Necessary and Sufficient Convex Condition for the Stabilization of Linear Sampled-data Systems under Poisson Sampling Process

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***Abstract*— This work presents a control design method for linear sampled-data systems whose random sampling intervals form a Poisson process. Unlike a previous result in the literature, the proposed stabilization conditions, based on linear feedbacks of both the state and the past input values, are necessary and sufficient for the mean exponential stability of the system. Moreover, such non-conservative conditions correspond to linear matrix inequalities, implying then that the stabilization problem can be efficiently addressed through semidefinite programming. As a second contribution, the characterization and optimization of the mean exponential convergence rate of the closed-loop system is given in form of a generalized eigenvalue problem. A numerical example illustrates the theoretical results.**

***Index Terms*— Sampled-data control, random sampling, mean exponential stability, Linear Matrix Inequalities.**

I. Introduction

In many real-world applications, continuous-time plants are controlled by digital devices in a sampled-data fashion [1]. These sampled-data systems are often implemented through a network, where communication protocols are responsible for the transmission of data between computers, actuators and sensors [2]. The use of a shared network for different purposes has several advantages, including flexibility and easy of maintenance [2]. On the other hand, due to imperfections on the communication channels, the control loop is commonly subject to time-varying, uncertain, sampling intervals, i.e. aperiodic sampling [3].

In this context, the analysis and design of aperiodic sampled-data systems have been the focus of many works in the last years [4]. In [5], for instance, a time-delay method based on Lyapunov-Krasovskii funcionals is considered. A similar idea is developed in [6], where the looped-functional method is introduced. Alternatively, a discrete-time approach can be considered, where the behavior of the system's state at the sampling instants is modeled through difference inclusions [7]–[11].

A common feature of the aforementioned references is that they consider a non-stochastic framework, where hard bounds are given for the time-varying sampling interval. However, taking into account that this assumption may not be realistic, some recent works have addressed the stability analysis and stabilization of linear sampled-data systems subject to a random sampling interval, where the corresponding distribution function has possibly unbounded support [12]–[16]. In particular, the works in [13]–[16] address the stabilization problem (in a stochastic sense) of the discrete-time model that describes the evolution of the system's state at the sampling instants. However, even if closely related, the stability of this discrete-time model is not equivalent to the stability of the corresponding (continuous-time) sampled-data system, as remarked, for instance, in [12, Pg. 222] and [17, Pg. 610].

The aim of this work is to propose a control design method that guarantees the stabilization of the *continuous-time* system, as it is done for instance in [12]. Our approach is, however, based on a convex condition – a Linear Matrix Inequality (LMI). Moreover, compared to [12], our method provides a stabilization condition that is *non-conservative* with respect to the considered control law, which is also more general. To derive this condition, we use an instrumental result from [18], which deals with the stability analysis of impulsive renewal systems but does not consider the control design problem. As in [19] (which does not consider the control design problem either) and [12], we deal with the case of Poisson sampling, i.e. the sampling intervals form a sequence of independent and identically distributed (i.i.d.) random variables with exponential distribution. Furthermore, as a second contribution, we will show how to adapt the approach to optimize the speed of convergence of the trajectories of the closed-loop system to the origin. This is done through a generalized eigenvalue problem [20, Section 2.2.3], which can be efficiently solved numerically.

The paper is organized as follows. Section II contains basic definitions and the problem formulation. Section III presents the main results related to the necessary and sufficient stabilization condition and to the control design method. A detailed comparison between our approach and the one of

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[12] is presented in Section IV. Section V shows a numerical example. At last, some concluding remarks end the paper.

Notation. $P[\cdot]$ denotes probability and $\mathbb{E}[\cdot]$ expectation. For $f:\mathbb{R}\to\mathbb{R}^n$, $f(t^-)\triangleq\lim_{\tau\mapsto t,\tau<t} f(\tau)$ if the limit exists. Given the square matrices A and B, $\lambda_{max}(A)$ ($\lambda_{min}(A)$) is the maximal (minimal) real part of the eigenvalues of A and $\mathrm{Diag}(A,B)$ is a block diagonal matrix formed by A and B. The symbol $\star$ denotes a symmetric block when applied as an entry of a matrix and $\succ$ ($\succeq$) characterizes positive (semi)-definiteness of a symmetric matrix. $I_n\in\mathbb{R}^{n\times n}$ is the identity matrix. $\|\cdot\|$ denotes the induced 2-norm of a matrix or the Euclidean norm of a vector. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ are, respectively, the real and imaginary parts of the argument. Given a matrix (or vector) $A\in\mathbb{R}^{m\times n}$, $A_{(i)}$ is its i-th row and A^T its transpose.

II. Problem Formulation

Consider the continuous-time plant described by the following linear model:

$$\dot{x}_p(t)=A_px_p(t)+B_pu(t) \tag{1}$$

where $x_p\in\mathbb{R}^{n_p}$ and $u\in\mathbb{R}^m$ are the state and the input of the plant, respectively. Matrices A_p and B_p have appropriate dimensions and are constant. It is assumed that the control input is updated at the time instants t_k and kept constant (by means of a zero-order-hold) for all $t\in[t_k,t_{k+1})$, being given by:

$$u(t)=K_px_p(t_k^-)+K_uu(t_k^-),\quad \forall t\in[t_k,t_{k+1}), \tag{2}$$

where K_p and K_u are matrices of appropriate dimensions. Notice that the control law is based not only on the sampled value of the state x_p but also on the value of the last control input applied to the plant. The term $K_uu(t_k^-)$ has already showed its benefits in [11], where the stabilization problem is solved in a non-stochastic framework.

