Chapter 2

# Exponential stability for hybrid systems with nested saturations

## 2.1. Introduction

Hybrid systems are systems with both continuous-time and discrete-time dynamics. Recently, the interest on hybrid systems has been growing, see [BRA 98, LIB 03, GOE 04, SUN 05, PRI 07, GOE 09, GOE 12], due to the increasing application of digital devices for the control of real systems, like chemical processes, communications and automotive systems, and also for their flexibility, which allows to overcome some fundamental limitations of classical control [BEK 04, PRI 10, FIC 12b, FIC 12c, FIC 12a, PRI 13].

We consider here the problem of characterizing both local and global exponential stability for hybrid systems with nested saturations. The proposed method is based on set-theory and invariance and provides computation-oriented conditions for determining estimations of the domain of attraction for this class of nonlinear hybrid systems. Set-theory and invariance in control have been widely employed in recent years to characterize the stability properties of linear and nonlinear systems, see [BER 72, GUT 86, GIL 91, BLA 94, KOL 98, BLA 99] and the monograph [BLA 08]. The peculiarity of this approach is that convex analysis and optimization techniques can be often employed to compute the Lyapunov functions and the estimations of the domain of attraction. For instance, the issue of estimating the domain of attraction for saturated systems, in continuous-time and discrete-time, has been dealt with considering ellipsoids, see [Gom 01, HU 02a, HU 02b, ALA 05], and polytopes, in [ALA 06].

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A first contribution of this work is the geometrical characterization of saturated functions. Parameterized set valued maps which are local extensions of the saturated and nested saturated functions are given. Such results permit to characterize contractivity of ellipsoids and to determine quadratic Lyapunov functions candidates by means of convex constraints. Some results present in literature for continuous-time, as [HU 02a, ALA 05], and discrete-time saturated systems, see [HU 02b], are improved or recovered as particular cases of our approach, see [FIA 11a]. The results are applied here also to obtain computationally suitable conditions for local and global asymptotic stability for hybrid systems with simple and nested saturations. Such conditions result in convex optimization problems and provide also ellipsoidal estimations of the domain of attraction, see also [FIA 12b]. On the other hand, as the resulting quadratic function is allowed to increase during the jump, the standard conditions for exponential stability are not satisfied, see [TEE 11]. Nevertheless the solution of the proposed convex problem is proved to ensure exponential stability for hybrid systems with simple and nested saturation. Moreover, a class of exponential Lyapunov functions related to the quadratic one is characterized. Finally, the computation-oriented conditions for local and global exponential stability are applied to numerical examples of saturated hybrid systems.

*Notation.* Given $n \in \mathbb{N}$, denote $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. The nonnegative real are denoted $\mathbb{R}_+$ Given $A \in \mathbb{R}^{n \times m}$, $A_i$ with $i \in \mathbb{N}_n$ denotes its $i$-th row, $A_{(j)}$ with $j \in \mathbb{N}_m$ its $j$-th column and $A_{i,j}$ the entry of the $i$-th row and $j$-th column of $A$. The identity matrix of order $n$ is denoted $I_n$, the null $m \times n$ matrix is $0_{m\times n}$. Given the matrix $P = P^T > 0$, define the ellipsoid $\mathscr{E}(P) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$. Given $D, E \subseteq \mathbb{R}^n$ the Minkowski set addition is defined as $D + E = \{z = x + y \in \mathbb{R}^n : x \in D,\ y \in E\}$. Given the set $D$ and $\alpha \geq 0$, denote the set $\alpha D = \{\alpha x :\ x \in D\}$, $\mathrm{co}(D)$ is its convex hull, $\mathscr{S}(D)$ are the subsets of $D$, $\mathscr{K}(D)$ are the convex compact subsets of $D$ and $\mathscr{K}^0(D)$ are the convex compact subsets of $D$ with $0 \in \mathrm{int}\,(D)$. Given the finite set $J \subseteq \mathbb{N}_m$, we denote $\bar{J} = \mathbb{N}_m \backslash J$ with $m \in \mathbb{N}$. The symbol $*$ stands for symmetric block.

### 2.2. Problem statement

Consider the closed-loop saturated hybrid system, represented by using the hybrid framework introduced in [ZAC 05, NES 08], whose continuous-time dynamics is given by

$$\left\{ \begin{array}{l} \dot{x} = \hat{g}(x) = \hat{A}x + \hat{B}\varphi(\hat{K}x), \\ \dot{\tau} = 1, \end{array} \right. \tag{2.1}$$

valid if $(x, \tau) \in \mathscr{G}$, where $x \in \mathbb{R}^n$ is the state, and the discrete-time dynamics is

$$\left\{ \begin{array}{l} x^+ = \tilde{g}(x) = \tilde{A}x + \tilde{B}\varphi(\tilde{K}x), \\ \tau^+ = 0, \end{array} \right. \tag{2.2}$$

if $(x, \tau) \in \mathscr{J}$. Regions $\mathscr{G}$ and $\mathscr{J}$ are referred to as the flow and jump sets, respectively. Function $\varphi : \mathbb{R}^a \to \mathbb{R}^a$ denotes the saturation, i.e. $\varphi_i(y) = \mathrm{sgn}(y_i)\min\{|y_i|, 1\}$, for every $i \in \mathbb{N}_a$, with $y \in \mathbb{R}^a$. The saturation bounds can be considered equal to 1, without loss of generality. Sets $\mathscr{F}$ and $\mathscr{J}$ are assumed to be defined as

$$\begin{aligned} \mathscr{G} &= \{(x,\tau) \in \mathbb{R}^{n+1} : \; x^T M x \geq 0, \;\text{ or }\; \tau < \rho\}, \\ \mathscr{J} &= \{(x,\tau) \in \mathbb{R}^{n+1} : \; x^T M x \leq 0, \;\text{ and }\; \tau \geq \rho\}, \end{aligned} \tag{2.3}$$

where $M = M^T \in \mathbb{R}^{n \times n}$ and $\rho \geq 0$, as in [GRO 93]. Different kinds of flow and jump regions can be defined by (2.3), like the reset conditions used in reset control as studied in [ZAC 05, NES 08, TAR 11a]. Furthermore, choosing $M = M^T > 0$ (or $M = M^T < 0$), the formulation (2.3) permits to restrict the dynamics to a continuous-time (resp. discrete-time) system, see also [FIA 11a].

**Remark 1** *The variable $\tau$ represents the time passed from the last jump. Its introduction, together with the parameter $\rho \geq 0$, permits to define a lower bound on the time interval between two successive jumps. The presence of such a bound, which will be referred to as "temporal regularization", can be used to prevent having an infinite number of jumps in a finite time interval, i.e. Zeno solutions [GOE 04], which should be avoided in real applications. Notice that conditions on the state $x$ ensuring the system flowing for a certain amount of time, used in some applications of hybrid systems theory, consist in determining implicitly a positive value of $\rho$. Hereafter the knowledge of the value of $\rho$ will be used to allow the potential Lyapunov function to increase during a jump. This leads to more general results than those obtained imposing its decreasing during both the flow and the jumps. This case can be recovered by posing $\rho = 0$.*

The presence of nested saturations are also considered to obtain a more general model (see [TAR 06]). In fact, the presence of a further saturation between the plant output and the controller input is a realistic assumption, considering that bounds on the measurements are often present. In this case, the continuous-time dynamics of the hybrid system becomes

$$\begin{cases} \dot{x} = \hat{g}(x) = \hat{A}x + \hat{B}\varphi(\hat{K}x + \hat{E}\varphi(\hat{F}x)), \\ \dot{\tau} = 1, \end{cases} \tag{2.4}$$

and, analogously, the discrete-time dynamics is

$$\begin{cases} x^+ = \tilde{g}(x) = \tilde{A}x + \tilde{B}\varphi(\tilde{K}x + \tilde{E}\varphi(\tilde{F}x)) \\ \tau^+ = 0. \end{cases} \tag{2.5}$$

The objectives can be summarized as follows.

**Problem 1** *Given the flow and jump sets, $\mathscr{G}$ and $\mathscr{J}$, determine an ellipsoidal region $\Omega = \mathscr{E}(P)$, with $P = P^T > 0$, as large as possible, such that the origin is locally exponentially stable for the saturated hybrid system (2.1)-(2.3), or for the hybrid system with nested saturations (2.3)-(2.5), within $\Omega$.*

In order to develop generic conditions, set-theory will be exploited to deal with hybrid systems. Some of the employed properties related to set-theory and invariance are presented in Section 2.3.

### 2.2.1. *Saturated reset systems*

A particularly interesting subclass of saturated hybrid systems is given by reset systems. Consider the following plant

$$\begin{cases} \dot{x}_p &= A_p x_p + B_p u_p, \\ y_p &= C_p x_p, \end{cases} \tag{2.6}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state, $y_p \in \mathbb{R}^p$ is the output and $u_p \in \mathbb{R}^{m_c}$ is the input of the plant.

**Remark 2** *The plant is assumed to have pure continuous-time dynamics, as in the classical reset systems framework. Nothing prevents to consider the more general case of a plant with hybrid nature, provided that the flow and jump sets of the overall closed-loop reset system can be expressed as in (2.3).*

Associated to system (2.6), we consider a hybrid controller whose state is $x_c \in \mathbb{R}^{n_c}$. The controller is described by continuous-time dynamics

$$\begin{cases} \dot{x}_c &= A_c x_c + B_c u_c, \\ y_c &= C_c x_c + D_c u_c, \\ \dot{\tau} &= 1, \end{cases} \tag{2.7}$$

if $(x_p, x_c, \tau) \in \mathscr{F}$, and

$$\begin{cases} x_c^+ &= A_d x_c + B_d u_d, \\ \tau^+ &= 0, \end{cases} \tag{2.8}$$

if $(x_p, x_c, \tau) \in \mathscr{J}$, where $x_c \in \mathbb{R}^{n_c}$ is the state of the controller at time $t$, $x_c$ the state after a jump and $y_c \in \mathbb{R}^{m_c}$ is the controller output. Variables $u_c \in \mathbb{R}^p$ and $u_d \in \mathbb{R}^{m_d}$ are the inputs of the continuous-time and the discrete-time dynamics of the controller, respectively. The signal $u_d$ is function of the controller state.

