Abstract

We propose time-frequency methods to filter dispersive guided wave signals. Guided waves occur in acoustical propagation (oceanic waveguides), geophysics (layered medium) or optics (dielectric optical waveguides). In waveguides, signals can be decomposed into normal modes which contain information on environmental parameters and source localization. As modes present non-linear time-frequency evolution, modal filtering is not possible with conventional tools. To overcome the difficulty presented by these non-linearities, we have develop matched tools: matched frequency and time-frequency representations and the modal filterings associated with these representations. The tools developed are based on unitary equivalence principle. Performance and robustness of different proposed modal filters are evaluated and compared. All these tools can be used for both source localization and environmental inversion.
Matched representations and filters for guided waves

INTRODUCTION

In many propagation cases (oceanic shallow water environments, geophysics layered media, dielectric optical fibres...), the medium is modelled by a waveguide. In this waveguide, the Helmholtz general equation (for the acoustic field) or Maxwell equations (for the electromagnetic field) establish a relationship between field values and frequencies. Signals can be decomposed into modes [1], [2], [3] which have dispersive propagation properties. Modal energy repartition is characterized by particular patterns in the time-frequency plane according to dispersive non-linear laws (layout does not follow a straight line in the time-frequency plane). The modal characteristics contain informations for source localization and geoacoustic inversion (modal energy is linked to the source depth, modal phase and time-frequency patterns are linked to the source range and to the environmental parameters). Many methods have already been developed to estimate source depth and range using the modal properties [4], [5], [6]. Dealing with modal properties implies to carry out a modal filtering beforehand. To filter modes, some classical methods use the modal orthogonality properties in the vertical direction [6] if signals are recorded on a vertical array of sensors. Recently, a new modal filtering technique has been proposed using an horizontal array with a large number of sensors and a frequency-wavenumber transform [4].

We focus here on a single sensor configuration for which the conventional modal filtering can not be applied. The only way to realise modal filtering is thus to integrate modal time-frequency characteristics. Recent studies in underwater acoustics or optical science have proposed methods to extract some modal characteristics by using conventional time-frequency methods [5], [7]. Differently, in our approach we obtain all the modal characteristics. Actually source localization and geoacoustic inversion techniques being based on several characteristics (modal energy, phase, dispersive time-frequency patterns), we have thus developed modal filtering techniques. Depending on the geoacoustic configuration, modes are not always distinguishable when using many of the conventional Time-Frequency Representations (TFR). Filtering techniques are thus not applicable for many source-sensor distances due to the closeness of modes in the time-frequency plane. In the present paper, we elaborate different types of representations (frequency, linear time-frequency, adaptive time-frequency) and their associated modal filtering matched to guided signals. In practical situations, these tools can be used on a single signal when the source is a pulse or of short duration. All the tools are based on unitary equivalence principle [8].

We first present some basic aspects of the propagation theory (section I): medium models, modal decomposition, dispersive laws. We then show and justify the unsatisfactory results proposed by conventional filtering (matched filtering, conventional time-frequency filtering) due to the non-linear time-frequency modal structures (section II). We then present the developed tools:

- The unitary operator, following the methods exposed in [8] and [9]. It is matched to the two-layer Pekeris waveguide model and avoids non-linear structure problems by transforming the representation domain (section III).
- The developed representations: one-dimensional Pekeris Fourier Transform, two-dimensional Pekeris Short Time Fourier Transform (STFT), modal Pekeris STFT and modal adaptive TFR (section IV).
- The developed filtering techniques based on those representations: linear Pekeris FT filtering, linear Pekeris STFT filtering and non-linear modal adaptive filtering (section V).

We finally assess the filtering performances of the developed techniques and we show their complementarity in respect to the noise level and the environment parameters uncertainty (section VI). The specific filtering methods developed based on representations with the unitary equivalence formalism are efficient. These filtering methods allow access to modes even when several environmental or localization parameters are unknown. Filtered modes can then be used to perform geoacoustic inversion (environmental parameters estimation) [5] or source localization [10].

All the representations shown have been applied to a real pressure signal registered in the North Sea. The source of this signal is an air-gun shot with a 3 dB bandwidth between 15 and 90 Hz.

I. GUIDED PROPAGATION

We describe the guided propagation in an underwater acoustical configuration. It can be extended for any
guided wave propagation. Let us consider an underwater environment with axial symmetry around the depth $Oz$ axis. In this configuration, propagation study is made in the plane $(u_r, u_z)$ (Figure 1). We note $V(z)$ the sound (in water) or compressional (in bottom) velocity and $\rho(z)$ the density in the medium. The acoustic pressure $P(r, z, t)$ recorded at $(r, z)$ is expressed by: $P(r, z, t) = p(r, z) \exp(-jwt)$ with $\omega$ the circular frequency and $p(r, z)$ satisfying the general Helmholtz equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial z} \left( \frac{1}{\rho(z)} \frac{\partial p}{\partial z} \right) + \frac{\omega^2}{c^2} p = - \frac{\delta(r)\delta(z - z_s)}{2\pi r}.$$  

In shallow water waveguides (Figure 1), boundary conditions must be taken into account: total reflection at the sea surface ($z = 0$) and reflection properties depending on environmental parameters at the water-bottom interface ($z = D$). We focus on the Pekeris waveguide [11] which is a classical fluid two layers model: a water layer of depth $D$ over a half space representing the bottom (sediment, rock...). Layers are homogeneous, constant sound velocities and densities are considered: $V(z) = V_1$, $\rho(z) = \rho_1$ for the water layer $z < D$ and $V(z) = V_2$, $\rho(z) = \rho_2$ for the bottom $z > D$.

In the Pekeris waveguide, bottom influence is taken into account and expressed by a phase coefficient $\Phi(t)$ depending on the grazing angle, it is thus more realistic than the ‘perfect’ (or ideal) model which does not take into account the bottom properties (total reflection on the bottom). By separating the $r$ and $z$ variable in the pressure and introducing it in the Helmholtz equation $??$, we obtain the ‘modal equation’. Depending on the boundary conditions, this equation is a Sturm-Liouville eigenvalue problem [12]. Solutions of this problem are a set of modes indexed by $m$, associated with their eigenfunctions $\psi_m(z)$ which correspond to the $z$ dependent part of the pressure and their eigenvalue $k_{rm}$ which correspond to the horizontal component of wavenumber. This horizontal component is linked to the wavenumber $k_m$ by $k_m = (k^2_{rm} + k^2_{zm})^{1/2} = \omega/V_1$. Those modes are like the modes of a vibrating string. We can note that for a ‘perfect’ waveguide (perfectly reflection on the bottom), the set of modes is complete and the number of modes is infinite. For the Pekeris waveguide, the problem is more complex: we obtain a mixed wavenumber spectrum. The horizontal wavenumber spectrum can then be divided in two parts:

- $0 < k_r < \omega/V_2$ corresponding to the 'continuous spectrum';
- $\omega/V_2 < k_r < \omega/V_1$ corresponding to the 'discrete spectrum'.