By convention $t_0=0$ and the difference between two successive sampling instants is denoted by $\delta_k\triangleq t_{k+1}-t_k$. It is assumed that $\{\delta_k\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with exponential distribution:

$$F(s)\triangleq P[\delta_k\le s]=1-e^{-\lambda s},\quad \forall k\in\mathbb{N},\ \forall s\ge 0, \tag{3}$$

where $\lambda>0$ and $\mathbb{E}[\delta_k]=1/\lambda$. Thus, the sampling process

$$N_t\triangleq\sup\{k\in\mathbb{N}:\ t_k\le t\} \tag{4}$$

is a Poisson process of intensity $\lambda>0$, where [21, Pg. 37]

$$P[N_t=n]=e^{-\lambda t}\frac{(\lambda t)^n}{n!}. \tag{5}$$

The Poisson process has been successfully used in the literature to model the stochastic behavior of networked control systems under random sampling [12], [19]. It should be noticed that, since $\mathbb{E}[N_t]<\infty$ (cf. [22, Pg. 186]) and since $\mathbb{E}[N_t]<\infty$ only if $P[N_t=\infty]=0$ (cf. [22, Pg. 2]), there is zero probability of an infinite number of samplings occurring in finite time (i.e. zero probability of Zeno behavior).

Denoting $x\triangleq[x_p^T\ u^T]^T\in\mathbb{R}^n, n\triangleq n_p+m$, the dynamics (1)-(2) can be described by the following impulsive system [11]:

$$\begin{cases}\dot{x}(t)=f(x(t))\triangleq A_cx(t), & \forall t\ge 0, t\ne t_k, x(0)=x_0 \quad (6a)\\ x(t_k)=g(x(t_k^-))\triangleq A_dx(t_k^-),\ \forall k\ge 1 & \quad (6b)\end{cases}$$

where $A_d\triangleq A_r+B_rK$ and

$$A_c\triangleq\begin{bmatrix}A_p & B_p\\ 0 & 0\end{bmatrix}\in\mathbb{R}^{n\times n},\ A_r\triangleq\begin{bmatrix}I_{n_p} & 0\\ 0 & 0\end{bmatrix}\in\mathbb{R}^{n\times n},$$
$$B_r\triangleq\begin{bmatrix}0\\ I_m\end{bmatrix}\in\mathbb{R}^{n\times m},\ K\triangleq\begin{bmatrix}K_p & K_u\end{bmatrix}\in\mathbb{R}^{m\times n}. \tag{7}$$

There are alternative impulsive system representations for (1)-(2). The choice (6), different from the one of [12, Eq. 14], will play a fundamental role to obtain a necessary and sufficient convex condition for the stabilization of the system.

Definition 1: The equilibrium point $x=0$ of (6) is *mean exponentially stable* (MES) if there exist scalars $c,\gamma_0>0$ such that for every initial condition $x_0=x(0)\in\mathbb{R}^n$:

$$\mathbb{E}[\|x(t)\|^2]\le ce^{-\gamma_0 t}\|x_0\|^2,\ \forall t\ge 0 \tag{8}$$

where $\gamma_0>0$ will be referred to as a *decay rate* of the trajectories of the system.

In particular, it is possible to show that (6) cannot be MES if the pair (A_p,B_p) in (1) is not stabilizable. Note that in this case the states corresponding to the uncontrollable part of (1) will behave deterministically.

We can now state the problem we focus on in this work.

Problem 1 (Mean square stabilization): Given the intensity $\lambda>0$ of the Poisson sampling process, characterize in a non-conservative way all the linear feedback gains K such that the resulting closed-loop system is MES.

Additionally, as a second contribution, constructive conditions will be provided to maximize the parameter γ_0 in (8). Moreover, the case where the intensity $\lambda>0$ of the Poisson process is not perfectly known will also be addressed.

A. Piecewise deterministic Markov processes

System (6) has a deterministic behavior except for the jumps that occur at the random sampling times. From the memoryless property of the Poisson process [21, Pg. 36], it follows that $x(t)$ is a Markov process. More precisely, system (6) belongs to the class of piecewise deterministic Markov processes (PDMP) defined in [21, Section 24]. This fact allows to establish the following key result, which is a particular case of [18, Theorem 1] and does not require the functions $f(x)$ and $g(x)$ in (6) to be necessarily linear.

