Mean Square Exponential Stabilization of Sampled-Data Systems Subject to Actuator Nonlinearities, Random Sampling, and Packet Dropouts

Daniel Denardi Huff, Mirko Fiacchini, and João Manoel Gomes da Silva Jr

***Abstract*—This work deals with the mean square exponential stabilization of sampled-data linear systems subject to sector-bounded actuator nonlinearities and to aperiodic sampling intervals, which are assumed to be Erlang-distributed random variables. The possibility of packet dropouts is also taken into account and modeled by a Bernoulli process. Linear matrix inequality (LMI) conditions are proposed to design a stabilizing state-feedback controller for the system. Moreover, it is shown that the method leads to necessary and sufficient stabilization conditions in the absence of actuator nonlinearities. The results are derived using the framework of piecewise deterministic Markov processes, a subclass of stochastic hybrid systems.**

***Index Terms*—Piecewise deterministic Markov processes, random sampling, sampled-data control, sector-bounded nonlinearities.**

I. Introduction

Sampled-data control is present whenever a continuous-time system is controlled by a digital device [1]. These control loops are often implemented through a network, where communication protocols are used for the transmission of data [2]. While the use of a shared network has some advantages, like flexibility and simplicity of maintenance, there are also some drawbacks [2]. Among them, the presence of uncertainties in the sampling interval of the system, i.e., *aperiodic sampling*, which directly affects the closed-loop performance [3].

The stability analysis and control design for sampled-data systems subject to aperiodic sampling have been the subject of many recent studies (see the survey [4] and the references therein). These works consider different approaches and tools to tackle the problem, such as the time-delay approach [5], the use of looped-functionals [6], and discrete-time approaches [7], [8], [9], [10], [11].

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The aforementioned references present, as a common feature, a nonstochastic framework, where hard bounds are imposed for the time-varying sampling interval. However, since this assumption may not be realistic, some recent works have considered the case where the sampling interval is a random variable with possibly unbounded support [12], [13], [14], [15], [16]. In particular, the authors in [13], [14], [15], and [16] proposed conditions for the stabilization (in a stochastic sense) of the discrete-time model that describes the behavior of the system state at the sampling instants. However, although closely related, the stability of this discrete-time model is not equivalent to the stability of the corresponding (continuous-time) sampled-data system, as remarked, for instance, in [17, Pg. 610].

Thus, the aim of the present work is to propose a control design method which guarantees the stabilization (in the mean square sense) of the *continuous-time* system. As in [14], it is assumed that the random sampling intervals have an Erlang distribution, which includes the exponential distribution (considered, for instance, in [12] and [18]) as a particular case. Moreover, the possibility of packet dropouts is explicitly considered and modeled, as in [15], through a Bernoulli distribution. Unlike our previous work [18], which dealt only with the exponential distribution and the linear case, we also consider that the sampled-data system is subject to actuator nonlinearities which satisfy a sector condition. This includes, for instance, saturation, deadzone, and quantization.

To derive our results, we use, as in [19] for instance, the framework of Piecewise Deterministic Markov Processes (PDMPs) [20], which can be viewed as a subclass of stochastic hybrid systems (SHSs) [21]. The proposed Lyapunov-based stabilization conditions are posed in terms of LMIs and can, therefore, be easily solved in practice using off-the-shelf semidefinite programming solvers. Moreover, we show that the proposed approach leads to necessary and sufficient stabilization conditions in the linear case, i.e., without actuator nonlinearities.

The rest of this article is organized as follows. Section II presents basic definitions and proposes an equivalent SHS representation for the closed-loop sampled-data system. Section III presents the main results related to the control design method. The extension of the results to consider phase-type distributions is discussed in Section IV. Section V shows a numerical example. Finally, Section VI concludes this article.

Notation: $P[\cdot]$ denotes probability, and $\mathbb{E}[\cdot]$ expectation. $X \sim Y$ means that the random variables X and Y have the same distribution, and $X \sim \mathscr{E}(\nu, \lambda)$ means that X has the Erlang distribution of degree ν and rate λ. $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{N}_m = \{i \in \mathbb{N} : 1 \leq i \leq m\}$. For $f : \mathbb{R} \to \mathbb{R}^n$, $f(t^-) \triangleq \lim_{\tau \mapsto t, \tau < t} f(\tau)$ if the limit exists. Given matrices A and B, $\mathrm{Diag}(A, B)$ is a block diagonal matrix formed by them. If A is square, $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) is the maximal (minimal) real part of the eigenvalues of A. The symbol $\star$ denotes a symmetric block when applied as an entry of a matrix and $\succ$ ($\succeq$) defines positive (semi)-definiteness of a symmetric matrix. $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix. $\|\cdot\|$ denotes the induced two-norm of a matrix or the Euclidean norm of a vector. Given a matrix $A \in \mathbb{R}^{m\times n}$, $A_{(i)}$ is its ith row and A^T its transpose. We sometimes identify a matrix $A \in \mathbb{R}^{m\times n}$ with the corresponding linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Given $f : \mathbb{R} \to \mathbb{R}$ and finite scalars $\underline{d}$ and $\bar{d}$, $f \in \sec[\underline{d}, \bar{d}]$ means that the graph of $f(\cdot)$ lies inside the sector formed by the lines $g_1(x) = \underline{d}x$ and $g_2(x) = \bar{d}x$, with $\bar{d} \geq \underline{d}$.

II. Problem Formulation

Consider the continuous-time plant described by the following linear model:

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t) \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ and $u \in \mathbb{R}^m$ are the state and the input of the plant, respectively. Matrices A_p and B_p have appropriate dimensions and are constant. The control input is updated at the *sampling instants* t_j and kept constant (by means of a zero-order-hold) for all $t \in [t_j, t_{j+1})$ according to the law

$$\begin{aligned} &u(t) = u(t_j), \text{ for } t \in [t_j, t_{j+1}) \\ &u(t_j) = \begin{cases} \phi(K_p x_p(t_j^-) + K_u u(t_j^-)), & \text{if no packet dropout} \\ u(t_j^-), & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

where K_p and K_u are matrices of appropriate dimensions and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ denotes an actuator nonlinearity (e.g., saturation, deadzone, quantization, etc.). Note in (2) that $u(t)$ is updated at time t_j only if the corresponding packet of measurement data from the sensors is not lost due to some misbehavior of the network, otherwise the controller maintains the current input until the next sampling instant t_{j+1}. It is assumed that ϕ is a measurable (though not necessarily continuous) function. It is also decentralized, i.e.,

$$\phi(\zeta) = [\phi_1(\zeta_{(1)})\, \phi_2(\zeta_{(2)}) \,\ldots\, \phi_m(\zeta_{(m)})],\ \zeta \in \mathbb{R}^m \tag{3}$$

and is elementwise sector bounded (see Notation), that is

$$\phi_i(\cdot) \in \sec[\underline{d}_i, \bar{d}_i],\ \forall i \in \mathbb{N}_m. \tag{4}$$

Note that (2) is based not only on the sampled value of the state x_p but also on the value of the last control input applied to the plant, where the use of the term $K_u u(t_j^-)$ has already showed its benefits in a nonstochastic framework [11]. The probability of packet dropout is modeled by a Bernoulli process $\{\alpha_j\}_{j\in\mathbb{N}_+}$ (i.e., a sequence of i.i.d. Bernoulli random variables) with $P[\alpha_j = 0] = \mu_0 \in (0, 1)$ and $P[\alpha_j = 1] = \mu_1 \triangleq 1 - \mu_0$, where $\alpha_j = 0$ means that a packet dropout occurs at the sampling instant t_j. The events of packet dropout for each t_j are mutually independent between them.