The 'continuous spectrum' corresponds to waves partially transmitted in the bottom. It leads to leaky modes which decay exponentially with range. We will thus neglect them as we work with a long range configuration. The 'discrete spectrum' corresponds to waves totally reflected in the water layer. It contains the eigenvalues $\{k_{rm}\}$ of the Sturm-Liouville problem associated with eigenfunction $\psi_m(z) = \sin(k_{zm} z)$. This incomplete set of modes is a good approximation at long range and the pressure signal can thus be written [13]:

$$p(r, z) = \sum_{m=1}^{M} A_m \psi_m(z_s) \psi_m(z) \frac{\exp(jk_{rm} r)}{\sqrt{k_{rm} r}}.$$  

Due to the numerous reflexions, guides are dispersive environments. Modes are characterised by the dispersive relationship $\nu_g(\nu)$ between their group velocity and frequency. With knowledge of $\nu_g(\nu)$ and the source-sensor distance $R$ the energy distribution of modes in the time-frequency $\nu(t)$ plane coming from a pulse source can be characterised [1]. This energy follows a nonlinear dispersive curve $\nu_m(t)$ in the time-frequency plane (Figure 2). In the Pekeris case, it is not possible to obtain a direct formulation of the dispersion relation $\nu(t)$. To approach one, we make the approximation: $\nu_g \nu_\phi = V_1^2$ with $\nu_g$ the group velocity of acoustic waves and $\nu_\phi$ the phase velocity. This approximation, valid for the 'perfect' model, has no theoretical base for the Pekeris waveguide. Nevertheless in practical cases, as we will see, it is efficient. We note the modified waveguide Pekeris in italic. With this approximation, the dispersion relation

![Fig. 1. Two-layer waveguide](image1)

![Fig. 2. Theoretical plot of the first seven modes of a Pekeris (dashed lines) and approximated Pekeris (continuous lines) pressure signal for a source-sensor distance $R = 10$ km](image2)
becomes
\[ \nu_m^{Pek}(t) = \frac{(2m - 1 + \frac{2\Phi(t)}{\pi})}{4D[t^2 - (R/V_1)^2]^{1/2}} V_1 t \]  
where the phase shift \( \Phi(t) \) introduced by the bottom layer is
\[ \Phi(t) = \begin{cases} \arctan \left( \frac{\rho_1 V_1 \left( \frac{m R}{V_1^2} \right) - 1}{\rho_2 V_2 \left( 1 - \frac{R}{V_1} \right)^2} \right)^{1/2}, & \text{for } R \leq t \leq \frac{RV_2}{V_1} \\ 0, & \text{for } t \geq \frac{RV_2}{V_1} \end{cases} \]  
These relations are defined on the time domain \( D_f = \left[ \frac{R}{V_1}, +\infty \right] \). Modes are limited in frequency by the cutoff frequency \( \nu_c(m) = \frac{(2m-1)V_2}{4D} \). The instantaneous frequency is the derivative of the instantaneous phase. For the Pekeris waveguide
\[ \phi_m^{Pek}(t) = 2\pi \int \nu_m^{per,f}(u) du = \frac{2\pi}{4D} \left( (2m-1)V_1 \xi(t) \right) \]
with
\[ \xi(t) = \left[ t^2 - (R/V_1)^2 \right]^{1/2} \]
\[ v(t) = (\alpha R^2 - V_1^2 t^2)^{1/2} \]
and
\[ \alpha = \frac{V_2^2}{V_1}, \quad \gamma = \frac{V_1 \rho_1}{V_2 \rho_2} \]
As the pressure signal is a sum of modes, when the source is a pulse, it can be described as
\[ p(t) = \sum_m f_m(t)e^{j\phi_m(t)} \]  
where \( f_m(t) \) is a smoothing function characterising the modal energy evolution during time. The number of modes under the sum is determined by the frequency range. For the studied North Sea signal, the sensor is laid on the bottom. The environmental parameters are: sound velocity in water \( V_1 = 1520 \text{ m.s}^{-1} \), compression velocity in the bottom \( V_2 = 1875 \text{ m.s}^{-1} \), densities in water and bottom \( \rho_1 = 1000 \text{ kg/m}^3 \) and \( \rho_2 = 2000 \text{ kg/m}^3 \), water depth \( D = 130 \text{ m} \), source depth \( z_s = 18 \text{ m} \), and source-distance sensor \( R = 5000 \text{ m} \). In this condition, 8 modes \( (m = 1, \ldots, 8) \) are present. Those parameters are typical for a shallow water configuration.

Modal decomposition and dispersion laws describe in this section remain valid in optics [2], [3]. Acoustic propagation must then be replaced by electromagnetic one, sound speed by light celerity and density by index of refraction. All the tools presented in following sections can thus find application in optics. Modal decomposition (called Floquet’s modes) with same laws of dispersion remains also valid for scattering on strip arrays configuration [14], [7].

II. Conventional time-varying filtering methods

The aim of this paper is to provide an efficient method to filter modes. As modes are non-stationary structures, a time-varying filter must be used (figure 3 represents the Fourier spectrum of the North Sea signal: in frequency representation, modes are not separated and do not allow the use of a stationary filtering). Filtered modes could then be used to estimate environmental parameters or source localization in oceanic engineering or optics. As a result, we could not use methods which need the full knowledge of the environment and source, matched filtering is not suitable to our issue. The Wigner-Hough method [15] must also be adapted to the accurate dispersive law and is thus no more suitable. A second solution is to use a time-frequency based filtering. To calibrate such a filter, a representation is preliminary used, on which modes must be separated. Time-frequency domain is a priori appropriate because modes are theoretically separated (Section I and figure 2). However this section shows that no conventional time-frequency representation (TFR) provides this separation.

In the literature, numerous TFR methods (linear and bilinear) are presented to overcome TFR inherent limitations, namely time-frequency uncertainty and interferences between structures. As for the acoustic waveguide, the signal contains several modes and each mode is a non-linear structure in the time-frequency plane (layout does not follow a line in this plane). Consequently, conventional TFR give unsatisfactory results: mode identification is impossible in the TFR and prevents mode filtering.

To illustrate it, various conventional time-frequency representations have been applied to the North Sea signal:
- methods from the Cohen’s class: from the Wigner-Ville Distribution (WVD) [16] to the adaptive optimal-kernel TFR [17];
- constant bandwidth atomic decompositions: the Short Time Fourier Transform (STFT) and the Chirplet Transform [18];
- frequency dependent atomic decompositions: the Continuous Wavelet Transform and the S-Transform [19];
- adaptive atomic decompositions derived from the Matching Pursuit algorithm [20]. The chirplet dictionary used has four degrees of freedom (time, frequency and scale shift, constant frequency modulation rate [21]).

For each class of representations, some TFR provide better separability than others, it is the case for optimal-kernel TFR (Figure 4 left), the Chirplet Transform (Figure 4 right), the S-transform (Figure 5 left) and the adaptive TFR based on a chirplet dictionary (Figure 5 right). But the separability shown by the latter TFR remains unsatisfactory. Consequently, no conventional time-varying filtering can be used.
Wavelet Transform has been proposed [24]. Later, this developed [22] [23] and a specific unitary equivalent a warped Cohen’s class for hyperbolic laws has been frequency laws have been developed with this technique: TFR matched to specific non-linear time-unitary operators and unitary equivalence principles can be used. Based on this operator we have then develop unitarily equivalent representations and an adaptive TFR based on the Matching Pursuit algorithm with unitarily equivalent atoms.