Theorem 1: Assume that $f(x)$ and $g(x)$ in (6) are globally Lipschitz and consider $V:\mathbb{R}^n\to\mathbb{R}, V\in\mathscr{C}^1$, such that

$$\mathbb{E}\left[\sum_{t_k\le p}|V(x(t_k))-V(x(t_k^-))|\right]<\infty,\ \forall p\in\mathbb{N},\ \forall x_0\in\mathbb{R}^n, \tag{9}$$

where $x(t_0^-)=x(t_0)=x(0)$ by convention. Then, for $t\ge 0$,

$$\mathbb{E}[V(x(t))]=V(x_0)+\mathbb{E}\left[\int_0^t\mathfrak{U}V(x(s))\mathrm{d}s\right],\ \forall x_0\in\mathbb{R}^n \tag{10}$$

where

$$\mathfrak{U}V(x)\triangleq\nabla V^T(x)f(x)+\lambda\left(V(g(x))-V(x)\right) \tag{11}$$

and $\nabla V(x)$ is the gradient of $V(x)$.

Relation (10) is known as the Dynkin's formula [21, Pg. 31] and can be intuitively interpreted as a stochastic version of the fundamental theorem of calculus. Note that the first term of the right-hand side of (11) is the usual time derivative of $V(x(t))$ along the trajectories of $\dot{x}(t) = f(x(t))$ while the second term accounts for the jumps at the sampling instants. Theorem 1 will be later used in the proof of Lemma 2.

III. Main results

In order to solve Problem 1, we will use the following lemma, adapted from [18, Theorem 7] to the particular case of interest, i.e. model (6). This result can be proved by applying Theorem 1 to a quadratic function of the state $x(t)$.

Lemma 1: Assume that

$$2\lambda_{max}(A_c) < \lambda \tag{12}$$

holds. Then system (6) is MES if and only if there exists $P \in \mathbb{R}^{n\times n}, P = P^T \succ 0$, such that

$$A_c^T P + PA_c - \lambda P + \lambda A_d^T PA_d \prec 0. \tag{13}$$

Remark 1: Since $\lambda > 0$, it follows from the particular structure of A_c in (7) that (12) is equivalent to $2\lambda_{max}(A_p) < \lambda$.

Using Lemma 1, we derive next our main result.

Theorem 2: There exists $K \in \mathbb{R}^{m\times n}$ such that the resulting closed-loop system (6) is MES if and only if there exist $W \in \mathbb{R}^{n\times n}, W = W^T \succ 0$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} WA_c^T + A_cW - \lambda W & \star \\ \lambda(A_rW + B_rY) & -\lambda W \end{bmatrix} \prec 0 \tag{14}$$

where K and Y are related by the equation $K = YW^{-1}$.

Proof: The proof is divided into two steps: (I) show that the statement of Lemma 1 still holds even if assumption (12) is not imposed; (II) show the equivalence between (13) and (14) using reversible operations. The result of the theorem will then follow directly from (I), (II) and Lemma 1.

Step (I): For the sufficiency part of the result, it suffices to note that (13) directly implies that $A_c^T P + PA_c - \lambda P \prec 0$ and, hence, (12) holds automatically.

For the necessity part of the result, we have to show that (13) holds if system (6) is MES. If assumption (12) holds, then it suffices to apply Lemma 1. Thus, let us focus on the nontrivial case where (12) does not hold. It turns out, however, that this case will never happen, as we will prove next. In other words, system (6) cannot be MES if $2\lambda_{max}(A_c) \geq \lambda$. More precisely, we will show that

$$\mathbb{E}\left[\|x(t)\|^2\right] \not\to 0 \quad \text{as} \quad t \to \infty \tag{15}$$

for an appropriately chosen initial condition $x(0)$ if $2\lambda_{max}(A_c) \geq \lambda$. Denote by $\sigma = a + jb, a, b \in \mathbb{R}$, an eigenvalue of A_c (possibly complex but not necessarily) which attains the maximum $\lambda_{max}(A_c)$, i.e. $a = \lambda_{max}(A_c)$, and by $v = v_1 + jv_2, v_1, v_2 \in \mathbb{R}^n$, a corresponding eigenvector. Consider, as initial condition of (6), $x(0) = \mathrm{Re}(v)$, where $\mathrm{Re}(v) \neq 0$ without loss of generality. Then, recalling that t_1 is the first sampling instant, it follows that, for $t \geq 0$:

$$\begin{aligned} \mathbb{E}\left[\|x(t)\|^2\right] &= \mathbb{E}\left[\|x(t)\|^2 | t_1 > t\right] P[t_1 > t] \\ &\qquad + \mathbb{E}\left[\|x(t)\|^2 | t_1 \leq t\right] P[t_1 \leq t] \\ &\geq \mathbb{E}\left[\|x(t)\|^2 | t_1 > t\right] P[t_1 > t] \\ &= \mathbb{E}\left[\|\mathrm{Re}(e^{\sigma t} v)\|^2 | t_1 > t\right] P[t_1 > t], \end{aligned} \tag{16}$$

where the last equality follows from the fact that $x(t) = \mathrm{Re}(e^{\sigma t} v)$ before the first sampling instant t_1, since it must satisfy (6a). Taking into account that the expression $||\mathrm{Re}(e^{\sigma t} v)||^2$ is deterministic for a fixed t, the expectation operator can be omitted and, then, after some algebraic manipulations, (16) becomes

$$\begin{aligned} \mathbb{E}\left[\|x(t)\|^2\right] &\geq \|\mathrm{Re}(e^{\sigma t} v)\|^2 P[t_1 > t] \\ &= e^{2\lambda_{max}(A_c)t} \|\cos(bt)v_1 - \sin(bt)v_2\|^2 P[t_1 > t], \end{aligned}$$

where we replaced $a = \lambda_{max}(A_c)$. At last, recalling that $t_1 = \delta_0$ by definition and using (3), one concludes that $P[t_1 > t] = 1 - P[\delta_0 \leq t] = e^{-\lambda t}$, i.e.

$$\mathbb{E}\left[\|x(t)\|^2\right] \geq e^{(2\lambda_{max}(A_c)-\lambda)t} \|\cos(bt)v_1 - \sin(bt)v_2\|^2 \overset{t\to\infty}{\not\to} 0,$$

which implies (15). Therefore, (8) does not hold for the chosen initial condition and system (6) cannot be MES if $2\lambda_{max}(A_c) \geq \lambda$, as we wanted to show.