By convention $t_0 = 0$ and the difference between two successive sampling instants—the *sampling interval*—is denoted by $\delta_j \triangleq t_{j+1} - t_j$. It is assumed that $\{\delta_j\}_{j\in\mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random variables with the Erlang distribution (see details, for instance, in [22, pp. 87–89]) of *degree* $\nu \in \mathbb{N}_+$ and *rate* $\lambda \in \mathbb{R}_+$, i.e., $\delta_j \sim \mathscr{E}(\nu, \lambda)$. The corresponding probability density function is given by [22, p. 87]

$$f_\delta(s) \triangleq \frac{\lambda^\nu s^{\nu-1} e^{-\lambda s}}{(\nu - 1)!},\ s \geq 0. \tag{5}$$

The Erlang distribution has been used to model the stochastic behavior of networked control systems under random sampling in [14]. As illustrated in Fig. 1, it allows to model an event whose probability density function is concentrated around a value, which may represent a nominal constant sampling interval (or period) δ in the ideal case, where there is no uncertainties on δ_k induced by the network. In particular, note that the mean and the variance of the sampling interval $\delta_j \sim \mathscr{E}(\nu, \lambda)$ are, respectively, ν/λ and ν/λ^2. Moreover, the exponential distribution, considered for instance in [12] and [18], corresponds to the particular case in which $\nu = 1$, as it can be seen from (5). Fig. 1 shows the shape of $f_\delta(s)$ for different values of ν and λ.

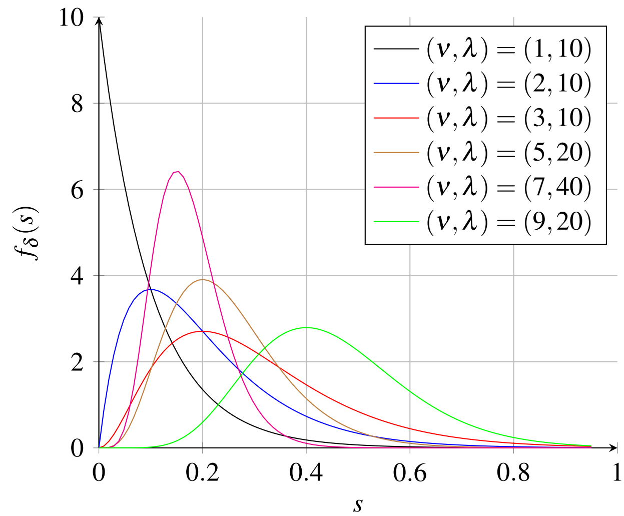

Fig. 1. Probability density function (5) of the Erlang distribution for different values of the parameters ν and λ.

It will be convenient to define $\underline{D} \triangleq \mathrm{Diag}(\underline{d}_1, \ldots, \underline{d}_m)$, $\overline{D} \triangleq \mathrm{Diag}(\bar{d}_1, \ldots, \bar{d}_m)$, $D \triangleq \overline{D} - \underline{D}$, and $\bar{\phi}(\zeta) \triangleq \phi(\zeta) - \underline{D}\zeta$. From these definitions and (4), the ith component of the nonlinearity $\bar{\phi}$ belongs to the sector $\sec[0, \bar{d}_i - \underline{d}_i]$. In other words, $\bar{\phi}$ satisfies the following sector condition, adapted from [23, Lemma 1.4].

Lemma 1: Given a diagonal matrix $T \in \mathbb{R}^{m\times m}, T \succeq 0$

$$\bar{\phi}^T(\zeta) T (D\zeta - \bar{\phi}(\zeta)) \geq 0,\ \forall \zeta \in \mathbb{R}^m. \tag{6}$$

□

Consider now $x \triangleq [x_p^T\, u^T]^T \in \mathbb{R}^n$, $n \triangleq n_p + m$. The dynamics (1) and (2) can be described by the following impulsive system (see also [11]):

$$\dot{x}(t) = A_c x(t), \qquad \forall t \geq 0,\ t \neq t_j,\ \forall j \in \mathbb{N}_+ \tag{7a}$$

$$x(t_j) = \begin{cases} g(x(t_j^-)), & \text{if } \alpha_j = 1, \\ x(t_j^-), & \text{if } \alpha_j = 0, \end{cases} \qquad \forall j \in \mathbb{N}_+ \tag{7b}$$

$$g(x) \triangleq A_d x + B_r \bar{\phi}(Kx) \tag{7c}$$

where $x(0) = x_0 \in \mathbb{R}^n$, $A_d \triangleq A_r + B_r \underline{D} K$ and

$$A_c \triangleq \begin{bmatrix} A_p & B_p \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad A_r \triangleq \begin{bmatrix} I_{n_p} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$
$$B_r \triangleq \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{n\times m}, \quad K \triangleq \begin{bmatrix} K_p & K_u \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Definition 1: The equilibrium point $x = 0$ of (7) is *mean exponentially stable* (MES) if there exist constants $c, \gamma > 0$ such that for every initial condition $x_0 \in \mathbb{R}^n$

$$\mathbb{E}[\|x(t)\|^2] \leq c e^{-\gamma t} \|x_0\|^2,\ \forall t \geq 0 \tag{8}$$

where $\gamma > 0$ will be referred to as a *decay rate* of the trajectories of the system. □

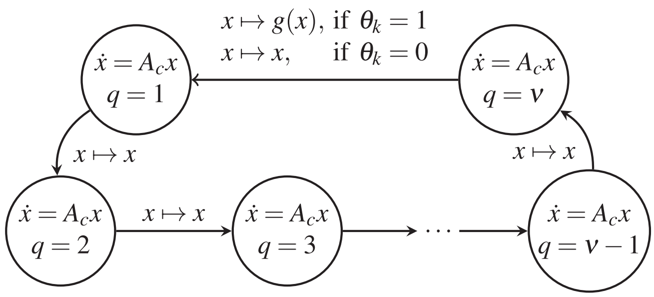

Fig. 2. Graphical representation of the SHS (10).

From Markov's inequality $P[\|x(t)\| > r] \leq \frac{\mathbb{E}[\|x(t)\|^2]}{r^2}$ [20, p. 12], the decay rate corresponds to a measure of how fast the probability of $\|x(t)\|$ being large decays with time.

The problem we focus on in this work can now be stated.

Problem 1 (Mean square exponential stabilization): Given the parameters of the Erlang and of the Bernoulli distributions (i.e., ν, λ, and μ_0), provide convex conditions for the design of the feedback gain K such that the resulting closed-loop system (7) is MES. □

A. Equivalent SHS Representation

Next, we present an SHS representation for (7), where we use the fact that an Erlang-distributed random variable $X \sim \mathscr{E}(\nu, \lambda)$ of degree ν is statistically equivalent to the sum of ν mutually independent exponentially distributed random variables $X_i \sim \mathscr{E}(1, \lambda)$, i.e., (cf. [24, Exercise 23.2]):

$$X \sim \sum_{i=1}^{\nu} X_i. \tag{9}$$

This feature, that has also been used in [25] to deal with semi-Markov jump linear systems with Erlang-distributed dwell times, will be important to derive convex conditions for the stabilization of the closed-loop system.

