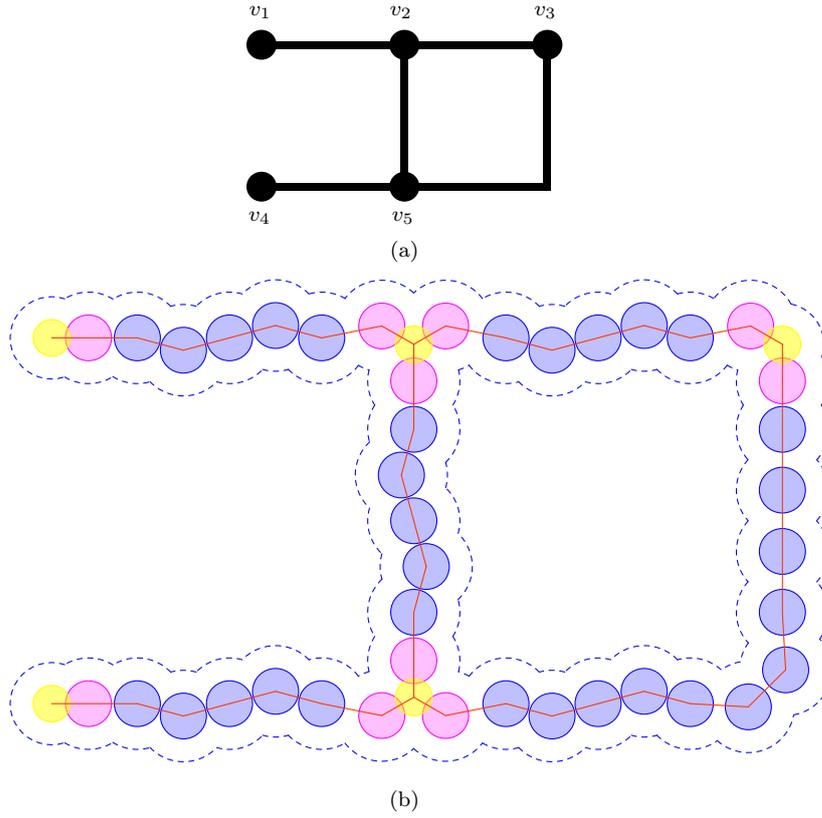


Fig. 6: Conversion from (a) an orthogonal grid drawing of a graph  $G$  with 5 vertices and 5 edges to (b) the collection of disks  $\mathcal{S}(G, \delta, \varepsilon)$  with 5 (yellow) ghost disks, 10 (purple) extreme rotulae and 31 (blue) regular rotulae. Ghost disks have radii  $\lambda\varepsilon$  with  $\lambda = 0.8$ . The  $\delta$ -dilation of the union of disks is bounded by the dashed blue lines.

of  $e$  with disks in  $S^{\oplus\delta}$ , while making sure that the extreme rotula to cover and its neighbour are one of these pairs (see Figure 7). This gives two possible coverings of the regular rotulae of  $e$  (one for each extreme rotulae) which we shall refer to as *canonical* for  $e$ . Furthermore, properties (iii) and (viii) guarantee that any disk containing a regular rotula will only contain rotulae belonging to the same edge gadget. Therefore it is necessary and sufficient to use  $\ell(e)$  disks in  $S^{\oplus\delta}$  to cover the regular rotulae of an edge gadget  $e$ , and these  $\ell(e)$  disks exclusively cover rotulae of  $e$ . However, one of the two extreme rotulae of  $e$  will not be covered by these  $\ell(e)$  disks and one extra disk is required to cover all rotulae of  $e$  for a total of  $\ell(e) + 1$  disks. Unlike the previous  $\ell(e)$  disks, by (v) this extra disk may be used to cover extreme rotulae of several edges assuming these extreme rotulae are all linked to the same ghost. We define  $L = \sum_{e \in E} \ell(e)$  which is the smallest number of disks needed to cover all regular rotulae and which can be achieved by taking for each edge, one of its two possible canonical coverings.

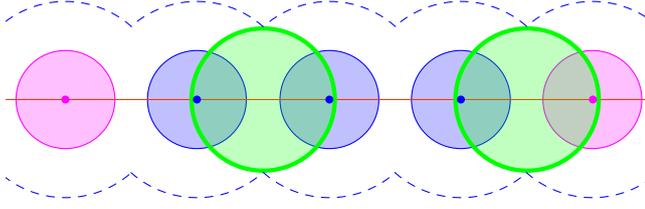


Fig. 7: A canonical covering of an edge gadget with two green disks. Same color convention as in Figure 6.b.

