Stabilizability and control co-design for discrete-time switched linear systems

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Abstract In this work we deal with the stabilizability property for discrete-time switched linear systems. First we provide a constructive necessary and sufficient condition for stabilizability based on set-theory and the characterization of a universal class of Lyapunov functions. Such a geometric condition is considered as the reference for comparing the computation-oriented sufficient conditions. The classical BMI conditions based on Lyapunov-Metzler inequalities are considered and extended. Novel LMI conditions for stabilizability, derived from the geometric ones, are presented that permit to combine generality with convexity. For the different conditions, the geometrical interpretations are provided and the induced stabilizing switching laws are given. The relations and the implications between the stabilizability conditions are analyzed to infer and compare their conservatism and their complexity. The results are finally extended to the problem of the co-design of a control policy, composed by both the state feedback and the switching control law, for discrete-time switched linear systems. Constructive conditions are given in form of LMI that are necessary and sufficient for the stabilizability of systems which are periodic stabilizable.

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1 Introduction

Switched systems are characterized by dynamics that may change along the time among a finite number of possible dynamical behaviors. Each behavior is determined by a mode and the active one is selected by means of a function of time, or state, or both, and referred to as switching law. The interest that such a kind of systems rose in the last decades lies in their capability of modeling complex real systems, as embedded or networked ones, and also for the theoretical issues involved, see [22, 24, 32].

Several conditions for stability have been proposed in the literature based on: switched Lyapunov functions [11]; the joint spectral radius [18]; path-dependent Lyapunov functions [21]; and the variational approach [25]. If the existence of polyhedral, hence convex, Lyapunov functions has been proved to be necessary and sufficient for stability [26, 4], convex functions result to be conservative for switched systems with switching law as control input, see [6, 32]. Thus, nonconvex functions must be considered for addressing stabilizability. Sufficient conditions for stabilizability have been provided in literature, mainly based on min-switching policies introduced in [34], developed in [22, 20] and leading to Lyapunov-Metzler inequalities [17, 16]. The fact that the existence of a min-switching control law is necessary and sufficient for exponential stabilizability has been claimed in [32]. In the same work, as well as in [6], it has been proved that the stabilizability of a switched system does not imply the existence of a convex Lyapunov function. Thus, for stabilizability, nonconvex Lyapunov functions might be considered, see for instance [17, 32].

We present here some recent results, mostly based on set-theory and convex analysis, on stabilizability and control co-design for switched linear systems, see [13, 14, 15]. We first propose a stabilizability approach based on set-theory and invariance, see [2, 19, 5]. A geometric necessary and sufficient condition for stabilizability and sufficient one for non-stabilizability of discrete-time linear switched systems are presented in [13]. A family of nonconvex, homogeneous functions is proved to be a universal class of Lyapunov functions for switched linear systems.

The geometric condition in [13] might, nonetheless, result to be often computationally unaffordable, although such a computational complexity appears to be inherent to the problem itself, hence unavoidable. In the literature, computation-oriented sufficient conditions for stabilizability have been provided that are based on min-switching policies and lead to nonconvex control Lyapunov functions in form of minimum of quadratics. Such functions are obtained as solutions to Lyapunov-Metzler BMI conditions, [17, 1], and through an LQR-based iterative procedure, [32]. New LMI conditions for stabilizability, which could conjugate computational affordability with generality, are proposed here, see [14]. The LMI conditions are proved to admit a solution if and only if the system is periodic stabilizable. Moreover, we provide geometrical and numerical insights on different stabilizability conditions to quantify their conservatism and the relations between them and with the necessary and sufficient ones.

The problem of co-designing both the switching law and the control input is even more involved than the problem of stabilizability of autonomous switched systems. This problem has been addressed in several works based on Lyapunov-Metzler BMI conditions, as in [12], or on techniques based on LQR control approximation in [35, 36, 1]. Constructive LMI conditions are given here that are necessary and sufficient for the stabilizability of systems which are periodic stabilizable, [15]. The conditions are constructive and provide the switching law and a family of state feedback gains stabilizing the system, although their complexity grows combinatorially with the maximal length of modes sequences considered.