We first suppose that magnitude limitations on the plant input and on the discrete-time dynamics input are present. Such assumptions are modeled by introducing saturations on the inputs, that is

$$u_p = \varphi(y_c), \qquad u_d = \varphi(x_c). \tag{2.9}$$

The controller input is the plant output, that is $u_c = y_p$. The continuous-time controller (2.7) is supposed to stabilize system (2.6), in absence of the saturation of the plant input, i.e. with $u_p = y_c$.

**Remark 3** *Classical reset systems, whose discrete-time dynamics consists essentially in setting the state of the controller to the value of* $0$*, are recovered by posing* $A_d = 0_{n_c \times n_c}$*,* $B_d = 0_{n_c \times m_d}$*. Our will is to consider a more general problem, whose solution could apply to a wider class of systems, as in [PRI 10], and then also to reset systems as a particular case.*

Considering the state vector defined as $x = (x_p, x_c) \in \mathbb{R}^n$, where $n = n_p + n_c$, the overall closed-loop saturated hybrid system is then given by (2.1)-(2.2) with

$$\begin{aligned}
\hat{A} &= \begin{bmatrix} A_p & 0_{n_p \times n_c} \\ B_c C_p & A_c \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_p \\ 0_{n_c \times m_c} \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} D_c C_p & C_c \end{bmatrix}, \\
\tilde{A} &= \begin{bmatrix} I_{n_p} & 0_{n_p \times n_c} \\ 0_{n_c \times n_p} & A_d \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0_{n_p \times n_c} \\ B_d \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0_{n_c \times n_p} & I_{n_c} \end{bmatrix}.
\end{aligned}$$

If also saturations on the plant outputs are present, i.e.

$$u_c = \varphi(y_p),$$

then nested saturation are present and the system is given as in (2.4)-(2.5). Consider, in fact, the continuous-time dynamics of the plant and the controller

$$\left\{ \begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_p \varphi(y_c), \\ y_p &=& C_p x_p, \end{array} \right. \qquad \left\{ \begin{array}{rcl} \dot{x}_c &=& A_c x_c + B_c \varphi(y_p), \\ y_c &=& C_c x_c + D_c \varphi(y_p). \end{array} \right.$$

where the dynamics of $\tau$ has been neglected. Notice that the nested saturation appears when the expression of $y_c$ is used in the plant dynamics and then we have that the overall continuous-time system is

$$\left\{ \begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_p \varphi\big(C_c x_c + D_c \varphi(C_p x_p)\big), \\ \dot{x}_c &=& A_c x_c + B_c \varphi(C_p x_p). \end{array} \right.$$

Hence, posing $x = (x_p, x_c)$, we have that the continuous-time dynamics of the system has the form (2.4) with:

$$\hat{A} = \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_p & 0 \\ 0 & B_c \end{bmatrix},$$
$$\hat{K} = \begin{bmatrix} 0 & C_c \\ C_p & 0 \end{bmatrix}, \quad \hat{E} = \begin{bmatrix} D_c \\ 0 \end{bmatrix}, \quad \hat{F} = \begin{bmatrix} C_p & 0 \end{bmatrix}.$$

Analogous definitions of matrices $\tilde{A}, \tilde{B}, \tilde{K}, \tilde{E}$ and $\tilde{F}$ lead to the representation of the discrete-time dynamics with nested saturations (2.5). Considering for instance a saturated reset as discrete-dynamics, as for the examples analyzed in Section 2.6, that is

$$\begin{cases} x_p^+ &= x_p, \\ x_c^+ &= x_c + \varphi(-x_c), \end{cases}$$

we have that it is equivalent to equation (2.5) (or more simply equation (2.2) since no nested saturation affects the discrete-time dynamics) with

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\tilde{K} = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Clearly, the representation (2.3)-(2.5) encloses also the case of simple saturation (2.1)-(2.3).

For reset systems, the output of the plant and the output of the controller are assumed to be one-dimensional, i.e. $p = m_c = 1$, and the jump depends on the sign of their product, see [ZAC 05, LOQ 07]. Then, in this case, $\mathscr{G}$ and $\mathscr{J}$ are given by (2.3) with $M = C^T T C$ where

$$T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad C = \begin{bmatrix} C_p & 0_{1\times n_c} \\ D_c C_p & C_c \end{bmatrix} \in \mathbb{R}^{2\times n}.$$

## 2.3. Set theory and invariance for nonlinear systems: brief overview

The concept of invariance has become fundamental for the analysis and design of control systems. The importance of invariant sets in control is due to stability and robustness implicit properties of these regions of the state space. An invariant set for a given dynamic system is a region of the state space such that the trajectory generated by the system remains confined in the set if the initial condition lies within it.

A notable pioneering contribution on invariance for dynamic systems is [BER 72]. Many well established results regarding invariance and related topics have been provided in literature: for instance on the maximal invariant set contained in a set, see [GUT 86, GIL 91, BLA 94, KOL 98, BLA 99]; and the minimal invariant set, see

[RAK 05, ONG 06]. A first important survey paper on invariance is [BLA 99], followed by the monograph [BLA 08], which gathers many of the results presented up to the actuality on invariance and set-theory in control. Invariance is also widely employed to ensure convergence of model predictive control, see [MAY 00].

Although there are many results which can be used for characterizing and computing invariant sets for linear systems, in the case of nonlinear systems few general results are available. Methods for obtaining ellipsoidal and parallelotopic invariant sets for nonlinear model predictive control, are proposed in [MAG 01, CAN 03], using LDI (Linear Difference Inclusions). The computation of ellipsoidal invariant sets for linear systems with particular static nonlinear feedbacks, such as piecewise affine and saturation, has been addressed in the works [Gom 99, HU 04, TAR 11b]. Methods to obtain polytopic invariant sets are proposed for saturated systems, [ALA 06] and for Lur'e systems, [ALA 09]. The problem of computing polytopic invariant sets for general nonlinear systems is addressed in [BRA 05], using interval arithmetic, and in [FIA 10b], using DC functions (i.e. expressible as the difference of convex functions). Nevertheless, there is still a clear gap between the importance of invariance in control and systems analysis and the availability of practical invariant sets computation methods. More recently, an approach based on convexity and difference inclusions has been proposed for characterizing invariance for nonlinear systems, see [FIA 10a, FIA 12a]. The underlining ideas of such an approach, developed mainly for discrete-time nonlinear systems and recalled here, are employed in the following sections to characterize invariance, contractivity and exponential stability for hybrid systems with nested saturations.

### 2.3.1. *Invariance for Convex Difference Inclusions*

A modeling framework for representing and approximating nonlinear and uncertain discrete-time systems has been introduced in [FIA 10a, FIA 12a]. The systems taken into account are named Convex Difference Inclusions (CDI) systems and are characterized by a particular class of set valued maps as dynamic functions. CDI systems are tightly related to differential and difference inclusions. A deep and exhaustive analysis of such models, and of their properties, is provided in the works of Aubin and co-authors, see [AUB 84, AUB 90, AUB 91]. The set valued map determining a CDI system is bounded by a set of convex functions and such that, given a point in the state space, its image through the map is a convex and compact set. Let the system be

$$x^+ \in \mathscr{G}(x), \tag{2.10}$$

where $x \in \mathbb{R}^n$ is the state, $x^+$ is the successor and $\mathscr{G}(\cdot)$ is a set valued map on $\mathbb{R}^n$, that is a function which relates a set to every point $x \in \mathbb{R}^n$. In particular we consider set valued dynamic functions such that $\mathscr{G}(x) \in \mathscr{K}(\mathbb{R}^n)$, for any $x \in \mathbb{R}^n$, and the graph of $\mathscr{G}(\cdot)$ is determined by a set of functions convex with respect to $x$, as stated below.

**Assumption 1** *Given the set valued map $\mathscr{G} : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ determining the system dynamics (2.10) and considering the function $\check{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as*

$$\check{F}(x, \eta) = \sup_{z \in \mathscr{G}(x)} \eta^T z, \tag{2.11}$$

*assume that $\check{F}(\cdot, \eta)$ is convex on $\mathbb{R}^n$ and $\check{F}(0, \eta) = 0$, for all $\eta \in \mathbb{R}^n$.*

We provide here the definition of support function, a useful tool when dealing with convex closed sets.

**Definition 1** *Given $D \subseteq \mathbb{R}^n$, the support function of $D$ at $\eta \in \mathbb{R}^n$ is*

$$\phi_D(\eta) = \sup_{x \in D} \eta^T x.$$

Among the properties of support function, see [ROC 70, SCH 93], we have that set inclusion conditions can be given in terms of linear inequalities involving the support functions.

**Property 1** *Given a closed, convex set $D \subseteq \mathbb{R}^n$, then $x \in D$ if and only if $\eta^T x \leq \phi_D(\eta)$, for all $\eta \in \mathbb{R}^n$. Given also $C \subseteq \mathbb{R}^n$, then $C \subseteq D$ if and only if $\phi_C(\eta) \leq \phi_D(\eta)$, for all $\eta \in \mathbb{R}^n$.*

Notice that, under Assumption 1 and for any $x \in \mathbb{R}^n$, the value $\check{F}(x, \eta)$ is the support function at $\eta \in \mathbb{R}^n$ of the set $\mathscr{G}(x)$ and is convex with respect to $x$. Furthermore, by convexity and compactness of $\mathscr{G}(x)$ for every $x \in \mathbb{R}^n$, we have that

$$\mathscr{G}(x) = \{z \in \mathbb{R}^n : \ \eta^T z \leq \check{F}(x, \eta), \forall \eta \in \mathbb{R}^n\}.$$

An alternative definition of CDI systems could be given in terms of the Minkowski set addition.

**Proposition 1** *The set valued map $\mathscr{G}(\cdot)$ determining the system dynamics (2.10) satisfies Assumption 1 if and only if $\mathscr{G} : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ is such that*

$$\mathscr{G}(\alpha x^1 + (1-\alpha)x^2) \subseteq \alpha \mathscr{G}(x^1) + (1-\alpha)\mathscr{G}(x^2),$$

*for every $\alpha \in [0, 1]$ and every $x^1, x^2 \in \mathbb{R}^n$, and $\mathscr{G}(0) = \{0\}$.*

A set valued map $\mathscr{G} : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ is a local extension of function $f : \mathbb{R}^n \to \mathbb{R}^n$ on $D \subseteq \mathbb{R}^n$ if

$$f(x) \in \mathscr{G}(x), \quad \forall x \in D.$$

From properties of the support functions, if $\mathscr{G}$, extension of $f$ on $D$ is such that $\mathscr{G}(x)$ is closed and convex for $x \in D$, then

$$\eta^T f(x) \le \check{F}(x, \eta), \quad \forall \eta \in \mathbb{R}^n,$$

for all $x \in D$.