Step (II): Applying the Schur's complement to (13) and replacing $A_d = A_r + B_rK$, one gets

$$\begin{bmatrix} A_c^T P + PA_c - \lambda P & \star \\ A_r + B_rK & -(\lambda P)^{-1} \end{bmatrix} \prec 0 \tag{17}$$

Next, defining $W \triangleq P^{-1}$ and left and right multiplying the resulting inequality by $\mathrm{Diag}(W, \lambda I_n)$, one has

$$\begin{bmatrix} WA_c^T + A_cW - \lambda W & \star \\ \lambda(A_rW + B_rKW) & -\lambda W \end{bmatrix} \prec 0 \tag{18}$$

which corresponds to (14) with $Y \triangleq KW$. ■

Remark 2: The use of the feedback term $K_u u(t_k^-)$ in (2) allows to obtain a LMI condition in (14) with the change of variable $Y = KW$, which would not be possible considering $K_u = 0$ without imposing a particular structure on matrix W. However, this does not mean that the use of $K_u \neq 0$ is necessary to stabilize the system, that is, it may exist stabilizing solutions with $K_u = 0$ verifying (14).

Next we present some extensions of the main result.

A. Optimization of the decay rate

It is possible to adapt condition (13) in order to ensure a decay rate γ_0 in (8), according to the next lemma.

Lemma 2: System (6) is MES with decay rate $\gamma_0 > 0$ if

$$\exists P = P^T \succ 0: \ A_c^T P + PA_c - \lambda P + \lambda A_d^T PA_d \preceq -\gamma P \tag{19}$$

with $\gamma = \gamma_0$. Moreover, (6) is MES with decay rate $\gamma_0 > 0$ only if (19) holds for all $\gamma \in [0, \gamma_0)$.

Proof: The proof, inspired by the reasoning in [18], is shown in the Appendix. ■

Remark 3: In the result above, if (6) is MES with decay rate $\gamma_0 > 0$, then (19) holds for all γ smaller than and arbitrarily close to γ_0. The result does not guarantee that (19) holds with $\gamma = \gamma_0$ because the matrix $P = P(\gamma)$ goes unbounded as $\gamma \to \gamma_0$ (see the proof of the lemma for the details).

Using Lemma 2, we obtain the following result.

Theorem 3: There exists $K \in \mathbb{R}^{m\times n}$ such that the resulting closed-loop system (6) satisfies (8) with decay rate $\gamma_0 > 0$ if there exist $W \in \mathbb{R}^{n\times n}, W = W^T \succ 0$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} WA_c^T + A_c W + (\gamma_0 - \lambda)W & \star \\ \lambda(A_r W + B_r Y) & -\lambda W \end{bmatrix} \preceq 0 \tag{20}$$

where K and Y are related by the equation $K = YW^{-1}$.

Proof: Considering now Lemma 2, it is analogous to the one of Theorem 2. ∎

Using the result above, it is possible to stabilize system (6) and to maximize its decay rate through the following optimization problem:

$$\begin{aligned} &\max_{W,R,\gamma_0} \gamma_0 \\ &\quad \text{subject to: (20)} \end{aligned} \tag{21}$$

Note that (21) can be solved using the bisection method on γ, since (20) is a LMI for a fixed value of this parameter and since the values of γ for which (20) can be satisfied correspond to a convex subset (an interval) of the real line.

B. Uncertain sampling rate

In this section we assume that the intensity λ of the Poisson sampling process is not exactly known but satisfies

$$0 < \lambda_{lb} \le \lambda \le \lambda_{ub} \tag{22}$$

for some bounds λ_{lb} and λ_{ub}. Then, it is possible to provide sufficient, though not necessary, stabilization conditions for (6). Consider the following corollary of Theorem 3.

Corollary 1: Assume that (22) holds. Then there exists $K \in \mathbb{R}^{m\times n}$ such that the resulting closed-loop system (6) is MES with decay rate $\gamma_0 > 0$ if there exist $W \in \mathbb{R}^{n\times n}, W = W^T \succ 0$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} WA_c^T + A_c W + (\gamma_0 - \lambda)W & \star \\ \lambda(A_r W + B_r Y) & -\lambda W \end{bmatrix} \preceq 0, \ \forall \lambda \in \{\lambda_{lb}, \lambda_{ub}\} \tag{23}$$

where K and Y are related by the equation $K = YW^{-1}$.