Consider $(q, x) \in \mathbb{N}_\nu \times \mathbb{R}^n$ and a sequence $\{\theta_k\}_{k \in \mathbb{N}_+}$ with the same distribution of $\{\alpha_j\}_{j \in \mathbb{N}_+}$. The proposed SHS model, illustrated in Fig. 2, is given by

$$(\dot{q}(t), \dot{x}(t)) = (0,\, A_c x(t)),\ \forall t \geq 0, t \neq r_k, \forall k \in \mathbb{N}_+ \tag{10a}$$

$$(q(r_k), x(r_k)) = \Psi(q(r_k^-), x(r_k^-), \theta_k), \quad \forall k \in \mathbb{N}_+ \tag{10b}$$

$$\Psi(q, x, \theta) \triangleq \begin{cases} (q+1, x), & \text{if } q < \nu \\ (1, g(x)), & \text{if } (q, \theta) = (\nu, 1) \\ (1, x), & \text{if } (q, \theta) = (\nu, 0) \end{cases} \tag{10c}$$

where $\{r_k\}_{k \in \mathbb{N}}$ is the sequence of *reset* (or *jump*) *times*, with $r_0 = 0$, and $(q_0, x_0) \triangleq (q(0), x(0))$. Note from (10b) that there is a one-to-one relationship between r_k and θ_k, even if the value of $\Psi(q, x, \theta)$ does not depend on θ for $q < \nu$. Moreover, $\{\rho_k\}_{k \in \mathbb{N}} \triangleq \{r_{k+1} - r_k\}_{k \in \mathbb{N}}$ is considered to be a sequence of i.i.d. random variables with $\rho_k \sim \mathscr{E}(1, \lambda)$ exponentially distributed, i.e., the counting process

$$N_t \triangleq \sup\{k \in \mathbb{N} : r_k \leq t\} \tag{11}$$

is a Poisson process [20, p. 37], [22, p. 378] of rate $\lambda > 0$. Since ρ_k has the exponential distribution [22, p. 88]:

$$P[\rho_k \leq s] = 1 - e^{-\lambda s}, \quad \forall k \in \mathbb{N}, \forall s \geq 0. \tag{12}$$

The sampling instants t_j of (7) are represented by the transition of (10) from $q = \nu$ to $q = 1$ while the other jumps of (10), which do not affect $x(t)$, allow to express the behavior of system (7) through exponentially distributed random variables $\rho_k \sim \mathscr{E}(1, \lambda)$. Indeed, we claim that $x(t)$ in (7) is statistically equivalent to $x(t)$ in (10) if $q_0 = 1$. To understand why, let us denote by $\{\bar{t}_j\}_{j \in \mathbb{N}}$ the sequence of times at which the transitions from $q = \nu$ to $q = 1$ of (10) take place. Then, it suffices to notice that, if $q_0 = 1$, $\{\bar{t}_j\}_{j \in \mathbb{N}}$, given in this case by $\{\bar{t}_j\}_{j \in \mathbb{N}} = \{r_{\nu k}\}_{k \in \mathbb{N}}$, has the same distribution of $\{t_j\}_{j \in \mathbb{N}}$ in (7). This is a direct consequence of the discussion at the beginning of this subsection and of the way these sequences are defined

$$\begin{aligned} \{\bar{t}_{j+1} - \bar{t}_j\}_{j \in \mathbb{N}} &= \{r_{\nu(k+1)} - r_{\nu k}\}_{k \in \mathbb{N}} \\ &= \left\{ \sum_{i=\nu k}^{\nu(k+1)-1} \rho_i \right\}_{k \in \mathbb{N}} \underbrace{\sim}_{(9)} \{\delta_j\}_{j \in \mathbb{N}} = \{t_{j+1} - t_j\}_{j \in \mathbb{N}}. \end{aligned}$$

These facts allow to address Problem 1 considering model (10), which involves the exponential distribution, as detailed in the next section.

Remark 1: Note that model (10) is slightly more general than (7), since the time elapsed until the first transition from $q = \nu$ to $q = 1$ can have any one of the distributions $\bar{t}_1 \sim \mathscr{E}(\hat{\nu}, \lambda)$, $\hat{\nu} \in \mathbb{N}_\nu$, depending on the initial condition of $q(t)$. □

III. Mean Square Exponential Stabilization

The main results of the work are presented next. Section III-A considers the nonlinear case and Section III-B shows that in the linear case, i.e., when $\phi(K_p x_p + K_u u) = K_p x_p + K_u u$, the proposed stabilization conditions are nonconservative, i.e., they are necessary and sufficient.

A. General Case

According to the definitions in [20, Sec. 24], system (10) belongs to the class of PDMPs (which is a subclass of SHSs, cf. [21, Table 1]), where the state space is given by $E \triangleq \mathbb{N}_\nu \times \mathbb{R}^n$ and the boundary set $\Gamma \triangleq \emptyset$ is empty. Moreover, see [20, Sec. 24]:

1) The flow map is given by $\Phi_q(t, x) = e^{A_c t} x$ for all q.
2) The jump rate $\lambda : E \to \mathbb{R}_{\geq 0}$ is constant, i.e., $\lambda(q, x) \equiv \lambda$, where λ is the rate of the Poisson process N_t.
3) The transition measure Q is defined using (10b) and (10c).
4) Assumption 24.4 of [20], which regards the expectation of the counting process N_t, is indeed satisfied since N_t is a Poisson process, and thus, $\mathbb{E}[N_t] = \lambda t < \infty$ [22, p. 378].

Given a function $V(q, x)$, these facts allow to establish the following result, which is closely related to [26, Th. 2].

Theorem 1: Consider system (10) and a continuously differentiable function $V : E \to \mathbb{R}$ such that

$$\mathbb{E}\left[\sum_{r_k \leq T} |V(q(r_k), x(r_k)) - V(q(r_k^-), x(r_k^-))| \right] < \infty \tag{13}$$

$\forall T \geq 0$, $\forall (q_0, x_0) \in E$, where $q(r_0^-) = q(r_0) = q_0$ by convention and similarly for $x(r_0^-)$. Then, for $t \geq 0$ and for all $(q_0, x_0) \in E$

$$\mathbb{E}[V(q(t), x(t))] = V(q_0, x_0) + \mathbb{E}\left[\int_0^t \mathfrak{U} V(q(s), x(s)) \mathrm{d}s \right] \tag{14}$$

where

$$\mathfrak{U} V(q, x) \triangleq \frac{\partial V(q, x)}{\partial x} A_c x + \lambda \left(Q V(q, x) - V(q, x) \right), \tag{15}$$

$$Q V(q, x) \triangleq \begin{cases} V(q+1, x), & \text{if } q < \nu, \\ \mu_1 V(1, g(x)) + \mu_0 V(1, x), & \text{if } q = \nu. \end{cases} \tag{16}$$

□

Proof: The result follows from [20, Th. 26.14] and [20, Remark 26.16], which guarantee, under condition (13), that [20, eq. 14.17] holds [which corresponds to (14)]. ■

Relation (14) is known as the Dynkin's formula [20, p. 33] and can be intuitively interpreted as a stochastic version of the fundamental theorem of calculus. The first term of the right-hand side of (15) is just the usual time derivative of $V(q,x)$ along the trajectories of $\dot{x}(t) = A_c x(t)$ while the second term accounts for the jumps at the reset times. Note also that one of the transitions of (10) (the jump from $q=\nu$ to $q=1$) is stochastic and depends on a Bernoulli random variable (that is why there are two terms in (16)).

