Some Topics on Nonlinear Moving-Horizon Observers

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For an updated version

www.lag.ensieg.inpg.fr/alamir/summer_school_no.pdf
Moving-Horizon?

- \( J = \int_{t-T}^{t} [\text{Output pred. err. (}\tau\text{)]} d\tau \)

- internal state:
  Estimate \( z(t) \) of \( x(t - T) \)

- Update \( z(t) \):
  \( z^+ = \varphi(z, J) \)

- The cost \( J(t) \) is explicitly/implicitly used in the innovation process (correction)
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Overview

State estimation: An optimization problem
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(Dynamic & non convex)
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Local minima

Computation time
Overview

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(Dynamic & non convex)

Local minima

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Singularities avoidance
heuristics
Overview

State estimation: An optimization problem

\(\text{(Dynamic \& non convex)}\)

- Local minima
- Computation time

- Singularities avoidance heuristics
- Real-time implementation

- Ideal Continuous Case (Gradient-based)
- Discrete general setting

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Some definitions and notations (1): The System

Uncertainty & noise free system

\[ x(t) = X(t, t_0, x_0) \]
\[ y(t) = h(t, x(t)) \]

Uncertain and noisy system

\[ x(t) = X(t, t_0, x_0, w^t_{t_0}) \]
\[ y(t) = h(t, x(t)) + v(t) \]

Constraints

- \( x(t) \in X(t) \subset \mathbb{R}^n \)
- \( w(t) \in W(t) \subset \mathbb{R}^{nw} \) Uncertainties/Disturbances.
- \( v(t) \in V(t) \subset \mathbb{R}^{ny} \) Measurement noise
Definitions and notations (2): Measurements-compatible configurations

Consider

- Time interval \([t - T, t]\)
- Measurement profile \(y_{t-T}^t\)
- \((\xi, w) \in X(t - T) \times [\mathbb{R}^{n_w}[t-T,t] \)

Notation

\((\xi, w) \in C(t, y_{t-T}) \)”
Definitions and notations (2): Measurements-compatible configurations

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- $(\xi, w) \in X(t - T) \times [\mathbb{R}^{n_w}]^{[t-T,t]}$

$(\xi, w)$ is $(y_{t-T}^t)$-compatible

if for all $\sigma \in [t - T, t]$: 

1. $w(\sigma) \in W(\sigma)$,
2. $X(\sigma, t - T, \xi, w) \in X(\sigma)$,
3. $y_{t-T}^t(\sigma) - Y(\sigma, t-T, \xi, w) \in V(\sigma)$. 

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Notation

\[
(\xi, w) \in C(t, y_{t-T}^t)
\]
Definitions and notations (2): Measurements-compatible configurations

Consider

- Time interval $[t - T, t]$
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if for all $\sigma \in [t - T, t]$:

1. $w(\sigma) \in \mathbb{W}(\sigma)$,
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3. $y_{t-T}^t(\sigma) - Y(\sigma, t-T, \xi, w) \in \mathbb{V}(\sigma)$.

Notation

$(\xi, w) \in \mathbb{C}(t, y_{t-T}^t)$
Definitions and notations (2): Measurements-compatible configurations

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- \((\xi, w) \in X(t - T) \times \mathbb{R}^{nw}[t - T, t]\)

\((\xi, w)\) is \((y_{t-T}^t)\)-compatible if for all \(\sigma \in [t - T, t]\):

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2. \(X(\sigma, t - T, \xi, w) \in X(\sigma),\)
3. \(y_{t-T}^t(\sigma) - Y(\sigma, t - T, \xi, w) \in V(\sigma).\)

Notation

\((\xi, w) \in C(t, y_{t-T}^t)\) if the corresponding trajectory

1. meets the constraints
2. explains the measurements
The finite horizon observation problem

The finite horizon observation problem

Choose \( T > 0 \) and use at each \( t \), the available information:

1. System equations
2. Past measurements \( y^t_{t-T} \),
3. Constraints and
4. Some additional exogenous knowledge.

in order to produce an estimation \( \hat{x}(t) \) of the current state \( x(t) \).
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Find \((\xi, w) \rightarrow \hat{x}(t) = X(t, t - T, \xi, w)\)
The finite horizon observation problem

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Find \((\xi, w)\) \[\Rightarrow\] \[\hat{x}(t) = X(t, t - T, \xi, w)\]

The set of candidate estimates \( \hat{x}(t) \):

\[ \Omega_t = \left\{ X(t, t - T, \xi, w) \mid (\xi, w) \in C(t, y_{t-T}^t) \right\}. \]
The need for additional knowledge

\[ \Omega_t = \left\{ X(t, t - T, \xi, w) \mid (\xi, w) \in \mathbb{C}(t, y_t^T) \right\}. \]

- Either \( \Omega_t = \{ x(t) \} \), for instance because
  - \( W = \{0\} \), \( V = \{0\} \) and
  - The system has no indistinguishable states

\[
\int_{t-T}^{t} \| Y(\sigma, t - T, x^{(1)}) - Y(\sigma, t - T, x^{(2)}) \|^2 d\sigma \geq \alpha(\| x^{(1)} - x^{(2)} \|)
\]

for all \( t \geq 0 \) and all \( (x^{(1)}, x^{(2)}) \in X(t - T) \times X(t - T) \).
The need for additional knowledge

\[ \Omega_t = \left\{ X(t, t - T, \xi, w) \mid (\xi, w) \in \mathbb{C}(t, y^t_{t-T}) \right\}. \]

- Or \( \Omega_t \neq \{x(t)\} \), and a selection must be made by solving

\[ P(t) : \min_{(\xi, w) \in \Omega_t} J(t, \xi, w) \rightarrow (\hat{\xi}(t), \hat{\mathbf{w}}(t)) \]

estimation: \( \hat{x}(t) = X(t, t - T, \hat{\xi}(t), \hat{\mathbf{w}}(t)) \)
Temporal Parametrization (1)

Solve $P(t) : \min_{(\xi, w) \in C(t)} J(t, \xi, w)$

In many textbooks, the following parametrization is suggested for $w$:

$$p_w := \{w(k\tau)\}_{k=k_0}^{k_0+N-1} \in \mathbb{W}(k_0) \times \cdots \times \mathbb{W}(k_0+N-1) \subset \mathbb{R}^{n_w \cdot N}$$

- Decision variable $(\xi, p_w)$ of dimension $n + N \cdot n_w$
- Too rich spectral content $increasing$ uselessly $\Omega_t$
- High sensitivity to the knowledge of $\mathbb{W}(\cdot)$.

*Unrealistically too many possible interpretations of the measurements*
Temporal parametrization (2)

Solve $P(t) : \min_{(\xi, w) \in C(t)} J(t, \xi, w)$

Use a reduced dimensional parametrization

$$w(t) = W(t, p_w) \ ; \ p_w \in \mathbb{P}.$$  

Solve $P(t) : \min_{(\xi, p_w) \in C(t)} J(t, \xi, W(\cdot, p_w)) =: J(t, \xi, p_w) \rightarrow (\hat{\xi}(t), \hat{p}_w(t))$

$$\hat{x}(t) = X(t, t - T, \hat{\xi}(t), W(\cdot, \hat{p}_w(t)))$$

- $\bar{x} := (x^T, p_w^T)^T \in \mathbb{R}^n \times \mathbb{R}^{np}$  
- $\dot{p}_w = 0$

New uncertainty-free extended state estimation problem.
Analytic vs optimization based observer

Analytic observers

(System) \[ \dot{x} = f(x) ; \quad y = h(x) \]

(Observ) \[ \dot{\hat{x}} = f(\hat{x}) + K(\hat{x}, y) \]

Try to show asymptotic convergence of \( e = x - \hat{x} \) governed by

\begin{align*}
\dot{x} &= f(x) \\
\dot{e} &= f(x) - f(x - e) - K(x - e, h(x))
\end{align*}

Very Hard Task

- Need for structural properties
- Coordinate transformation
- Constructive assumptions
- Observability \( \neq \) Existence of observer
Analytic vs optimization based observer

**Analytic observers**

(System) \( \dot{x} = f(x) \); \( y = h(x) \)

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\]

**Very Hard Task**

- Need for structural properties
- Coordinate transformation
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**optimization based observers**

Rely on the implication

\[
\left\{ J(t, \xi) \to 0 \right\} \Rightarrow \left\{ X(t, t - T, \xi) \to x(t) \right\}
\]

+ No need to study the dynamic of \( e \)
+ No need for structural assumptions
+ Observability \( \Leftrightarrow \) Observer
+ Handling constraints on the state

**Potential problems**

- Global convergence ?
- Computation time ?
Forthcoming issues

Global convergence?

- No generic and definitive solution . . .!
- Heuristics for singularities avoidance

Computation time?