*Property 4* If  $G$  has a vertex cover  $C \subseteq V$ , then  $S$  has a  $(\delta, \varepsilon)$ -ball approximation  $\mathcal{B}$  with  $|\mathcal{B}| = L + |C|$ .

*Proof* By (i), we know that any  $(\delta, \varepsilon)$ -ball approximation of  $S$  is a collection of disks contained in  $S^{\oplus \delta}$  whose union covers the rotulae of all edges of  $G$ . To build such a  $(\delta, \varepsilon)$ -ball approximation, we use property (vii) and start by selecting for each vertex  $v \in C$  a disk that covers all extreme rotulae linked to the ghost  $g_v$ . This gives a set  $\mathcal{C}$  of  $|C|$  disks that cover at least one extreme rotula per edge. Using the appropriate canonical covering of each edge, that is, the one excluding the extreme rotula already covered by  $\mathcal{C}$ , we complete the set  $\mathcal{C}$  into a  $(\delta, \varepsilon)$ -ball approximation adding  $L$  more disks.  $\square$

*Property 5* If  $S$  has a  $(\delta, \varepsilon)$ -ball approximation  $\mathcal{B}$ , then  $G$  has a vertex cover  $C$  with  $|C| \leq |\mathcal{B}| - L$ .

*Proof* Without loss of generality, we may assume that all disks in  $\mathcal{B}$  cover at least one rotula. Indeed, if a disk  $b$  does not cover any rotula, it can be removed from  $\mathcal{B}$  while keeping the property that  $\mathcal{B}$  is a  $(\delta, \varepsilon)$ -ball approximation. Starting from  $\mathcal{B}$ , we first deduce a  $(\delta, \varepsilon)$ -ball approximation  $\mathcal{R}$  of  $S$  having the property that it contains one of the two canonical coverings of each edge  $e \in E$ . For  $e \in E$ , let  $\mathcal{B}_e = \{b \in \mathcal{B} \mid b \text{ contains a regular rotula of } e\}$ . Note that for different edges the  $\mathcal{B}_e$  are disjoint and that  $|\mathcal{B}_e| \geq \ell(e)$ . Given an extreme rotula  $c_{xy}$ , we denote by  $\mathcal{C}_{xy}$  the canonical covering that contains  $c_{xy}$  and all the regular rotulae of edge  $\{x, y\}$ . Initializing  $\mathcal{R}$  to  $\mathcal{B}$ , we modify  $\mathcal{R}$  as follows, applying the following transformation to each edge  $e$  of  $G$ :

- If disks in  $\mathcal{B}_e$  cover no extreme rotula, we replace  $\mathcal{B}_e$  by one of the two canonical coverings of  $e$ .
- If disks in  $\mathcal{B}_e$  cover 1 extreme rotula, we replace  $\mathcal{B}_e$  by the canonical covering of  $e$  that covers this extreme rotula.
- If disks in  $\mathcal{B}_e$  contains both extreme rotulae of  $e$ , then  $|\mathcal{B}_e| \geq n(e) + 1$ . We choose arbitrarily one of the two extreme rotulae, say  $c_{xy}$ , and let  $b$  be a disk in  $S^{\oplus \varepsilon}$  containing the other extreme rotula  $c_{yx}$  but no regular rotula. We replace  $\mathcal{B}_e$  by  $\mathcal{C}_{xy} \cup \{b\}$ .

Each of these substitutions preserves the  $\varepsilon$ -covering property and does not increase the cardinality of the resulting collection of disks. Consider the disks of  $\mathcal{R}$  that do not contain any regular rotula,  $\mathcal{C} = \mathcal{R} \setminus (\cup_{e \in E} \mathcal{B}_e)$ . By construction,

$|\mathcal{C}| = |\mathcal{R}| - L \leq |\mathcal{B}| - L$  and  $\mathcal{C}$  covers exactly half of the extreme rotulae (one for each edge). Let  $C$  be the set of vertices  $v \in V$  whose ghost  $g_v$  is linked to an extreme rotula covered by a disk in  $\mathcal{C}$ . We claim that  $C$  is a vertex cover of  $G$  and that its cardinality satisfies  $|C| \leq |\mathcal{C}|$ . All  $b \in \mathcal{C}$  must contain at least one extreme rotula, thus  $C$  is empty if and only if  $\mathcal{C}$  is empty. In this particular case,  $G$  has no edges because otherwise  $\mathcal{R}$  would only cover half of the extreme rotulae. It follows that in this case the empty set is a vertex cover of  $G$ . Assume now that  $\mathcal{C}$  is not empty. By construction, any  $b \in \mathcal{C}$  covers only extreme rotulae, which by (vi) are all linked to the same ghost  $g_v$ . It is thus possible to map each  $b \in \mathcal{C}$  to one vertex  $v \in C$ , showing that  $|C| \leq |\mathcal{C}|$ . To see that  $C$  is a vertex cover, recall that  $\mathcal{C}$  covers exactly one of the two extreme rotulae of each edge. The definition of  $C$  thus ensures it contains at least one endpoint of each edge.  $\square$