Now we are ready to state our main stabilization result, whose proof is in Appendix A.

Theorem 2: If there exist a scalar $\gamma > 0$, a matrix $Y \in \mathbb{R}^{m\times n}$, positive definite matrices $W_q \in \mathbb{R}^{n\times n}, \forall q \in \mathbb{N}_\nu$, and a diagonal positive definite matrix $S \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} A_c W_q + W_q A_c^T + (\gamma-\lambda) W_q & \star \\ W_q & -\frac{W_{q+1}}{\lambda} \end{bmatrix} \preceq 0, \ \forall q < \nu, \tag{17a}$$

$$\begin{bmatrix} A_c W_\nu + W_\nu A_c^T + (\gamma-\lambda) W_\nu & \star & \star & \star \\ DY & -2S & \star & \star \\ A_r W_\nu + B_r \underline{D} Y & B_r S & -\frac{W_1}{\lambda\mu_1} & \star \\ W_\nu & 0 & 0 & -\frac{W_1}{\lambda\mu_0} \end{bmatrix} \preceq 0 \tag{17b}$$

then, for $K = Y W_\nu^{-1}$, system (7) is MES with decay rate γ. □

The proof of Theorem 2, based in Theorem 1, considers a Lyapunov function of the form $V(q,x) = x^T P_q x$ for the SHS (10). This choice is not arbitrary but motivated by the fact that it leads to nonconservative (i.e., necessary and sufficient) stabilization conditions in the linear case, as we will show next.

Remark 2: Until now we have considered that $\mu_0 \in (0,1)$. If $\mu_0 = 0$ (i.e., zero probability of packet dropouts), then LMI (17b) is replaced by

$$\begin{bmatrix} A_c W_\nu + W_\nu A_c^T + (\gamma-\lambda) W_\nu & \star & \star \\ DY & -2S & \star \\ A_r W_\nu + B_r \underline{D} Y & B_r S & -\frac{W_1}{\lambda} \end{bmatrix} \preceq 0. \tag{18}$$

□

Remark 3: From the upper left blocks of the matrices in (17), we conclude that (17) can be satisfied only if $2\bar{\lambda}(A_c) < \lambda$, where $\bar{\lambda}(A_c)$ denotes the largest real part of the eigenvalues of A_c. This observation is in accordance with [17, condition (5)], which provides a necessary condition for the mean exponential stability of impulsive renewal systems in the linear case. □

B. Linear Case

Consider the case where $\phi(\zeta) = \zeta$ and (7c) reduces to

$$g(x) = A_d x = (A_r + B_r K)x. \tag{19}$$

The following theorem provides necessary and sufficient conditions for the mean square exponential stabilization of the closed-loop system in this case.

Theorem 3: There exists a gain K such that the system (7) with $g(x)$ given by (19) is MES if and only if there exist a matrix $Y \in \mathbb{R}^{m\times n}$ and positive definite matrices $W_q^L \in \mathbb{R}^{n\times n}, \forall q \in \mathbb{N}_\nu$, such that

$$\begin{bmatrix} A_c W_q^L + W_q^L A_c^T - \lambda W_q^L & \star \\ W_q^L & -\frac{W_{q+1}^L}{\lambda} \end{bmatrix} \prec 0, \ \forall q < \nu \tag{20a}$$

$$\begin{bmatrix} A_c W_\nu^L + W_\nu^L A_c^T - \lambda W_\nu^L & \star & \star \\ A_r W_\nu^L + B_r Y & -\frac{W_1^L}{\lambda\mu_1} & \star \\ W_\nu^L & 0 & -\frac{W_1^L}{\lambda\mu_0} \end{bmatrix} \prec 0 \tag{20b}$$

with $K = Y(W_\nu^L)^{-1}$. □

Proof: See Appendix B. ■

Remark 4: As in Theorem 2, it is possible to guarantee a specific decay rate $\gamma^L > 0$ in (8) replacing (20) by

$$\begin{bmatrix} A_c W_q^L + W_q^L A_c^T + (\gamma^L - \lambda) W_q^L & \star \\ W_q^L & -\frac{W_{q+1}^L}{\lambda} \end{bmatrix} \preceq 0, \ \forall q < \nu, \tag{21a}$$

$$\begin{bmatrix} A_c W_\nu^L + W_\nu^L A_c^T + (\gamma^L - \lambda) W_\nu^L & \star & \star \\ A_r W_\nu^L + B_r Y & -\frac{W_1^L}{\lambda\mu_1} & \star \\ W_\nu^L & 0 & -\frac{W_1^L}{\lambda\mu_0} \end{bmatrix} \preceq 0. \tag{21b}$$

□

Remark 5: If $\mu_0 = 0$ and $\nu = 1$ (i.e., zero probability of packet dropouts and exponentially distributed sampling intervals), constraints (20) reduce, after application of the Schur's complement and denoting $P = (W_1^L)^{-1}$, to

$$PA_c + A_c^T P + \lambda(A_d^T P A_d - P) \prec 0$$

which is a known necessary and sufficient condition for the mean square exponential stability of system (7) with $g(x)$ given by (19) when the sampling intervals have the exponential distribution (see [12, Th. 5.1] or [19, Th. 7]). □

IV. Extension to Phase-Type Distributions

Phase-type distributions, considered for instance in [27] in the context of aperiodic sampled-data control, can accurately approximate any given probability distribution on $(0,\infty)$ [28, Th. 4.2, Ch. 3]. It turns out that they are closely related to the Erlang distribution, which is actually a particular case of them. The cumulative distribution function of a random variable Y with the phase-type distribution of order p, parametrized by $\Sigma \in \mathbb{R}^{p\times p}$ and $\sigma \in \mathbb{R}^{1\times p}$, is given by [28, Ch. 3, p. 83]

$$P[Y \leq y] = F(y) = 1 - \sigma e^{\Sigma y}\mathbf{1}$$

where $\mathbf{1} = [1\dots1]^T$. As explained in [28, Ch. 3, pp. 82–83], Y can be defined in terms of a continuous-time Markov chain, where the time elapsed until the next state transition has the exponential distribution with an intensity/rate that depends on the states involved in the transition. More precisely, Y can be defined in terms of a Markov chain with a finite state space $\{1,2,\dots,p+1\}$, with one absorbing state (labeled here $p+1$) and p transient states (labeled from 1 to p). The corresponding transition rate matrix $\Lambda \in \mathbb{R}^{(p+1)\times(p+1)}$ has the form

$$\Lambda = \begin{bmatrix} \Sigma & -\Sigma\mathbf{1} \\ 0 & 0 \end{bmatrix}.$$

The variable Y can then be interpreted as the time that the state of the Markov chain takes to reach the absorbing state $p+1$ after initialized at time zero with initial state probability distribution $[\sigma\ 0]^T$.

Notice now that the state $q(t)$ of the SHS model (10) can also be interpreted in terms of a continuous-time Markov chain, with state space $\{1,\dots,\nu\}$. This (cyclic) Markov chain is used to emulate a sampling interval δ_j with the Erlang distribution. However, with the appropriate modifications in (10), it would also be possible to emulate a sampling interval with a phase-type distribution, resulting in a more complex chain and more complex expressions in (15) and (16). This fact leads to believe that it is possible to extend the results of the present work to this type of distribution.