- Differential form of optimization based observer
- Real-Time iterations / Optimal choice of updating period
Ideal discrete-time estimation scheme

\[ \xi^{(i+1)} = S(\xi^{(i)}) \]

\[ \hat{\xi}(t_k-1) = \text{prediction based on } \hat{\xi}(t_k-1) \]

\[ \hat{\xi}(t_k) = \lim_{i \to \infty} \xi^{(i)} \]

\[ X(\cdot, t_k-N, \hat{\xi}(t_k)) \]

\[ \hat{x}(t_k) \]
**Ideal discrete-time estimation scheme**

\[
\hat{x}(t_k) = X(t_k, t_{k-N}, \hat{\xi}(t_k))
\]

\[
\hat{\xi}(t_{k-1}) = \xi^{(0)} = \text{prediction based on } \hat{\xi}(t_{k-1})
\]

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\hat{\xi}(t_k) = \lim_{i \to \infty} \xi^{(i)}
\]

\[
\xi^{(i+1)} = S(\xi^{(i)})
\]
Ideal discrete-time estimation scheme

\[
\hat{\xi}(t_k) = \arg \min_{\xi \in \mathcal{X}(t_k-N)} \left[ J(t_k, \xi) \right] := \sum_{i=k-N}^{k} \| y(t_i) - Y(t_i, t_{k-N}, \xi) \|_{Q_i(k)}^2
\]

Initial Guess \( \hat{\xi}^{(0)} \)

Present time

Past

\( \hat{\xi}(t_k) = \lim_{i \to \infty} \hat{\xi}^{(i)} \)

\( \hat{\xi}(t_{k-1}) \)

Iterations

\( \hat{\xi}^{(i+1)} = S(\hat{\xi}^{(i)}) \)
Ideal discrete-time estimation scheme

\[ J^*(t_k, \xi) := \| \xi - \xi^{(0)} \| Q_0 + \sum_{i=k-N}^{k} \| y(t_i) - Y(t_i, t_{k-N}, \xi) \| Q_i(k)^2 \]
Ideal discrete-time estimation scheme

\[ \xi^{(i+1)} = S(\xi^{(i)}) \]

\[ \hat{\xi}(t_{k-1}) \]

\[ \xi^{(0)} = X(t_{k-N}, t_{k-N-1}, \hat{\xi}(t_{k-1})) \]
Ideal discrete-time estimation scheme

\[ \hat{\xi}(t_k) = \xi^{(N_{\text{max}})} = S_{N_{\text{max}}}(\xi^{(0)}, t_k, y_{t_{k-N}}) \]

In practice:

\[ \hat{\xi}(t_k) = \lim_{i \to \infty} \xi^{(i)} \]

\[ \xi^{(0)} = \text{prediction based on } \hat{\xi}(t_{k-1}) \]

Past

\[ \tau_s \]

Present time

\[ \hat{\xi}(t_{k-1}) \]

\[ t_{k-N-1}, t_{k-N}, t_k \]
Ideal discrete-time estimation scheme

In practice:
\[ \hat{\xi}(t_k) = \xi^{(N_{\text{max}})} = \bar{S}^{N_{\text{max}}}(\hat{\xi}(t_{k-1}), t_k, y_{t_k}^{t_{k-N}}) \]
State estimation is a very particular optimization problem

Particular feature of the state estimation related optimization problem

$x(t_{k-N})$ is the unique global minimum of ALL the optimization problems:

$$\hat{\xi}(t_k) = \arg\min_{\xi \in \mathbb{X}(t_{k-N})} \left[ J(t_k, \xi) \right] := \sum_{i=k-N}^{k} \| y(t_i) - Y(t_i, t_{k-N}, \xi) \|_{Q_i(k)}^2$$

that may be obtained by changing the positive definite weighting matrices $Q_i(k)$. 
State estimation is a very particular optimization problem

Particular feature of the state estimation related optimization problem

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that may be obtained by changing the positive definite weighting matrices \( Q_i(k) \).

Let us take the following family subset of weighting choices:

\[ Q_i(k) = \gamma^{k-i} \cdot q_i \cdot I_{n_y} \quad \text{s.t.} \quad q_i > 0 \quad \text{and} \quad \sum_i q_i = 1 \quad (1) \]

- \( \gamma \in ]0, 1] \) Forgetting factor
- \( I_{n_y} \) identity matrix in \( \mathbb{R}^{n_y \times n_y} \)
- Notations: \( \bar{q} = (q_1, q_2, \ldots, q_N)^T \), \( J_\bar{q}(t_k, \xi) \), \( S_{\bar{q}}^{N_{\max}}(\xi(0), t_k, y_{t_{k-N}}) \)
Crossing singularity by swapping the weighting vectors

\[ J_{\bar{q}}(t_k, \xi) \]

\[ J_{\bar{q}^{(1)}}(t_k, \xi) \]

Generally, \( J_{\bar{q}}(1) \) and \( J_{\bar{q}}(2) \) have no reasons to share the same local minima.

They do share the same global minimum \( x(t_k - N) \).

Think about an infinite number of \( J_{\bar{q}}(t_k, \xi) \) (randomly generated).

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Crossing singularity by swapping the weighting vectors

- Generally, $J_{\bar{q}}^{(1)}(t_k, \xi)$ and $J_{\bar{q}}^{(2)}(t_k, \xi)$ have no reasons to share the same local minima.
- They do share the same global minimum $x(t_{k-N})$.
Crossing singularity by swapping the weighting vectors

- Generally, $J_{\bar{q}(1)}(t_k, \cdot)$ and $J_{\bar{q}(2)}(t_k, \cdot)$ have no reasons to share the same local minima.
- They do share the same global minimum $x(t_{k-N})$
- Think about an infinite number of $J_{\bar{q}}(t_k, \cdot)$ (randomly generated)
This suggests

The crossing singularities heuristic

\[ \bar{q} \leftarrow \frac{1}{n_y} (1 \ 1 \ \ldots \ 1) \]

\[ \hat{\xi}(t_k) \leftarrow X(t_k-N, t_k-N-1, \hat{\xi}(t_k-1)) \]

for \( i = 1 : N_{\text{trials}} \)

\[ \hat{\xi}(t_k) \leftarrow S_{\bar{q}}^{N_{\text{max}}} (\hat{\xi}(t_k), t_k, y_{t_k-N}^{t_k}) \]

Generate randomly new admissible \( \bar{q} \)

end

\[ \hat{x}(t_k) \leftarrow X(t_k, t_k-N, \hat{\xi}(t_k)) \]
This suggests

**The crossing singularities heuristic**

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\hat{\xi}(t_k) \leftarrow X(t_{k-N}, t_{k-N-1}, \hat{\xi}(t_{k-1}))
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for \(i = 1 : N_{\text{trials}}\)

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\hat{\xi}(t_k) \leftarrow S_{\bar{q}}^{N_{\text{max}}} (\hat{\xi}(t_k), t_k, y_{t_k-N}^{t_k})
\]

Generate randomly new admissible \(\bar{q}\)

end

\[
\hat{x}(t_k) \leftarrow X(t_k, t_{k-N}, \hat{\xi}(t_k))
\]

- This is not a multiple initial guess trials

- Implementation constraint

\[
N_{\text{trial}} \times N_{\text{max}} \times \tau_{\text{iter}} \leq \tau_s
\]

- The Trade-off is problem dependent
Example: State estimation of terpolymerization reactors

- Produce polymer from multi-monomer
- Controlling the final properties need the state to be estimated
- State: Polymer composition ↔ Monomers concentrations
- Complex equations
- Unknown dynamics
- High gain observers need tremendous simplifications to give rather poor performance

Coll. Nida Sheibat-Othman & Sami Othman
(LAGEP,Lyon)
Example: terpolymerization reactors (The mathematical model)

\[ \dot{N}_i = Q_i - R_{Pi} \quad i = 1, 2, 3 \]
Example: terpolymerization reactors (The mathematical model)

\[ \dot{N}_i = Q_i - R_{P_i} \quad i = 1, 2, 3 \]

\[ R_{P_i} = \mu [M_i^P](k_{p1i}P_1^P + k_{p2i}P_2^P + k_{p3i}P_3^P) \]
Example: terpolymerization reactors (The mathematical model)

\[ \dot{N}_i = Q_i - R_{Pi} \quad i = 1, 2, 3 \]

\[ R_{Pi} = \mu [M_i^P](k_{p1i}P_1^P + k_{p2i}P_2^P + k_{p3i}P_3^P) \]

where

\[ P_1^P = \frac{\alpha}{\alpha + \beta + \gamma} \quad ; \quad P_2^P = \frac{\beta}{\alpha + \beta + \gamma} \quad ; \quad P_3^P = 1 - P_1^P - P_2^P \]
Example: terpolymerization reactors (The mathematical model)

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in which

\[ \alpha = [M_1^P](k_{p21}k_{p31}[M_1^P] + k_{p21}k_{p32}[M_2^P] + k_{p31}k_{p23}[M_3^P]) \]

\[ \beta = [M_2^P](k_{p12}k_{p31}[M_1^P] + k_{p12}k_{p32}[M_2^P] + k_{p13}k_{p32}[M_3^P]) \]

\[ \gamma = [M_3^P](k_{p13}k_{p21}[M_1^P] + k_{p21}k_{p23}[M_2^P] + k_{p13}k_{p23}[M_3^P]) \]
Example: terpolymerization reactors (The mathematical model)

\[ \dot{N}_i = Q_i - R_{Pi} \quad i = 1, 2, 3 \]

\[ R_{Pi} = \mu [M_i^P](k_{p1i}P_1^P + k_{p2i}P_2^P + k_{p3i}P_3^P) \]

where

\[ P_1^P = \frac{\alpha}{\alpha + \beta + \gamma} \quad ; \quad P_2^P = \frac{\beta}{\alpha + \beta + \gamma} \quad ; \quad P_3^P = 1 - P_1^P - P_2^P \]

The \([M_i^P]\) depend in the state according to:

\[
[M_i^P] = \begin{cases} 
(1 - \phi_p^P)N_i \\
\sum_j \frac{N_j MW_j}{\rho_j} \\
\sum_j MW_j \left( \frac{N_j^T - N_j}{\rho_{j,h}} + \frac{N_j}{\rho_j} \right) 
\end{cases}
\]

(Phase II) (Phase III)
Example: terpolymerization reactors (The mathematical model)

\[ \dot{N}_i = Q_i - R_{P_i} \quad i = 1, 2, 3 \]

\[ R_{P_i} = \mu [M_i^P] (k_{p1i} P_1^P + k_{p2i} P_2^P + k_{p3i} P_3^P) \]

- \( \mu \) plays a crucial role
- The dynamic of \( \mu \) is unknown
Example: terpolymerization reactors (The mathematical model)

\[ \dot{N}_i = Q_i - R_{Pi} \quad i = 1, 2, 3 \]

\[ R_{Pi} = \mu [M_i^P](k_{p1i} P_1^P + k_{p2i} P_2^P + k_{p3i} P_3^P) \]

- \( \mu \) plays a crucial role
- The dynamic of \( \mu \) is unknown

- Measurement

The overall monomer conversion measured by calorimetry:

\[ y = \frac{\sum_{i=1}^{3} MW_i (N_i^T - N_i)}{\sum_{j=1}^{3} MW_j N_j^T} \]
Example: State reconstruction of terpolymerization reactors (Validation)

1. Simulation results
2. Experimental results
Example: State reconstruction of terpolymerization reactors (Validation)

Simulation results

\[
\begin{align*}
\dot{N} &= \begin{pmatrix} 1 + d_1 & 0 & 0 \\ 0 & 1 + d_2 & 0 \\ 0 & 0 & 1 + d_3 \end{pmatrix} \cdot f(x, u) \\
\dot{\mu} &= 0 \\
y &= (1 + \nu) \cdot h(x)
\end{align*}
\]

- The state \( x := (N_1, N_2, N_3, \mu) \in \mathbb{R}^4_+ \)
- The uncertainties

\[
\begin{align*}
d_i(k) &= d_{\text{max}} \cdot r_i(k) \\
\nu(k) &= \nu_{\text{max}} \cdot r_\nu(k)
\end{align*}
\]

- \( r_i \) and \( \nu \) randomly chosen in \([-1, +1]\)
Simulation results

Figure: Observer behavior under model uncertainty given by (2)-(2) with $d_{\text{max}} = 10\%$ and no measurement noise ($\nu_{\text{max}} = 0$). The observation horizon is $N = 10$ and the number of trials for the singularity crossing scheme is $N_{\text{trials}} = 4$. Initial state of the observer is $\hat{x}(0) = \text{diag}(0.8, 1.3, 1.3) \cdot x(0)$ and $\mu_{\text{obs}}(0) = 0.8\mu_{\text{model}}$. 