#### 4.2 Practical construction

Given as input a graph  $G$  and real non-negative numbers  $\delta$  and  $\varepsilon$  such that  $\delta + \varepsilon > 0$ , we build a collection  $\mathcal{S}$  fulfilling properties (i) to (viii). To do so, we rely on the following result.

**Theorem 4 ([9])** *There is a linear time and space algorithm to draw a connected at most cubic graph on an orthogonal grid.*

Given a drawing of  $G$  on an orthogonal grid, we now describe a way to convert it into an appropriate collection of disks; see Figure 6.a for an example of such a drawing. To perform the conversion, we fix the size of the grid to  $8(\delta + \varepsilon)$  so that we can fit square blocks of size  $4(\delta + \varepsilon) \times 4(\delta + \varepsilon)$  as in Figure 8. Note that in Figures 6 to 10, we have  $\delta = \varepsilon^2$ .

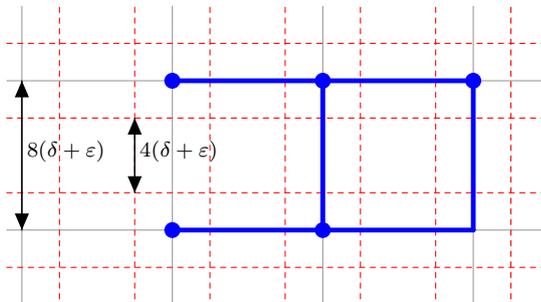


Fig. 8: Drawing (in blue) of a cubic graph on an orthogonal grid and grid division into blocks. Gray lines represent the grid and dashed red lines are the blocks.

There are two different ways in which the blocks meet the graph drawing: the block either contains one vertex or only a portion of one edge. Blocks

<sup>2</sup> See [www.gipsa-lab.grenoble-inp.fr/~isabelle.sivignon/recherche\\_en.html](http://www.gipsa-lab.grenoble-inp.fr/~isabelle.sivignon/recherche_en.html) for Geogebra files with parameterized generic constructions for any values of  $\delta$  and  $\varepsilon$

containing a vertex only vary depending on the number and layout of incident edges. Similarly, blocks containing a portion of an edge vary depending on whether the edge bends within the block or not. In each case, we convert the graph drawing covered by the block into a set of disks that satisfies properties (i) to (viii). For blocks containing a vertex, see Figure 9 for the four subcases. Similarly for edges, we have two subcases. However, recall that edge gadgets must have an odd number of rotulae. To achieve this, we use the fact that every edge necessarily crosses at least one block in a straight line and provide an odd and an even conversion for this type of block. The three block conversions are presented in Figure 10.

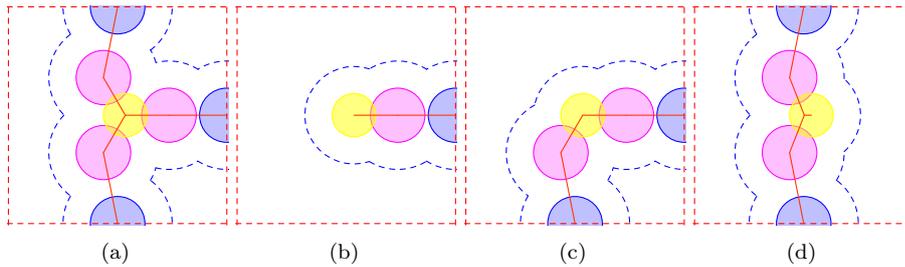


Fig. 9: Block conversions for a vertex of degree (a) 3, (b) 1, (c) 2 in a bend and (d) 2 in a straight line. The red dashed square delimits the block. Same color convention as in Figure 6.b.