**Corollary 1** *Let Assumption 1 hold for a given map $\mathscr{G}$. Function $\check{F}(\cdot,\cdot)$ as in (2.11), are convex with respect to $x$ and such that*

$$\eta^T f(x) \le \check{F}(x, \eta),$$

*for every $x \in D$ and $\eta \in \mathbb{R}^n$ and every $f$ such that $\mathscr{G}$ is an extension on $D \subseteq \mathbb{R}^n$.*

The fact that Assumption 1 holds for the dynamic function of a system allows us to exploit features inherited by properties of convex functions and convex sets. Some useful properties are listed below.

– Set relations, such as set inclusion, involving the image of a state $x$ through the set valued map, i.e., $\mathscr{G}(x)$, for any $x \in \mathbb{R}^n$, can often be posed as a set of convex constraints. For systems as in (2.10) and under Assumption 1, condition of inclusion of the successor state can be imposed through a set of convex constraints, which can yield to convex problems, efficiently solvable, see [ROC 70, BEN 01, BOY 04].
– Convexity related properties of the dynamic set valued function, in particular convexity of functions $\check{F}(\cdot, \eta)$, for all $\eta \in \mathbb{R}^n$, permits to infer features shared by all the elements of a set by means of conditions involving only a subset of elements, possibly finite.
– Assuming that the effect of the parametric uncertainty or the nonlinearity are bounded by convex functions is not very restrictive. The family of dynamic systems under analysis encloses a large class of functions. Many methods to approximate nonlinear systems lead to systems with a structure that can be reduced to CDI systems, as defined in (2.10). This means that, given a generic system defined by a real valued function $f(\cdot)$, it is often possible to determine a CDI system with function $\mathscr{G}(\cdot)$ for which Assumption 1 holds and $\mathscr{G}(\cdot)$ is an extension of $f(\cdot)$. Therefore, any invariant set for the approximating CDI system is also an invariant set for the nonlinear one.

– In the case where the system presents a form of CDI systems as in (2.10), with Assumption 1, the results presented are quite strong: the maximal invariant set, for instance, can be well approximated. Recall that computation of the maximal (robust) invariant set can be an hard task also for linear systems, for nonlinear systems few general results have been provided in literature.

In [FIA 10a, FIA 12a], it is proved that, many desirable properties, typical of linear systems, are valid also in the context of CDI ones. The main results are briefly recalled hereafter.

First, it can be proved that, as for linear systems, necessary and sufficient conditions for invariance and $\lambda$-contractivity of convex sets exists, for CDI systems. In particular, such conditions are given by convex constraints. Moreover, they are boundary conditions, that is, they involve only the elements on the boundary, just a finite number of points (the vertices) in case of polytopic sets. Recall that such very desirable properties do not hold for generic nonlinear systems.

**Theorem 1 ([FIA 12a])** *Let Assumption 1 hold for the set-valued map $\mathscr{G}(\cdot)$ determining the system dynamics (2.10) with state constraint set $X$ convex, closed and $0 \in int\,(X)$. Given $\lambda \in [0,1]$, a set $\Omega \in \mathscr{K}^0(X)$ is a contractive set for system (2.10) if and only if*

$$\check{F}(x,\eta) \leq \lambda \phi_\Omega(\eta), \quad \forall x \in \partial\Omega, \quad \forall \eta \in \mathbb{R}^n. \tag{2.12}$$

The necessary and sufficient condition is given by a set of convex constraints, involving only the boundary of the set $\Omega$. Moreover, as for the linear systems, every contractive set induces a local Lyapunov function, since the contractivity of $\Omega$ implies the contractivity of $\alpha\Omega$ for all $\alpha \in [0,1]$.

**Proposition 2 ([FIA 12a])** *Let Assumption 1 hold for the set-valued map $\mathscr{G}(\cdot)$ determining the system dynamics (2.10) with state constraint set $X$ convex, closed and $0 \in$ int$(X)$. Every contractive set $\Omega \in \mathscr{K}^0(X)$ with contracting factor $\lambda \in [0,1)$ induces local Lyapunov function in $\mathscr{S}(\Omega)$ for the system (2.10).*

Hence, the convexity conditions given by Assumption 1 permits one to extend many properties valid in linear context to the nonlinear one. It is also important to stress that many nonlinear systems admit CDI representations or can be approximated by CDI systems, see [FIA 10a, FIA 12a].

– Every system $x^+ = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ twice differentiable in $D = \{x \in \mathbb{R}^n : \|x - x_0\|_2 < r\}$, with $r > 0$, admits a CDI approximation determined by an extension of $f$. Any invariant, contractive set and local Lyapunov function in $D$ for the CDI system, is so also for the nonlinear one.

– A popular way of approximating nonlinear and uncertain systems is given by Linear Difference Inclusion (LDI) systems, see [BOY 04, GUR 95]. The LDI systems form a subclass of the CDI ones, in particular of those whose convex bounding functions are piecewise linear. Hence, using an LDI system to approximate a nonlinear one is a way of generating a CDI extension. Nonetheless CDI provides a more general modeling framework, as not every CDI system admits an LDI representation.
– Generalized saturated systems, introduced in [TAR 11b], are a family of systems including many common static nonlinearities and are easily extendible by CDI systems. A linear system in closed-loop with a (possibly time-varying) static function $\varphi(y,k)$ such that

$$-\Gamma(-y) \leq \varphi(y,k) \leq \Gamma(y), \quad \forall y \in \mathbb{R}^p,\ \forall k \in \mathbb{N}, \tag{2.13}$$

where $\Gamma(y) = \max\{\mu(y+\sigma),\ -y_0\}$ and $k \in \mathbb{N}$ is a generalized saturated system. Such functions permit to represent common static nonlinear functions as saturation plus dead-zone, hysteresis, saturation etc.

This means that the results valid for CDI systems can be used to obtain invariant sets and contractive sets for a wide class of nonlinear systems. As a matter of fact, the analysis of a CDI system can be considered as the analysis of families of systems, since any nonlinear system bounded by a CDI one shares important invariance related properties with the CDI system.

## 2.4. Quadratic stability for saturated hybrid systems

In this section it is shown, first, that the image of the state $x \in \mathbb{R}^n$ through a saturated function $g(x)$ is contained within a set explicitly obtainable. The resulting set valued map is proved to satisfy the properties required for determining CDI systems, see Assumption 1 and Proposition 1. This result permits one to geometrically characterize quadratic stability for saturated hybrid systems, as well as for continuous-time and discrete-time systems.

### 2.4.1. *Set valued extensions of saturated functions*

The following theorem is enunciated for nonlinear functions of the type $g(x) = Ax + B\varphi(Kx)$, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $K \in \mathbb{R}^{m\times n}$. The theorem can be employed to prove results for both the continuous-time and the discrete-time dynamics, and then applied to hybrid systems.

**Theorem 2** *Given a function $g(x) = Ax + B\varphi(Kx)$, the ellipsoid $\Omega = \mathscr{E}(P)$, with $P \in \mathbb{R}^{n\times n}$ and $P = P^T > 0$, and $H(i,J) \in \mathbb{R}^{1\times n}$ such that $|H(i,J)x| \leq 1$ for all $x \in \Omega$, for* *every $J \subseteq \mathbb{N}_m$ and every $i \in J$, then we have $g(x) \in G(x)$ for all $x \in \Omega$, where*

$$G(x) = co(\{N(J)x \in \mathbb{R}^n : J \subseteq \mathbb{N}_m\}),$$

*and*

$$N(J) = A + \sum_{i \in \bar{J}} B_{(i)} K_i + \sum_{i \in J} B_{(i)} H(i,J). \tag{2.14}$$

**Proof:** See [FIA 12b] for the proof. □

The meaning of Theorem 2 is that, for all $x \in \Omega$, the image $g(x)$ is contained in the polytope $G(x)$, whose vertices are known.

**Remark 4** *Among the set of matrices $N(J)$, with $J \subseteq \mathbb{N}_m$, there are some which can be neglected. In fact, any of them represent a combination of saturations on the plant inputs. As proved in [FIA 11b], not every combination is admissible but only the subsets $J \in \mathscr{N}(\Omega)$ where*

$$\mathscr{N}(\Omega) = \{J \subseteq \mathbb{N}_m : \ \exists x \in \Omega, \eta \in \mathbb{R}^n \ s.t.\ i \in J \ \Leftrightarrow \ \eta^T B_{(i)} K_i x < -|\eta^T B_{(i)}|\} \cup \{\emptyset\},$$

*are combination of saturated inputs that occur in at least a $x \in \Omega$. Then, the computational complexity can be reduced by removing the unnecessary $J$ from the analysis. In [FIA 11b] it is proved also that such complexity reduction might be remarkable in some case. Analogous considerations could be done concerning the combinations of saturated plant inputs and outputs and then also in the case of nested saturated systems the computational complexity could be reduced.*

We consider now the functions that present nested saturations, that is $g(x) = Ax + B\varphi(Kx + E\varphi(Fx))$, with $E \in \mathbb{R}^{m\times p}$ and $F \in \mathbb{R}^{p\times n}$. The analysis applies then to both discrete-time and continuous-time linear systems with nested saturations, as well as to hybrid ones.