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Simulation results

Figure: Observer behavior under model uncertainty given by (2)-(2) with $d_{\text{max}} = 10\%$ and in the presence of measurement noise ($\nu_{\text{max}} = 0.01$). The observation horizon is $N = 15$ and the number of trials for the singularity crossing scheme is $N_{\text{trials}} = 4$. $\mu_{\text{obs}}(0) = 0.8\mu_{\text{model}}$. Note that concerning the output, only the true output and the estimated one are shown, measurement noise is not presented. This scenario uses a tolerance $\varepsilon = 10^{-8}$ for the optimization subroutine.
Simulation results

Figure: Comparison between the observer behavior when $N_{\text{trials}} = 1$ and $N_{\text{trials}} = 4$ under the scenario depicted on figure 2. Note how the singularity cross mechanism enables to avoid drops in the estimation quality when the observer encounters a singular situation. This scenario uses a tolerance $\varepsilon = 10^{-8}$ for the optimization subroutine.
Simulation results

Figure: Computation times needed to achieve the state estimation depicted on figure 2. Note that an explicit upper bound has been imposed in the internal loop of the optimizer in order to deliver the best estimation that can be obtained within the available computation time defined by the sampling period (30 seconds). This scenario uses a tolerance $\varepsilon = 10^{-8}$ for the optimization subroutine.
Experimental results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_p^P$</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>$MW_1$</td>
<td>128.2</td>
<td>(g/mol)</td>
</tr>
<tr>
<td>$MW_2$</td>
<td>100.12</td>
<td>(g/mol)</td>
</tr>
<tr>
<td>$MW_3$</td>
<td>86.09</td>
<td>(g/mol)</td>
</tr>
<tr>
<td>$\rho_1$</td>
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<td>(g/cm$^3$)</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0.94</td>
<td>(g/cm$^3$)</td>
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<tr>
<td>$\rho_3$</td>
<td>0.93</td>
<td>(g/cm$^3$)</td>
</tr>
<tr>
<td>$\rho_{1,h}$</td>
<td>1.08</td>
<td>(g/cm$^3$)</td>
</tr>
<tr>
<td>$\rho_{2,h}$</td>
<td>1.15</td>
<td>(g/cm$^3$)</td>
</tr>
<tr>
<td>$\rho_{3,h}$</td>
<td>1.17</td>
<td>(g/cm$^3$)</td>
</tr>
<tr>
<td>$k_{p11}$</td>
<td>$4.5 \times 10^5$</td>
<td>(cm$^3$/mol/s)</td>
</tr>
<tr>
<td>$k_{p22}$</td>
<td>$1.28 \times 10^6$</td>
<td>(cm$^3$/mol/s)</td>
</tr>
<tr>
<td>$k_{p33}$</td>
<td>$4.26 \times 10^6$</td>
<td>(cm$^3$/mol/s)</td>
</tr>
<tr>
<td>$r_{12}$</td>
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<tr>
<td>$r_{21}$</td>
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<td></td>
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<td>$r_{13}$</td>
<td>6.635</td>
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<tr>
<td>$r_{31}$</td>
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<td>$r_{23}$</td>
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<td></td>
</tr>
<tr>
<td>$r_{32}$</td>
<td>0.07</td>
<td></td>
</tr>
</tbody>
</table>

Table: Parameter values of the terpolymerization of BuA/MMA/VAc (used in the experimental validation)

Table: Recipe of the terpolymerization of BuA/MMA/VAc

| Component                                      | Charge (g) |
Experimental results: $N_{\text{trials}} = 10$

Figure: Experimental validation with $N_{\text{trials}} = 10$ and tolerance threshold $\varepsilon = 10^{-3}$. The same scenario is depicted on figure 6 where $N_{\text{trials}} = 1$ is used. The computation time is given in seconds.
Experimental results: $N_{\text{trials}} = 1$

Figure: Experimental validation with $N_{\text{trials}} = 1$ and tolerance threshold $\varepsilon = 10^{-3}$. The same scenario is depicted on figure 5 where $N_{\text{trials}} = 10$ is used. The computation time is given in seconds.
More general formulation for singularities avoidance

Consider a general simulator

\[ x(t) = X(t, t_0, x_0) \quad ; \quad y(t) = h(x(t)) \]
More general formulation for singularities avoidance

Consider a general simulator

\[ x(t) = X(t, t_0, x_0) \; ; \; y(t) = h(x(t)) \]

Classical observer related optimization problem

\[
\min_{z \in \mathbb{X}(t-T)} \left[ J(t, z) \right] := \int_{t-T}^{t} \| Y(\tau, t-T, z) - y(\tau) \|^2 d\tau
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\[
\min_{z \in X(t-T)} \left[ J(t, z) \right] := \int_{t-T}^{t} \| Y(\tau, t-T, z) - y(\tau) \|^2 \, d\tau
\]

Ideally: \( \hat{z}(t) = x(t-T) \) is the unique global minimum for ALL cost functions

\[
J_i(t, z) = \int_{t-T}^{t} \left[ \Phi_i(\tau) \cdot \Psi_i(\epsilon_y(\tau, z)) \right] d\tau
\]

for all

- \( \Phi_i : [t-T, t] \rightarrow \mathbb{R}_+ \)
- \( \Psi_i(\cdot) \) positive definite function defined on \( \mathbb{R}^{n_y} \)
Typical scheme behind the intuition
Let us try the special choice

\[ J_i(t, z) = \int_{t-T}^{t} \left[ \Phi_i(\tau) \cdot \Psi_i(\epsilon_y(\tau, z)) \right] d\tau \]

with

\[ \Psi_i(y) = y^T y \]

\[ \Phi_i(\tau) = \frac{1}{2} \left[ T_i\left( \frac{2\tau}{T} - 1 \right) + 1 \right] \]

where for \( i \in \{1, \ldots, N\} \), \( T_i \) stands for the \( i \)th Tchebychev polynomials of the first kind, namely:

\[ T_0(x) = 1 \quad ; \quad T_1(x) = x \]

\[ T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x) \]
Example: Recombinant Escherichia Coli

\[ \dot{X} = \mu(S) X - k_d \exp\left(-\frac{k_p}{P}\right) X \]

\[ \dot{S} = -y_s \mu X - k_m X \]

\[ \dot{P} = y_p \mu(S) \frac{I}{I + k_l} X - k_d \exp\left(-\frac{k_p}{P}\right) P \]

- \( X \): *E. Coli* strain
- \( S \): substrate glycerol
- \( P \): intracellular product
  \( \beta \)-galactosidase protein
- \( \mu \) is the growth rate

\[ \mu(S) = \frac{\mu_m S}{k_s + S} \]

**Output measurement:**

Light produced by the bioluminescence:

\[ L = y_l \cdot \mu(S) \frac{I}{I + k_l} XP \]
Example: Recombinant Escherichia Coli

\[
\begin{align*}
\dot{X} &= \mu(S)X - k_d \exp\left(-\frac{k_p}{P}\right)X \\
\dot{S} &= -y_s \mu X - k_m X \\
\dot{P} &= y_p \mu(S) \frac{I}{I + k_l} X - k_d \exp\left(-\frac{k_p}{P}\right)P
\end{align*}
\]

- \(X\): *E. Coli* strain
- \(S\): substrate glycerol
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<table>
<thead>
<tr>
<th>param.</th>
<th>Val.</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_m)</td>
<td>0.49</td>
<td>(h^{-1})</td>
</tr>
<tr>
<td>(k_s)</td>
<td>0.06</td>
<td>(g/l)</td>
</tr>
<tr>
<td>(k_p)</td>
<td>0.047</td>
<td>(g/l)</td>
</tr>
<tr>
<td>(k_d)</td>
<td>0.005</td>
<td>(g/l)</td>
</tr>
<tr>
<td>(k_m)</td>
<td>0.21</td>
<td>(h^{-1})</td>
</tr>
<tr>
<td>(k_l)</td>
<td>0.03</td>
<td>(g/h)</td>
</tr>
<tr>
<td>(y_s)</td>
<td>0.75</td>
<td>(g\text{ cell}/g\text{ glycerol})</td>
</tr>
<tr>
<td>(y_p)</td>
<td>0.32</td>
<td>(g\text{ protein}/\beta\text{-galactosidase})</td>
</tr>
<tr>
<td>(y_l)</td>
<td>17.6</td>
<td>(U/\beta\text{-galactosidase})</td>
</tr>
</tbody>
</table>

Output measurement:

Light produced by the bioluminescence:

\[
L = y_l \cdot \mu(S) \frac{I}{I + k_l} XP
\]
- $T = 10$
- $x(t - T) = (0.08, 2, 0.1)$
- $z_2 = x_2(t - T), z_3 = x_3(t - T)$
Evolution of the cost functions \( \{J_i\}_{i=0}^4 \)

- \( T = 15 \)
- \( x(t - T) = (0.05, 3, 0.1) \)
- \( z_2 = x_2(t - T), z_3 = x_3(t - T) \)
Some general formalism

- Assume some solver iteration:

\[ z^{(i+1)} = S(z^{(i)}, J(\cdot)) \]

- Denote multiple-iteration map

\[ z^{(i+r)} = S^{(r)}(z^{(i)}, J) \]

- Define the \((z^{(0)}, J)\)-solver path by

\[ \{ S^{(r)}(z^{(0)}, J) \}_{r \in \mathbb{N}} \]
Definition: N-Safely redundant optimization problem

The optimization problem is called *N-safely redundant* iff

1. There exists a finite sequence of *N* cost functions *J*<sub>*i*</sub> defined on *R*<sup>*n*</sup> that admits *x*(<sub>*t* − *T*</sub>) as a global minimum.