IV. Comparative analysis

The mean square stabilization problem of system (1) has already been dealt with in [12] considering the control law

$$u(t) = K_p x_p(t_k^-), \quad \forall t \in [t_k, t_{k+1}), \tag{24}$$

which is a particular case of (2). It should be noticed that the results in [12] are derived from a different impulsive system, where the error $e(t) \triangleq x_p(t) - x_p(t_{N_t})$ between $x_p(t)$ and its last sampled value until time t $x_p(t_{N_t})$ is considered as an additional state variable (see [12, Eq. 14]). In this case, due to a different problem structure, it is not possible to obtain non-conservative convex conditions for the computation of K_p. In our method, this problem is overcome by considering a different impulsive representation and a more generic control law, which also depends on the past value of the control input. We briefly recall in this section the method of [12], which provides only a partial solution to the stabilization problem, since it is based on the assumption below.

Assumption 1: There exist positive definite matrices $R = R^T$ and $P = P^T$ which solve the algebraic Riccati equation

$$A_p^T P + PA_p - 2PB_p R^{-1} B_p^T P = -\alpha P, \ \alpha > 0, \tag{25}$$

and such that $(A_p - B_p R^{-1} B_p^T P)$ is Hurwitz. Moreover, the matrix $P = P(\alpha)$ has the property that for some $p > \frac{2}{3}$

$$\limsup_{\alpha \downarrow 0} \frac{\lambda_{max}(P)}{\alpha^p} < \infty, \tag{26}$$

i.e. $\lambda_{max}(P) = O(\alpha^p)$ as $\alpha \downarrow 0$.

This assumption is rather strong, as recognized in [12, Pg. 239]. Indeed, in general, Assumption 1 is not expected to hold for systems where A_p has eigenvalues on the open right-half complex plane (see Remark 6.4 in [12]). Note that even verifying Assumption 1 is not a simple task. It is necessary not only to solve the Riccati equation (25) to obtain the matrix $P = P(\alpha) \succ 0$, but also to analyze property (26). It is possible to get some insight about the behavior of the ratio $\lambda_{max}(P)/\alpha^p$ computing it for a grid of values of the pair $(\alpha, p) \in (0, \infty) \times (\frac{2}{3}, \infty)$. However, this numerical procedure does not formally guarantee that (26) holds.

In case Assumption 1 is satisfied, the stabilizing control law is given by the result below [12, Theorem 6.5].

Lemma 3: Suppose that Assumption 1 holds. Then there exists $\alpha > 0$ (sufficiently small) such that the feedback gain

$$K_p = -R^{-1} B_p^T P(\alpha) \tag{27}$$

where $P(\alpha)$ solves (25) renders the closed-loop system composed by (1) and (24) mean exponentially stable.

Notice that Lemma 3 is non-constructive in the sense that it does not provide a valid value for $\alpha > 0$, which difficults the use of this result in practice. On the other hand, our method does not have this limitation and can be easily applied, since it relies on semidefinite programming problems. Moreover, the non-conservative result of Theorem 2 is valid in the general case and not only for the class of systems which satisfy Assumption 1.

V. Numerical Example

Consider the following system matrices taken from [11]:

$$A_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B_p = \begin{bmatrix} 0 \\ -5 \end{bmatrix}, \tag{28}$$

where $\lambda = 3$. Assumption 1 was numerically tested and does not seem to hold, what is in accordance with Remark 6.4 of [12], since one of the eigenvalues of A_p is equal to 1. Thus, the stabilization result of [12], i.e. Lemma 3, cannot be applied in this case. On the other hand, solving (21), one obtains the following feedback gain:

$$K = \begin{bmatrix} 0.2536 & 0.2574 & 0.0032 \end{bmatrix}, \tag{29}$$

which renders the closed-loop system mean exponentially stable with decay rate $\gamma_0 = 0.49$. The corresponding Lyapunov matrix P is given by:

$$P = W^{-1} = \begin{bmatrix} 0.0668 & 0.0665 & -0.221 \\ 0.0665 & 0.0683 & -0.222 \\ -0.221 & -0.222 & 0.885 \end{bmatrix}. \tag{30}$$

Figure 1 shows several trajectories of the closed-loop system in the x_p-subspace for some realizations of the sequence $\{\delta_k\}_{k\in\mathbb{N}}$, where the initial conditions, depicted by blue circles, satisfy $\|x_p(0)\| = 1$. Moreover, the control law (2) is initialized with $u(0^-) = 0$. As one should expect, the trajectories converge to the origin. We recall that $x_p(t)$ is continuous. On the other hand, its derivative is discontinuous at the sampling instants (represented by black circles) because of the update of the control input $u(t)$ using the law (2).

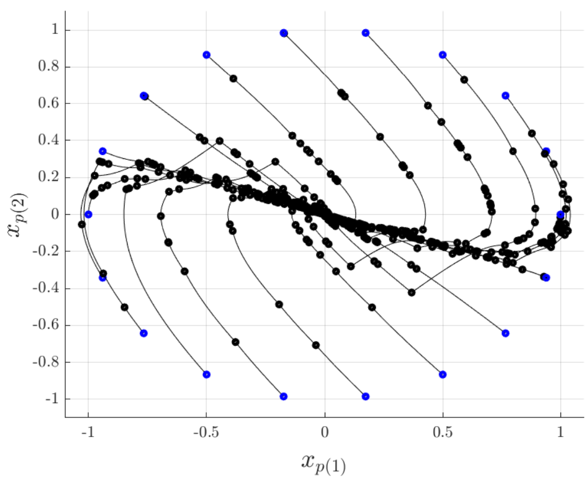

Fig. 1. Trajectories of the closed-loop system (28)-(29) in the x_p-subspace, where the initial conditions $x_p(0)$ are depicted by blue circles and the sampling instants t_k by black ones.