**Theorem 3** *Given a function $g(x) = Ax + B\varphi(Kx + E\varphi(Fx))$, consider the ellipsoid $\Omega = \mathscr{E}(P)$, with $P \in \mathbb{R}^{n\times n}$ and $P = P^T > 0$, $H(j,J) \in \mathbb{R}^{1\times n}$ such that $|H(j,J)x| \leq 1$ for every $J \subseteq \mathbb{N}_m$ and $j \in J$, $L(i,I(k)) \in \mathbb{R}^{1\times n}$ such that $|L(i,I(k))x| \leq 1$ for every $k \in \mathbb{N}_m$, every $I(k) \subseteq \mathbb{N}_p$ and $i \in I(k)$, for all $x \in \Omega$. Then we have $g(x) \in S(x)$ for all $x \in \Omega$, where*

$$S(x) = co(\{Q(J,\boldsymbol{I})x \in \mathbb{R}^n : J \subseteq \mathbb{N}_m, I(k) \subseteq \mathbb{N}_p, k \in \mathbb{N}_m\}),$$

*where $\boldsymbol{I} = \{I(1), I(2), \dots, I(m)\}$ and*

$$Q(J,\boldsymbol{I}) = A + \sum_{j \in \bar{J}} B_{(j)} \left( K_j + \sum_{i \in \bar{I}(j)} E_{j,i} F_i + \sum_{i \in I(j)} E_{j,i} L(i, I(j)) \right) + \sum_{j \in J} B_{(j)} H(j,J).$$

**Proof:** See [FIA 12b] for the proof. □

Notice that the image bounding condition for nested saturations involves the existence of a set $I(k)$ for every $k \in \mathbb{N}_m$, besides of $J$. There are $2^m$ possible sets $J$ (each one representing a subset of $\mathbb{N}_m$) and $2^p$ possibilities of every $I(k)$, with $k \in \mathbb{N}_m$. Hence there are $2^{(p+1)m}$ different values of $Q(J, \mathbf{I})$, although some of them lead to redundant or non-admissible selections and could be neglected, see Remark 4.

**Remark 5** *The set valued maps $G(\cdot)$ and $S(\cdot)$, defined in Theorems 2 and 3, are local extensions on the ellipsoid $\Omega$ of the saturated and nested saturated functions, respectively. Moreover, they satisfy the convexity related properties characterizing a CDI, posed in Assumption 1. Consider, in fact, $G(\cdot)$ and the function $g(x) = Ax + B\varphi(Kx)$ (analogous considerations hold for $S(\cdot)$ and the function with nested saturations). The map $G : \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ is an extension of the saturated function $g$ over $\Omega$. Furthermore, for every $x \in \Omega$, we have that*

$$\check{F}(x, \eta) = \sup_{z \in G(x)} \eta^T z = \max_{J \subseteq \mathbb{N}_m} \eta^T N(J)x$$

*with $\eta \in \mathbb{R}^n$, is convex in x, being the pointwise maximum of a family of convex functions, see [BOY 04]. Moreover, $\check{F}(0, \eta) = \{0\}$ for all $\eta \in \mathbb{R}^n$.*

As illustrated in [FIA 11a], applying Theorem 2 to continuous-time and discrete-time systems permits one to recover or extend results form literature, for instance those presented in [HU 02a, HU 02b, ALA 05]. The results can be also extended to the case of presence of nested saturation, as shown below.

### 2.4.2. *Continuous-time quadratic stability*

The application of the result provided in Theorem 2 to the case of continuous-time systems leads to a condition for local quadratic stability of the saturated system. The obtained result recovers the one provided in [ALA 05], which, in turn, generalizes the condition presented in [HU 02a]. The proof of the following proposition can be found in [FIA 11a].

**Proposition 3** *Given the continuous-time dynamics $\hat{g}(x) = \hat{A}x + \hat{B}\varphi(\hat{K}x)$ in (2.1), consider the ellipsoid $\Omega = \mathscr{E}(P)$, with $P \in \mathbb{R}^{n\times n}$ and $P = P^T > 0$, the matrix $Q \in \mathbb{R}^{n\times n}$ with $Q = Q^T > 0$ and $\hat{H}(i, I) \in \mathbb{R}^{1\times n}$ such that $|\hat{H}(i, I)x| \leq 1$ for all $x \in \Omega$, for every $I \subseteq \mathbb{N}_{m_c}$ and every $i \in I$. If*

$$\hat{N}(I)^T P + P\hat{N}(I) \leq -Q, \tag{2.15}$$

*for all* $I \subseteq \mathbb{N}_{m_c}$*, with*

$$\hat{N}(I) = \hat{A} + \sum_{i \in \bar{I}} \hat{B}_{(i)} \hat{K}_i + \sum_{i \in I} \hat{B}_{(i)} \hat{H}(i,I), \tag{2.16}$$

*then* $\Omega$ *is an ellipsoidal estimation of the domain of attraction and* $V(x) = x^T P x$ *is a local Lyapunov function in* $\Omega$ *for system (2.1).*

The conditions for global quadratic stability, are presented here for the continuous-time system (2.1), see [FIA 12b].

**Corollary 2** *Given the continuous-time dynamics* $\hat{g}(x) = \hat{A}x + \hat{B}\varphi(\hat{K}x)$ *in (2.1), consider* $P, Q \in \mathbb{R}^{n \times n}$ *with* $P = P^T > 0$ *and* $Q = Q^T > 0$*. If (2.15) holds with*

$$\hat{N}(I) = \hat{A} + \sum_{i \in \bar{I}} \hat{B}_{(i)} \hat{K}_i, \tag{2.17}$$

*for every* $I \subseteq \mathbb{N}_{m_c}$*, then* $V(x) = x^T P x$ *is a global Lyapunov function for system (2.1).*

**Proof:** The result follows from Proposition 3 with $\hat{H}(i,I) = 0_{1 \times n}$, for all $I \subseteq \mathbb{N}_{m_c}$ and $i \in I$. □

Notice that exponential stability of the open-loop part of the system (2.1), is a necessary condition for global exponential stability, in fact, given by constraint (2.15) with $I = \mathbb{N}_{m_c}$ and $\bar{I} = \emptyset$ in (2.16). Also asymptotic stability of the closed-loop system in absence of saturations, implied by condition (2.15) with $I = \emptyset$ and $\bar{I} = \mathbb{N}_{m_c}$ in (2.16), is necessary.

Analogous results for the case of continuous-time systems with nested saturations (2.4) are stated in the following theorem.

**Theorem 4** *Given the continuous-time dynamics (2.4), consider the ellipsoid* $\Omega = \mathscr{E}(P)$*, with* $P \in \mathbb{R}^{n \times n}$ *and* $P = P^T > 0$*, the matrix* $Q \in \mathbb{R}^{n \times n}$ *with* $Q = Q^T > 0$ *and* $\hat{H}(j,J) \in \mathbb{R}^{1 \times n}$ *such that* $|\hat{H}(j,J)x| \leq 1$ *for every* $J \subseteq \mathbb{N}_{m_c}$ *and* $j \in J$*;* $\hat{L}(i,I(k)) \in \mathbb{R}^{1 \times n}$ *such that* $|\hat{L}(i,I(k))x| \leq 1$ *for every* $k \in \mathbb{N}_{m_c}$*, every* $I(k) \subseteq \mathbb{N}_{p_c}$ *and* $i \in I(k)$*, for all* $x \in \Omega$*. If*

$$\hat{Q}(J,\boldsymbol{I})^T P + P\hat{Q}(J,\boldsymbol{I}) \leq -Q, \tag{2.18}$$

*with* $\boldsymbol{I} = \{I(1), I(2), \ldots, I(m_c)\}$*, where* $\hat{Q}(J,\boldsymbol{I})$ *is defined as*

$$\hat{Q}(J,\boldsymbol{I}) = \hat{A} + \sum_{j \in \bar{J}} \hat{B}_{(j)} \left( \hat{K}_j + \sum_{i \in \bar{I}(j)} \hat{E}_{j,i} \hat{F}_i + \sum_{i \in I(j)} \hat{E}_{j,i} \hat{L}(i,I(j)) \right) + \sum_{j \in J} \hat{B}_{(j)} \hat{H}(j,J), \tag{2.19}$$

*for all $J \subseteq \mathbb{N}_{m_c}$, $I(k) \subseteq \mathbb{N}_{p_c}$, $k \in \mathbb{N}_{m_c}$, then $\Omega$ is an ellipsoidal estimation of the domain of attraction and $V(x) = x^T Px$ is a local Lyapunov function in $\Omega$ for system (2.4).*

*Furthermore, $V(x) = x^T Px$ is a global Lyapunov function for the continuous-time system with nested saturations (2.4) if conditions (2.18) hold with*

$$\hat{Q}(J,\boldsymbol{I}) = \hat{A} + \sum_{j\in\bar{J}} \hat{B}_{(j)} \left( \hat{K}_j + \sum_{i\in\bar{I}(j)} \hat{E}_{j,i}\hat{F}_i \right), \tag{2.20}$$

*for all $J \subseteq \mathbb{N}_{m_c}$, $I(k) \subseteq \mathbb{N}_{p_c}$, $k \in \mathbb{N}_{m_c}$ where $\boldsymbol{I} = \{I(1), I(2), \dots, I(m_c)\}$.*

### 2.4.3. *Discrete-time quadratic stability*

Analogous results hold for the case of discrete-time systems presenting saturations on the loop. Actually, Theorem 2 yields also a condition for quadratic stability for discrete-time saturated systems. The results presented in [HU 02b], are particular cases, more conservative, of our results, see the proof in [FIA 11a].

**Proposition 4** *Given the discrete-time dynamics $\tilde{g}(x) = \tilde{A}x + \tilde{B}\varphi(\tilde{K}x)$ in (2.2), consider the ellipsoid $\Omega = \mathscr{E}(P)$, with $P \in \mathbb{R}^{n\times n}$ and $P = P^T > 0$, the matrix $Q \in \mathbb{R}^{n\times n}$ with $Q = Q^T > 0$, and $\tilde{H}(j,J) \in \mathbb{R}^{1\times n}$, such that $|\tilde{H}(j,J)x| \leq 1$ for all $x \in \Omega$, for every $J \subseteq \mathbb{N}_{m_d}$ and every $j \in J$. If*

$$\tilde{N}(J)^T P \tilde{N}(J) - P \leq -Q, \tag{2.21}$$

*for all $J \subseteq \mathbb{N}_{m_d}$, with*

$$\tilde{N}(J) = \tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)}\tilde{K}_j + \sum_{j\in J} \tilde{B}_{(j)}\tilde{H}(j,J), \tag{2.22}$$

*then $\Omega$ is an ellipsoidal estimation of the domain of attraction and $V(x) = x^T Px$ is a local Lyapunov function in $\Omega$ for system (2.2).*

A condition for global exponential stability of the origin for the discrete-time saturated systems follows. The proof is avoided since similar to that one of Corollary 2.

**Corollary 3** *Given the discrete-time dynamics $\tilde{g}(x) = \tilde{A}x + \tilde{B}\varphi(\tilde{K}x)$ in (2.2), consider the matrices $P, Q \in \mathbb{R}^{n\times n}$ with $P = P^T > 0$ and $Q = Q^T > 0$. If (2.21) holds with*

$$\tilde{N}(J) = \tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)}\tilde{K}_j, \tag{2.23}$$

*for every $J \subseteq \mathbb{N}_{m_d}$, then $V(x) = x^T Px$ is a global Lyapunov function for system (2.2).*

The conditions for local and global exponential stability are stated in the following theorem for the case in which nested saturations are present, i.e. for system (2.5).