III. UNITARY OPERATORS AND UNITARY EQUIVALENCE

To overcome problems linked to non-linear structures, unitary operators and unitary equivalence principles can be used. TFR matched to specific non-linear time-frequency laws have been developed with this technique: a warped Cohen’s class for hyperbolic laws has been developed [22] [23] and a specific unitary equivalent basis decomposition (called fan basis) of STFT and Wavelet Transform has been proposed [24]. Later, this problematic has been generalised [8] and all the unitary equivalent possibilities have been described (unitary equivalent atomic decompositions, unitary equivalent affine and Cohen’s class). The case of matched TFR (Cohen’s and affine classes) to dispersive signals using warping operators has been studied in [9]. More recently, warping techniques have also been developed to adapt Matching Pursuit to non-linear multiple structures [25]. In this present paper, we first present the development of a unitary operator matched to Pekeris guided waves. Based on this operator we have then develop unitarily equivalent representations and an adaptive TFR based on the Matching Pursuit algorithm with unitarily equivalent atoms.

A. Principles

A unitary operator $U$ is a linear transformation from a Hilbert space into another preserving the energy and the inner product by isometry

$$\begin{align*}
\|Us\|^2 &= \|s\|^2 \\
< Ug, Uh > &= < g, h >
\end{align*}$$

where the inner product $< g, h > = \int g(u)h^*(u)du$.

In the present paper, conventional unitary operators are used:

- **Time shift**: $( Tu_x)(t) = x(t - u)$ with eigenfunction $u^T(t) = e^{j2\pi ct}$. Note that the Fourier Transform $F$ can be expressed as $(Fx)(\nu) = < x, u^T_\nu >$
- **Frequency shift**: $( F_f x)(t) = x(t)e^{j2\pi ft}$
- **Dilation**: $( D_a x)(t) = |a|^{1/2} x(at)$
- **Axe Warping operator**: $W_w$ warps the time axis of a given signal $x$, $( W_w x)(t) = |w'(t)|^{1/2} x[w(t)]$ where $w'(t)$ is the derivative of $w(t)$.
- **Modulation operator**: $( M_{m \nu} x)(t) = x(t)e^{j2\pi mt}$ where $m(t)$ represents an invertible modulation function.

The representation of a signal $x$ can be achieved in the natural time dimension associated with the time unitary operator $T$ and the coordinate $t$ or in the natural frequency dimension associated with the frequency unitary operator $F$ and the coordinate $\nu$. Unitary transformation can be seen as a change of basis which can turn a difficult problem into an easy one. To change the basis of a signal, we apply the unitary transformation $x \rightarrow Ux$. This procedure can be extended to processings or analysis systems (representations, filters...), $P = PU$ is said unitarily equivalent to $P$. The unitary transformation $U$ maps the physical quantities in $P x$ represented by the natural unitary operator $T$, $F$ or $T - F$ into new quantities in $PUx$ represented by the unitarily equivalent operators $\tilde{T} = U^{-1}TU$, $\tilde{F} = U^{-1}FU$ and $\tilde{T} - \tilde{F}$ (corresponding to the unitary equivalent time, frequency or time-frequency dimensions). This leads to the following definition [8]: two operators $A$ and $\tilde{A}$ are unitarily equivalent if $\tilde{A} = U^{-1}AU$ (with $U$ the unitary...
transformation). It is also possible to apply the unitary operator $U^{-1}$ on $\mathcal{P}U$ as a post-processing to come back in the natural represented domain ($T$, $F$ or $T - F$) creating a $U^{-1}\mathcal{P}U$ processing which is also unitarily equivalent to $\mathcal{P}$. In the following sections, we will use the unitary equivalence of conventional unitary operators, of decomposition basis, of time-frequency representations and of time-invariant filters.

**B. Unitary operator for Pekeris guided waves**

Our goal is to develop a unitary operator to map Pekeris modes into pure frequencies in the unitary equivalent domain and thus to obtain linear modal structures. Using equation 5, the modal phase in the Pekeris waveguide can be written as

$$\phi_{m}^{P}(t) = 2\pi(\nu_{c}(m)x(t) + \chi(t))$$  \hspace{1cm} (11)

with

$$\chi(t) = \begin{cases} \frac{1}{2\pi} \left[V_{1}\xi(t)\Phi(t) + \frac{2\sqrt{\alpha_{1}}}{\sqrt{1-\gamma^{2}}} \left[v(t) - \frac{\sqrt{\alpha_{1}}}{\sqrt{1-\gamma^{2}}}\right]ight] & \text{for } \frac{R_{V_{1}}}{V_{1}} \leq t \leq \frac{RV_{2}}{V_{1}} \\ 0 & \text{for } t \geq \frac{RV_{2}}{V_{1}} \end{cases}$$  \hspace{1cm} (12)

The developed operator has to compensate the two phase terms $\xi(t)$ and $\chi(t)$. $\chi(t)$ does not depend on the mode number $m$. To compensate it, we introduce the modulation operator $M_{q}$ described by the modulation function $q(t)$ as

$$q(t) = -\chi(t)$$  \hspace{1cm} (13)

The Pekeris Modulation Operator is then

$$(M_{q}x)(t) = x(t)e^{j2\pi q(t)}$$  \hspace{1cm} (14)

Note that $u_{b}^{M_{q}} = \delta(t - a)$ and $\lambda_{a}^{M_{q}} = e^{j2\pi q(a)}$ are respectively eigenfunction and eigenvalue of $M_{q}$.

To compensate the first phase term $\nu_{e}(m)\xi(t)$, we introduce a time warping operator $W_{w}$ described by the warping function $w(t)$

$$w(t) = \xi^{-1}(t) = \left( t^{2} + \frac{R_{V_{1}}^{2}}{V_{1}^{2}} \right)^{1/2}$$  \hspace{1cm} (15)

The Pekeris warping operator is then

$$(W_{w}x)(t) = \left| w'(t) \right|^{1/2}x[w(t)] = \left| \frac{t}{w(t)} \right|^{1/2}x[w(t)]$$  \hspace{1cm} (16)

Note that $u_{a}^{W_{w}} = t^{1/2}e^{j2\pi t}$ and $\lambda_{a}^{W_{w}} = e^{j2\pi t}$ are respectively eigenfunction and eigenvalue of $W_{w}$.

The unitary operator $O_{w,q}$ used to obtain Pekeris modes which are linear is called Pekeris operator. It is the combination of $W_{w}$ and $M_{q}$ operators

$$O_{w,q}(t) = (W_{w}M_{q}x)(t) = \left| \frac{t}{w(t)} \right|^{1/2}x[w(t)]e^{j2\pi q[w(t)]]}$$  \hspace{1cm} (17)

Using this operator, we can introduce several tools by unitary equivalence. The Pekeris time shift operator $T_{\tau}$ is defined by

$$(T_{\tau}x)(t) = (O_{w,q}^{-1}T\nu_{c}(m)x)(t) = \left| \frac{t}{w^{-1}(t) - \tau} \right|^{1/2}x[(w^{-1}(t) - \tau) + \tau]$$  \hspace{1cm} (18)

The Pekeris frequency shift operator $F_{\tau}$ is described by

$$(F_{\tau}x)(t) = (O_{w,q}^{-1}F\nu_{c}(m)x)(t) = x(t)e^{j2\pi f_{w}(t)}$$  \hspace{1cm} (19)

Their respective eigenfunctions are $u_{f}^{F}(t) = O_{w,q}^{-1}u_{c}(t)$ and

$$u_{c}^{F}(t) = O_{w,q}^{-1}u_{b}(t) = \left| \frac{t}{w^{-1}(t)} \right|^{1/2}e^{j2\pi c(w^{-1}(t) - q(t))}$$

$u_{c}^{F}(t)$ is the generalised Pekeris impulse [9] and corresponds to the expression of the Pekeris mode $m$ if $c = \nu_{c}(m)$. The Pekeris pressure signal can be written as a sum of these smoothed impulses $p_{P}(t) = \sum_{m}h_{P}^{m}(t)u_{c}^{m}(t)$.