2. There exists a solver *S* and a finite integer *r*<sup>*</sup> ∈ *N* such that:

\[
\Delta(z) := \min_{i \in \{1, \ldots, N\}} \left[ J_0(S^{(r^*)}(z, J_i)) - \gamma J_0(z) \right] \leq 0
\]

(2)

holds for some \( \gamma \in [0, 1] \) and all *z* ∈ *Z*. Moreover:

\[ S^{(r^*)}(z, J_{i^*}) \in Z \]

where *i*<sup>*</sup> is the optimal argument of the minimization invoked in (2).♥
Evolution of the cost functions \( \{J_i\}_{i=0}^4 \)
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Evolution of the cost functions $\{J_i\}_{i=0}^3$
Algorithm $A_1$

0. Initialization $z^{(0)}$ initial guess, $\sigma \leftarrow 0$

1. while $(J_0(z^{(\sigma)}) > \varepsilon)$ do

End while
Algorithm $A_1$

0. Initialization $z^{(0)}$ initial guess, $\sigma \leftarrow 0$

1. while ($J_0(z^{(\sigma)}) > \varepsilon$) do
   1.1 $i \leftarrow 1$; again $\leftarrow$ true
   1.2 while (again & $i \leq N$) do
      1.2.1 $\xi^{(\sigma,i)} \leftarrow S^o(z^{(\sigma)}, J_i)$
      1.2.2 again $\leftarrow \left( J_0(\xi^{(\sigma,i)}) > \gamma J_0(z^{(\sigma)}) \right)$
      1.2.3 If again then $i \leftarrow i + 1$
      1.2.4 Else $\sigma \leftarrow \sigma + 1$, $z^{(\sigma)} \leftarrow \xi^{(\sigma,i)}$
   End while
End while
Discussion

- The scheme holds regardless the optimizer $S$
  - Gradient-based iteration
  - SQP
  - Multiple shooting
  - Non smooth (simplex, powell’s, etc.)
Discussion

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- Easily usable in a parallel architecture
  [parallel $(z^{(0)}, J_i)$ path-solvers exploration]
• The scheme holds regardless the optimizer $S$
  
  • Gradient-based iteration  
  • SQP  
  • multiple shooting  
  • non smooth (simplex, powell’s, etc.)

• Easily usable in a parallel architecture  
  [parallel $(z^{(0)}, J_i)$ path-solvers exploration]

• $\neq$ a multiple initial guess scheme  
  [It’s the cost function that changes not the present solution]
Discussion

- The scheme holds regardless the optimizer $S$
  - Gradient-based iteration
  - SQP
  - multiple shooting
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- Easily usable in a parallel architecture
  [parallel \((z^{(0)}, J_i)\) path-solvers exploration]

- $\neq$ a multiple initial guess scheme
  [It’s the cost function that changes not the present solution]

- Price: Loose optimality that is \textit{loosely defined}
Back to the non-ideal real-time iterations

\[ \hat{\xi}(t_k) = \arg\min_{\xi \in X} (t_k - N) \]

\[ \hat{\xi}(t_k) = S_{\text{max}}(\xi(0), t_k, y_{t_k} t_k - N) \]

\[ \xi(0) = X(t_k - N, t_k - N - 1, \hat{\xi}(t_k - 1)) \Rightarrow \text{Implicit updating rule} \]

\[ \hat{\xi}(t_k) = F(t_k, \hat{\xi}(t_k - 1), y_{t_k - T}) \]

Is there a differential form of this updating rule?

\[ \frac{d\xi}{dt}(t_k) = f(t_k, \xi, y_{t_k - T}) \]

→ Differential form for moving horizon observers
Back to the non-ideal real-time iterations

Initial Guess $\hat{\xi}(t_k) = \arg\min_{\xi \in X(t_k-N)} J(t_k, \xi)$
Back to the non-ideal real-time iterations

Initial Guess $\hat{\xi}^{(0)}$

- $\hat{\xi}(t_k) = \arg\min_{\xi \in X(t_k-N)} J(t_k, \xi)$
- $\hat{\xi}(t_k) = S_{\max}^{N}(\hat{\xi}^{(0)}, t_k, y_{t_k-N})$
Back to the non-ideal real-time iterations

Initial Guess $\xi^{(0)}$

- $\hat{\xi}(t_k) = \arg\min_{\xi \in \mathbb{X}(t_k)} J(t_k, \xi)$
- $\hat{\xi}(t_k) = S^{N_{\text{max}}}(\xi^{(0)}, t_k, y^{t_k}_{t_k-N})$
- $\xi^{(0)} = X(t_{k-N}, t_{k-N-1}, \hat{\xi}(t_k))$

Definitions
The Observation Pb.
Analytic vs optim.
Singularities avoidance
Differential form
RT Optim.
Further readings
Conclusion
M.H. Diagnosis
Back to the non-ideal real-time iterations

\[ \hat{\xi}(t_k) = F(t_k, \hat{\xi}(t_k-1), y_{t_k}^{t_k-N}) \]

⇒ Implicit updating rule

\[ \hat{\xi}(t_k) = \text{arg} \min_{\xi \in \mathcal{X}(t_k-N)} J(t_k, \xi) \]

- \( \hat{\xi}(t_k) = S^{N_{\text{max}}} (\xi^{(0)}, t_k, y_{t_k}^{t_k-N}) \)
- \( \xi^{(0)} = \mathcal{X}(t_k-N, t_k-N-1, \hat{\xi}(t_k-1)) \)
Back to the non-ideal real-time iterations

$\hat{\xi}(t_k) = F(t_k, \hat{\xi}(t_{k-1}), y_{t_{k-N}}^t)$

⇒ Implicit updating rule

Is there a differential form of this updating rule?

$$\frac{d\xi}{dt}(t) = f(t, \xi, y_{t-T}^t)$$
Back to the non-ideal real-time iterations

⇒ Implicit updating rule

\[ \hat{\xi}(t_k) = F(t_k, \hat{\xi}(t_{k-1}), y_{t_k}^{t_{k-N}}) \]

Is there a differential form of this updating rule?

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→ Differential form for moving horizon observers

Initial Guess \( \xi^{(0)} \)

- \( \hat{\xi}(t_k) = \arg \min_{\xi \in X(t_k-N)} J(t_k, \xi) \)
- \( \hat{\xi}(t_k) = S_{Nmax}(\xi^{(0)}, t_k, y_{t_k}^{t_{k-N}}) \)
- \( \xi^{(0)} = X(t_k-N, t_{k-N-1}, \hat{\xi}(t_{k-1})) \)
Differential Form of Moving-Horizon Observers: Outline

For system of the form $\dot{x} = f(t, x)$:

\[
\begin{align*}
\dot{\xi}(t) &= f(t - T, \xi(t)) + c(t, \xi(t)) \\
\hat{x}(t) &= X(t, t - T, \xi(t))
\end{align*}
\]

- The correction term

\[
c(t, \xi) := \gamma \left[ \frac{J_{\xi}^T(t, \xi)}{\|J_{\xi}\|^2 + \varepsilon} \right] \left[ -|\Delta_{t-T}(\epsilon_y(\cdot, \xi))| - [1 + \phi(t, \xi)] \sqrt{J} \right]
\]

- Post-Stabilization technique $\rightarrow$ improve (Sampling period)/Precision ratio

- Alstom-Transport Patent
Preliminary definitions & assumptions

System model

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t)) \\
y(t) &= h(t, x(t))
\end{align*}
\]

Notations

- \(X(t, t_0, x_0)\): State evolution
- \(Y(t, t_0, x_0)\): Output evolution
Preliminary definitions & assumptions

System model

\[ \dot{x}(t) = f(t, x(t)) \]
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The cost function

\[ J(t, \xi) = \int_{t-T}^{t} \| Y(\tau, t - T, \xi) - y(\tau) \|^2 d\tau \]

NOTA:
Dependence w.r.t \( y_{t-T} \) are implicitly assumed through dependence on \( t \)
Preliminary definitions & assumptions

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Dependence w.r.t \(y_{t-T}\) are implicitly assumed through dependence on \(t\)

- if \(\xi = x(t - T)\) then \(J(t, \xi) = 0\)
- Let \(\xi\) be a dynamic variable \(\xi(t)\)
Preliminary definitions & assumptions

System model

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NOTA:
Dependence w.r.t \( y_{t-T} \) are implicitly assumed through dependence on \( t \)

- if \( \xi = x(t - T) \) then \( J(t, \xi) = 0 \)
- Let \( \xi \) be a dynamic variable \( \xi(t) \)
- Look for a dynamic on \( \xi \):
  \[
  \dot{\xi}(t) =? 
  \]
  such that \( \lim_{t \to \infty} \xi(t) = x(t - T) \)
Preliminary definitions & assumptions

System model

\[ \dot{x}(t) = f(t, x(t)) \]
\[ y(t) = h(t, x(t)) \]

Notations

- \( X(t, t_0, x_0) \): State evolution
- \( Y(t, t_0, x_0) \): Output evolution

The cost function

\[ J(t, \xi) = \int_{t-T}^{t} \| Y(\tau, t - T, \xi) - y(\tau) \|^2 d\tau \]

Observer

\[ \dot{\xi}(t) =? \]
\[ \hat{x}(t) = X(t, t - T, \xi(t)) \]

if \( \xi = x(t - T) \) then \( J(t, \xi) = 0 \)

Let \( \xi \) be a dynamic variable \( \xi(t) \)

Look for a dynamic on \( \xi \):

\[ \dot{\xi}(t) =? \]

such that \( \lim_{t \to \infty} \xi(t) = x(t - T) \)
Reformulating the condition on $\dot{\xi}(t)$

By the very definition of observability, the two following formulations are equivalent