V. Numerical Example

Consider the system matrices

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -330.46 & -12.15 & -2.44 & 0 \\ 0 & 0 & 0 & 1 \\ -812.61 & -29.87 & -30.10 & 0 \end{bmatrix}, \; B_p = \begin{bmatrix} 0 \\ 2.71762 \\ 0 \\ 6.68268 \end{bmatrix}$$

taken from a cart–spring–pendulum system, which has been fully described in [29] and has also been used in [30] and [23, Example 8.3], for instance. The state is given by $x_p = [p \quad \dot{p} \quad \beta \quad \dot{\beta}]^T$, where $p(t)$ is the linear position of the cart and $\beta(t)$ is the angular position of the pendulum. The control input $u(t)$ is the voltage applied to the armature of the DC motor of the cart, which is limited in amplitude by the value $\pm 5\,V$. Since $u(t)$ is constrained in magnitude, it can be modeled by using the standard saturation nonlinearity $\mathrm{sat}(\cdot)$, i.e., $\phi(\cdot) = \mathrm{sat}(\cdot)$ in (2). In this case, the sector condition (4) is verified with $\overline{D} = I_m$ and $\underline{D} = 0$ [23].

Consider the parameters $\nu = 3$ and $\lambda = 10$ to model the stochastic sampling effects. The corresponding Erlang distribution is depicted in Fig. 1. Note that it allows to model a stochastic sampling interval whose probability density function is concentrated around 0.2 approximately. Consider also the parameters $\mu_0 = 0.05$ and $\mu_1 = 0.95$, which means that the probability of packet dropout at each sampling instant is of 5%.

The feedback matrix $K = [K_p \; K_u]$ will be computed such that the closed-loop system composed by (1) and (2) is MES. Moreover, as a second control objective, the gain K will be designed to maximize the decay rate of the trajectories of the linear model composed by (1) and (2) with $\phi(\cdot)$ replaced by the identity function, which corresponds to the behavior of the nonlinear closed-loop system when the control input does not saturate. Combining the results of Theorems 2 and 3 and Remark 4, the following optimization problem is proposed:

$$\begin{aligned} \max_{\gamma^L, Y, S, W_q, W_q^L} \quad & \gamma^L \\ \text{subject to:} \quad & (17), (21), \\ & S \succ 0 \,(\text{diagonal}), \\ & W_q \succ 0, W_q^L \succ 0, \forall q \in \mathbb{N}_\nu, \\ & W_\nu = W_\nu^L \end{aligned} \qquad (22)$$

where $\gamma > 0$ in (17) is fixed a priori. Then, the resulting feedback gain is given by $K = Y(W_\nu)^{-1} = Y(W_\nu^L)^{-1}$. Note that the use of a common Lyapunov matrix $W_\nu = W_\nu^L$ allows to construct an optimization problem with LMI constraints for each fixed value of γ^L. More precisely, (22) corresponds to a generalized eigenvalue problem [31, Sec. 2.2.3] and can be solved using bisection on γ^L and a semidefinite programming algorithm. The results in this example were obtained using the solver SeDuMi [32] and the parser YALMIP [33].

Solving (22) with $\gamma = 0.01$, one obtains $\gamma^L = 0.0225$ and

$$K = \begin{bmatrix} 12.87 & 0.24 & -0.21 & 0.032 & -0.00076 \end{bmatrix} \qquad (23)$$

which makes the closed-loop system composed by (1) and (2) MES. Fig. 3 shows several trajectories of the closed-loop system in the subspace of $[p(t)\,\beta(t)]^T$ for different initial conditions (depicted by blue circles, with $\dot{p}(0) = 0$ and $\dot{\beta}(0) = 0$) and different realizations of the sequences $\{\delta_j\}_{j\in\mathbb{N}}$ and $\{\alpha_j\}_{j\in\mathbb{N}_+}$, where the black circles represent the sampling instants. As it can be seen, the trajectories converge to the origin. Moreover, Fig. 4 presents the states $p(t)$ and $\beta(t)$ as a function of time for a single realization of the sequences $\{\delta_j\}_{j\in\mathbb{N}}$ and $\{\alpha_j\}_{j\in\mathbb{N}_+}$.

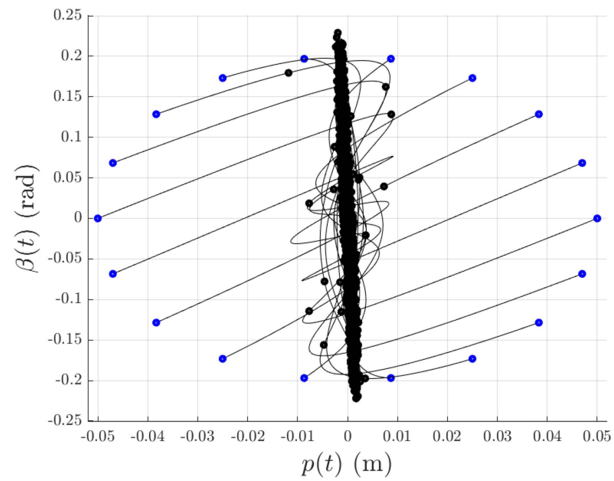

Fig. 3. Trajectories of the closed-loop system composed by (1) and (2) with $\phi(\cdot) = \mathrm{sat}(\cdot)$ and K given by (23) in the subspace of $[p(t)\,\beta(t)]^T$, where the initial conditions are depicted by blue circles and the state at the sampling instants t_j by black ones.

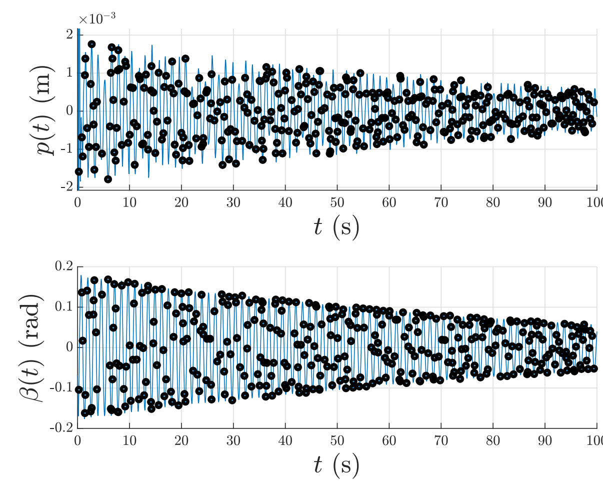

Fig. 4. Response of the states $p(t)$ and $\beta(t)$ of the closed-loop system composed by (1) and (2) with $\phi(\cdot) = \mathrm{sat}(\cdot)$ and K given by (23), where the values at the sampling instants t_j are depicted by black circles.

VI. Conclusion

In this work, LMI conditions are proposed for the mean square exponential stabilization of randomly sampled linear systems subject to control input nonlinearities and packet dropouts, where the sampling intervals are considered to be Erlang-distributed random variables (as in [14]) and the possibility of packet dropouts is modeled through a Bernoulli distribution (as in [15]). Unlike [13], [14], [15], and[16], which focus on the discrete-time trajectories of the system, our method formally guarantees the exponential stabilization of the *continuous-time* system. Moreover, the proposed stabilization conditions are necessary and sufficient in the linear case, i.e., in the absence of actuator nonlinearities.

As a future work, it would be interesting to consider other (and more general) distribution functions for the sampling interval of the system. In particular, as discussed in Section IV, it is in principle possible to extend the results for phase-type distributions, which are closely related to the Erlang distribution and allow to approximate arbitrarily well any probability distribution on $(0, \infty)$ [28, Th. 4.2]. The problem of fitting a phase-type distribution to a given probability distribution is addressed, for instance, in [34] via a maximum likelihood approach.