$F_{\tau}$ induces a Pekeris modal shift on pressure signal (with the corresponding shift $m_{s} = \frac{2D}{V_{1}} + \frac{1}{2}$). As a result, the Pekeris frequency shift is also a Pekeris modal shift. The Pekeris frequency $F_{\tau}$ and the Pekeris modal dimension are thus equivalent. Modal dispersive structures are linked by the Pekeris frequency shift operator $F_{\tau}$ (or Pekeris modal shift operator) $u_{c}^{m}(t) = u_{c}^{m}(x_{1}(t))u_{c}^{m}(x_{2}(t))$.

**IV. Pekeris representations**

If the signal perfectly fits Pekeris guided waves, each mode is mapped into a pure frequency in the Pekeris frequency dimension $F$. This frequency is the cutoff frequency $\nu_{c}(m) = \frac{2m - 2m_{c}}{4D/V_{1}}$ of the mode $m$. As a result, each mode becomes a linear structure. However, the source is never a pulse, the Pekeris model never perfectly corresponds to the real medium and parameters are partially unknown. As a consequence, modes are not perfectly located on their pure theoretical frequencies in $\bar{F}$. We have thus developed representations taking into account the linearisation of modal structures but also allowing degrees of freedom. We first developed a one dimensional representation based on the Fourier Transform (FT), then a two dimensional TFR based on the STFT. We finally developed an adaptive TFR fitted to the Pekeris model.
A. Pekeris Fourier Transform

The first representation developed is called Pekeris Fourier Transform: it is the application of the conventional FT on the transformed signal. This Pekeris FT measures the Pekeris frequency $\tilde{F}$ content and not $F$ content. We use the $\mu$ coordinate in the $\tilde{F}$ dimension (which is equivalent to the modal dimension). Using equation 10, we obtain the Pekeris FT

$$\langle F^{Pek}(x)(\mu) = \langle O_{w,q}x, u^T_{\mu} \rangle = \langle x, O_{w,q}^{-1}u^T_{\mu} \rangle = \langle x, u^T_{\mu} \rangle = \int_{D_f} \left| \frac{1}{w(t)} \right| x e^{-j2\pi(\mu w^{-1}(t)-q(t))} dt \rangle$$

If the signal perfectly fits Pekeris guided waves, modes are strictly separated in $\tilde{F}$ domain. Matched filter and Pekeris FT is a perfectly matched calibration. WVD is not matched because structures are multiple. Frequency dependent atomic decompositions (S-Transform, CW) are not matched because modes need the same time-frequency resolution whatever their frequency. Since modes are narrow-band structures around a pure frequency, STFT or Gabor expansion are matched to this configuration. Mathematically these two tools are identical. The Gabor expansion is used with a sparsity approach. The sparsity is a drawback for our application (bad resolution of the representation and inversion limitations). Since signals are of short duration in our practical configuration, sparsity is not an objective and we prefer the STFT formalism with great oversampling degree which yields to negligible reconstruction errors ('formal identity' is respected). This is why we adopt the continuous formalism along this paper. We thus apply STFT (noted $S$) in the transformed domain $\tilde{T}$ to obtain the Pekeris STFT. Decomposition atoms are chosen to separate modes (favouring the frequency resolution). It does not measure joint time-frequency $T-\tilde{F}$ content but joint $\tilde{T}-\tilde{F}$ content. We use coordinates $(\tau, \mu)$ in the $\tilde{T}-\tilde{F}$ dimension. Pekeris STFT is

$$\langle S^{Pek}(x)(\tau, \mu) = \langle SO_{w,q}x, \tau, \mu \rangle = \langle x, O_{w,q}^{-1}b_{\tau,\mu} \rangle \rangle$$

where $b_{\tau,\mu}$ are the time-frequency atoms (similar to Gabor logon) of the STFT defined by

$$b_{\tau,\mu}(u) = (F_{\tau}T_{\mu}g)(u) = g(u-t)e^{j2\pi nu}$$

Pekeris STFT and conventional STFT are unitary equivalent. Pekeris STFT satisfies the properties:

- invertibility: $x = \langle S^{Pek}x, (O_{w,q}^{-1}b_{\tau,\mu})^* \rangle$
- energy conservation $\langle \int_\mu |S^{Pek}(x(\mu))|^2 d\mu = \int_\mu |x(t)|^2 dt \rangle$
- covariance by Pekeris time and frequency shift $\tilde{T}_u$ and $\tilde{F}_v$

$$\langle S^{Pek}(\tilde{T}_u \tilde{F}_v x)(\tau, \mu) = \langle S^{Pek}(x)(\tau-u, \mu-f) \rangle$$

The conventional STFT results from the decomposition of the signal on the basis $\{b_{\tau,\mu}\}$. This decomposition is done on a regular tiling of the time-frequency plane (Left panel of figure 8). Pekeris STFT is the decomposition of the signal on the transformed basis $\{O_{w,q}^{-1}b_{\tau,\mu}\}$ which leads to a non-regular tiling of projection (Right panel of figure 8). Basis decomposition $\{b_{\tau,\mu}\}$ and $\{O_{w,q}^{-1}b_{\tau,\mu}\}$ are unitarily equivalent.

$S^{Pek}$ remains in the $\tilde{T}-\tilde{F}$ domain ($\tau - \mu$ plane). To come back in the $t - \nu$ plane and build the Modal Pekeris STFT $S_{M}^{Pek}$, the inverse operator $O_{w,q}^{-1}$ must be applied on the Pekeris STFT. To find the appropriate relation between the two coordinate systems $(\tau, \mu)$ and $(t, \nu)$, the following system of equations must be solved [8]