**Theorem 5** *Given the discrete-time dynamics (2.5), consider the ellipsoid* $\Omega = \mathscr{E}(P)$, *with* $P \in \mathbb{R}^{n\times n}$ *and* $P = P^T > 0$, *the matrix* $Q \in \mathbb{R}^{n\times n}$ *with* $Q = Q^T > 0$ *and* $\tilde{H}(j,J) \in \mathbb{R}^{1\times n}$ *such that* $|\tilde{H}(j,J)x| \leq 1$ *for every* $J \subseteq \mathbb{N}_{m_c}$ *and* $j \in J$; $\tilde{L}(i,I(k)) \in \mathbb{R}^{1\times n}$ *such that* $|\tilde{L}(i,I(k))x| \leq 1$ *for every* $k \in \mathbb{N}_{m_d}$, *every* $I(k) \subseteq \mathbb{N}_{p_d}$ *and* $i \in I(k)$, *for all* $x \in \Omega$. *If*

$$\tilde{Q}(J,\boldsymbol{I})^T P \tilde{Q}(J,\boldsymbol{I}) - P \leq -Q, \tag{2.24}$$

*with* $\boldsymbol{I} = \{I(1), I(2), \ldots, I(m_d)\}$, *where* $\tilde{Q}(J,\boldsymbol{I})$ *is defined as*

$$\tilde{Q}(J,\boldsymbol{I}) = \tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)} \left( \tilde{K}_j + \sum_{i\in\bar{I}(j)} \tilde{E}_{j,i}\tilde{F}_i + \sum_{i\in I(j)} \tilde{E}_{j,i}\tilde{L}(i,I(j)) \right) + \sum_{j\in J} \tilde{B}_{(j)}\tilde{H}(j,J), \tag{2.25}$$

*for all* $J \subseteq \mathbb{N}_{m_d}$, $I(k) \subseteq \mathbb{N}_{p_d}$, $k \in \mathbb{N}_{m_d}$. *then* $\Omega$ *is an ellipsoidal estimation of the domain of attraction and* $V(x) = x^T Px$ *is a local Lyapunov function in* $\Omega$ *for system (2.5).*

*Furthermore,* $V(x) = x^T Px$ *is a global Lyapunov function for the continuous-time system with nested saturations (2.5) if conditions (2.24) hold with*

$$\tilde{Q}(J,\boldsymbol{I}) = \tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)} \left( \tilde{K}_j + \sum_{i\in\bar{I}(j)} \tilde{E}_{j,i}\tilde{F}_i \right), \tag{2.26}$$

*for all* $J \subseteq \mathbb{N}_{m_d}$, $I(k) \subseteq \mathbb{N}_{p_d}$, $k \in \mathbb{N}_{m_d}$ *where* $\boldsymbol{I} = \{I(1), I(2), \ldots, I(m_d)\}$.

### 2.4.4. *Exponential stability for saturated hybrid systems*

The presented results are employed to state conditions for exponential stability for hybrid systems with saturations, possibly nested, [FIA 11a, FIA 12b]. First the case of simple saturations (2.1)-(2.3) is considered. We impose the decreasing of the candidate Lyapunov function $V(x) = x^T Px$ along the continuous trajectories. Moreover, we have to ensure that the variation of $V(x)$ during a jump plus the variation during a flowing interval of $\rho$, is negative. This, with the temporal regularization, would imply that $V(x)$ is decreasing between two successive jumps. The resulting condition is less conservative than imposing the decreasing of $V(x)$ also during the jump. In the following, $m_c$ and $m_d$ are the number of columns of $\hat{B}$ and $\tilde{B}$, $p_c$ and $p_d$ those of $\hat{E}$ and $\tilde{E}$. Notice that the case of functions increasing along flow trajectories and decreasing during jumps, as well as more general cases, could be considered, see also [HES 08, GOE 12]. The following result is stated with no proof, see [FIA 12b].

**Theorem 6** *Given the hybrid system (2.1)-(2.3), consider the ellipsoid* $\Omega = \mathscr{E}(P)$, *with* $P \in \mathbb{R}^{n\times n}$ *and* $P = P^T > 0$, $\hat{H}(i,I) \in \mathbb{R}^{1\times n}$ *and* $\tilde{H}(j,J) \in \mathbb{R}^{1\times n}$ *such that* $|\hat{H}(i,I)x| \leq 1$ *and* $|\tilde{H}(j,J)x| \leq 1$*, for all* $x \in \Omega$*, for every* $I \subseteq \mathbb{N}_{m_c}$ *and* $i \in I$*,* $J \subseteq \mathbb{N}_{m_d}$ *and* $j \in J$*,* $\lambda > 0$*, and* $\sigma \geq 0$*. If*

$$\hat{N}(I)^T P + P\hat{N}(I) \leq -2\lambda P, \tag{2.27}$$

$$\tilde{N}(J)^T e^{-\lambda\rho I_n} P e^{-\lambda\rho I_n} \tilde{N}(J) - \sigma M < P, \tag{2.28}$$

*where* $\hat{N}(I)$ *and* $\tilde{N}(J)$ *are defined as*

$$\hat{N}(I) = \hat{A} + \sum_{i\in\bar{I}} \hat{B}_{(i)}\hat{K}_i + \sum_{i\in I} \hat{B}_{(i)}\hat{H}(i,I), \qquad \tilde{N}(J) = \tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)}\tilde{K}_j + \sum_{j\in J} \tilde{B}_{(j)}\tilde{H}(j,J), \tag{2.29}$$

*for all* $I \subseteq \mathbb{N}_{m_c}$ *and* $J \subseteq \mathbb{N}_{m_d}$*, then* $\Omega$ *is an ellipsoidal estimation of the domain of attraction and the origin is locally asymptotically stable for the hybrid system (2.1)-(2.3).*

A condition for global asymptotic stability is stated for hybrid systems (2.1)-(2.3).

**Corollary 4** *Consider the hybrid system (2.1)-(2.3) and* $P \in \mathbb{R}^{n\times n}$ *with* $P = P^T > 0$*,* $\lambda > 0$ *and* $\sigma \geq 0$*. If (2.27) and (2.28) hold with*

$$\hat{N}(I) = \hat{A} + \sum_{i\in\bar{I}} \hat{B}_{(i)}\hat{K}_i, \qquad \tilde{N}(J) = \tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)}\tilde{K}_j, \tag{2.30}$$

*for every* $I \subseteq \mathbb{N}_{m_c}$ *and* $J \subseteq \mathbb{N}_{m_d}$*, then the origin is globally asymptotically stable for the hybrid system (2.1)-(2.3).*

**Proof:** The result follows from Theorem 6 with $\hat{H}(i,I) = \tilde{H}(j,J) = 0_{1\times n}$, for all $I \subseteq \mathbb{N}_{m_c}$, $J \subseteq \mathbb{N}_{m_d}$, $i \in I$ and $j \in J$. □

Notice that asymptotic stability of the systems $\dot{x} = \hat{A}x$ and $x^+ = \tilde{A}x$ is a necessary condition for global asymptotic stability of system (2.1)-(2.3), in fact, given by constraints (2.27) and (2.28) with $I = \mathbb{N}_{m_c}$ (then $\bar{I} = \emptyset$) and $J = \mathbb{N}_{m_d}$ (thus $\bar{J} = \emptyset$) in (2.30). Also asymptotic stability of $\dot{x} = (\hat{A} + \hat{B}\hat{K})x$ and $x^+ = (\tilde{A} + \tilde{B}\tilde{K})x$, implied by conditions (2.27) and (2.28) with $I = \emptyset$ and $J = \emptyset$ in (2.30), is necessary.

Analogous results for the case of nested saturations (2.3)-(2.5) are stated in the following theorem.

**Theorem 7** *Given the hybrid system with nested saturations (2.3)-(2.5), consider the ellipsoid $\Omega = \mathscr{E}(P)$, with $P \in \mathbb{R}^{n\times n}$ and $P = P^T > 0$, $\lambda > 0$ and $\sigma \geq 0$. Assume there exist: $\hat{H}(j,J) \in \mathbb{R}^{1\times n}$ such that $|\hat{H}(j,J)x| \leq 1$ for every $J \subseteq \mathbb{N}_{m_c}$ and $j \in J$; $\hat{L}(i,I(k)) \in \mathbb{R}^{1\times n}$ such that $|\hat{L}(i,I(k))x| \leq 1$ for every $k \in \mathbb{N}_{m_c}$, every $I(k) \subseteq \mathbb{N}_{p_c}$ and $i \in I(k)$, for all $x \in \Omega$; $\tilde{H}(u,U) \in \mathbb{R}^{1\times n}$ such that $|\tilde{H}(u,U)x| \leq 1$ for every $U \subseteq \mathbb{N}_{m_d}$ and $u \in U$; $\tilde{L}(v,V(l)) \in \mathbb{R}^{1\times n}$ such that $|\tilde{L}(v,V(l))x| \leq 1$ for every $l \in \mathbb{N}_{m_d}$, every $V(l) \subseteq \mathbb{N}_{p_d}$ and $v \in V(l)$, for all $x \in \Omega$, such that:*

$$\hat{Q}(J,\boldsymbol{I})^T P + P\hat{Q}(J,\boldsymbol{I}) \leq -2\lambda P, \tag{2.31}$$

$$\tilde{Q}(U,\boldsymbol{V})^T e^{-\lambda\rho I_n} P e^{-\lambda\rho I_n} \tilde{Q}(U,\boldsymbol{V}) - \sigma M < P, \tag{2.32}$$

*with $\boldsymbol{I} = \{I(1), I(2), \ldots, I(m_c)\}$ and $\boldsymbol{V} = \{V(1), V(2), \ldots, V(m_d)\}$, where $\hat{Q}(J,\boldsymbol{I})$ and $\tilde{Q}(U,\boldsymbol{V})$ are defined as in (2.19) and (2.25), for all $J \subseteq \mathbb{N}_{m_c}$, $I(k) \subseteq \mathbb{N}_{p_c}$, $k \in \mathbb{N}_{m_c}$ and all $U \subseteq \mathbb{N}_{m_d}$, $V(l) \subseteq \mathbb{N}_{p_d}$, $l \in \mathbb{N}_{m_d}$. Then $\Omega$ is an ellipsoidal estimation of the domain of attraction and the origin is locally asymptotically stable in $\Omega$ for the hybrid system (2.3)-(2.5).*

**Proof:** This result can be proved by using reasonings analogous to those of Theorem 6 and Corollary 4 and employing the results from Theorem 3. □

Also a condition for global asymptotically stability can be given.