**Formulation 1**

Look for a dynamic on $\xi$:

$$\dot{\xi}(t) = ?$$

such that $\lim_{t \to \infty} \xi(t) = x(t - T)$

**Formulation 2**

Look for a dynamic on $\xi$:

$$\dot{\xi}(t) = ?$$

such that $\lim_{t \to \infty} J(t, \xi(t)) = 0$
Formulation 2

Look for a dynamic on $\xi$:

$$\dot{\xi}(t) = ?$$

such that $\lim_{t \to \infty} J(t, \xi(t)) = 0$

Observer

$$\dot{\xi}(t) = ?$$
$$\hat{x}(t) = X(t, t - T, \xi(t))$$

Conditions on the dynamic of $\dot{\xi}$

1. **Consistency**

   $$\dot{\xi}(t) = f(t - T, \xi(t)) + \underbrace{c(t, \xi(t))}_{\text{correction term } O(J(t, \xi(t)))}$$

2. **Convergence**

   $$\lim_{t \to \infty} J(t, \xi(t)) = 0$$
Consistency

\[ \dot{\xi}(t) = f(t - T, \xi(t)) + c(t, \xi(t)) \]

Convergence

\[
\lim_{t \to \infty} J(t, \xi(t)) = 0
\]
Consistency

\[ \dot{\xi}(t) = f(t - T, \xi(t)) + c(t, \xi(t)) \]

Convergence

\[ \lim_{{t \to \infty}} J(t, \xi(t)) = 0 \]

\[ \frac{dJ}{dt}(t, \xi(t)) = J_t(t, \xi(t)) + [J_\xi(t, \xi(t))] \dot{\xi} \]
Consistency

\[ \dot{\xi}(t) = f(t - T, \xi(t)) + c(t, \xi(t)) \]

Convergence

\[ \lim_{t \to \infty} J(t, \xi(t)) = 0 \]

\[ \frac{dJ}{dt}(t, \xi(t)) = J_t(t, \xi(t)) + [J_\xi(t, \xi(t))] \dot{\xi} \]

\[ \frac{dJ}{dt} = J_t(t, \xi(t)) + [J_\xi(t, \xi(t))] \cdot [f(t - T, \xi(t)) + c(t, \xi(t))] \]
Consistency

\[ \dot{\xi}(t) = f(t - T, \xi(t)) + c(t, \xi(t)) \]

Convergence

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\[ \frac{dJ}{dt}(t, \xi(t)) = J_t(t, \xi(t)) + [J_\xi(t, \xi(t))] \dot{\xi} \]

\[ \frac{dJ}{dt} = J_t(t, \xi(t)) + \left[ J_\xi(t, \xi(t)) \right] \cdot \left[ f(t - T, \xi(t)) + c(t, \xi(t)) \right] \]

This writes

\[ \frac{dJ}{dt} = \left. \frac{dJ}{dt} \right|_{c(\cdot, \cdot) = 0} + \left[ J_\xi(t, \xi(t)) \right] \cdot c(t, \xi(t)) \]
\[
\frac{dJ}{dt} = \frac{dJ}{dt}_{c(\cdot,\cdot)\equiv 0} + \left[ J_{\xi}(t, \xi(t)) \right] \cdot c(t, \xi(t))
\]

**Lemma**

The correction-free time derivative of \( J \) satisfies:

\[
\frac{dJ}{dt}_{c(\cdot,\cdot)\equiv 0} \leq |\epsilon_y(t, \xi(t)) - \epsilon_y(t - T, \xi(t))| + \left[ \phi(t, \xi(t)) \right] \cdot \sqrt{J(t, \xi(t))}
\]

where

\[
J(t, \xi) = \int_{t-T}^{t} \| Y(\tau, t - T, \xi) - y(\tau) \|^2 d\tau
\]

\[
\epsilon_y(\tau, \xi(t)) = Y(\tau, t - T, \xi(t)) - y(\tau) \quad \forall \tau \in [t - T, t]
\]

\[
\left[ \phi(t, \xi(t)) \right] = \sup_{\tau \in [t - T, t]} \left| \frac{dY}{dt}_{c\equiv 0}(\tau, t - T, \xi(t)) - \dot{y}(\tau) \right|
\]
Therefore

The correction-free time derivative of \( J \) satisfies:

\[
\frac{dJ}{dt} \bigg|_{c(\cdot, \cdot) \equiv 0} \leq |\Delta^t_{t-T}(\epsilon(\cdot, \xi(t)))| + \phi(t, \xi(t)) \cdot \sqrt{J(t, \xi(t))}
\]

where

- \( \Delta^t_{t-T}(\cdot, \xi(t)) = \epsilon_y(t, \xi(t)) - \epsilon_y(t - T, \xi(t)) \) and

- \( \phi(t, \xi(t)) \)

are computable quantities
\[
\frac{dJ}{dt} \leq |\Delta_{t-T}(\epsilon_y(\cdot, \xi(t)))| + \left[\phi(t, \xi(t))\right] \cdot \sqrt{J} + \left[J_{\xi}(t, \xi(t))\right] \cdot c(t, \xi(t))
\]

Therefore

The correction-free time derivative of \( J \) satisfies:

\[
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- \( \left[\phi(t, \xi(t))\right] \)

are computable quantities
The ability to decrease $J$ depend on the rank of $J_\xi$

**Uniform Global Regularity Assumption**

There is a K-function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following inequality holds:

$$\|J_\xi(t, \xi)\|^2 \geq \gamma(J(t, \xi))$$

for all $(t, \xi)$

- Non constructive assumption (only for proof)
- Locally linked to the observability of the linearized system
\[
\frac{dJ}{dt} \leq |\Delta^t_{t-T}(\epsilon_y(\cdot, \xi(t)))| + \left[\phi(t, \xi(t))\right] \cdot \sqrt{J} + \left[J_\xi(t, \xi(t))\right] \cdot c(t, \xi(t))
\]

The ability to decrease \( J \) depend on the \textit{rank of} \( J_\xi \)

**Uniform Global Regularity Assumption**

There is a K-function \( \Upsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that the following inequality holds:

\[
\|J_\xi(t, \xi)\|^2 \geq \Upsilon(J(t, \xi))
\]

for all \((t, \xi)\)

This suggests the following correction term

\[
c(t, \xi(t)) := \gamma \left[ \frac{J_\xi^T(t, \xi(t))}{\|J_\xi\|^2 + \varepsilon} \right] \left[ -|\Delta^t_{t-T}(\epsilon_y(\cdot, \xi(t)))| - \left[1 + \phi(t, \xi(t))\right] \sqrt{J} \right]
\]
\[
\frac{dJ}{dt} \leq |\Delta^t_{t-\tau}(\epsilon_y(\cdot, \xi(t)))| + \left[ \phi(t, \xi(t)) \right] \cdot \sqrt{J} + \left[ J_{\xi}(t, \xi(t)) \right] \cdot c(t, \xi(t))
\]

The ability to decrease $J$ depend on the \textit{rank of $J_{\xi}$}

\textbf{Uniform Global Regularity Assumption}

There is a $K$-function $\Upsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the following inequality holds:

\[
\| J_{\xi}(t, \xi) \|^2 \geq \Upsilon(J(t, \xi))
\]

for all $(t, \xi)$

This suggests the following correction term

\[
c(t, \xi(t)) := 1 \left[ \frac{J_{\xi}^T(t, \xi(t))}{\| J_{\xi} \|^2 + 0} \right] \left[ -|\Delta^t_{t-\tau}(\epsilon_y(\cdot, \xi(t)))| - [1 + \phi(t, \xi(t))] \sqrt{J} \right]
\]
\[
\frac{dJ}{dt} \leq -\sqrt{J(t, \xi(t))}
\]

The ability to decrease \( J \) depend on the rank of \( J_\xi \)

**Uniform Global Regularity Assumption**

There is a K-function \( \Upsilon : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the following inequality holds:

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\|J_\xi(t, \xi)\|^2 \geq \Upsilon(J(t, \xi))
\]

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This suggests the following correction term

\[
c(t, \xi(t)) := 1 \left[ \frac{J_\xi^T(t, \xi(t))}{\|J_\xi\|^2 + 0} \right] \left[ -\Delta_{t-T}^t(\epsilon_y(\cdot, \xi(t))) - [1 + \phi(t, \xi(t))] \sqrt{J} \right]
\]
\[
d\frac{dJ}{dt} \leq |\Delta^T_{t-\tau}(\epsilon_y(\cdot, \xi(t)))| + \left[\phi(t, \xi(t))\right] \cdot \sqrt{J} + \left[J_\xi(t, \xi(t))\right] \cdot c(t, \xi(t))
\]

The ability to decrease \(J\) depend on the rank of \(J_\xi\)

**Uniform Global Regularity Assumption**

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\|J_\xi(t, \xi)\|^2 \geq \Upsilon(J(t, \xi))
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for all \((t, \xi)\)

This suggests the following correction term

\[
c(t, \xi(t)) := \gamma \left[\frac{J_\xi^T(t, \xi(t))}{\|J_\xi\|^2 + \varepsilon}\right] \left[-|\Delta^T_{t-\tau}(\epsilon_y(\cdot, \xi(t)))| - [1 + \phi(t, \xi(t))] \sqrt{J}\right]
\]
More generally ($\gamma > 1$ and $\varepsilon \neq 0$), one has the following implication

$$\left\{ \Upsilon(J(t, \xi(t))) > \frac{\varepsilon}{\gamma - 1} \right\} \Rightarrow \left\{ \dot{J}(t, \xi(t)) < 0 \right\}.$$
More generally ($\gamma > 1$ and $\varepsilon \neq 0$), one has the following implication

$$\left\{ \Upsilon(J(t, \xi(t))) > \frac{\varepsilon}{\gamma - 1} \right\} \Rightarrow \left\{ \dot{J}(t, \xi(t)) < 0 \right\}.$$ 

therefore, the following set

$$\mathcal{A}_J := \left\{ (t, \xi) \mid J(t, \xi) \leq \Upsilon^{-1}\left(\frac{\varepsilon}{\gamma - 1}\right) \right\}$$

is invariant and globally attractive set.
More generally ($\gamma > 1$ and $\varepsilon \neq 0$), one has the following implication

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is invariant and globally attractive set.