$$\begin{cases}
\tau = \sigma(\mu, t) \\
\mu = \zeta(\tau, \nu)
\end{cases}$$

Fig. 6. Pekeris FT modulus of the North Sea signal
with
\[ \sigma(\mu, t) = -\frac{1}{2\pi} \frac{\partial}{\partial \mu} \arg(F_{\omega, q} u^T_t)(\mu) \]  
(26)
and
\[ \zeta(\tau, \nu) = \frac{1}{2\pi} \frac{\partial}{\partial \tau} \arg(O_{\omega, q} u^T_\tau)(\nu) \]  
(27)
The modal Pekeris STFT is then
\[ S^{Pek}(t, \nu) = (O_{\omega, q} S^{Pek} x)(t, \nu) = (S^{Pek} x)(w^{-1}(t), w([w^{-1}(t)][\nu + q'(t)])) \]  
(28)
It can be written as the projection in the natural time domain \( T \)
\[ (S^{Pek} x)(t, \nu) = \int x(u) b^{Pek}_{t, \nu}(u) du \]  
(29)
with
\[ b^{Pek}_{t, \nu}(u) = g^{Pek}_t(u) \]
\[ \exp \left( j2\pi \left[ \frac{w^{-1}(u)w^{-1}(t)}{2D\pi} - q(u) \right] \right) \]  
(30)
and
\[ g^{Pek}_t(u) = \left| \frac{u}{w^{-1}(u)} \right|^{1/2} g \left( w^{-1}(u) - w^{-1}(t) \right) \]  
(31)
Modal Pekeris STFT satisfies the properties:
- invertibility: \( x = \langle S^{Pek} x, b^{Pek}_f \rangle \)
- energy conservation \( \int \int_{\mathbb{T}_\nu} |S^{Pek} x(\mu)|^2 d\tau d\mu = \int \int_{\mathbb{F}_f} |x(t)|^2 dt \)
- covariance by Pekeris time and frequency shift \( \mathbb{T}_u \) and \( \mathbb{F}_f \)
\[ (S^{Pek} \mathbb{T}_u \mathbb{F}_f x)(\tau, \mu) = (S^{Pek} x)(\tau - u, \mu - f) \]  
(32)
Both \( S^{Pek} \) and \( S^{Pek} \) can be interpreted as the decomposition of the signal on a basis fitted to theoretical modes which leads to the tiling represented in figure 8 where \( S^{Pek} \) is represented in the transformed \( \mathbb{T} - \mathbb{F} \) domain whereas \( S^{Pek} \) is represented in the natural \( \mathbb{T} - \mathbb{F} \) domain. As the basis are matched to Pekeris dispersion laws, the time-frequency uncertainty can be mapped along the Pekeris time dimension \( \mathbb{T} \) if the window function \( g(u) \) is long. In this case, the representation is precise in the Pekeris frequency (equivalently modal) dimension \( \mathbb{F} \). Modes are thus well separated as one can see in figure 7.

It is possible to extend this procedure to the WVD and all the TFRs of the Cohen’s Class. We do not present the results here because TFRs from Cohen’s class are not all invertible, and because the WVD does not provide satisfactory results due to the multi-structure configuration.

C. Modal adaptive TFR with Matching Pursuit matched to guided waves configuration

1) Matching Pursuit principle: The Matching Pursuit algorithm has been introduced by Mallat et al. [20]. The objective is to find the smallest and best matched basis for a given signal. In this way, the best atomic projection of the signal in a given dictionary of functions \( D = \{ \psi_m \} \) is searched. The Matching Pursuit is an iterative algorithm. For each iteration \( i \), the function \( \psi_i(t) \) which best matches \( x_i(t) \) is found. The coefficient \( c_i \) is then
\[ \left\{ \begin{array}{ll} k = \text{arg max} |x_i, \psi_m| & \\
 c_i = \langle x_i, \psi_k \rangle & 
\end{array} \right. \]  
(33)
where \( x_0(t) = 0 \) and the residual signal at the \( i \)th iteration (for \( i \geq 1 \)) is expressed by
\[ x_i(t) = x_{i-1}(t) - c_{i-1} \psi_{i-1}(t) \]  
(34)
After a number \( N \) of iterations, the signal can be written as
\[ x(t) = \sum_{i=0}^{N-1} c_i \psi_i(t) + (R_N x)(t) \]  
(35)
with residue \( (R_N x)(t) = x_N(t) \). This matched procedure provides a collection of atoms \( \Psi_N = \{ \psi_i, i = 0 : N \} \) which are best matched to the signal \( x(t) \). If the dictionary is complete, \( x(t) \) will exactly be decomposed so that \( (R_N x)(t) = 0 \).

2) Choice of dictionary: Conventional chirplet dictionaries do not match non-linear structures (Figure 5). To solve this problem, Papandreou et al. [25] proposed a method integrating a warping operator \( U \) matched to a dispersive law \( \xi(t) \). Depending on \( \xi(t) \), the dictionary can contain sinusoidal functions, linear...
chirps, hyperbolic chirps etc. The dictionary is formed by warped, warped frequency shifted, dilated and translated versions of a window \( \psi(t) = g(t)e^{j2\pi t} \): \( \{ \psi(t)_\xi,f,d,u = T_d \tilde{F}_\chi U_\xi \psi(t) \} \).

We use the same principle: namely a dictionary of functions matched to non-linear structures. As for the continuous atomic Pekeris TFR, atom construction is based on the dispersive laws of the environment. To formulate it, we use the unitary operator \( O_{w,q} \). Moreover, in natural environments, three elements must be taken into account:
- the source is never a pulse which implies a spreading of modes;
- the model never fits perfectly the real environment;
- environmental parameters are partially unknown.
Consequently, parameters variations are introduced in the dictionary of atoms:
- atoms are defined locally thanks to a dilation \( D_\mu \);
- geoaoustic parameters variations (range, velocities, densities) around their \textit{a priori} supposed values is allowed, a parameters vector \( \bar{p}[r_1,v_1,v_2,\rho_1,\rho_2] \) is integrated in functions \( w(t) \) and \( g(t) \);
- to cover the entire time-frequency plane, atoms can be shifted on Pekeris time and frequency dimensions by the action of \( \tilde{F}_f \) and \( \tilde{T}_u \).

Finally, the dictionary is formulated by:
\[
\psi(t)_{\xi,f,d,u}^{Pek} = (\tilde{T}_u \tilde{F}_f O_{w,q}^{-1} D_{\mu} g(t)e^{j2\pi t})(t) = |a|^{1/2} \left| \frac{1}{w_{\bar{\xi}}(t)} \right|^{1/2} g(a[w_{\bar{\xi}}^{-1}(t) - \tau]) \exp j2\pi(a + f)[w_{\bar{\xi}}^{-1}(t) - \tau] - q_{\bar{\xi}}(t)
\]
with \( \kappa = [f,\tau] \) the Pekeris time-frequency shift vector.

Note that in optical science, a Matching Pursuit algorithm for ‘perfect’ guided waves has been proposed in [7]. However, it neither uses unitary operator formulation nor proposes any TFR of the signal. Our global Matching Pursuit method is generalising previous methods [7] and [25] to allow more adaptability.

3) Time-Frequency Representation of the decomposition: The modal adaptive TFR is the sum of unitary equivalent Wigner-Ville Distributions \( \mathcal{W} \) of selected atoms [20]
\[
\mathcal{P}_{\xi,f,d,u}^{Pek}(t,\nu) = \sum_i |c_i|^2 (O_{w,q}(\xi)) \mathcal{W} \psi(t)_{\xi,f,d,u}^{Pek}(t,\nu)
\]
It can not be represented in the transformed \( \tilde{T} - \tilde{F} \) domain because this domain depends on geoaoustic parameters which can change due to their integration on atoms. Figure 9 presents the TFR of the modal adaptive Pekeris decomposition, modes are actually separated.

V. MODAL FILTERING

For single sensor configuration, we mentioned in introduction two existing methods of modal characteristic filtering.

Potty et al. [5] estimate group velocity with scalograms in a geoaustical inversion context. The first limitation regards the filtered modal characteristic: it is only the modal group speed. The second limitation regards the use of scalograms: distance needs to be large to distinguish modes in the time-frequency plane with scalograms (it is impossible in our North Sea configuration, see CWT modulus 5 which is the squared root version of the scalogram). The third limitation is the spreading of auto-terms in scalograms which implies error on the time arrival estimation. A way to improve this geoaustical inversion would precisely be to use our modal filtering methods: group velocity would be better estimated in a single structure configuration and source-sensor ranges can be shorter.