Another idea consists in dealing with the case where only local stabilization (in a probabilistic sense) around the origin is possible. Moreover, the use of observer-based control laws could also be a topic of future research. Some works propose, for instance, the use of a continuous-discrete state observer for aperiodic sampled-data systems (e.g., [35], [36]). It would be interesting to apply this idea in a stochastic framework.

Appendix A
Proof of Theorem 2

We will show that $(q(t), x(t))$ given by (10) with $K = YW_\nu^{-1}$ satisfies (8) for all initial conditions (q_0, x_0), where $c > 0$ will be appropriately chosen and $\gamma > 0$ is given by the statement of the theorem. Then, property (8) will also hold for system (7), according to the reasoning in Section II-A, implying that (7) is MES. A time-varying function $W(q, x, t)$ will be considered, where, since $(q(t), x(t))$ is a PDMP, $(q(t), x(t), t)$ is also a PDMP [20, p. 84]. Thus, for a function of the form

$$W(q, x, t) = e^{\gamma t} V(q, x) \tag{24}$$

the Dynkin's formula analogous to (14) holds with (cf. [20, p. 84])

$$\mathfrak{U}W(q, x, t) = e^{\gamma t}(\gamma V(q, x) + \mathfrak{U}V(q, x)) \tag{25}$$

as long as (13) is satisfied (with $V(\cdot)$ replaced by $W(\cdot)$). Let us now show that (13) indeed holds for $W(q, x, t)$ given by (24) with

$$V(q, x) \triangleq x^T P_q x \tag{26}$$

where $P_q \triangleq W_q^{-1} \succ 0, \forall q \in \mathbb{N}_\nu$. Define $\operatorname{Proj}_x \Psi$ as the projection of the $(\mathbb{N}_\nu \times \mathbb{R}^n)$-valued function Ψ onto $\mathbb{R}^n$ and denote by $\Psi^k(\cdot)$ the map $x \mapsto \operatorname{Proj}_x \Psi((q(r_k^-), x), \theta_k), \forall k \geq 1$. Note that $x(t)$ in (10) can be expressed by

$$x(t) = e^{A_c(t - r_{N_t})} \circ \Psi^{N_t} \circ e^{A_c(r_{N_t} - r_{N_t - 1})} \circ \cdots \\ \cdots \circ \Psi^2 \circ e^{A_c(r_2 - r_1)} \circ \Psi^1 \circ e^{A_c r_1} x_0 \tag{27}$$

where N_t, defined in (11), counts the number of resets until time t and $\circ$ denotes the composition of functions. Let

$$c_1 \triangleq \|A_c\|,$$
$$c_2 \triangleq \max\{1, \|A_d\| + \|B_r\|\|D\|\|K\|\}$$
$$\bar{c} \triangleq \max_{q \in \mathbb{N}_\nu} \|P_q\|$$

and note from (7c), (10c) and the sector condition (6) satisfied by $\bar{\phi}$ that $\|\operatorname{Proj}_x \Psi(q, x, \theta)\| \leq c_2 \|x\|, \forall (q, x, \theta) \in E \times \{0, 1\}$. Then, from (27), one concludes that $\|x(t)\| \leq c_2^{N_t} e^{c_1 t} \|x_0\|$. Moreover, for $k \geq 1$

$$\|x(r_k)\| \leq c_2^k e^{c_1 r_k} \|x_0\|,$$
$$\|x(r_k^-)\| \leq c_2^{k-1} e^{c_1 r_k} \|x_0\| \leq c_2^k e^{c_1 r_k} \|x_0\|. \tag{28}$$

Thus, given $T \in \mathbb{R}_+$ and $(q_0, x_0) \in E$, one has

$$\begin{aligned}
&\mathbb{E}\left[\sum_{r_k \leq T} |W(q(r_k), x(r_k), r_k) - W(q(r_k^-), x(r_k^-), r_k)|\right] \\
&\leq \mathbb{E}\left[\sum_{r_k \leq T} e^{\gamma r_k}\left(x^T(r_k) P_{q(r_k)} x(r_k) + x^T(r_k^-) P_{q(r_k^-)} x(r_k^-)\right)\right] \\
&\underbrace{\leq}_{(28)} \mathbb{E}\left[\sum_{r_k \leq T} e^{\gamma r_k} 2\bar{c}(c_2^k e^{c_1 r_k} \|x_0\|)^2\right] \\
&\leq \mathbb{E}\left[\sum_{r_k \leq T} e^{\gamma T} 2\bar{c}(c_2^k e^{c_1 T} \|x_0\|)^2\right] \\
&= e^{T(\gamma + 2c_1)} 2\bar{c}\|x_0\|^2 \mathbb{E}\left[\sum_{k=0}^{N_T} c_2^{2k}\right] \triangleq C\mathbb{E}\left[\sum_{k=0}^{N_T} c_2^{2k}\right] \\
&= C\sum_{j=0}^{\infty}\left(P[N_T = j]\sum_{k=0}^{j} c_2^{2k}\right) = C\sum_{k=0}^{\infty}\left(c_2^{2k}\sum_{j=k}^{\infty} P[N_T = j]\right) \\
&= C\sum_{k=0}^{\infty}\left(c_2^{2k} P[N_T \geq k]\right) = C\sum_{k=0}^{\infty}\left(c_2^{2k} P[r_k \leq T]\right) < \infty
\end{aligned}$$

where the order of summation was changed in the third-to-last equality, and the last inequality follows from [37, Th. 3.3.1]. Consequently, (13) holds, as we wanted to show, and Theorem 1 can indeed be applied to $W(q, x, t)$ defined by (24) and (26), in which case

$$\mathfrak{U}V(q, x) = 2x^T P_q A_c x + \lambda\left(x^T P_{q+1} x - x^T P_q x\right), \text{if } q < \nu \tag{29a}$$

$$\mathfrak{U}V(q, x) = 2x^T P_\nu A_c x + \lambda\left(\mu_1 g^T(x) P_1 g(x) + \mu_0 x^T P_1 x - x^T P_\nu x\right), \quad \text{if } q = \nu. \tag{29b}$$

Next, inequalities (17) will be used to show that

$$\mathfrak{U}W(q, x, t) \leq 0, \ \forall (q, x, t) \in \mathbb{N}_\nu \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}. \tag{30}$$

Left and right multiplying (17b) by $\operatorname{Diag}(P_\nu, T, I_n, I_n)$ (where $T \triangleq S^{-1}$) and then applying the Schur's complement, one gets

$$\begin{bmatrix} P_\nu A_c + A_c^T P_\nu + (\gamma - \lambda) P_\nu & \star \\ TDK & -2T \end{bmatrix} + \begin{bmatrix} A_d & B_r \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} \lambda\mu_1 P_1 & 0 \\ 0 & \lambda\mu_0 P_1 \end{bmatrix} \begin{bmatrix} A_d & B_r \\ I_n & 0 \end{bmatrix} \preceq 0.$$

Then, left and right multiplying the relation above by $[x^T \ \bar{\phi}^T(Kx)]$ and its transpose, respectively, applying Lemma 1 and combining the resulting inequality to (7c), (25), and (29b), one concludes that (30) holds for all $(q, x, t) \in \{\nu\} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. In a similar manner, it is possible to left and right multiply LMI (17a) by $\operatorname{Diag}(P_q, I_n)$ and, then, to apply the Schur's complement. Finally, using (29a), it follows that (30) holds for all $(q, x, t) \in \mathbb{N}_{\nu-1} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$.