But according to the definition of observability

$$\int_{t-T}^{t} \| Y(\sigma, t - T, x^{(1)}) - Y(\sigma, t - T, x^{(2)}) \|^2 d\sigma \geq \alpha(\| x^{(1)} - x^{(2)} \|)$$
More generally \((\gamma > 1\) and \(\varepsilon \neq 0)\), one has the following implication

\[
\left\{ \Upsilon(J(t, \xi(t))) > \frac{\varepsilon}{\gamma - 1} \right\} \Rightarrow \left\{ \dot{J}(t, \xi(t)) < 0 \right\}.
\]

therefore, the following set

\[
A_J := \left\{ (t, \xi) \mid J(t, \xi) \leq \Upsilon^{-1}\left(\frac{\varepsilon}{\gamma - 1}\right) \right\}
\]

is invariant and globally attractive set.

Taking \(x^{(1)} = x(t - T)\) and \(x^{(2)} = \xi(t)\)

\[
\int_{t-T}^{t} \|Y(\sigma, t - T, x^{(1)}) - Y(\sigma, t - T, x^{(2)})\|^2 \, d\sigma \geq \alpha(\|x^{(1)} - x^{(2)}\|)
\]
More generally ($\gamma > 1$ and $\varepsilon \neq 0$), one has the following implication

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One has

$$J(t, \xi(t)) \geq \alpha(\|x(t - T) - \xi(t)\|)$$
More generally ($\gamma > 1$ and $\varepsilon \neq 0$), one has the following implication

$$\left\{ \Upsilon(J(t, \xi(t))) > \frac{\varepsilon}{\gamma - 1} \right\} \Rightarrow \left\{ \dot{J}(t, \xi(t)) < 0 \right\}.$$

therefore, the following set

$$A_J := \left\{ (t, \xi) \mid J(t, \xi) \leq \Upsilon^{-1}\left(\frac{\varepsilon}{\gamma - 1}\right) \right\}$$

is invariant and globally attractive set.

one has

$$\alpha^{-1}\left(J(t, \xi(t))\right) \geq \|x(t - T) - \xi(t)\|$$
More generally ($\gamma > 1$ and $\varepsilon \neq 0$), one has the following implication

$$\left\{ \Upsilon(J(t, \xi(t))) > \frac{\varepsilon}{\gamma - 1} \right\} \Rightarrow \left\{ \dot{J}(t, \xi(t)) < 0 \right\}.$$ 

therefore, the following set

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is invariant and globally attractive set.

one has

$$\alpha^{-1} \circ \Upsilon^{-1}(\frac{\varepsilon}{\gamma - 1}) \geq \alpha^{-1}(J(t, \xi(t))) \geq \|x(t - T) - \xi(t)\|$$

asymptotically
More generally ($\gamma > 1$ and $\epsilon \neq 0$), one has the following implication

$$\left\{ \gamma(J(t, \xi(t))) > \frac{\epsilon}{\gamma - 1} \right\} \Rightarrow \left\{ \dot{J}(t, \xi(t)) < 0 \right\}.$$ 

Therefore, the following set

$$A_J := \left\{ (t, \xi) \mid J(t, \xi) \leq \gamma^{-1}\left(\frac{\epsilon}{\gamma - 1}\right) \right\}$$

is invariant and globally attractive set.

One has

$$\alpha^{-1} \circ \gamma^{-1}\left(\frac{\epsilon}{\gamma - 1}\right) \geq \alpha^{-1}(J(t, \xi(t))) \geq \|x(t - T) - \xi(t)\|$$

asymptotically.

$$\lim_{(\epsilon/\gamma) \to 0} \left[ \lim_{t \to \infty} \|\hat{x}(t) - x(t)\| \right] = 0.$$
Convergence result

If the following conditions hold:

1. The maps are continuously differentiable
2. The system is uniformly observable
3. The uniform regularity assumption is satisfied

then for any a priori fixed desired precision $\eta > 0$ on the state estimation error, there is a sufficiently high ratio $\gamma/\varepsilon$ such that the dynamic system given by:

$$
\dot{\xi}(t) = f(t - T, \xi(t)) + c(t, \xi(t)) \\
\hat{x}(t) = X(t, t - T, \xi(t))
$$

where the correction term $c(t, \xi)$ is given by:

$$
c(t, J) := \gamma \left[ \frac{J^T(\xi(t))}{\| J_\xi \|^2 + \varepsilon} \right] \left[ - | \Delta_{t-T}(\xi(t)) | - [1 + \phi(t, \xi(t))] \right] \sqrt{J}
$$

leads to an estimation error that is asymptotically lower than $\eta$. 

Mazen Alamir
Summer school, September 2007, Grenoble.
The Post Stabilization Technique

\[ \dot{\xi}(t) = f(t - T, \xi(t)) + c(t, \xi(t)) \]

\[ c(t, J) := \gamma \left[ \frac{J^T(t, \xi(t))}{\|J\xi\|^2 + \varepsilon} \right] \left[ - |\Delta_t \tau(\cdot, \xi(t))| - [1 + \phi(t, \xi(t))] \sqrt{J} \right] \]
The Post Stabilization Technique

\[ \dot{\xi}(t) = f_c(t, \xi(t), J_\xi(t)) \]
\[ \hat{x}(t) = X(t, t - T, \xi(t)) \]
\[ J(t, \xi(t)) = 0 \text{ (Ideally)} \]
The Post Stabilization Technique

\[ \dot{\xi}(t) = f_c(t, \xi(t), J_{\xi}(t)) \]
\[ \hat{x}(t) = X(t, t - T, \xi(t)) \]
\[ J(t, \xi(t)) = 0 \text{ (Ideally)} \]

Bad News: The computation of the r.h.s of the observer equation is expensive.

[Integration of a differential system of order \(n(n + 1)\)]

→ For a given sampling period, need for lower order integration methods
The Post Stabilization Technique

\[
\dot{\xi}(t) = f_c(t, \xi(t), J_\xi(t)) \\
\hat{x}(t) = X(t, t - T, \xi(t)) \\
J(t, \xi(t)) = 0 \quad \text{(Ideally)}
\]

**Good News:** Ideally, DAE’s with invariant submanifold

→ There are techniques (Ascher, Num. Alg. 1997) to accurately integrate with lower order methods
Integrate over $[t_k, t_{k+1}]$ starting from the initial condition $(t_k, \xi(t_k))$

$$\dot{\xi}(t) = f_c(t, \xi(t), J\xi(t_k)) \quad ; \quad t \in [t_k, t_{k+1}]$$

to obtain $\tilde{\xi}(t_{k+1})$
1. Integrate over \([t_k, t_{k+1}]\) starting from the initial condition \((t_k, \xi(t_k))\)

\[
\dot{\xi}(t) = f_c(t, \xi(t), J_\xi(t_k)) \quad ; \quad t \in [t_k, t_{k+1}]
\]

to obtain \(\tilde{\xi}(t_{k+1})\)

2. Correct the \textit{rough} approximation \(\tilde{\xi}(t_{k+1})\) by projection

\[
\xi(t_{k+1}) = \tilde{\xi}(t_{k+1}) - \frac{J_\xi(t_{k+1}, \tilde{\xi}(t_{k+1}))}{\|J_\xi(t_{k+1}, \tilde{\xi}(t_{k+1}))\|^2 + \nu} \cdot J(t_{k+1}, \tilde{\xi}(t_{k+1}))
\]

to obtain the update \(\xi(t_{k+1})\)
Convergence analysis

Regardless the order of the integration scheme, one has

\[
\lim_{k \to \infty} J(\xi(t_k)) = O(\tau_s^4)
\]

[Alamir: Int. J. Contr. 1999]

1. Integrate over \([t_k, t_{k+1}]\) starting from the initial condition \((t_k, \xi(t_k))\)

\[
\dot{\xi}(t) = f_c(t, \xi(t), J\xi(t_k)) \quad ; \quad t \in [t_k, t_{k+1}]
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\]

to obtain the update \(\xi(t_{k+1})\)
The benefit from the post-stabilization technique

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sin(x_1) - 0.2x_1 \cos(x_1x_2) \\
y &= x_1 + x_2
\end{align*}
\]

Almost no need for Post-stabilization step

\(\tau_s = 0.1\)

Post-stabilization is mandatory to keep precision under high sampling period.

\(\tau_s = 0.4\)
The Moving-Horizon Observer in Tilting Trains

**Context**

- Coll. Alstom-Transport (Villuerbanne)
- 1997-1999
- Patented solution

**Outline**

- Tilting train control problem
- Estimation problem
- Appl. of the Diff. Form. of MHO
Why tilting trains?

\[ \delta(r) + \alpha \]

\[ \bar{\delta}(r) \]

Centrifugal acceleration

\[ \frac{V^2 \rho(r)}{g} \]

By construction (comfort)

\[ \bar{\delta}(r) = \tan^{-1}\left(\frac{V^2 \rho(r)}{g}\right) \]

If \( V > V_{\text{nom}} \), we need an additional inclinaison \( \alpha_d \)

\[ \alpha_d = \tan^{-1}\left(\frac{V^2(t) \rho(r(t))}{g \cdot \bar{\delta}(r(t))}\right) \]

Tilting trains enables higher velocities on old rails while maintaining almost the same comfort level.
Why tilting trains?

- Centrifugal acceleration $V^2 \rho(r)$

\[ \tilde{\delta}(r) + \alpha \]
Why tilting trains?

- Centrifugal acceleration $V^2 \rho(r)$
- By construction (comfort)

$$\bar{\delta}(r) = \tan^{-1}\left(\frac{V_{\text{nom}}^2 \cdot \rho(r)}{g}\right)$$
Why tilting trains?

- Centrifugal acceleration $V^2 \rho(r)$
- By construction (comfort)
  \[
  \bar{\delta}(r) = \tan^{-1}\left( \frac{V_{nom}^2 \cdot \rho(r)}{g} \right)
  \]
- If $V > V_{nom}$, we need an additional inclinaison $\alpha_d$

\[
\alpha_d(V(t), r(t)) := \tan^{-1}\left( \frac{1}{g} \cdot V^2(t) \cdot \rho(r(t)) \right) - \bar{\delta}(r(t))
\]
Why tilting trains?