**Corollary 5** *The origin is locally asymptotically stable for the hybrid system with nested saturations (2.3)-(2.5) if conditions (2.31)-(2.32) hold where $\hat{Q}(J,\boldsymbol{I})$ and $\tilde{Q}(U,\boldsymbol{V})$ are defined as in (2.20) and (2.26), for all $J \subseteq \mathbb{N}_{m_c}$, $I(k) \subseteq \mathbb{N}_{p_c}$, $k \in \mathbb{N}_{m_c}$ and all $U \subseteq \mathbb{N}_{m_d}$, $V(l) \subseteq \mathbb{N}_{p_d}$, $l \in \mathbb{N}_{m_d}$, where $\boldsymbol{I} = \{I(1), I(2), \ldots, I(m_c)\}$ and $\boldsymbol{V} = \{V(1), V(2), \ldots, V(m_d)\}$.*

Function $V(x)$ in Theorems 6 and 7 and Corollaries 4 and 5 are not necessarily decreasing along the trajectories of systems (2.1)-(2.3) and (2.3)-(2.5), due to the jumps. Nevertheless, there exist $k > 0$ and $\sigma > 0$ such that

$$V(x(t)) \leq k e^{-\sigma(t+j)} V(x_0), \qquad \forall t \geq 0, \tag{2.33}$$

where $j$ is the number of jumps occurred before $t$. This means that the origin in $\mathbb{R}^n$ is locally (if it holds for all $x_0 \in \Omega$) or globally (if valid over $\mathbb{R}^n$) exponentially stable for the hybrid saturated systems, as $V(x)$ is a norm if $P$ is positive definite. Proving exponential stability and providing an exponential Lyapunov function in the space of $(x, \tau)$ are the objectives of the following section.

### 2.4.5. *Exponential Lyapunov functions for saturated hybrid systems*

In this section the exponential Lyapunov functions and the exponential stability of the origin are considered for the hybrid systems with nested saturation, (2.3)-(2.5), as it is the more general case. Given a trajectory of the system (2.3)-(2.5), we introduce, for notational convenience, the following definition

$$\begin{cases} x^-(t_j) = x(t_j), \\ x^+(t_j) = \tilde{g}(x(t_j)), \end{cases}$$

if $t_j$ is a jump instant. That is, $x^-(t_j)$ denotes the state before and $x^+(t_j)$ the state after the $j$-th jump. We also assume that $x^+(t) = x^-(t) = x(t)$ if the system is flowing at $t$.

**Proposition 5** *The system (2.3)-(2.5) jumps at most once in the time intervals $[t, t+\rho)$ and $(t, t+\rho]$ for every $t \geq 0$.*

**Proof:** Notice in fact that, denoting with $t_i$ the time of the $i$-th jump, the system flows on the time interval $(t_i, t_i + \rho)$, from the definition of the flow and jump sets, see (2.3). Then, for every $[t, t+\rho)$, no more than a jump can occur. Analogously for the interval $(t, t+\rho]$. □

Given the positive definite matrix $P$ as in Theorem 7, consider the quadratic function

$$V(x) = x^T P x, \tag{2.34}$$

and recall that the set $\Omega$ is a level set of such a function, in particular

$$\Omega = \mathscr{E}(P) = \{x \in \mathbb{R}^n : x^T P x \leq 1\}.$$

We recall that $V(x)$ is a norm of $x \in \mathbb{R}^n$ provided the matrix $P$ is positive definite, and such that there exist positive $\alpha$, $\beta$

$$\alpha \|x\|_2^2 \leq V(x) \leq \beta \|x\|_2^2, \qquad \forall x \in \mathbb{R}^n, \tag{2.35}$$

with $\alpha$ and $\beta$ minimal and maximal eigenvalue of $P$, for instance.

**Proposition 6** *If the hypotheses of Theorem 7 hold then there exists $\theta \in [0, 1)$ such that*

$$\begin{array}{l} V(x^-(t+\rho)) \leq \theta V(x^-(t)), \\ V(x^+(t+\rho)) \leq \theta V(x^+(t)), \end{array} \tag{2.36}$$

*for all $x(t) \in \Omega$ if a jump occurred at $t_j \in [t, t+\rho]$.*

**Proof:** The condition (2.32), which holds being among the hypotheses of Theorem 7, is equivalent to the existence of $\theta \in [0,1)$ such that

$$\tilde{Q}(U,\mathbf{V})^T e^{-\lambda\rho I_n} P e^{-\lambda\rho I_n} \tilde{Q}(U,\mathbf{V}) - \sigma M \leq \theta P. \tag{2.37}$$

Then, following the lines of the proof of Theorem 7, see [FIA 12b], one has that $x^-(t_j+\rho)^T P x^-(t_j+\rho) \leq \theta x^-(t_j)^T P x^-(t_j)$ and $x^+(t_j+\rho)^T P x^+(t_j+\rho) \leq \theta x^+(t_j)^T P x^+(t_j)$, for all $x(0) \in \Omega$, with $t_j$ jumping instant. Analogously, we have also that

$$x(t+\rho)^T P x(t+\rho) \leq \theta x(t)^T P x(t),$$

if the system is flowing at $t$, provided a jump is occurred in the interval $(t, t+\rho)$. In fact, no more than one jump is possible in such an interval, see Proposition 5. The result follows from the definition of the function $V(x)$ as in (2.34). □

Supposing that Theorem 7 holds and given the positive $\theta \in [0,\,1)$ as in Proposition 6, a rate useful to bound the decreasing along the flow and the at the jump is defined.

**Definition 1** *Let $\theta \in [0,\,1)$ as in the Proposition 6. Define $\delta \in \mathbb{R}$ such that*

$$0 < \delta < \min\left\{2\lambda, -\frac{\ln\theta}{\rho}\right\}, \tag{2.38}$$

*with $\lambda \in \mathbb{R}$ as in Theorem 7.*

Notice that from $\lambda > 0$ and $\theta < 1$, it follows that

$$\theta < e^{-\delta\rho}, \qquad\qquad e^{-2\lambda\rho} < e^{-\delta\rho}, \tag{2.39}$$

from the definition (2.38).

**Corollary 6** *If the hypotheses of Theorem 7 hold then there exists $\theta \in [0,\,1)$ such that*

$$\begin{aligned} V(x^-(t+\rho)) &\leq e^{-\delta\rho} V(x^-(t)),\\ V(x^+(t+\rho)) &\leq e^{-\delta\rho} V(x^+(t)), \end{aligned}$$

*for all $x(t) \in \Omega$, with $\delta$ as in (2.38).*

**Proof:** If a jump occurs in $[t, t+\rho]$, then from Proposition 6 and (2.39), we have

$$\begin{aligned} V(x^-(t+\rho)) &\leq \theta V(x^-(t)) \leq e^{-\delta\rho} V(x^-(t)),\\ V(x^+(t+\rho)) &\leq \theta V(x^+(t)) \leq e^{-\delta\rho} V(x^+(t)), \end{aligned}$$

otherwise the system flows in $[t, t+\rho]$ and it follows

$$V(x(t+\rho)) \leq e^{-2\lambda\rho} V(x(t)) \leq e^{-\delta\rho} V(x(t)),$$

as proved for the Theorem 7, see [FIA 12b]. □

**Remark 6** *Notice that no assumption on the value of $\tau(0)$ has been done. Then, the condition $\tau(t) > \rho$ could hold at every instant preceding the first jump. Thus, the first jump could occur at any instant, in general.*

Notice first that, from Proposition 3.29 in [GOE 12], the Theorem 7 provides a sufficient condition for the local exponential stability of the origin in $\mathbb{R}^n$.

**Proposition 7** *If the hypotheses of Theorem 7 hold then the origin in $\mathbb{R}^n$ is locally exponentially stable in $\Omega$ for the system (2.3)-(2.5).*

**Proof:** Consider the quadratic function given by (2.34) where $P$ is such that the hypotheses of the theorem are satisfied. Notice that also (2.35) holds being $P$ positive definite. Then we have that

$$\langle \nabla V(x), \hat{g}(x) \rangle \leq -2\lambda V(x), \qquad \forall x \in \Omega,$$

while flowing and

$$V(\tilde{g}(x)) < e^{2\lambda\rho} V(x), \qquad \forall x \in \Omega,$$

while jumping, which is equivalent to the existence of $0 < \varepsilon < \lambda$ such that

$$V(\tilde{g}(x)) \leq e^{2(\lambda-\varepsilon)\rho} V(x), \qquad \forall x \in \Omega,$$

being $V(x)$ a quadratic function. Denoting with $j = j(t)$ the number of jumps occurred before $t$, we have that $j\rho \leq t + \rho$ from the temporal regularization assumption and the Remark 6. Then from Proposition 3.29 in [GOE 12] we have

$$\begin{aligned} -2\lambda t + 2(\lambda-\varepsilon)\rho j &= -2\lambda t + 2(\lambda - \tfrac{\varepsilon}{2})\rho j - \varepsilon\rho j \\ \leq -2\lambda t + 2(\lambda - \tfrac{\varepsilon}{2})t + 2(\lambda - \tfrac{\varepsilon}{2})\rho - \varepsilon\rho j &= 2(\lambda - \tfrac{\varepsilon}{2})\rho - \varepsilon(t+j), \end{aligned}$$

which is a sufficient condition for locally exponentially stability in $\Omega$, since (2.35) holds. □

Then, there exist $k > 0$ and $\sigma > 0$ such that (2.33) is satisfied for all $x_0 \in \Omega$. Finally, a class of Lyapunov function satisfying the sufficient conditions for local exponential stability of the closed set

$$\mathscr{A} = \{0\} \times \mathbb{R}_+, \tag{2.40}$$

are given. Such functions are defined in the space of $(x, \tau)$ that is in $\mathbb{R}^n \times \mathbb{R}$ and are parametrized with respect to a function $\gamma(\tau)$.