McClure et al. [7] use the matching pursuit algorithm to decompose a signal scattered by a target in optics. It does not realise modal filtering as the dictionary elements are directly the sum of modes. It is thus similar to matched filtering with parameters variation. The approach is similar to our matching pursuit approach but with more limitations:
- waveguide considered are perfectly reflecting and less realistic than a Pekeris waveguide;
- parameters evolution allowed are: source-sensor range, time shift (\( T \) operator) and waveguide parameters;
- time-varying evolution of those parameters is not allowed (the projection is not local). This aspect seems less important in optics than for underwater acoustics.

We developed invertible representations and allow modal structures separation in the representation domain. We have then developed filters based and these representations:
- frequency filters in the transform domain based on Pekeris FT \( \mathcal{F}^{Pek} \) (Section IV-A);
- continuous atomic time-frequency filters in the natural or time-frequency domain based on Pekeris STFT and modal Pekeris STFT \( \mathcal{S}^{Pek} \) (Section IV-B);
- atom selections in the set of atoms found by the Matching Pursuit approach on \( \mathcal{P}^{Pek} \) (Section IV-C.1).

The two first filters are linear time-varying and unitarily equivalent to conventional filter. They follow the generic equation
\[
(Hx)(t) = \int_u h(t,u)x(u)du
\]
where \( h(t,u) \) is the transfer function of the filter expressed in the time domain \( T \). The third one is a non-linear filter. In this section, we present these filtering operators.
A. Mode identification

The first step to filter modes is to identify their number. To this purpose, representation is used as an interpretation tool. Developed representations are such that we can distinguish modes. If all the parameters are known, modes location can be accurately determined (for instance, near their cutoff frequency in the Pekeris frequency domain) and identification is thus easy. But in practical configurations, at least one parameter is unknown. These errors do not prevent the modal separation in the representation if it is well chosen (as we will see in section VI-B), but it causes significant errors on the modal location depending on the erroneous parameters. It is thus impossible to identify modes based on their absolute location. However, modes localization follows with time and frequency in the TFR. Two limitations must be taken into account:

1) the modal energy is not constant and can be near zero;
2) a signal can be added to the modal decomposition: noise or other structures.

The first limitation can be avoided following this process:

1) identification of the first mode: it is always energetic and is the smallest frequency mode, it can thus be identified;
2) evaluation of the modal distance \( S \) between two consecutive modes: two energetic consecutive modes are present in practical configurations;
3) evaluation of the modal distance between to distinguishable modes: we can then avoid problems linked to non energetic modes.

Following this process, the \( n \)th first modes can be easily identified in the North Sea signal using developed representations (Figure 6 and 7). If a signal is added onto the modes (noise or deterministic structures), it could be impossible to identify modes (depending on the noise level or structure locations). To conclude, mode recognition can not be automated because the modal absolute location is not accurate. However a simple ‘hand-made’ recognition is possible thanks to relative modal location. The only real limitation comes from the unwanted signal: noise or added structures.

B. Linear Pekeris FT filtering

As we have observed, modes are separated and identifiable in the Pekeris FT. Consequently, we can apply conventional bandpass frequency filters in the transformed domain \( \mathcal{F} \) such as Butterworth or Chebybitchev filters. These filters \( \mathcal{F} \) are characterised by their transfer functions in the frequency domain \( \mathcal{F}(\mu) \) and in the time domain \( \mathcal{F}(\tau, u) = \mathcal{F}(\tau - u) \). Applied in the Pekeris time domain \( \mathcal{T} \), the filtering is: \( \mathcal{T}(y)(\tau) = \int \mathcal{F}(\tau - u)x(u)du \).

**Calibration:** Filters \( \mathcal{F} \) are characterised by their central frequency and their band. To determine the central frequency, Pekeris FT \( \mathcal{F} \) representation is used as an interpretation tool. The peak which corresponds to the mode we want to filter is first determined, the maximum of this peak is then taken for the central frequency filter. Band determination is based on the peak decreasing.

**Filter formulation:** We want to characterise the Pekeris FT filter \( \mathcal{F} \) which is a combination of the Pekeris FT and the filter \( \mathcal{F} \). The filter applied in the natural time domain is

\[
(\mathcal{F}^{\text{Pek}}x)(t) = (\mathcal{O}_{w,q}^{-1} \mathcal{F} \mathcal{O}_{w,q})(t) = \int \left| \frac{v}{w^{-1}(v)} \right|^{1/2} \left| \frac{t}{w^{-1}(t)} \right|^{1/2} h(w^{-1}(t) - w^{-1}(v)) \exp(j2\pi[q(v) - q(t)]x(v)dv)
\]

Filters \( \mathcal{F}^{\text{Pek}} \) and \( \mathcal{F} \) are unitarily equivalent. The transfer function \( h^{\text{Pek}}(t, u) \) of the Pekeris FT filter \( \mathcal{F}^{\text{Pek}} \) is expressed by

\[
h^{\text{Pek}}(t, u) = \left| \frac{u}{w^{-1}(u)} \right|^{1/2} \left| \frac{t}{w^{-1}(t)} \right|^{1/2} \exp(j2\pi[q(u) - q(t)])
\]

C. Linear Pekeris continuous atomic time-frequency filtering

Since modes are separated in \( S^{\text{Pek}} \) and \( SM^{\text{Pek}} \), a conventional time-frequency filtering can be developed based on these TFR. The first step is the determination of the time-frequency filtered region \( Z \) associated with the time-frequency mask \( M(t, \nu) \) by

\[
M(t, \nu) = \begin{cases} 
1, & (t, \nu) \in Z \\
0, & (t, \nu) \notin Z
\end{cases}
\]

The three most conventional time-frequency filterings are a non-linear filtering linked to WVD [26], the Weyl filtering [27] and the atomic filtering linked to the method of linear atomic decomposition (STFT, Gabor, CWT etc.). WVD is not efficient, we can not apply the non-linear method. Weyl filtering requires the accurate localization of auto-terms. As Pekeris STFT spreads auto-terms due to atomic decomposition, the Weyl filtering is not adapted to our configuration [28]. As we...
develop atomic TFR which are STFT based, we present the STFT filtering \cite{29} which can be extended for any continuous invertible atomic TFR defined by
\[
(T \mathcal{F} R x)(t, \nu) = \langle x, b_{t, \nu}^T \mathcal{F} R \rangle
\] (42)

Continuous atomic time-frequency filtering consists in masking the TFR before coming back in the time domain by inverse $T \mathcal{F} R^{-1}$ application. $\mathcal{H}^T \mathcal{F} R$ is
\[
(\mathcal{H}^T \mathcal{F} R x)(t) = (T \mathcal{F} R^{-1} h_{t, \nu} T \mathcal{F} R x)(t)
= \int M(t', \nu')(T \mathcal{F} R x)(t', \nu') b_{t', \nu'}^{T \mathcal{F} R}(t) dt' d\nu'
\] (43)
which implies the transfer function formulation of this time-variant filter
\[
h^{R \mathcal{F} R}(t, u)
= \int M(t', \nu') b_{t', \nu'}^{T \mathcal{F} R}(u) b_{t, \nu}^{T \mathcal{F} R}(t) dt' d\nu'
\] (44)
for the STFT decomposition, $b_{t, \nu}(u) = g(u - t) \exp(-j2\pi \nu u)$. In our case, $S^{Pek}$ and $S^{M^{Pek}}$ present separated modal structures used to determine the filtering zone with a conventional image segmentation technique, the watershed algorithm \cite{30}, combined to a pre-processing geodesic reconstruction \cite{31} to smooth local extrema.