Thus, applying Theorem 1 and relation (30), one has, for $t \geq 0$

$$\begin{aligned}
\mathbb{E}[W(q(t), x(t), t)] &= W(q_0, x_0, 0) + \mathbb{E}\left[\int_0^t \mathfrak{U}W(q(s), x(s), s)\mathrm{d}s\right] \\
&\leq W(q_0, x_0, 0), \ \forall (q_0, x_0) \in E.
\end{aligned} \tag{31}$$

Substituting (24) and (26) in (31), one then gets, for $t \geq 0$

$$e^{\gamma t}\mathbb{E}[x^T(t)P_{q(t)}x(t)] \leq x_0^T P_{q_0} x_0,\ \forall (q_0, x_0) \in E.$$

At last, defining $\underline{c} \triangleq \min_{q\in\mathbb{N}_\nu} \lambda_{\min}(P_q) > 0$ and $c \triangleq \bar{c}/\underline{c}$, we conclude after some algebraic manipulations that $\mathbb{E}[\|x(t)\|^2] \leq ce^{-\gamma t}\|x_0\|^2$ for all initial conditions, i.e., (8) holds, as we wanted to show.

Appendix B
Proof of Theorem 3

The proof of the sufficiency part of the result is analogous to the one of Theorem 2 *mutatis mutandis* and will be omitted. Next, we prove the necessity part. In view of the arguments in Section II-A, we consider system (10) in the proof, i.e., we show that constraints (20) can be satisfied for appropriately chosen matrices Y and $W_q^L, q \in \mathbb{N}_\nu$, if system (10) satisfies (8) for some feedback gain $K \in \mathbb{R}^{m\times n}$ and for all initial conditions.

Consider a function $V : E \to \mathbb{R}_{\geq 0}$ defined by

$$V(q_0, x_0) \triangleq \mathbb{E}_{(q_0,x_0)}\left[\int_0^\infty \|x(s)\|^2 \mathrm{d}s\right] \tag{32}$$

where $\mathbb{E}_{(q_0,x_0)}$ emphasizes that the initial condition considered is $(q(0), x(0)) = (q_0, x_0)$ with probability one (the subscript will be omitted from now on). From (8), $V(q_0, x_0)$ is indeed well defined (i.e., it is finite). More precisely, interchanging expectation with integral operations, one has

$$V(q_0, x_0) = \int_0^\infty \mathbb{E}\left[\|x(s)\|^2\right]\mathrm{d}s \leq \int_0^\infty ce^{-\gamma s}\|x_0\|^2\mathrm{d}s = c\|x_0\|^2/\gamma < \infty.$$

Let us show that $V(q_0, x_0) = x_0^T P_{q_0} x_0$ for appropriately chosen matrices $P_q, q \in \mathbb{N}_\nu$. Note that in the linear case, the solution $x(t)$ of (10) depends linearly on x_0. In other words, (27) reduces to an expression of the form

$$x(t) = \Phi(t, q_0)x_0$$

where the (random) transition matrix of the system $\Phi(t, q_0)$ depends on the initial condition q_0, as explicitly shown in the notation. Thus, substituting the relation above in (32), one obtains

$$V(q_0, x_0) = x_0^T\left(\int_0^\infty \mathbb{E}[\Phi^T(s, q_0)\Phi(s, q_0)]\mathrm{d}s\right)x_0 \triangleq x_0^T P_{q_0} x_0. \tag{33}$$

Let us prove that $P_q = P_q^T$ is positive definite for all $q \in \mathbb{N}_\nu$. Consider again (32) and note that

$$V(q_0, x_0) = \mathbb{E}\left[\int_0^\infty \|x(s)\|^2\mathrm{d}s\right] \geq \mathbb{E}\left[\int_0^{r_1} \|x(s)\|^2\mathrm{d}s\right] = \mathbb{E}\left[\int_0^{r_1} \|e^{A_c s}x_0\|^2\mathrm{d}s\right] \geq \|x_0\|^2\mathbb{E}\left[\int_0^{r_1} e^{-2\|A_c\|s}\mathrm{d}s\right]$$

where we used the fact that $x(t) = e^{A_c t}x_0$ before the first reset time r_1 of (10), and the lower bound for $\|x(t)\|$ comes from [38, Exercise 3.17]. Fix now a constant (deterministic) value $\bar{r} > 0$ and observe that

$$\begin{aligned} V(q_0, x_0) &\geq \|x_0\|^2\mathbb{E}\left[\int_0^{r_1} e^{-2\|A_c\|s}\mathrm{d}s\right] \\ &= \|x_0\|^2\left(\mathbb{E}\left[\int_0^{r_1} e^{-2\|A_c\|s}\mathrm{d}s\Big|r_1 > \bar{r}\right]P[r_1 > \bar{r}] + \mathbb{E}\left[\int_0^{r_1} e^{-2\|A_c\|s}\mathrm{d}s\Big|r_1 \leq \bar{r}\right]P[r_1 \leq \bar{r}]\right) \\ &\geq \|x_0\|^2\mathbb{E}\left[\int_0^{r_1} e^{-2\|A_c\|s}\mathrm{d}s\Big|r_1 > \bar{r}\right]P[r_1 > \bar{r}] \\ &\geq \|x_0\|^2\mathbb{E}\left[\int_0^{\bar{r}} e^{-2\|A_c\|s}\mathrm{d}s\Big|r_1 > \bar{r}\right]P[r_1 > \bar{r}]. \end{aligned}$$

As $r_1 = \rho_0$ by definition, from (12) it follows that $P[r_1 > \bar{r}] = P[\rho_0 > \bar{r}] = e^{-\lambda\bar{r}}$, which implies that

$$V(q_0, x_0) \geq \|x_0\|^2\int_0^{\bar{r}} e^{-2\|A_c\|s}\mathrm{d}s e^{-\lambda\bar{r}} = L\|x_0\|^2 \tag{34}$$

with $L \triangleq e^{-\lambda\bar{r}}\int_0^{\bar{r}} e^{-2\|A_c\|s}\mathrm{d}s > 0$. Comparing (34) and (33), it follows that $P_q \succ 0, \forall q \in \mathbb{N}_\nu$, as claimed.

Note now from (33), (19) and definitions (15) and (16) that

$$\mathfrak{U}V(q, x) = x^T M_q x,\ \forall (q, x) \in E \tag{35}$$

where we replaced (q_0, x_0) by (q, x) and

$$M_q \triangleq \begin{cases} 2P_q A_c + \lambda(P_{q+1} - P_q), & \text{if } q < \nu, \\ 2P_\nu A_c + \lambda(\mu_1 A_d^T P_1 A_d + \mu_0 P_1 - P_\nu), & \text{if } q = \nu. \end{cases} \tag{36}$$

Applying [20, Th. 32.2] to (32), we also know that

$$\mathfrak{U}V(q, x) = -\|x\|^2,\ \forall (q, x) \in E. \tag{37}$$

Combining (35)–(37), one has

$$P_q A_c + A_c^T P_q + \lambda(P_{q+1} - P_q) = -I_n \prec 0,\ \forall q < \nu$$

$$P_\nu A_c + A_c^T P_\nu + \lambda(\mu_1 A_d^T P_1 A_d + \mu_0 P_1 - P_\nu) = -I_n \prec 0.$$

From (19) and the Schur's complement, the inequalities above are equivalent to (20) with $W_q^L \triangleq P_q^{-1}, \forall q \in \mathbb{N}_\nu$, and $Y \triangleq KW_\nu^L$, which ends the proof.