**Proposition 8** *If the hypotheses of Theorem 7 hold then for every $\mathscr{C}^1$ function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\gamma(0) = 0$,*

$$\frac{d\gamma(\tau)}{d\tau} = \begin{cases} 1, & \text{if } \tau \le \rho, \\ 0, & \text{if } \tau \ge \eta, \end{cases} \tag{2.41}$$

*with $\eta > \rho$ and $0 \le \frac{d\gamma(\tau)}{d\tau} \le 1$ for all $\tau \in \mathbb{R}_+$ and every $\delta$ such that*

$$2\lambda - \varepsilon < \delta < 2\lambda, \tag{2.42}$$

*with*

$$\varepsilon = -\frac{\ln\theta}{\rho} > 0, \tag{2.43}$$

*with $\theta$ satisfying (2.37), then the function $\bar{V} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ defined as*

$$\bar{V}(x,\tau) = e^{\delta\gamma(\tau)} V(x), \tag{2.44}$$

*is an exponential Lyapunov function in $\Omega \times \mathbb{R}_+$ and the set $\mathscr{A}$ as in (2.40) is locally exponentially stable for the system (2.3)-(2.5).*

**Proof:** We prove that $\bar{V}$ satisfies the sufficient condition for being a local exponential Lyapunov function in $\mathscr{A}$, as stated in [TEE 11]. That is, we prove that there exist positive real numbers $\alpha_1$, $\alpha_2$, $p$, $\sigma$ such that

$$\alpha_1 \|(x,\tau)\|^p_{\mathscr{A}} \le \bar{V}(x,\tau) \le \alpha_2 \|(x,\tau)\|^p_{\mathscr{A}}, \tag{2.45}$$

where $\|(x,\tau)\|_{\mathscr{A}} = \min_{y\in\mathscr{A}} \|(x,\tau) - y\|_2$ with $\mathscr{A}$ as in (2.40), and

$$\langle \nabla\bar{V}(x,\tau), (\hat{g}(x), 1)\rangle \le -\sigma\bar{V}(x,\tau), \qquad \forall (x,\tau) \in (\Omega\times\mathbb{R}_+)\cap\mathscr{G}, \tag{2.46}$$

$$\bar{V}(\tilde{g}(x), 0) \le e^{-\sigma}\bar{V}(x,\tau), \qquad \forall (x,\tau) \in (\Omega\times\mathbb{R}_+)\cap\mathscr{J}, \tag{2.47}$$

hold.

By construction $\gamma(\tau)$ is a monotonically non-decreasing function bounded above, i.e. the real number $\Gamma = \sup_{\tau\in\mathbb{R}_+}\gamma(\tau)$ is positive, finite and such that $\gamma(\tau) \le \Gamma$ for all $\tau \ge 0$. Moreover $\Gamma \ge \rho$ from (2.41). Then

$$\bar{V}(x,\tau) \le e^{\delta\Gamma} V(x), \qquad \forall (x,\tau) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Thus condition (2.45) is satisfied since $\|(x,\tau)\|_{\mathscr{A}} = \|x\|_2$ and then it is sufficient to choose $p = 2$ and $\alpha_1 = \beta_1$, $\alpha_2 = e^{\delta\Gamma}\beta_2$, with $\beta_1, \beta_2$ the minimal and maximal eigenvalues of $P$, respectively.

Concerning (2.46), and since $\gamma(\tau) \le \tau$ for all $\tau \ge 0$, we have

$$\begin{aligned} \langle \nabla\bar{V}(x,\tau), (\hat{g}(x), 1)\rangle &= \delta\tfrac{d\gamma(\tau)}{d\tau}\dot{\tau}e^{\delta\gamma(\tau)}V(x) + e^{\delta\gamma(\tau)}\dot{V}(x) \\ &\le \delta e^{\delta\gamma(\tau)}V(x) + e^{\delta\gamma(\tau)}\dot{V}(x) \le (\delta - 2\lambda)e^{\delta\gamma(\tau)}V(x), \end{aligned}$$

for all $(x,\tau)\in(\Omega\times\mathbb{R}_+)\cap\mathscr{G}$, with $(\delta-2\lambda)<0$ from (2.42). From (2.37), (2.43) and $\tau\geq\gamma(\tau)\geq\rho$ at every jumping instant, it follows

$$\begin{aligned}\bar{V}(\tilde{g}(x),0)-\bar{V}(x,\tau)&=V(\tilde{g}(x))-e^{\delta\gamma(\tau)}V(x)\leq e^{2\lambda\rho}\theta V(x)-e^{\delta\gamma(\tau)}V(x)\\&=\left(e^{2\lambda\rho}\theta e^{-\delta\gamma(\tau)}-1\right)e^{\delta\gamma(\tau)}V(x)=\left(e^{2\lambda\rho-\varepsilon\rho-\delta\gamma(\tau)}-1\right)e^{\delta\gamma(\tau)}V(x)\\&\leq\left(e^{(2\lambda-\varepsilon-\delta)\rho}-1\right)e^{\delta\gamma(\tau)}V(x)=e^{(2\lambda-\varepsilon-\delta)\rho}\bar{V}(x,\tau)-\bar{V}(x,\tau),\end{aligned}$$

for all $(x,\tau)\in(\Omega\times\mathbb{R}_+)\cap\mathscr{J}$, and $e^{(2\lambda-\varepsilon-\delta)\rho}<1$ from (2.42). Then, conditions (2.46) and (2.47) are satisfied for all positive $\sigma$ such that

$$\sigma\leq\min\{2\lambda-\delta,\ (\delta-2\lambda+\varepsilon)\rho\},$$

holds. □

### 2.5. Computational issues

Some computation-oriented considerations on how to practically obtain a quadratic functions $V(x)$ ensuring exponential stability of the origin for systems (2.1)-(2.3) and (2.3)-(2.5) are provided. First, we propose a formulation of the condition provided by Theorem 6 which can be reduced in LMI form by fixing the value of $\lambda$.

**Proposition 9** *Consider the hybrid system (2.1)-(2.3). Suppose that there exist $W\in\mathbb{R}^{n\times n}$ with $W=W^T>0$, $\lambda>0$, $\hat{Z}(i,I)\in\mathbb{R}^{1\times n}$ and $\tilde{Z}(j,J)\in\mathbb{R}^{1\times n}$ for every $I\subseteq\mathbb{N}_{m_c}$, $i\in I$, $J\subseteq\mathbb{N}_{m_d}$ and $j\in J$, such that conditions*

$$\begin{aligned}&\Big(\hat{A}W+\sum_{i\in\bar{I}}\hat{B}_{(i)}\hat{K}_iW+\sum_{i\in I}\hat{B}_{(i)}\hat{Z}(i,I)+\lambda W\Big)\\&+\Big(W\hat{A}^T+\sum_{i\in\bar{I}}W\hat{K}_i^T\hat{B}_{(i)}^T+\sum_{i\in I}\hat{Z}(i,I)^T\hat{B}_{(i)}^T+\lambda W\Big)\leq 0,\end{aligned}\tag{2.48}$$

$$\begin{bmatrix} W & \big(W\tilde{A}^T+\sum_{j\in\bar{J}}W\tilde{K}_j^T\tilde{B}_{(j)}^T+\sum_{j\in J}\tilde{Z}(j,J)^T\tilde{B}_{(j)}^T\big)e^{-\lambda\rho I_n}\\ * & W\end{bmatrix}>0,\tag{2.49}$$

$$\begin{bmatrix}1 & \hat{Z}(i,I)\\ * & W\end{bmatrix}\geq 0,\quad\forall i\in I,\qquad\begin{bmatrix}1 & \tilde{Z}(j,J)\\ * & W\end{bmatrix}\geq 0,\quad\forall j\in J,\tag{2.50}$$

*are satisfied for every $I\subseteq\mathbb{N}_{m_c}$ and $J\subseteq\mathbb{N}_{m_d}$. Then set $\Omega=\mathscr{E}(P)$, with $P=W^{-1}$, is an ellipsoidal estimation of the domain of attraction and the origin in $\mathbb{R}^n$ is locally exponentially stable in $\Omega$ for the hybrid system (2.1)-(2.3) can be determined.*

**Proof:** The proposition stems from Theorem 6. In fact, it can be proved, using standard matrix inequalities manipulation techniques, that (2.48)-(2.50) imply the conditions of the theorem, with $W = P^{-1}$, $\hat{Z}(i,I) = \hat{H}(i,I)W$ and $\tilde{Z}(j,J) = \tilde{H}(j,J)W$, for every $I \subseteq \mathbb{N}_{m_c}$ and $i \in I$, $J \subseteq \mathbb{N}_{m_d}$ and $j \in J$. The only difference is that condition (2.28), concerning $x \in \Omega$ and $(x,\tau) \in \mathscr{J}$, is relaxed in (2.49) imposing the condition on jumps for all $x \in \Omega$. Finally, (2.50) assures that $|\hat{H}(i,I)x| \leq 1$ and $|\tilde{H}(j,J)x| \leq 1$, for all $x \in \Omega$, every $I \subseteq \mathbb{N}_{m_c}$ and $J \subseteq \mathbb{N}_{m_d}$. □

Recall that although functions $V(x)$ in Theorems 6 and 7 and Proposition 9 do not decrease along the trajectories, local exponential Lyapunov functions exists.

**Remark 7** *As stated in the proof of Proposition 9, the condition on the variation of the value of $V(x)$ during the jump is imposed over the whole set $\Omega$, although it could have been restricted to the set $\mathscr{J}$. In fact, the term $\sigma M$ in (2.28) is not present in (2.49). This introduces some conservativeness, but permits to formulate the related problem in LMI form, fixing $\lambda$. Removing this source of conservativeness is a possible future improvement.*

The result provided in Proposition 9 can be used to pose an optimization problem to maximize the size of $\Omega$ and hence to provide a solution to Problem 1.