Pekeris STFT filter formulation: The Pekeris STFT filter $\mathcal{H}^{Pek}$ is a combination between the $S^{Pek}$ decomposition and the masking function $M^{Pek}$ determined on $S^{Pek}$. The filter applied in the transformed domain $\mathcal{F} - \mathcal{F}$ is a conventional STFT filter $\mathcal{H}^S$ with transfer function $h^S(t, u)$ which satisfy respectively equations 43 and 44 (with $b_{t, \nu}^{T \mathcal{F} R} = b_{t, \nu}^{Pek}$). In the natural domain $\mathcal{T} - \mathcal{F}$, the Pekeris STFT filter is expressed by
\[
(\mathcal{H}^{Pek} x)(t) = (O_{w, q}^{-1} \mathcal{H}^S O_{w, q} x)(t)
= \int |v|^{1/2} (\frac{t}{w^{-1}(v)} - \frac{t}{w^{-1}(t)})^{1/2} h^S(w^{-1}(t), w^{-1}(v)) \exp(j2\pi [q(v) - q(t)]) x(v) dv
\] (45)
Filter $\mathcal{H}^S$ and $\mathcal{H}^{Pek}$ are unitarily equivalent. We can thus formulate the transfer function $h^{Pek}(t, u)$ of the Pekeris STFT filter $\mathcal{H}^{Pek}$
\[
\begin{align*}
h^{Pek}(t, u) &= \left| \frac{u}{w^{-1}(u)} \right|^{1/2} \left| \frac{w^{-1}(t)}{w^{-1}(u)} \right|^{1/2} \\
h^S(w^{-1}(t), w^{-1}(u)) \exp(j2\pi [q(u) - q(t)])
\end{align*}
\] (46)

Modal Pekeris STFT filter formulation: The Pekeris STFT filter $\mathcal{H}^{PekSM}$ is a combination between the modal decomposition $S^{M^{Pek}}$ and the masking function $M^{PekSM}$ determined on $S^{M^{Pek}}$. $S^{M^{Pek}}$ is invertible and is a continuous atomic decomposition (cf. equation 42) expressed in the natural time-frequency domain $\mathcal{T} - \mathcal{F}$, it satisfies equation 42. The Pekeris STFT filter $\mathcal{H}^{PekSM}$ and its transfer function $h^{PekSM}(t, u)$ satisfy respectively equations 43 and 44 (with $b_{t, \nu}^{T \mathcal{F} R} = b_{t, \nu}^{Pek}$ satisfying equations 30 and 31). Finally, filter $\mathcal{H}^{PekSM}$ and the transfer function $h^{PekSM}(t, u)$ are expressed by equations 47 and 48.

D. Non-linear modal adaptive atomic filtering

We have developed a non-linear adaptive atomic filtering based on the modal adaptive TFR $\mathcal{P}^{Pek}$. This TFR provides a set of atoms $\Psi_N = \{ \psi_i, i = 0 : N \}$ which best match the signal (Section IV-C). $\mathcal{P}^{Pek}$ is used as a decision tool to determine atoms $\{ \psi_j \}$ in the collection $\Psi_N$ which correspond to the filtered modes. The mode $x_m$ is the sum of these chosen atoms
\[
x_m(t) = \sum_j c_j \psi_j(t)
\] (49)
This operation is a non-linear filtering. Atoms choice is done manually, the user decides the belonging of each atom to the mode using the TFR. They can be divided into three categories:

1) the atom is associated with mode $m$ if there is no doubt on its belonging;
2) the atom does not belong to the filtered mode (other mode or noise), it is rejected;
3) the atom is ‘ambiguous’ if there is a doubt on its belonging, two possibilities of belonging are associated with the atom, $m_1$ and $m_2$ if the doubt is between two modes or $m_1$ and noise if the doubt is on the belonging of the atom to signal or noise.

VI. Filtering performances in an acoustical underwater configuration

In this last section, we evaluate filtering performances of the developed techniques in four configurations. The difference between the configurations will consist in the presence of noise in the data and the degree of knowledge of the environmental parameters: known parameters imply accurate parametrisation of the Pekeris operator, unknown or partially known parameters imply bad parametrisation. Filtering performances are measured by the Mean Square Error (MSE) in % between theoretical mode $x_m$ and filtered mode $\hat{x}_m$
\[
MSE = 100 \frac{\sum_{i=1}^m |x_m(i) - \hat{x}_m(i)|^2}{\sum_{i=1}^m |x_m(i)|^2}
\] (50)
This evaluation is realised on synthetic data with the underwater configuration of the North Sea data for different ranges $R$. Data is simulated in a real Pekeris (not modified) waveguide with a pulse source. Environmental parameters are those of the North Sea configuration (see section I). The source duration is 3.45 ms and its 3 dB bandwidth 24 – 84.5 Hz, filtered mode is $m = 3$. Concerning the Pekeris continuous atomic time-frequency filtering, it can be performed equivalently with Pekeris STFT filtering $\mathcal{H}^{Pek}$ or modal Pekeris STFT filtering $\mathcal{H}^{PekSM}$ since $\mathcal{S}^{Pek}$ and $\mathcal{M}^{Pek}$ are equivalent. We choose $\mathcal{H}^{Pek}$ which is easier to implement.

All the developed tools are based on the modified Pekeris waveguide (Section II) whose properties are integrated in the Pekeris operator $O_{w, q}$. We have also developed
\[
(\mathcal{H}^{\text{PeksM},x}(t) = \int \mathcal{M}^{\text{PeksM}}(t, \nu)(\mathcal{S}^{\text{Peks}})(t', \nu') \left| \frac{t}{w^{-1}(t)} \right|^{1/2} \exp\left( j2\pi \left[ \nu w^{-1}(t') w^{-1}(t) - \frac{V_t\Phi(t')}{2D\pi} t' - q(t) \right] \right) dt'd\nu' \]

(47)

\[
\mathcal{H}^{\text{PeksM}}(t, u) = \int \int \left[ \frac{t}{w^{-1}(t)} \right]^{1/2} \left[ \frac{u}{w^{-1}(u)} \right]^{1/2} \mathcal{M}^{\text{PeksM}}(t', \nu') g\left( w^{-1}(t') - w^{-1}(u) \right) g^*(w^{-1}(u) - w^{-1}(t')) \exp\left( j2\pi \left[ \nu' w^{-1}(t') (w^{-1}(t) - w^{-1}(u)) - \frac{V_t\Phi(t')}{2D\pi} (w^{-1}(t) - w^{-1}(u)) + [q(t) - q(u)] \right] \right) dt'd\nu' \]

(48)

---

Fig. 10. Performances without noise and with accurate parametrisation (‘perfect’, Pekeris FT and STFT filterings)

Fig. 11. Performances without noise and with accurate parametrisation (Pekeris FT and STFT filterings and modal adaptive filtering)

Fig. 12. Performances without noise and with bad parametrisation

Fig. 13. Performances with noise and accurate parametrisation

Fig. 14. Performances with noise and bad parametrisation

---

The ‘perfect’ operator (matched to perfectly reflecting waveguides, see section I) and all the representations and filterings based on this model [32, 33, 28]. The ‘perfect’ operator has also been separately developed by Jiang et al. [34]. Measured performances using ‘perfect’ filters are always worse than Pekeris filters performances due to the more realistic modelisation of the Pekeris waveguide. The ‘perfect’ model performances are hence only shown in the first case, without noise and with accurate knowledge of the environment.
A. Performance without noise and with accurate parametrisation

In this first configuration, geoacoustic parameters \((V_1, V_2, \rho_1, \rho_2, D, \text{and } R)\) integrated in the Pekeris operator \(O_{w,q}\) are accurately known and no noise is added to the signal. Figures 10 and 11 present the performances in this configuration. Several observations can be made:

1) Pekeris filtering is more efficient than 'perfect' filtering due to the fact that it is more realistic (Figure 10).