VI. Conclusions

A new control design method for linear sampled-data systems under Poisson sampling, which is based on semidefinite programming, was presented. As discussed above, unlike [12], this approach provides a necessary and sufficient stabilization condition for the system and can be easily implemented numerically. Moreover, unlike [13]–[16], which focus on the discrete-time trajectories of the system, we formally guarantee the stability of the continuous-time system.

An idea of future work consists in considering measurement noise as well as other (and more general) distribution functions for the sampling interval of the system. Besides that, the possibility of packet dropouts can also be explicitly considered, as in [15], and, in principle, it is also possible to extend the approach to cope with input nonlinearities, like saturation and quantization, as in [14].

Appendix: Proof of Lemma 2

Since time-varying Lyapunov functions will be applied, it will be convenient to consider the augmented process $\bar{x}(t) \triangleq (x(t), v(t)) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, where $\dot{v}(t) = 1$, $v(0) = v_0 \in \mathbb{R}_{\geq 0}$ and $v(t) = v_0 + t$, which also is a PDMP [21, Pg. 84]. In particular, the dynamics of $\bar{x}(t)$ has the form (6) with $\bar{f}(\bar{x}) \triangleq (f(x), 1) = [(A_c x)^T \;\; 1]^T$ and $\bar{g}(\bar{x}) \triangleq (g(x), v) = [(A_d x)^T \;\; v]^T$. Thus, for a function

$$W(x,v) = x^T P x e^{\gamma v}, \; P = P^T \succeq 0, \; \gamma \geq 0, \tag{31}$$

the Dynkin's formula (10) (with the appropriate notation changes) holds with

$$\mathfrak{U}W(x,v) = x^T(PA_c + A_c^T P + \gamma P + \lambda A_d^T P A_d - \lambda P) x e^{\gamma v} \tag{32}$$

as long as (9) is satisfied. To show that (9) holds, notice first that $x(t)$ can be written as

$$x(t) = \Phi(t) x_0 \tag{33}$$

where

$$\Phi(t) \triangleq e^{A_c(t-t_{N_t})} A_d e^{A_c(t_{N_t} - t_{N_t-1})} \dots A_d e^{A_c(t_2 - t_1)} A_d e^{A_c t_1} \tag{34}$$

is the (random) transition matrix of the system and N_t, defined in (4), counts the number of samplings until time t. From these relations, it follows that

$$\|x(t)\| \leq c_2^{N_t} e^{c_1 t} \|x_0\| \tag{35}$$

where $c_1 \triangleq \|A_c\| \geq 0$ and $c_2 \triangleq \|A_d\| \geq 1$. Moreover, for $k \geq 1$,

$$\begin{aligned} \|x(t_k)\| &\leq c_2^k e^{c_1 t_k} \|x_0\| \\ \|x(t_k^-)\| &\leq c_2^{k-1} e^{c_1 t_k} \|x_0\| \leq c_2^k e^{c_1 t_k} \|x_0\|. \end{aligned} \tag{36}$$

Thus, given $p \in \mathbb{N}$ and $(x_0, v_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, one has

$$\begin{aligned} &\mathbb{E}\left[\sum_{t_k \leq p} |W(x(t_k), v(t_k)) - W(x(t_k^-), v(t_k^-))|\right] \\ &\leq \mathbb{E}\left[\sum_{t_k \leq p} e^{\gamma v(t_k)} (x^T(t_k) P x(t_k) + (x^T(t_k^-) P x(t_k^-))\right] \\ &\underbrace{\leq}_{(36)} \mathbb{E}\left[\sum_{t_k \leq p} e^{\gamma v(t_k)} 2\|P\| (c_2^k e^{c_1 t_k} \|x_0\|)^2\right] \\ &\leq \mathbb{E}\left[\sum_{t_k \leq p} e^{\gamma v(p)} 2\|P\| (c_2^k e^{c_1 p} \|x_0\|)^2\right] \\ &= e^{\gamma(v_0 + p + 2c_1 p)} 2\|P\| \|x_0\|^2 \mathbb{E}\left[\sum_{k=0}^{N_p} c_2^{2k}\right] \triangleq C \mathbb{E}\left[\sum_{k=0}^{N_p} c_2^{2k}\right] \\ &= C \sum_{j=0}^{\infty} \left(P[N_p = j] \sum_{k=0}^{j} c_2^{2k}\right) = C \sum_{k=0}^{\infty} \left(c_2^{2k} \sum_{j=k}^{\infty} P[N_p = j]\right) \\ &= C \sum_{k=0}^{\infty} \left(c_2^{2k} P[N_p \geq k]\right) = C \sum_{k=0}^{\infty} \left(c_2^{2k} P[t_k \leq p]\right) < \infty \end{aligned}$$

where the order of summation was changed in the third-to-last equality, and the last inequality follows from [22, Theorem 3.3.1]. Consequently, (9) holds, as we wanted to show, and Theorem 1 can indeed be applied to function (31).