- Centrifugal acceleration $V^2 \rho(r)$
- By construction (comfort)
  \[
  \bar{\delta}(r) = \tan^{-1}\left(\frac{V_{\text{nom}}^2 \cdot \rho(r)}{g}\right)
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\]

Tilting trains enables higher velocities on old rails while maintaining almost the same comfort level.
Tilting train: The Control Problem

\[ \alpha_d(V(t), r(t)) := \tan^{-1}\left(\frac{1}{g} \cdot V^2(t) \cdot \rho(r(t))\right) - \bar{\delta}(r(t)) \]

that depends at each instant \( t \) on

- The train’s velocity \( V(t) \)
- The curvilinear abscissa of the train on the rail \( r(t) \)
- The geometric characteristics of the rail \( \rho(r(t)) \) and \( \bar{\delta}(r(t)) \)
The time derivative of the reference

\[ \dot{\alpha}_d = \frac{\partial \alpha_d}{\partial V} \dot{V} + \frac{\partial \alpha_d}{\partial r} V \approx \frac{\partial \alpha_d}{\partial r} (V, r) V \]

is high precisely at high velocities when tilting is needed.
Tilting train: The Control Problem

The time derivative of the reference

\[ \dot{\alpha}_d = \frac{\partial \alpha_d}{\partial V} \dot{V} + \frac{\partial \alpha_d}{\partial r} V \approx \frac{\partial \alpha_d}{\partial r} (V, r) V \]

is high precisely at high velocities when tilting is needed.

- Need for an anticipative action (Predictive control)
- Any delay may give the exact inverse effect
- Anticipation needs \( r(t) \) to be known
- Estimating \( V(t) \) is a classical trouble in railways applications
The Estimation Problem

- **State**

  \[ x := \begin{pmatrix} \text{crvilinear abscissa } r \\ \text{relative error on velocity} \end{pmatrix} \]

- **Measurement**

  \[ y = \text{Yaw angular velocity} \]

  \[ V_m = \text{velocity sensor output} \]

  The map \( \rho(\cdot) \) is available after off-line careful crossing of the line under consideration

  [Paris-toulouse profile]
Computation of $J(t, \xi)$ and $J_\xi(t, \xi)$

**Sensitivity System**

\[
\begin{align*}
\dot{z}_1 &= (1 + z_2)V \\
\dot{z}_2 &= 0 \\
\dot{z}_3 &= \left[(1 + z_2)V\rho(z_1) - y(t)\right]^2 \\
\dot{A}(\tau) &= \begin{pmatrix}
0 & V(\tau) & 0 \\
0 & 0 & 0 \\
\chi_1(\tau) & \chi_2(\tau) & 0
\end{pmatrix} A
\end{align*}
\]

where

\[
\begin{align*}
\chi_1 &= \frac{\partial \dot{z}_3}{\partial z_1} \\
\chi_2 &= \frac{\partial \dot{z}_3}{\partial z_2}
\end{align*}
\]
Computation of $J(t, \xi)$ and $J_\xi(t, \xi)$

**Sensitivity System**

\[
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\dot{z}_1 &= (1 + z_2)V \\
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\dot{A}(\tau) &= \begin{pmatrix} 0 & V(\tau) & 0 \\ 0 & 0 & 0 \\ \chi_1(\tau) & \chi_2(\tau) & 0 \end{pmatrix} A
\end{align*}
\]

where

\[
\begin{align*}
\chi_1 &= \frac{\partial \dot{z}_3}{\partial z_1} = 2\left[(1 + z_2)V \cdot \rho(z_1) - y\right](1 + z_2)V \frac{\partial \rho}{\partial z_1}(z_1) \\
\chi_2 &= \frac{\partial \dot{z}_3}{\partial z_2} = 2\left[(1 + z_2)V \cdot \rho(z_1) - y\right] \cdot V \cdot \rho(z_1)
\end{align*}
\]
Computation of \( J(t, \xi) \) and \( J_\xi(t, \xi) \)

**Sensitivity System**

\[
\begin{align*}
\dot{z}_1 &= (1 + z_2)V \\
\dot{z}_2 &= 0 \\
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\dot{A}(\tau) &= \begin{pmatrix} 0 & V(\tau) & 0 \\ 0 & 0 & 0 \\ \chi_1(\tau) & \chi_2(\tau) & 0 \end{pmatrix} A
\end{align*}
\]

**Initial Conditions**

\[
\begin{align*}
z(t - T) &= (\xi^T, 0)^T \\
A(t - T) &= I_{3 \times 3}
\end{align*}
\]

\[
J(t, \xi) = z_3(t) \\
J_\xi(t, \xi) = (A_{31}(t), A_{32}(t))
\]

where

\[
\begin{align*}
\chi_1 &= \frac{\partial \dot{z}_3}{\partial z_1} = 2 \left[(1 + z_2) V \cdot \rho(z_1) - y\right] (1 + z_2) V \frac{\partial \rho}{\partial z_1}(z_1) \\
\chi_2 &= \frac{\partial \dot{z}_3}{\partial z_2} = 2 \left[(1 + z_2) V \cdot \rho(z_1) - y\right] \cdot V \cdot \rho(z_1)
\end{align*}
\]
Validation on the Paris-Toulouse line ($T = 5$ sec and $\tau_s = 0.4$ sec)
Validation on the Paris-Toulouse line ($T = 5 \text{ sec}$ and $\tau_s = 0.4 \text{ sec}$)

- **Velocity (dotted) vs Estimated velocity (solid)**
- **True $x_2$ (dotted) vs estimated $x_2$ (solid)**
- **open-loop position error (dotted) vs observer-related error (solid)**
Moving-Horizon Observers with Distributed Optimization

- The system
  
  \[
  x(t) = X(t, t_0, x_0), \\
y(t) = h(t, x(t)),
  \]

- Measurement acquisition period \(\tau_a\)
- Updating period \(\tau_u = N_u \cdot \tau_a\)
- Updating instants \(t_k = k \cdot \tau_u\)
- Observation horizon \(T = N \cdot \tau_a\)
- Cost function at instant \(t_k\): \(J(t_k, \xi)\)
Moving-Horizon Observers with Distributed Optimization

- The system
  \[ x(t) = X(t, t_0, x_0), \]
  \[ y(t) = h(t, x(t)), \]

- Measurement acquisition period \( \tau_a \)
- Updating period \( \tau_u = N_u \cdot \tau_a \)
- Updating instants \( t_k = k \cdot \tau_u \)
- Observation horizon \( T = N \cdot \tau_a \)
- Cost function at instant \( t_k \): \( J(t_k, \xi) \)
Moving-Horizon Observers with Distributed Optimization

\[ h(t_k - T, \xi(t_k)) \]

\[ T = N \cdot \tau_a \]

\[ \tau_u \]

\[ \tau_a \]

\[ t_k - T \quad t_k \quad t_{k+1} \]
Moving-Horizon Observers with Distributed Optimization

During the interval \([t_k, t_{k+1}]\), perform \(n\) iterations of some iterative process \(S\):

\[
\tilde{\xi}(t_k) = S^{(n)}(t_k, \xi(t_k), \tilde{y}_{t_k-T})
\]
Moving-Horizon Observers with Distributed Optimization

During the interval \([t_k, t_{k+1}]\), perform \(n\) iterations of some iterative process \(S\):

\[
\tilde{\xi}(t_k) = S^{(n)}(t_k, \xi(t_k), y_{t_k-T})
\]

Using \(\tilde{\xi}(t_k)\), update the value of \(\xi(t_{k+1})\) according to

\[
\xi(t_{k+1}) = X(t_{k+1} - T, t_k - T, \tilde{\xi}(t_k))
\]
Two opposite processes

The updating mechanism involves two opposite effects on $J(t_k, \xi(t_k))$:

1. A decreasing effect from the $n$-iterations of the optimization process
2. An increasing effect from the open loop prediction over $\tau_u = N\tau_a$. 
The iterative process $S$ is efficiently in the sense that there exists some efficiency map $\alpha_{\text{eff}} : \mathbb{N} \rightarrow [0, 1]$ such that for all $t$ and $\xi$, one has:

$$J\left(t, S^{(n)}(t, \xi, y_t^t - T)\right) \leq \alpha_{\text{eff}}(n) \cdot J(t, \xi)$$

where $\alpha(\cdot)$ is a decreasing function such that $\alpha(0) = 1$. 
Assumption: Open-loop behavior of the cost function

When using open-loop prediction, the only inequality one can guarantee is given by:

\[ J(t + \tau, X(t + \tau - T, t - T, \xi)) \leq \left[ J(t, \xi) \right] \cdot \vartheta(\tau) \]  

(3)
Moving-Horizon Observers with Distributed Optimization

A rather qualitative result

Under the assumptions above, the convergence of the distributed in time optimization based observer is guaranteed provided that the following inequality holds:

\[ \varpi(N_u) := \alpha_{\text{eff}} \left( E \left( \frac{N_u \tau_a}{\tau_{\text{iter}}} \right) \right) \cdot \vartheta(N_u \tau_a) < 1 \]  

(4)

Moreover, the convergence time is given by:

\[ t_r(N_u) \approx \left[ \frac{3N_u}{|\log(\varpi(N_u))|} \right] \cdot \tau_a \]  

(5)

where

✓ \( \tau_a \) is the measurement acquisition period
✓ \( N_u \tau_a \) is the updating period
✓ \( \tau_{\text{iter}} \) is the time necessary to perform one iteration of the process \( S \)
Moving-Horizon Observers with Distributed Optimization

Take the following example for illustration

\[ \alpha_{\text{eff}}(n) = \frac{D}{n^d + D} \quad ; \quad \vartheta(\tau) = \exp(\beta \cdot \tau) \]

Note that:

- \( d \uparrow \) increases the efficiency
- \( D \uparrow \) decreases the efficiency
- \( \alpha_{\text{eff}}(0) = 1 \)
- \( \beta \uparrow \) assumes high model discrepancy
Moving-Horizon Observers with Distributed Optimization

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Moving-Horizon Observers with Distributed Optimization

\[ \varpi(N_u) \]

\[ \frac{t_r(N_u)}{\tau_a} \]

**Figure:** Evolutions of the stability indicator \( \varpi(N_u) \) and the settling time \( t_r(N_u) \) vs the number of iterations \( N_u \) used to update the state estimation. \( D = 3, d = 1, \beta \cdot \tau_a = 0.3 \) and \( \tau_a/\tau_{iter} = 5 \). Under these conditions, stability cannot be guaranteed when more than 9 iterations are used. The optimal choice (in term of settling time) is the one where only one iteration is used to perform the updating.
Moving-Horizon Observers with Distributed Optimization

Figure: Evolutions of the stability indicator $\varpi(N_u)$ and the settling time $t_r(N_u)$ vs the number of iterations $N_u$ used to update the state estimation. $D = 50$, $d = 2$, $\beta \cdot \tau_a = 0.05$ and $\tau_a/\tau_{iter} = 5$. Under these conditions, while stability seems guaranteed regardless the number of iterations used to perform the updating, the use of 3 iterations gives the best result in term of settling time.
Moving-Horizon Observers with Distributed Optimization

The success and the quality of the Moving-Horizon-Observer depend on:

- The quality of the optimizer \((d, D)\)
- The quality of the model \((\beta)\)
- The iteration complexity \((\tau_{iter})\)
- The problem itself (The very existence of such parameters)

On-line identification of the problem parameters?