Notation: Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{j \in \mathbb{N} : 1 \leq j \leq n\}$. The Euclidean-norm in $\mathbb{R}^n$ is $\|x\|$. The i-th element of a finite set of matrices is denoted as A_i. We use the shortcut $P > 0$ (resp. $P \geq 0$) to define a symmetric positive definite (resp. semi-definite) matrix, i.e. such that $P = P^T$ and its eigenvalues are positive (resp. non-negative). Given $P \in \mathbb{R}^{n \times n}$ with $P > 0$, define $\mathscr{E}(P) = \left\{x \in \mathbb{R}^n : \ x^T P x \leq 1\right\}$. Given $\theta \in \mathbb{R}$, $R(\theta) \in \mathbb{R}^2$ is the rotation matrix of angle θ. The set of q switching modes is $\mathscr{I} = \mathbb{N}_q$, all the possible sequences of modes of length N is $\mathscr{I}^N = \prod_{j=1}^N \mathscr{I}$, and $|\sigma| = N$ if $\sigma \in \mathscr{I}^N$. Given $N, M \in \mathbb{N}$ with $N \leq M$, denote $\mathscr{I}^{[N:M]} = \bigcup_{i=N}^M \mathscr{I}^i$ and then $N_{\mathscr{I}} = \sum_{k=1}^N q^k$ is the number of elements in $\mathscr{I}^{[1:N]}$. Given $\sigma \in \mathscr{I}^N$, define: $\mathbb{A}_\sigma = \prod_{j=1}^N A_{\sigma_j} = A_{\sigma_N} \cdots A_{\sigma_1}$, and define $\prod_{j=M}^N A_{\sigma_j} = I$ if $M > N$. Given $a \in \mathbb{R}$, the maximal integer smaller than or equal to a is $\lfloor a \rfloor$. The set of Metzler matrices of dimension N, i.e. matrices $\pi \in \mathbb{R}^{N \times N}$ whose elements are nonnegative and $\sum_{j=1}^N \pi_{ji} = 1$ for all $i \in \mathbb{N}_N$, is $\mathscr{M}_N$. Throughout the chapter,

2 Stabilizability of discrete-time linear switched systems

Consider the discrete-time switched system

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law that, at any instant, selects the transition matrix among the finite set $\{A_i\}_{i \in \mathbb{N}_q}$, with $A_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathbb{N}_q$. Given the initial state x_0 and a switching law $\sigma(\cdot)$, we denote with $x_N^\sigma(x_0)$ the state of the system (1) at time N starting from x_0 by applying the switching law $\sigma(\cdot)$, that can be state-dependent, i.e. $\sigma(k) = \sigma(x(k))$ with slight abuse of notation.

Assumption 1 *The matrices A_i, with $i \in \mathbb{N}_q$, are nonsingular.*

Remark 1. Assumption 1 is not restrictive. In fact, the stable eigenvalues of the matrices A_i are beneficial from the stability point of view of the switched systems and poles in zero are related to the most contractive dynamics. Moreover, the results presented in the following can be extended to the general case with appropriate considerations.

A concept widely employed in the context of set-theory and invariance is the C-set, see [4, 5]. A C-set is a compact and convex set with $0 \in \mathrm{int}(\Omega)$. We define an analogous concept useful for our purpose. For this, we first recall that a set Ω is a star-convex set if there exists $x_0 \in \Omega$ such that every convex combination of x and x_0 belongs to Ω for every $x \in \Omega$.

Definition 1. A set $\Omega \subseteq \mathbb{R}^n$ is a C*-set if it is the union of a finite number of C-sets. The gauge function of a C*-set $\Omega \subseteq \mathbb{R}^n$ is $\Psi_\Omega(x) = \min_{\alpha \geq 0}\{\alpha \in \mathbb{R} : x \in \alpha\Omega\}$.

Notice that every C*-set is star-convex, i.e. there is $z \in \Omega$ such that every convex combination of x and z belongs to Ω for all $x \in \Omega$, but the converse is not true in general. Some basic properties of the C*-sets and their gauge functions are listed below, see also [30].