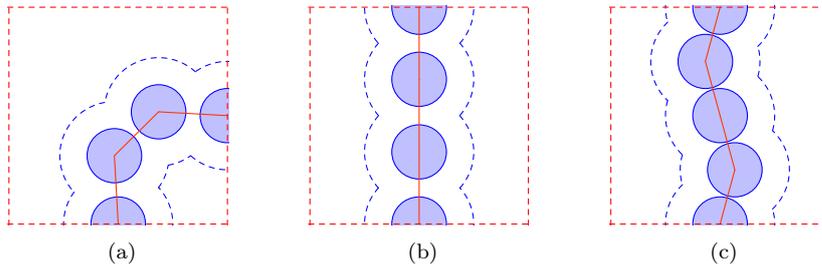


Fig. 10: Block conversions for (a) a bent edge, (b) an even and (c) odd straight edge. Same color convention as in Figure 9.

## 5 Conclusion

In our proof of NP-hardness, we have chosen carefully the set of disks composing the shape, so that its erosion is composed of a finite set of points (the rotulae). It follows that the reduction from vertex cover induces a set cover problem of a finite set of points (the rotulae), using an infinite range of balls

(the medial axis of the dilation). We notice however that it is possible to define a similar construction where the erosion of the shape is an infinite set of points too. Furthermore, a construction where the shape, its erosion and its dilation are arcwise-connected is also possible. To build such variants, it suffices to tune the radii of disks composing the gadgets, and distances between consecutive disks so that the eight properties (i)-(viii) are fulfilled. This does not change the overall result.

Ascertaining the computational complexity of a problem is often the first step before the design of algorithms that return either an optimal solution for a simple subclass of inputs, or an approximated solution for wider classes of inputs. In [28], we investigate the  $(\delta, \varepsilon)$ -ball approximation problem for the subclass of shapes  $S$  defined as a finite union of balls in  $\mathbb{R}^2$ , with an acyclic medial axis  $\text{MA}(S)$ . We design a greedy algorithm that computes an optimal  $(\delta, \varepsilon)$ -ball approximation of  $S$  in polynomial time with respect to the size of  $\text{MA}(S)$ . Whether or not this algorithm can be extended for acyclic finite unions of balls in  $\mathbb{R}^3$  remains an open question.

In the  $(\delta, \varepsilon)$ -ball approximation definition, the core set to cover and the outer set that must not be crossed over are defined through mathematical morphology operators. It is natural to investigate an even more general version of the problem:

**Problem 3** Given a shape  $S \subseteq \mathbb{R}^n$ , a subset  $S^-$  of  $S$  and a superset  $S^+$  of  $S$ , find the smallest collection of balls whose union  $X$  satisfies  $S^- \subseteq X \subseteq S^+$ .

When the shape is defined as a finite union of balls  $S = \bigcup \mathcal{S}$ , an obvious choice for  $S^-$  is to consider the union of balls obtained by deflating balls in  $\mathcal{S}$ , either decreasing each radius by a constant  $\varepsilon$  or multiplying each radius by a factor  $\mu < 1$  as in [27]. In both cases, keeping for  $S^+$  the  $\delta$ -dilation, it is not difficult to see that the associated decision problem is again NP-complete, using the same reduction. We ask what would be the complexity of Problem 3 for other choices of  $S^-$  and  $S^+$ .

## References

1. Agarwal, P.K., Pan, J.: Near-linear algorithms for geometric hitting sets and set covers. In: Proceedings of the thirtieth annual symposium on Computational geometry, p. 271. ACM (2014)
2. Amenta, N., Kolluri, R.K.: Accurate and efficient unions of balls. In: Proceedings of the Sixteenth Annual Symposium on Computational Geometry, SCG '00, pp. 119–128. ACM (2000)
3. Attali, D., Boissonnat, J.D., Edelsbrunner, H.: Stability and Computation of Medial Axes - a State-of-the-Art Report, pp. 109–125. Springer Berlin Heidelberg, Berlin, Heidelberg (2009)
4. Berberich, E., Eigenwillig, A., Hemmer, M., Hert, S., Mehlhorn, K., Schömer, E.: A Computational Basis for Conic Arcs and Boolean Operations on Conic Polygons, pp. 174–186. Springer Berlin Heidelberg (2002)
5. Bradshaw, G., O'Sullivan, C.: Adaptive medial-axis approximation for sphere-tree construction. ACM Transactions on Graphics (TOG) **23**(1), 1–26 (2004)