References

[1] K. J. Aström and B. Wittenmark, *Computer Controlled Systems. Theory and Design* (Information and System Sciences Series). Englewood Cliffs, NJ, USA: Prentice-Hall, 1984.

[2] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Syst. Mag.*, vol. 21, no. 1, pp. 84–99, Feb. 2001.

[3] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE Proc. IRE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[4] L. Hetel et al., "Recent developments on the stability of systems with aperiodic sampling: An overview," *Automatica*, vol. 76, pp. 309–335, 2017.

[5] E. Fridman, "A refined input delay approach to sampled-data control," *Automatica*, vol. 46, no. 2, pp. 421–427, 2010.

[6] A. Seuret, "A novel stability analysis of linear systems under asynchronous samplings," *Automatica*, vol. 48, no. 1, pp. 177–182, 2012.

[7] L. Hetel, A. Kruszewski, W. Perruquetti, and J.-P. Richard, "Discrete and intersample analysis of systems with aperiodic sampling," *IEEE Trans. Autom. Control*, vol. 56, no. 7, pp. 1696–1701, 2011.

[8] W. Lombardi, S. Olaru, S.-I. Niculescu, and L. Hetel, "A predictive control scheme for systems with variable time-delay," *Int. J. Control*, vol. 85, no. 7, pp. 915–932, 2012.

[9] H. Fujioka, "A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices," *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2440–2445, Oct. 2009.

[10] H. Fujioka and Y. Oishi, "A switched Lyapunov function approach to stability analysis of non-uniformly sampled-data systems with robust LMI techniques," in *Proc. 8th Asian Control Conf.*, 2011, pp. 1487–1491.

[11] D. D. Huff, M. Fiacchini, and J. M. Gomes da Silva Jr, "Stability and stabilization of aperiodic sampled-data systems subject to control input saturation: A set invariant approach," *IEEE Trans. Autom. Control*, vol. 67, no. 3, pp. 1423–1429, Mar. 2022.

[12] A. Tanwani, D. Chatterjee, and D. Liberzon, “Stabilization of deterministic control systems under random sampling: Overview and recent developments,” *Uncertainty Complex Netw. Syst.*, pp. 209–246, 2018.
[13] B. Shen, Z. Wang, and T. Huang, “Stabilization for sampled-data systems under noisy sampling interval,” *Automatica*, vol. 63, pp. 162–166, 2016.
[14] B. Shen, H. Tan, Z. Wang, and T. Huang, “Quantized/saturated control for sampled-data systems under noisy sampling intervals: A confluent Vandermonde matrix approach,” *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4753–4759, Sep. 2017.
[15] Z. Hu, H. Ren, F. Deng, and H. Li, “Stabilization of sampled-data systems with noisy sampling intervals and packet dropouts via a discrete-time approach,” *IEEE Trans. Autom. Control*, vol. 67, no. 6, pp. 3204–3211, Jun. 2022.
[16] H. Sun, J. Sun, and J. Chen, “Analysis and synthesis of networked control systems with random network-induced delays and sampling intervals,” *Automatica*, vol. 125, 2021, Art. no. 109385.
[17] D. J. Antunes, J. P. Hespanha, and C. J. Silvestre, “Volterra integral approach to impulsive renewal systems: Application to networked control,” *IEEE Trans. Autom. Control*, vol. 57, no. 3, pp. 607–619, Mar. 2012.
[18] D. D. Huff, M. Fiacchini, and J. M. Gomes da Silva Jr, “Necessary and sufficient convex condition for the stabilization of linear sampled-data systems under poisson sampling process,” *IEEE Control Syst. Lett.*, vol. 6, pp. 3403–3408, 2022.
[19] D. J. Antunes, J. P. Hespanha, and C. J. Silvestre, “Stability of networked control systems with asynchronous renewal links: An impulsive systems approach,” *Automatica*, vol. 49, no. 2, pp. 402–413, 2013.
[20] M. Davis, *Markov Models & Optimization*. London, U.K.: Chapman & Hall/CRC Monographs on Statistics and Applied Probability, 1993, vol. 49.
[21] A. R. Teel, A. Subbaraman, and A. Sferlazza, “Stability analysis for stochastic hybrid systems: A survey,” *Automatica*, vol. 50, no. 10, pp. 2435–2456, 2014.
[22] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes (International Edition)*, 4th ed. Maidenhead, U.K.: McGraw-Hill Education, 2002.
[23] S. Tarbouriech, G. Garcia, J. M. Gomes da Silva Jr, and I. Queinnec, *Stability and Stabilization of Linear Systems With Saturating Actuators*. Berlin, Germany: Springer, 2011.
[24] P. Billingsley, *Probability and Measure*, 3rd ed. Hoboken, NJ, USA: Wiley, 1995.
[25] M. Souza, M. De Almeida, A. R. Fioravanti, and O. L. V. Costa, “H2 output-feedback cluster control for continuous semi-Markov jump linear systems with erlang dwell times,” *IEEE Control Syst. Lett.*, vol. 7, pp. 109–114, 2023.
[26] J. P. Hespanha, “Modeling and analysis of networked control systems using stochastic hybrid systems,” *Annu. Rev. Control*, vol. 38, no. 2, pp. 155–170, 2014.
[27] D. J. Antunes, “Frequency-domain analysis of aperiodic control loops with identically distributed delays,” *Automatica*, vol. 151, 2023, Art. no. 110626.
[28] S. Asmussen, *Applied Probability and Queues*, 2nd ed. Berlin, Germany: Springer, 2003.
[29] G. Grimm, J. Hatfield, I. Postlethwaite, A. R. Teel, M. C. Turner, and L. Zaccarian, “Antiwindup for stable linear systems with input saturation: An LMI-based synthesis,” *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1509–1525, Sep. 2003.
[30] D. Dai, T. Hu, A. R. Teel, and L. Zaccarian, “Output feedback synthesis for sampled-data system with input saturation,” in *Proc. Amer. Control Conf.*, Baltimore, USA, 2010, pp. 1797–1802.
[31] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, USA: SIAM Studies in Applied Mathematics, 1994.
[32] J. F. Sturm, “Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones,” *Optim. Methods Softw.*, vol. 11, no. 1–4, pp. 625–653, 1999.
[33] J. Löfberg, “YALMIP: A toolbox for modeling and optimization in MATLAB,” in *Proc. CACSD Conf.*, Taipei, Taiwan, 2004, pp. 284–289.
[34] S. Asmussen, O. Nerman, and M. Olsson, “Fitting phase-type distributions via the EM algorithm,” *Scand. J. Statist.*, vol. 23, no. 4, pp. 419–441, 1996.
[35] F. Ferrante, F. Gouaisbaut, R. G. Sanfelice, and S. Tarbouriech, “State estimation of linear systems in the presence of sporadic measurements,” *Automatica*, vol. 73, pp. 101–109, 2016.
[36] A. Sferlazza, S. Tarbouriech, and L. Zaccarian, “Time-varying sampled-data observer with asynchronous measurements,” *IEEE Trans. Autom. Control*, vol. 64, no. 2, pp. 869–876, Feb. 2019.
[37] S. I. Resnick, *Adventures in Stochastic Processes*. Birkhäuser Boston, 1992.
[38] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, NJ, USA: Prentice-Hall, 2002.