2) Modal reconstruction is never perfect. The modified Pekeris model does not perfectly fit the Pekeris model, modes are not strictly separated in the Pekeris frequency dimension and filtering is thus not perfect.

3) Performances increase with \(R\). Modal structures are more spaced in the time-frequency plane and consequently filtering is easier. For short ranges \(R\) modes have their own frequency 'thickness' and they overlap each other.

4) The Pekeris FT filtering \(\mathcal{H}^{F_{Pek}}\) is more efficient than the Pekeris STFT filtering \(\mathcal{H}^{S_{Pek}}\). Even reduced, time-frequency uncertainty is present in the \(S^{Pek}\) and involves an overlapping between modes. This limitation is not present for \(F^{Pek}\) (Figure 10).

5) Modal adaptive filtering performances present uncertainties symbolised by error bars and coming from ambiguity on atoms belonging. Mean performance level is between \(\mathcal{H}^{F_{Pek}}\) and \(\mathcal{H}^{S_{Pek}}\) performances: the local projection introduces a time-frequency uncertainty but the adaptability of this technique makes it more efficient than the \(\mathcal{H}^{S_{Pek}}\) filtering (Figure 11).

B. Performance without noise and with bad parametrisation

In this second configuration, geoacoustic parameters integrated to the Pekeris operator \(O_{w,q}\) contain large errors: \(R_p = 0.3R, V_{1p} = 1.1V_1, V_{2p} = 0.9V_2, \rho_{1p} = 1.1\rho_1\) and \(\rho_{2p} = 0.9\rho_2\). These errors are considerable regarding to the usual knowledge of geoacoustic parameters. They are also chosen to avoid compensation phenomena between parameters. Figure 12 presents the performances in this configuration. Several observations can be made:

1) Concerning linear \(\mathcal{H}^{F_{Pek}}\) and \(\mathcal{H}^{S_{Pek}}\) filters, errors on some parameters only imply a compression or a dilation of the modal dimension (similarly on \(\mathcal{F}\) dimension) on \(F^{Pek}\) and \(S^{Pek}\) representations which does not affect the performances. This is the case for an error on depth \(D\).

2) Errors on some other parameters imply a distortion of modal structures. With a modal approach, representations are seen as decomposition on modal structures (in the whole signal for \(F^{Pek}\), locally for \(S^{Pek}\)). Decomposition of the signal on this type of erroneous structures maps the modes on a narrow band in the \(\mathcal{F}\) dimension rather than on pure frequency. As a consequence, modes overlap each other in the \(\mathcal{F}\) dimension and filtering performances of \(\mathcal{H}^{S_{Pek}}\) are degraded. \(S^{Pek}\) introduces a degree of freedom thanks to the two dimensional decomposition which partially compensate the bad parametrisation, filtering performances are thus weakly degraded and \(\mathcal{H}^{S_{Pek}}\) is more efficient than \(\mathcal{H}^{F_{Pek}}\).

3) Finally, performances of linear filterings are degraded but remain acceptable considering the large parameters errors. This is particularly true for the \(\mathcal{H}^{S_{Pek}}\) filter and it is a strength of this method: even distorted, projection functions remain matched to the signal and filtering is possible (except for a small range \(R = 1500\) m).

4) Modal adaptive filtering performances remain unchanged because parameters evolution is allowed for this filtering. This method is particularly adapted if the environment knowledge is bad.

C. Performance with noise and accurate parametrisation

In this configuration, parameters are accurately known and signal noise ratio is \(SNR = 0\) dB. Performances are presented in figure 13. Several observations can be made:

1) In comparison with the first configuration (without noise), performances are degraded but remain satisfactory at long range. Modal adaptive filtering is however less performant due to the numerous ambiguities on modes belonging implied by the noise.

2) For short ranges, performances of \(\mathcal{H}^{F_{Pek}}\) are degraded. Modes overlap each other as mentioned in the first configuration and one dimensional representation does not offer the degree of freedom of TFR.
D. Performance with noise and bad parametrisation

In this configuration, parameters errors are the same than in section VI-B and $SNR = 0 \, dB$. Performances are presented in figure 14. Performances of the Pekeris FT filtering are degraded by both bad parametrisation and noise (Section VI-B and VI-C). Modes are no more distinguishable in $H^{FTek}$ and modal filtering is impossible.

Modal adaptive filtering performances are unchanged because parameters evolution is allowed for this filtering. Performances of $H^{STFTek}$ are degraded but, as for the accurate parametrisation, located between low and high bound of modal adaptive filtering performances. Choice between these two techniques depends on the level of error on the parametrisation.

E. Performance of conventional STFT filtering

Starting from a sufficiently large source-sensor distance $R$, the time duration of the signal becomes longer and the time-frequency distance between modes increases. As a consequence, modes appear separated when using a conventional TFR and conventional time-varying filterings based on TFR can be applied. In the North Sea configuration, mode 3 appears separated from $R = 20000 m$ in STFT. We thus compare our developed filterings with the STFT conventional filtering starting from $R = 20000 m$ without noise. Results are presented in figure 15: conventional filtering is less efficient, even when the parametrisation of Pekeris STFT filtering is bad. However, it tends to converge for very large distances (> $50000 m$).

F. Conclusion on performances

In this section, we have established performances of filters and studied their robustness against noise and bad parametrisation. We have shown their complementarity thanks to different degree of freedom. This complementarity allows for the adaptability necessary to face knowledge missing or hypothesis limitations (source is never a pulse, parameters are partially unknown and the Pekeris model does not perfectly fit the medium). The choice between the different proposed filters should thus be:

- the Pekeris FT filtering in case of low noise level and good parametrisation;
- the Pekeris FT or the Pekeris STFT filtering in case of high noise level and good parametrisation;
- the modal adaptive filtering or the Pekeris STFT filtering in case of bad parametrisation.

CONCLUSION

In this study, we have developed matched methods to filter modes in waveguides. These modes are dispersive non-linear structures and do not allow the use of conventional time-frequency tools to perform filtering. We have first introduced a modified Pekeris waveguide model with dispersive properties to build a Pekeris unitary operator. Based on this operator and using unitary equivalence, frequency and time-frequency representations have been developed. Using these representations, modes are separated and can be identified. We have thus developed modal filtering techniques based on these representations: linear Pekeris FT and Pekeris STFT filterings and a non-linear modal adaptive filtering. We have studied the robustness against noise and bad knowledge of the geoacoustic parameters for an acoustical underwater configuration. Developed filters can be seen between conventional matched filters and conventional time-frequency filters (without a priori) with increasing degrees of freedom: the Pekeris FT filtering is very efficient but sensitive to parameter knowledge, the Pekeris STFT filtering is more robust to parameters uncertainty thanks to a local projection but introduces time-frequency limitations, and the modal adaptive filtering is insensitive to parameters knowledge but presents ambiguity phenomena. Finally, we have developed efficient and complementary modal filtering techniques which enable new source localization methods [10] or could improve geoacoustic inversion [5] with a single sensor or a sparse array of sensors.

REFERENCES