6. Brönnimann, H., Goodrich, M.T.: Almost optimal set covers in finite vc-dimension. *Discrete & Computational Geometry* **14**(4), 463–479 (1995)
7. Broutta, A., Coeurjolly, D., Sivignon, I.: Hierarchical discrete medial axis for sphere-tree construction. In: *International Workshop on Combinatorial Image Analysis*, pp. 56–67. Springer (2009)
8. Cabello, S., De Berg, M., Giannopoulos, P., Knauer, C., Van Oostrum, R., Veltkamp, R.C.: Maximizing the area of overlap of two unions of disks under rigid motion. *International Journal of Computational Geometry and Applications* **19**(06), 533–556 (2009)
9. Calamoneri, T., Petreschi, R.: An efficient orthogonal grid drawing algorithm for cubic graphs. In: *Comp. and Combinatorics*, pp. 31–40. Springer (1995)
10. Cazals, F., Dreyfus, T., Sachdeva, S., Shah, N.: Greedy geometric algorithms for collection of balls, with applications to geometric approximation and molecular coarse-graining. In: *Computer Graphics Forum*, vol. 33, pp. 1–17. Wiley Online Lib. (2014)
11. Chaussard, J., Couprie, M., Talbot, H.: Robust skeletonization using the discrete  $\lambda$ -medial axis. *Pattern Recognition Letters* **32**(9), 1384 – 1394 (2011)
12. Chazal, F., Lieutier, A.: The  $\lambda$ -medial axis. *Graphical Models* **67**(4), 304 – 331 (2005)
13. Chvatal, V.: A greedy heuristic for the set-covering problem. *Mathematics of operations research* **4**(3), 233–235 (1979)
14. Coeurjolly, D., Montanvert, A.: Optimal separable algorithms to compute the reverse euclidean distance transformation and discrete medial axis in arbitrary dimension. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **29**(3), 437–448 (2007)
15. Culver, T., Keyser, J., Manocha, D.: Accurate computation of the medial axis of a polyhedron. In: *Proceedings of the Fifth ACM Symposium on Solid Modeling and Applications*, SMA '99, pp. 179–190. ACM (1999)
16. Dey, T.K., Zhao, W.: Approximate medial axis as a voronoi subcomplex. *Computer-Aided Design* **36**(2), 195 – 202 (2004)
17. Edelsbrunner, H., Koehl, P.: The geometry of biomolecular solvation. *Combinatorial and computational geometry* **52**, 243–275 (2005)
18. Federer, H.: Curvature measures. *Trans. Amer. Math. Soc.* **93**, 418–491 (1959)
19. Feinauer, J., Spetl, A., Manke, I., Strege, S., Kwade, A., Pott, A., Schmidt, V.: Structural characterization of particle systems using spherical harmonics. *Materials Characterization* **106**, 123 – 133 (2015)
20. Garey, M.R., Johnson, D.S.: The rectilinear steiner tree problem is NP-complete. *SIAM Journal on Applied Mathematics* **32**(4), 826–834 (1977)
21. Har-Peled, S.: *Geometric approximation algorithms*, vol. 173. American mathematical society Boston (2011)
22. Hatcher, A.: *Algebraic topology*. Cambridge University Press (2002)
23. Hubbard, P.M.: Collision detection for interactive graphics applications. *IEEE Transactions on Visualization and Computer Graphics* **1**(3), 218–230 (1995)
24. Hubbard, P.M.: Approximating polyhedra with spheres for time-critical collision detection. *ACM Transactions on Graphics (TOG)* **15**(3), 179–210 (1996)
25. Johnson, D.S.: Approximation algorithms for combinatorial problems. *Journal of computer and system sciences* **9**(3), 256–278 (1974)
26. Kershner, R.: The number of circles covering a set. *American Journal of Mathematics* **61**(3), 665–671 (1939)
27. Miklos, B., Giesen, J., Pauly, M.: Discrete scale axis representations for 3d geometry. In: *ACM Transactions on Graphics (TOG)*, vol. 29, p. 101. ACM (2010)
28. Nguyen, T.B., Sivignon, I.: Epsilon-covering: a greedy optimal algorithm for simple shapes. In: *Canadian Conference on Computational Geometry*, pp. 187–194. Vancouver (2016)
29. Pach, J.: *New Trends in Discrete and Computational Geometry. Algorithms and Combinatorics*. Springer Berlin Heidelberg (2012)
30. Ranjan, V., Fournier, A.: Matching and interpolation of shapes using unions of circles. *Computer Graphics Forum* **15**(3), 129–142 (1996)
31. de Rezende, P.J., Miyazawa, F.K., Sasaki, A.T.: A {PTAS} for the disk cover problem of geometric objects. *Operations Research Letters* **41**(5), 552 – 555 (2013)
32. Serra, J.: *Image analysis and mathematical morphology*, v. 1. Academic press (1982)
33. Tóth, G.F.: *Handbook of Discrete and Computational Geometry*, chap. Packing and covering. Chapman & Hall, CRC (2004)