(Sufficiency part) Assume that (19) holds and define $W(x,v)$ as above, i.e. $W(x,v) \triangleq x^T P x e^{\gamma v}$. Left and right multiplying (19) by x^T and $x e^{\gamma v}$, respectively, one gets, for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$:

$$x^T(A_c^T P + PA_c - \lambda P + \lambda A_d^T P A_d) x e^{\gamma v} \leq -\gamma x^T P x e^{\gamma v},$$

which implies, according to (32), that $\mathfrak{U}W(x,v) \leq 0, \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Thus, applying Theorem 1, one has

$$\begin{aligned} \mathbb{E}[W(x(t), v(t))] &= W(x_0, v_0) + \mathbb{E}\left[\int_0^t \mathfrak{U}W(x(s), v(s)) \mathrm{d}s\right] \\ &\leq W(x_0, v_0), \; \forall (x_0, v_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}. \end{aligned} \tag{37}$$

Considering $v_0 = 0$ (which means that $v(t) = t$), (37) becomes

$$\mathbb{E}[x^T(t)Px(t)e^{\gamma t}] \leq x_0^T P x_0, \ \forall x_0 \in \mathbb{R}^n$$

or, equivalently, $\mathbb{E}[x^T(t)Px(t)] \leq x_0^T P x_0 e^{-\gamma t}, \ \forall x_0 \in \mathbb{R}^n$. Consequently, using the fact that P is positive definite, one concludes after some algebraic manipulations that $\mathbb{E}[\|x(t)\|^2] \leq \frac{\lambda_{max}(P)}{\lambda_{min}(P)} e^{-\gamma t}\|x_0\|^2, \ \forall x_0 \in \mathbb{R}^n$, and the result follows.

(Necessity part) Given $\gamma \in [0, \gamma_0)$, consider $W : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$W(x_0, v_0) \triangleq \mathbb{E}_{(x_0,v_0)}\left[\int_0^\infty \|x(s)\|^2 e^{\gamma v(s)} \mathrm{d}s\right] \tag{38}$$

where $\mathbb{E}_{(x_0,v_0)}$ emphasizes that the initial condition considered is $(x(0), v(0)) = (x_0, v_0)$ with probability one (the subscript will be omitted from now on). Since (8) holds (by assumption, (6) is MES with decay rate γ_0), $W(x_0, v_0)$ is indeed well defined (i.e. it is finite). More precisely, interchanging expectation with integral operations and, then, replacing $v(s) = v_0 + s$, one has

$$W(x_0, v_0) = \int_0^\infty \mathbb{E}\left[\|x(s)\|^2 e^{\gamma v(s)}\right] \mathrm{d}s = e^{\gamma v_0}\int_0^\infty \mathbb{E}\left[\|x(s)\|^2\right] e^{\gamma s}\mathrm{d}s. \tag{39}$$

At last, applying (8) one gets

$$W(x_0, v_0) \leq e^{\gamma v_0}\int_0^\infty c e^{-(\gamma_0-\gamma)s}\|x_0\|^2 \mathrm{d}s = \frac{c e^{\gamma v_0}}{\gamma_0 - \gamma}\|x_0\|^2 < \infty.$$

Moreover, note from (33) and (39) that

$$\begin{aligned} W(x_0, v_0) &= e^{\gamma v_0} x_0^T \left(\int_0^\infty \mathbb{E}[\Phi^T(s)\Phi(s)] e^{\gamma s}\mathrm{d}s\right) x_0 \\ &\triangleq e^{\gamma v_0} x_0^T P x_0 \end{aligned} \tag{40}$$

Let us find a lower bound for $W(x_0, v_0)$ to show that $P = P^T$ is positive definite. Consider again (39) and note that

$$\begin{aligned} W(x_0, v_0) &= e^{\gamma v_0}\mathbb{E}\left[\int_0^\infty \|x(s)\|^2 e^{\gamma s}\mathrm{d}s\right] \geq e^{\gamma v_0}\mathbb{E}\left[\int_0^{t_1} \|x(s)\|^2 \mathrm{d}s\right] \\ &= e^{\gamma v_0}\mathbb{E}\left[\int_0^{t_1} \|e^{A_c s}x_0\|^2 \mathrm{d}s\right] \geq e^{\gamma v_0}\|x_0\|^2 \mathbb{E}\left[\int_0^{t_1} e^{-2\|A_c\|s}\mathrm{d}s\right] \end{aligned}$$

where we used the fact that $x(t) = e^{A_c t}x_0$ before the first sampling time t_1 and applied Exercise 3.17 of [23], which provides a lower bound for $\|x(t)\|$. Fix now a constant (deterministic) value $t^* > 0$ and observe that