⇒ On-line adaptation of the updating rate?
Further readings

General Conclusion

- The progress in MHO ← progress of optimization tools

- MHO-related problem is NOT ONLY an optimization problem

- Promising direction:

  **Combine Analytic and Optimization Based Observers**

  (Let the MHO concentrate on the structure-free part of the problem)
Example of a selection index

\[ J(t, \xi, w) := \Gamma(t, \xi - \xi^*(t)) + \int_{t-T}^{t} L(w(\sigma), \varepsilon_y(\sigma)) \]

- \( \varepsilon_y(\sigma) = y_{t-T}^T(\sigma) - Y(\sigma, t - T, \xi, w) \) output prediction error
- \( \xi^*(t) \) condenses the past knowledge.
- For Kalman filter

\[ L(w, \varepsilon_y) = w^T Q^{-1} w + \varepsilon_y^T R^{-1} \varepsilon_y \]

\( \xi^*(t) \) Induced by the past estimate (discrete KF)
Handling abrupt behavior of uncertainties
Handling abrupt behavior of uncertainties

\( w(\cdot) \)
Handling abrupt behavior of uncertainties

\[ w(\cdot) \]

Time
Handling abrupt behavior of uncertainties
Handling abrupt behavior of uncertainties

\[ w(\cdot) \]

Time
Handling abrupt behavior of uncertainties
Non smooth behaviors can be parametrized

\[ W_2(\cdot, p_w^{(3)}) \]

\[ W_1(\cdot, p_w^{(2)}) \]

\[ W(\cdot, p_w) := \begin{cases} 
W_1(\tau, p_w^{(2)}) & \text{if } \tau \leq p_w^{(1)} \\
W_2(\tau, p_w^{(3)}) & \text{otherwise}
\end{cases} \]

\[ p_w := (p_w^{(1)}, p_w^{(2)}, p_w^{(3)}) \]
Definition of the phase II: Existence of monomer droplets

\[ N_1 \delta_1 + N_2 \delta_2 + N_3 \delta_3 - \frac{(1 - \phi_p^p)}{\phi_p^p} \sigma > 0 \]  

(6)

where

\[ \delta_i = MW_i \left( \frac{1}{\rho_i} + \frac{(1 - \phi_p^p)}{\rho_i, h \phi_p^p} \right), \quad i = 1, 2, 3 \]  

(7)

and

\[ \sigma = \sum_{j=1}^{3} \frac{MW_j N_j^T}{\rho_j, h} \]  

(8)
Example of dynamic evolution of $\mu$
\[
\frac{dY}{dt}\big|_{c\equiv0}(\tau, t - T, \xi(t)) = Y_{t2}(\tau, t - T, \xi(t)) + Y_{\xi}(\tau, t - T, \xi(t))f(t - T, \xi(t))
\]

Computing \(\frac{dY}{dt}\big|_{c\equiv0}(\tau, t - T, \xi(t))\) amounts to compute the sensitivity of solutions of ODE’s to initial conditions.

**Sensitivity Computation**

If \(X(t, x_0)\) is solution of \(\dot{x} = f(x)\) then \(\frac{\partial X}{\partial x_0}(\tau, x_0)\) is given by \(A(\tau) \in \mathbb{R}^{n \times n}\) where

\[
\dot{A}(\sigma) = \left[\frac{\partial f}{\partial X}(X(\sigma, x_0))\right]A(\sigma) \quad ; \quad A(0) = I_n
\]
Evolution of the curvature map \( \rho \) on a portion of the Paris-Toulouse line. This map is used in the validating scenarios.

Mazen Alamir
Summer school, September 2007, Grenoble.
Consider a dynamic simulator

\[ x(t) = X(t, x_0, p) \]
\[ y(t) = h(x(t), p) \in \mathbb{R} \]

- \( x \) state vector
- \( p \) a vector of parameter/faults
- \( y \) vector of measurements

\[ Y(k) := (y(k - N_m), \ldots, y(k - 1)) \in \mathbb{R}^{N_m} \]
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For \( N_m \) sufficiently high

\[ y(k) = F(Y(k), p) \]
For $N_m$ sufficiently high

\[ y(k) = F(Y(k), p) \]

- $\forall p \in P$, $F(\cdot, p) = F_p(\cdot) : \mathbb{R}^{N_m} \rightarrow \mathbb{R}$
- $F_p(\cdot)$ is unknown
- But $F_p(\cdot)$ is sensitive to variations on $p$

*Any graphical signature of $F_p(\cdot)$ is generically sensitive to variations on $p$.***
Graphical signatures generation

Any map

\[ S : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^2 \]

may be used as a *pensil* to draw a signature of \( F_p(\cdot) \) when applied to \((Y, F_p(Y))\)

A family of signatures can be defined

- Either by fixing \( N \) and changing \( S \)
- Or by *fixing* \( S \) and changing \( N \in \{n, \ldots, \infty\} \)
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The key intuition (1)

Signature $S_{N_1}$ when $p_1$ varies

Signature $S_{N_2}$ when $p_2$ varies

Signature $S_{N_3}$ when $p_3$ varies
The key intuition (2)

Road map

1. Find as many $N_i$’s as necessary such that all the $p_j$’s are discriminated.
   - The information is somewhere there!
   - There are degrees of freedom
   - Human brain’s classification skills

2. Translate geometrical deformations into mathematical expressions

3. Encode it on-line to perform identification and/or diagnosis
Example 1: Parameterized Van-der-Pol oscillator

Modified Vand-der-Pol oscillator

\[\begin{align*}
\dot{x}_1 &= p_1 x_2 \\
\dot{x}_2 &= -9x_1 + p_2 (1 - (x_1 + p_3)^2)x_2 \\
y &= x_1 
\end{align*}\]

with \( p \in [1, 2] \times [1.5, 3] \times [0, 0.1] \)

→ See illustrations.

Mazen Alamir
Summer school, September 2007, Grenoble.
Signatures vs Least Squares identification

prediction error related cost for 
\[ p = (1.2 \ 1.5 \ 0)^T \]

Output prediction error related cost for 
\[ p = (2 \ 2.5 \ 0.1)^T \]
Signatures vs Least Squares identification

Output prediction error related cost for 
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Output prediction error related cost for 
\( p = (2 \ 2.5 \ 0.1)^T \)

Signatures enable

- Decoupling
- Convexification
Example 2: Electronic Throttle Control System (ETC)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{J} \left[ -K_{sp}(x_1 + \theta_0) - K_f x_2 + (N K_t) x_3 \right] \\
\dot{x}_3 &= -\frac{1}{L_a} \left[ N K_b x_2 + R_a x_3 + u \right]
\end{align*}
\]

Problem: Estimate the coefficients: \( K_t, R_a, K_b \) and \( K_f \)

Measures: \( y = (\theta, i_a)^T \).

→ See illustrations.
Properties of a signature

- Let $S_N$ be a signature
- Let

$$C_N = \left\{ \xi_1(i), \xi_2(i) \right\}_i$$

be the corresponding 2D curves.
Properties of a signature

- Let $S_N$ be a signature
- Let

$$C_N = \left\{ \xi_1(i), \xi_2(i) \right\}_i$$

be the corresponding 2D curves.

- A property of $S_N$ is a scalar function $P(C_N)$.

Examples
- $P(C_N) = \max_i \{\xi_j(i)\}$, $\text{std}(\xi_j(\cdot))$, $\text{mean}(\xi_j(\cdot))$, $\max_i \{\xi_j(i)\} - \min_i \{\xi_j(i)\}$
- But also,

$$x_1(i^*) \quad \text{where} \quad i^* = \arg \max_i |\hat{y}(i)|$$

- etc.
A pair

\[(S_N, P)\]

of a signature and a property is called a coordinate.
A pair

\((S_N, P)\)

of a signature and a property is called a coordinate.

By choosing two coordinates \((S_{N_1}, P_1), (S_{N_2}, P_2)\), an experiment can be represented by one point in the 2D plane.
A pair

\((S_N, P)\)

of a signature and a property is called a \textit{coordinate}

By choosing two coordinates \((S_{N_1}, P_1), (S_{N_2}, P_2)\), an \textit{experiment} can be represented by one point in the 2D plane.
Using signature for faults classification

\[ y^{(1,1)}(\cdot), \ldots, y^{(1,n_1)}(\cdot) ; \text{ experiments / config 1} \]
\[ y^{(2,1)}(\cdot), \ldots, y^{(2,n_2)}(\cdot) ; \text{ experiments / config 2} \]
\[ y^{(3,1)}(\cdot), \ldots, y^{(3,n_3)}(\cdot) ; \text{ experiments / config 3} \]
\[ y^{(4,1)}(\cdot), \ldots, y^{(4,n_4)}(\cdot) ; \text{ experiments / config 4} \]

- These can be obtained
  - Either using real data collected after fault occurrence
  - Or using faithful models representing faulty configurations
Definitions

The Observation Pb.

Analytic vs optim.

Singularities avoidance

Differential form

RT Optim.

Further readings

Conclusion

M.H. Diagnosis

\[ P_2(C_1) \]

\[ P_1(C_1) \]
DiagSign

- Generic software
- Fast Prototyping of classification algorithms
- Automated/Manual Modes
- Cheap Computations

See more on www.diagsign.com