- 
34. Wein, R., Berberich, E., Fogel, E., Halperin, D., Hemmer, M., Salzman, O., Zukerman, B.: 2D Arrangements. CGAL: Computational Geometry Algorithms Library
  35. Wein, R., Zukerman, B.: Exact and efficient construction of planar arrangements of circular arcs and line segments with applications. Tech. rep., Tech. Rep. ACS-TR-121200-01, Tel Aviv University, Israel (2006)

## A An alternative way to check set inclusion

In this appendix, we provide an alternative polynomial time algorithm to check whether the erosion of a finite collection of disks is contained into the union of a finite collection of disks (see Section A.3). The algorithm relies on a characterization of the inclusion of one set into another. We give first a very general formulation (Lemma 5) before proposing a specialization to our case (Lemma 6).

### A.1 A very general result

We start by proving a very general result to characterize the inclusion of one set into another one. We assume very little on the two sets, only that they are subsets of a topological space. We show that it suffices to test “local” inclusions at the boundary points of the two sets.

**Lemma 5** *Let  $A$  and  $X \neq \emptyset$  be two subsets of a topological space. If*

$$(i) \partial A \subseteq \overline{X}$$

$$(ii) \forall x \in \partial X, \exists N_x \text{ an open neighbourhood of } x \text{ such that } N_x \cap A \subseteq X$$

*then  $A \subseteq X$ .*

*Proof* Letting  $H = A \cap X^c$ , we prove the lemma in two stages:

(a) First, we prove that  $\partial H \subseteq \partial X$ . Using  $\partial(A \cap B) \subseteq (\partial A \cap \overline{B}) \cup (\overline{A} \cap \partial B)$  and plugging  $B = X^c$ , we get  $\partial H \subseteq (\partial A \cap \overline{X^c}) \cup (\overline{A} \cap \partial X^c)$ . Since  $\partial A \subseteq \overline{X}$  and  $\partial X^c = \partial X$ , we deduce  $\partial H \subseteq \partial X$ .

(b) Second, we prove that  $H = \emptyset$ . Suppose for a contradiction that  $H \neq \emptyset$ . Since  $X$  is not empty,  $X^c$  is not the entire topological space, so that  $H$  cannot be the entire topological space neither. Thus,  $\partial H \neq \emptyset$  and we can pick a point  $x \in \partial H$ . Because  $\partial H \subseteq \partial X$ , we can apply (ii) on point  $x$ , showing that there is  $N_x$  a neighbourhood of  $x$  such that  $N_x \cap A \subseteq X$ . Since  $x \in \partial H$ , necessarily  $N_x \cap H \neq \emptyset$ . However,  $N_x \cap H = N_x \cap A \cap X^c \subseteq X \cap X^c = \emptyset$ , yielding a contradiction.  $\square$

### A.2 Specialization to finite union of disks

We now give a specialized version of Lemma 5 in which one of the set is a finite union of disks.

**Lemma 6** *Let  $A$  be a bounded subset of  $\mathbb{R}^2$  and consider  $X = \bigcup \mathcal{X}$  a finite union of disks. Let  $\mathcal{V}$  be the collection of vertices of  $\partial X$ . If*

$$(i) \partial A \subseteq X$$

$$(ii) \forall v \in \mathcal{V}, \exists N_v \text{ an open neighborhood of } v \text{ such that } N_v \cap A \subseteq X$$

*then  $A \subseteq X$ .*

Recall that a *hole* of a subset is a bounded connected component of its complement. The proof of the previous lemma relies on the following technical lemma.

**Lemma 7** *Let  $H$  be a hole of a finite union of disks  $S$ , and  $\mathcal{V}$  the collection of vertices of  $\partial S$ . We have  $\partial H \cap \mathcal{V} \neq \emptyset$ .*

*Proof* Let  $\mathcal{S}$  be a collection of disks such that  $S = \bigcup \mathcal{S}$ . Without loss of generality, we assume that  $\forall b \in \mathcal{S}, \partial b \cap \partial H \neq \emptyset$ . Indeed, by removing any disk that does not contribute to the boundary of  $H$ , the resulting union of disks still admits  $H$  as a hole. Notice that  $\partial H \subseteq \partial S$ . In fact, the boundary of  $H$  is formed by vertices and edges of  $\partial S$ . By contradiction, assume there are no vertices in  $\partial H$ . Then  $\partial H$  must be an union of complete circular arcs, thus disks in  $\mathcal{S}$  cannot intersect one another.  $S$  is an union of disjoint disks and  $S^c$  has only one connected component by Alexander duality Theorem [22]. This connected component is unbounded, contradicting the existence of  $H$ .  $\square$

*Proof of Lemma 6* Let  $H = A \cap X^c$ . The scheme of the proof is similar to the proof of Lemma 5, with only one extra stage in the middle:

(a) First, we prove that  $\partial H \subseteq \partial X$ . For this, we proceed as in the first stage of the proof of Lemma 5.