$$\begin{aligned} W(x_0, v_0) &\geq e^{\gamma v_0}\|x_0\|^2 \mathbb{E}\left[\int_0^{t_1} e^{-2\|A_c\|s}\mathrm{d}s\right] \\ &\geq e^{\gamma v_0}\|x_0\|^2 \mathbb{E}\left[\int_0^{t_1} e^{-2\|A_c\|s}\mathrm{d}s \Big| t_1 > t^*\right] P[t_1 > t^*] \\ &\geq e^{\gamma v_0}\|x_0\|^2 \mathbb{E}\left[\int_0^{t^*} e^{-2\|A_c\|s}\mathrm{d}s \Big| t_1 > t^*\right] P[t_1 > t^*] \\ &= e^{\gamma v_0}\|x_0\|^2 \int_0^{t^*} e^{-2\|A_c\|s}\mathrm{d}s e^{-\lambda t^*} = L e^{\gamma v_0}\|x_0\|^2 \end{aligned} \tag{41}$$

where we applied (3) (recall that $t_1 = \delta_0$ by definition) and $L \triangleq e^{-\lambda t^*}\int_0^{t^*} e^{-2\|A_c\|s}\mathrm{d}s > 0$. Comparing (40) and (41), it follows that $P \succ 0$, as claimed.

Applying Theorem 32.2 of [21] to (38), we conclude that

$$\mathfrak{U}W(x_0, v_0) = -\|x_0\|^2 e^{\gamma v_0} \tag{42}$$

Combining (42) (with (x_0, v_0) replaced by (x, v)) and (32) and since these eqs. hold for all $x \in \mathbb{R}^n$ and $v = 0$, it follows that $PA_c + A_c^T P + \gamma P + \lambda A_d^T P A_d - \lambda P = -I_n \prec 0$, that implies (19).

References

[1] K. J. Aström and B. Wittenmark, *Computer Controlled Systems. Theory and Design*, ser. Information and System Sciences Series. Prentice-Hall, 1984.

[2] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 21(1), pp. 84–99, 2001.

[3] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[4] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.-P. Richard, and S.-I. Niculescu, "Recent developments on the stability of systems with aperiodic sampling: An overview," *Automatica*, vol. 76, pp. 309 – 335, 2017.

[5] E. Fridman, "A refined input delay approach to sampled-data control," *Automatica*, vol. 46, no. 2, pp. 421–427, 2010.

[6] A. Seuret, "A novel stability analysis of linear systems under asynchronous samplings," *Automatica*, vol. 48, no. 1, pp. 177–182, 2012.

[7] L. Hetel, A. Kruszewski, W. Perruquetti, and J.-P. Richard, "Discrete and intersample analysis of systems with aperiodic sampling," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1696–1701, 2011.

[8] W. Lombardi, S. Olaru, S.-I. Niculescu, and L. Hetel, "A predictive control scheme for systems with variable time-delay," *International Journal of Control*, vol. 85, no. 7, pp. 915–932, 2012.

[9] H. Fujioka, "A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2440–2445, 2009.

[10] H. Fujioka and Y. Oishi, "A switched Lyapunov function approach to stability analysis of non-uniformly sampled-data systems with robust LMI techniques," in *8th Asian Control Conference (ASCC)*, 2011, pp. 1487–1491.

[11] D. D. Huff, M. Fiacchini, and J. M. Gomes da Silva Jr., "Stability and stabilization of aperiodic sampled-data systems subject to control input saturation: a set invariant approach," *IEEE Transactions on Automatic Control*, vol. 67, no. 3, pp. 1423–1429, 2022.

[12] A. Tanwani, D. Chatterjee, and D. Liberzon, "Stabilization of deterministic control systems under random sampling: Overview and recent developments," *Uncertainty in Complex Networked Systems*, pp. 209–246, 2018.

[13] B. Shen, Z. Wang, and T. Huang, "Stabilization for sampled-data systems under noisy sampling interval," *Automatica*, vol. 63, pp. 162–166, 2016.

[14] B. Shen, H. Tan, Z. Wang, and T. Huang, "Quantized/saturated control for sampled-data systems under noisy sampling intervals: A confluent Vandermonde matrix approach," *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4753–4759, 2017.

[15] Z. Hu, H. Ren, F. Deng, and H. Li, "Stabilization of sampled-data systems with noisy sampling intervals and packet dropouts via a discrete-time approach," *IEEE Transactions on Automatic Control (Early Access)*, 2021.

[16] H. Sun, J. Sun, and J. Chen, "Analysis and synthesis of networked control systems with random network-induced delays and sampling intervals," *Automatica*, vol. 125, pp. 1–7, 2021.

[17] D. J. Antunes, J. P. Hespanha, and C. J. Silvestre, "Volterra integral approach to impulsive renewal systems: application to networked control," *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 607–619, 2012.

[18] ——, "Stability of networked control systems with asynchronous renewal links: an impulsive systems approach," *Automatica*, vol. 49, no. 2, pp. 402–413, 2013.

[19] M. Tabbara and D. Nesic, "Input–output stability of networked control systems with stochastic protocols and channels," *IEEE Transactions on Automatic control*, vol. 53, no. 5, pp. 1160–1175, 2008.

[20] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, 1994.

[21] M. Davis, *Markov Models & Optimization*. London, UK: Chapman & Hall/CRC Monographs on Statistics and Applied Probability, 1993, vol. 49.

[22] S. I. Resnick, *Adventures in stochastic processes*. Birkhäuser Boston, 1992.

[23] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice-Hall, 2002.