**Remark 8** *A possible evaluation criterion is the maximization of the value of $\beta$ such that the polytope $\beta L = co(\{\beta v(k) \in \mathbb{R}^n : k \in \mathbb{N}_V\})$ is contained in the estimate $\Omega = \mathscr{E}(P)$, where $v(k) \in \mathbb{R}^n$, with $k \in \mathbb{N}_V$, are given points in the state space. The optimization problem results:*

$$\begin{array}{ll} \max\limits_{\beta,\lambda,\tilde{Z},\hat{Z},W} & \beta \\ s.t. & (2.48), (2.49), (2.50), \quad \forall I \subseteq \mathbb{N}_{m_c}, \ \forall J \subseteq \mathbb{N}_{m_d} \\ & \begin{bmatrix} 1 & \beta v(k)^T \\ * & W \end{bmatrix} < 0, \quad \forall k \in \mathbb{N}_V, \end{array} \tag{2.51}$$

*where, for sake of notational compactness, we denoted with $\tilde{Z}$ and $\hat{Z}$ the matrices $\hat{Z}(i,I)$ and $\tilde{Z}(j,J)$ for all $I \subseteq \mathbb{N}_{m_c}$ and $i \in I$, $J \subseteq \mathbb{N}_{m_d}$ and $j \in J$. Constraints (2.48)-(2.50) ensure that $V(x) = x^T P x$ yields local exponential stability of the origin in $\mathscr{E}(W^{-1})$ for the hybrid system, and the second set of constraints imposes that $\beta v(k) \in \mathscr{E}(W^{-1})$, for every $k \in \mathbb{N}_V$.*

Notice that, although the constraints (2.48) and (2.49) are not linear in the optimization variables, they are LMI for fixed values of $\lambda$. Then, in practice, the problem can be solved for different values of $\lambda > 0$, to obtain a guess of the maximal value of $\beta$. Notice also that $\lambda$ is a bound on the decreasing rate of the quadratic function along the trajectories of the continuous-time dynamics, then it could be considered as a design parameter and fixed beforehand. The LMI condition for global exponential stability for system (2.1)-(2.3) (and fixed $\lambda$) follows.

**Corollary 7** *Consider the hybrid system (2.1)-(2.3), matrix $P \in \mathbb{R}^{n\times n}$ with $P = P^T > 0$, $\lambda > 0$ and $\sigma \geq 0$. If conditions*

$$\begin{aligned} \big(\hat{A} + \sum_{i\in\bar{I}} \hat{B}_{(i)}\hat{K}_i\big)^T P + P\big(\hat{A} + \sum_{i\in\bar{I}} \hat{B}_{(i)}\hat{K}_i\big) \leq -2\lambda P, \\ \big(\tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)}\tilde{K}_j\big)^T e^{-\lambda\rho I_n} P e^{-\lambda\rho I_n} \big(\tilde{A} + \sum_{j\in\bar{J}} \tilde{B}_{(j)}\tilde{K}_j\big) - \sigma M < P, \end{aligned} \tag{2.52}$$

*are satisfied for every $I \subseteq \mathbb{N}_{m_c}$ and $J \subseteq \mathbb{N}_{m_d}$, then $V(x) = x^T P x$ yields global exponential stability of the origin in $\mathbb{R}^n$ for the hybrid system (2.1)-(2.3).*

**Remark 9** *The conditions for hybrid systems with nested saturations (2.3)-(2.5) can be easily recovered, by adequately modifying the terms $\hat{B}_{(i)}\hat{K}_i W$ in (2.48) and $\tilde{B}_{(i)}\tilde{K}_i W$ in (2.49), as well the terms $\hat{B}_{(i)}\hat{K}_i$ and $\tilde{B}_{(j)}\tilde{K}_j$ in (2.52).*

## 2.6. Numerical examples

The systems presented below can be expressed as in (2.1)-(2.3), or (2.3)-(2.5), by posing $x = (x_p, x_c)$, see for instance Section 2.2.1.

**Example 1** *We consider the linear one-dimensional unstable system, proposed in [TAR 11a], in closed-loop with a stabilizing reset PI controller:*

$$\begin{cases} \dot{x}_p &= 0.1x_p + \varphi(y_c), \\ y_p &= x_p, \end{cases} \qquad \begin{cases} \dot{x}_c &= -0.2y_p, \\ y_c &= x_c - 2y_p. \end{cases} \tag{2.53}$$

*The dynamics characterizing the reset behavior with saturation is $x_c^+ = x_c + \varphi(-x_c)$. The minimum time interval between two jumps is set to 2 seconds, that is $\rho = 2$.*

*We solve the optimization problem (2.51) where points $v(k)$, with $k \in \mathbb{N}_4$, are the vertices of the square set $L = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$, and for different values of $\lambda$. We found that the value of $\lambda = 0.02$ provides the best value (among those tested) of $\beta$, that is $\beta = 3.2689$ with*

$$P = \begin{bmatrix} 0.0409 & -0.0101 \\ * & 0.03241 \end{bmatrix}.$$

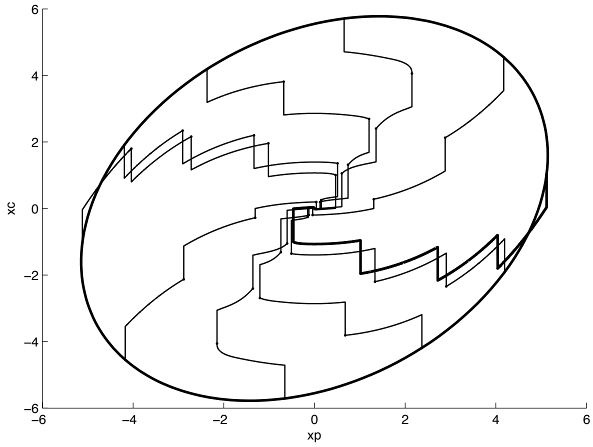


**Figure 2.1.** *Set $\Omega$ and trajectories of the saturated reset system.*

*The set $\Omega = \mathscr{E}(P)$ is an estimation of the domain of attraction of the reset system, regardless of the set $\{x \in \mathbb{R}^n : x^T M x \geq 0\}$. This can be noticed in Figure 2.1, where $\Omega$ is depicted with some trajectories of the system assuming that the jump can occur at any point of $\Omega$. Note in particular the trajectory marked in bold line with initial condition $x(0) = x_0^- = [5.1188\ 1.0376]^T$. With the first jump at time 0 the trajectory leaves $\Omega$, then $V(x)$ increases, i.e. $V(x_0^+) = 1.0686 > 1$. At the time of the second jump the state is contained in the ellipsoid, with $V(x(\rho^-)) = 0.9196 < 1$. Then $V(x)$ decreases between the two jumps, as ensured by Theorem 6.*

**Example 2** *The case of nested saturations is considered. A further saturation is added between the plant output and the controller input of the continuous-time dynamics of system (2.53):*

$$\begin{cases} \dot{x}_p &= 0.1x_p + \varphi(x_c - 2\varphi(x_p)), \\ \dot{x}_c &= -0.2\varphi(x_p), \end{cases}$$

*while the discrete-time behavior is the same as in Example 1. The solution of the optimization problem (2.51) adapted to the case of nested saturations and with $\lambda = 0.02$ leads to $\beta = 1.8922$. As expected, the further saturation entails a reduction of the size of the estimation of the domain of attraction, see Figure 2.2.*

**Example 3** *The condition for global exponential stability provided by Corollary 7 is applied to a multi-input system. Consider the system, inspired to the examples in work*

*[BEK 04] and references therein, whose dynamics are given by*

$$A_p = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} 4 & 0 \end{bmatrix},$$

*in closed-loop with continuous-time dynamical controller whose matrices are*

$$A_c = -3, \quad B_c = -1, \quad C_c = \begin{bmatrix} 0.1 \\ 0.22 \end{bmatrix}, \quad D_c = \begin{bmatrix} -0.0625 \\ -0.1250 \end{bmatrix}.$$

*We suppose that the controller discrete-time dynamics is a saturated reset, i.e.* $x_c^+ = x_c + \varphi(-x_c)$*, and the plant state performs an instantaneous rotation of* $\pi/4$ *radians, at any jump instant. Notice that exponential stability of both the open-loop and closed-loop continuous-time systems in absence of saturation, which are necessary conditions for global exponential stability, are ensured. Posing* $\rho = 0.5$ *and* $\lambda = 0.01$ *and supposing that the jump can occur at any* $x \in \mathbb{R}^n$*, conditions (2.52) are satisfied by*

$$P = \begin{bmatrix} 2.0972 & 0.0068 & -0.0113 \\ * & 2.1054 & -0.0056 \\ * & * & 1.8822 \end{bmatrix},$$

*for every* $I \subseteq \mathbb{N}_{m_c}$ *and* $J \subseteq \mathbb{N}_{m_d}$*. Then, from Corollary 7, the origin in* $\mathbb{R}^n$ *is globally exponentially stable for the saturated reset system .*

## 2.7. Conclusions

In this work we dealt with the problems of analyzing exponential stability and computing ellipsoidal estimations of the domain of attraction for hybrid systems with

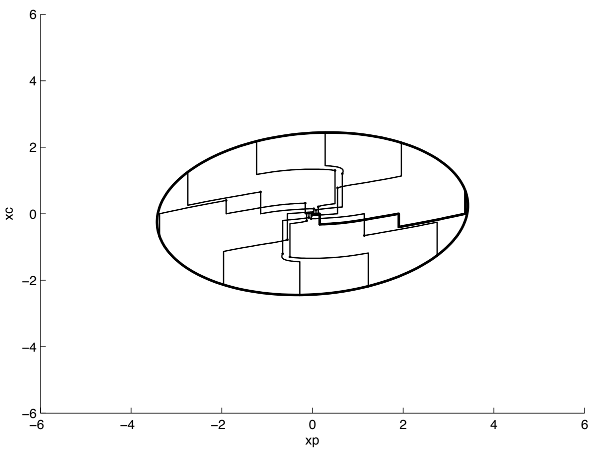


**Figure 2.2.** *Set* $\Omega$ *and trajectories of the reset system with nested saturations.*

nested saturations. The approach is based on set-theory and invariance. A geometrical characterization of the saturated functions is provided first, by determining a class of set valued local extensions. The results lead to computation-oriented conditions for quadratic stability, for continuous and discrete-time systems, and exponential stability for saturated hybrid systems. Estimations of the domain of attraction are also obtained, as well as exponential Lyapunov functions induced by the quadratic ones, for saturated hybrid systems.

An interesting forthcoming issue could be to exploit the hybrid loop to improve the performance of a controlled system in presence of exogenous signals. This could be achieved by designing the reset law and both the flow and jump sets. Furthermore, more general sets, as polytopes and generic convex sets, and more generic Lyapunov functions candidates, as the polyhedral ones, should be considered to generalize the approach.

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