(b) Second, we prove that  $H$  is either empty or an union of holes of  $X$ . The proof is given below.

(c) Last, we prove that  $H = \emptyset$ . For this, we proceed as in the last stage of the proof of Lemma 5, with the only difference being that we pick a point  $x$  on  $\partial H$  among vertices of  $\partial X$ . This is always possible thanks to Lemma 5 (stated below).

It remains to prove that  $H$  is either empty or an union of holes of  $X$ . Suppose  $H \neq \emptyset$  and let  $p \in H$ . Consider  $H_p$  the connected component of  $H$  that contains  $p$ . Likewise, let  $X_p^c$  be the connected component of  $X^c$  that contains  $p$ . We prove that  $H_p = X_p^c$ . Since  $A$  is bounded,  $H_p$  is bounded too and  $H_p$  is a hole of  $X$ , so that  $H$  is an union of holes.

Recall that  $H_p \subseteq X^c$  is connected by definition, and that it intersects the connected component  $X_p^c$  of  $X^c$ . Hence,  $H_p \subseteq X_p^c$ . Let us prove the converse inclusion  $X_p^c \subseteq H_p$ . Let  $q \in X_p^c$  and consider  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$  a path in  $X_p^c$  connecting  $p$  to  $q$ . Such a path exists because  $X_p^c$  is open and connected. We have  $\partial H_p \subseteq \partial H$ , hence  $\partial H_p \subseteq \partial X$ . We have  $\gamma \cap \partial H_p \subseteq \gamma \cap \partial X = \emptyset$ . We claim that  $q \in H_p$ . Indeed, suppose for a contradiction that  $\gamma$  connects  $p \in H_p$  to  $q \notin H_p$ . Then,  $\gamma$  would meet  $\partial H_p$ , which is impossible.  $\square$

### A.3 A polynomial time algorithm

We are now ready to describe an algorithm that tests in polynomial time whether  $A \subseteq X$ , where  $A$  is the erosion of a finite collection of disks and  $X$  is the union of a finite collection of disks. The algorithm hinges on Lemma 6 and its pseudocode is given in Algorithm 1. To describe it, let  $\mathcal{S}$  and  $\mathcal{B}$  be two finite collections of disks,  $\mathcal{C}$  a finite collection of circular arcs,  $x$  a point,  $\varepsilon \geq 0$  a non-negative real number. We introduce four functions:

**BOUNDARY** takes as input the pair  $(\mathcal{S}, \varepsilon)$  and returns the set of circular arcs that compose the boundary of the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$ .

**POINTLOCATION** takes as input the triplet  $(x, \mathcal{S}, \varepsilon)$  and returns **inner** if  $x \in \mathring{A}$ , **boundary** if  $x \in \partial A$  and **outer** if  $x \notin A$ , where  $A = (\bigcup \mathcal{S})^{\ominus \varepsilon}$ .

**INCLUSION** takes as input  $(\mathcal{C}, \mathcal{B})$  and returns **true** iff  $(\bigcup \mathcal{C}) \subseteq (\bigcup \mathcal{B})$ .

**LOCALINCLUSION** takes as input  $(v, \mathcal{S}, \varepsilon, \mathcal{B})$  where  $v$  lies on the boundary of both the erosion  $(\bigcup \mathcal{S})^{\ominus \varepsilon}$  and the union  $\bigcup \mathcal{B}$ . It returns **true** iff there exists an open neighborhood  $N_v$  of  $v$  such that  $N_v \cap (\bigcup \mathcal{S})^{\ominus \varepsilon} \subseteq (\bigcup \mathcal{B})$ .

Each of those four functions can be implemented in polynomial time and we leave to the reader the choice of an implementation.

```

Input:  $(\mathcal{S}, \varepsilon, \mathcal{B})$ .
if not INCLUSION(BOUNDARY( $\mathcal{S}, \varepsilon$ ),  $\mathcal{B}$ ) then return false;
forall the vertices  $v$  of the boundary of  $\bigcup \mathcal{B}$  do
    switch POINTLOCATION( $v, \mathcal{S}, \varepsilon$ ) do
        case outer return false;
        case boundary
            if not LOCALINCLUSION( $v, \mathcal{S}, \varepsilon, \mathcal{B}$ ) then return false ;
        end
    endsw
end
return true ;
    
```

**Algorithm 1:** An alternative way to test the inclusion  $(\bigcup \mathcal{S})^{\ominus \varepsilon} \subseteq \bigcup \mathcal{B}$ .

## B Interior of a cone

**Lemma 8** *Let  $u, c_1, \dots, c_k \in \mathbb{R}^n$  be  $k+1$  points. We have the inclusion*

$$\mathring{\text{coni}}(u, C) \subseteq \left\{ u + \sum_{i=1}^k \lambda_i (c_i - u) \mid \lambda_i \in \mathbb{R}, \lambda_i > 0 \right\}$$

In order to prove Lemma 8 we rely on the following technical result:

**Lemma 9** *Let  $c \in \mathring{\text{coni}}(u, C)$  and  $l \in \{1, \dots, k\}$ . There exist  $\lambda_1, \dots, \lambda_k \geq 0$ ,  $\lambda_l > 0$ , such that  $c - u = \sum_{i=1}^k \lambda_i (c_i - u)$ .*

*Proof* Let  $\mathbf{e}_0 = (c_l - u) / \|c_l - u\|$ . Consider a basis of unit vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  such that  $\sum_{i=0}^n \mathbf{e}_i = \mathbf{0}$ . To build such a basis, consider a regular  $n$ -simplex with barycenter at  $u$  and one vertex at  $c_l$  and take the vectors with origin  $u$  and end point the other vertices of the  $n$ -simplex. Let  $r > 0$  such that  $b(c, r) \subseteq \mathring{\text{coni}}(u, C)$ . For all  $j \in \{0, \dots, n\}$ , we have  $c + r\mathbf{e}_j \in \text{coni}(u, C)$ . In particular, we can find  $\lambda_{j1}, \dots, \lambda_{jk} \geq 0$  such that  $c - u + r\mathbf{e}_j = \sum_{i=1}^k \lambda_{ji} (c_i - u)$ . By summing over  $0 \leq j \leq n$  we obtain

$$(n+1)(c-u) + r\mathbf{0} = \sum_{i=1}^k \left( \sum_{j=0}^n \lambda_{ji} \right) (c_i - u).$$

Letting  $\lambda_i = \sum_{j=0}^n \lambda_{ji} / (n+1)$ , we can thus write  $c - u = \sum_{i=1}^k \lambda_i (c_i - u)$  with  $\lambda_i \geq 0$  because all  $\lambda_{ji} \geq 0$ . To prove that  $\lambda_l > 0$ , notice that

$$c - u + r\mathbf{e}_0 = \sum_{i=1}^k \lambda_i (c_i - u) + \frac{r}{\|c_l - u\|} (c_l - u).$$

On the other hand, we have  $c - u + r\mathbf{e}_0 = \sum_{i=1}^k \lambda_{0i} (c_i - u)$ . Equating the two expressions, we get  $\lambda_{0l} = \lambda_l + \frac{r}{\|c_l - u\|} > 0$ . Hence  $\lambda_l = \sum_{j=0}^n \lambda_{jl} / (n+1) > 0$ .  $\square$

We can now prove Lemma 8.

*Proof of Lemma 8* By Lemma 9, for each  $l \in \{1, \dots, k\}$  there are  $\lambda_{l1}, \dots, \lambda_{lk} \geq 0$ ,  $\lambda_{ll} > 0$ , such that  $c - u = \sum_{i=1}^k \lambda_{li} (c_i - u)$ . By summing over  $1 \leq l \leq k$  we have

$$k(c-u) = \sum_{i=1}^k \left( \sum_{l=1}^k \lambda_{li} \right) (c_i - u)$$

with  $\sum_{l=1}^k \lambda_{li} / k > 0$  for all  $i$ .  $\square$

---

## C To do list

- Authors ✓
- Abstract ✓
- Key words ✓
- Conclusion ✓
- Polish Appendix B ✓
- Lemma 1: move the fact that  $c_i$  and  $u$  are not aligned inside the proof. ✓
- Introduction ✓
- Captions ✓
- Dans figure 8, changer  $\varepsilon$  pour  $\varepsilon + \delta$ . ✓

- 
- A figure to illustrate Lemma 1
  - Mettre en annexe ce que donnent les figures 9 et 10 dans les deux cas limites  $\delta \rightarrow 0$  et  $\varepsilon \rightarrow 0$  ???
  - Acknowledgments (ANR, financement de la thèse et autres...) ????
  - Supprimer les commentaires dans les marges et la couleur dans le texte.