Property 1. Any C-set is a C*-set. Given a C*-set $\Omega \subseteq \mathbb{R}^n$, we have that $\alpha\Omega \subseteq \Omega$ for all $\alpha \in [0,1]$, and the gauge function $\Psi_\Omega(\cdot)$ is: homogeneous of degree one, i.e. $\Psi_\Omega(\alpha x) = \alpha\Psi_\Omega(x)$ for all $\alpha \geq 0$ and $x \in \mathbb{R}^n$; positive definite; defined on $\mathbb{R}^n$ and radially unbounded.

The gauge functions induced by C-sets have been used in the literature as Lyapunov functions candidates, see [3]. In particular, it has been proved that they provide a universal class of Lyapunov functions for linear parametric uncertain systems, [26, 4], and switched systems with arbitrary switching, [24]. We prove that the gauge functions induced by C*-sets form a universal class of Lyapunov function for switched systems with switching control law. For this, we provide a definition of Lyapunov function for the particular context, analogous to the one given in [4].

Definition 2. A positive definite continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a global control Lyapunov function for (1) if there exist a positive $N \in \mathbb{N}$ and a switching law $\sigma(\cdot)$, defined on $\mathbb{R}^n$, such that V is non-increasing along the trajectories $x_k^\sigma(x)$ and decreasing after N steps, i.e. $V(x_1^\sigma(x)) \leq V(x)$ and $V(x_N^\sigma(x)) < V(x)$, for all $x \in \mathbb{R}^n$.

Definition 2 is a standard definition of global control Lyapunov function except for the N-steps decreasing requirement. On the other hand, such a function implies the convergence of every subsequence in $j \in \mathbb{N}$ of the trajectory, i.e. $x_{i+jN}^\sigma(x)$ for all $i < N$, then also the convergence of the trajectory itself. This, with the stability assured by $V(x_1^\sigma(x)) \leq V(x)$, ensures global asymptotic stabilizability.

Definition 3. The system (1) is globally exponentially stabilizable if there are $c \geq 0$ and $\lambda \in [0,1)$ and, for all $x \in \mathbb{R}^n$, there exists a switching law $\sigma : \mathbb{N} \to \mathbb{N}_q$, such that

$$\|x_k^\sigma(x)\| \leq c\lambda^k \|x\|, \quad \forall k \in \mathbb{N}. \tag{2}$$

A periodic switching law is given by $\sigma(k) = i_{p(k)}$ and $p(k) = k - M\lfloor k/M \rfloor + 1$, with $M \in \mathbb{N}$ and $i \in \mathscr{I}^M$, which means that the sequence given by i repeats cyclically. We will consider conditions under which system (1) is stabilized by a periodic $\sigma(\cdot)$.

Definition 4. The system (1) is periodic stabilizable if there exist a periodic switching law $\sigma : \mathbb{N} \to \mathbb{N}_q$, $c \geq 0$ and $\lambda \in [0,1)$ such that (2) holds for all $x \in \mathbb{R}^n$.

For stabilizability the switching function might be state-dependent whereas for periodic stabilizability it must be not dependent on the state.

Lemma 1. *The system (1) is periodic stabilizable if and only if there exist $M \in \mathbb{N}$ and $i \in \mathscr{I}^M$ such that $\mathbb{A}_i$ is Schur.*

2.1 Geometric necessary and sufficient condition

It is proved in [26] that for an autonomous linear switched system, the origin is asymptotically stable if and only if there exists a polyhedral Lyapunov function, see also [4, 24]. Analogous results can be stated in the case where the switching sequence is a properly chosen selection, that is considering it as a control law. This contribution is based on the following algorithm.

Algorithm 1 *Computation of a contractive C^*-set for (1) satisfying Assumption 1.*

- ***Initialization****: given the C^*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;*
- ***Iteration*** *for $k \geq 0$: $\Omega_{k+1} = \bigcup_{i \in \mathbb{N}_q} \Omega_{k+1}^i$ with $\Omega_{k+1}^i = A_i^{-1} \Omega_k$ for all $i \in \mathbb{N}_q$;*
- ***Stop*** *if $\Omega \subseteq \mathrm{int}\Big(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j \Big)$; denote $\check{N} = k+1$ and*

$$\check{\Omega} = \bigcup_{j \in \mathbb{N}_{\check{N}}} \Omega_j. \tag{3}$$

From the geometrical point of view, Ω_{k+1}^i is the set of x mapped in Ω_k through A_i. Then Ω_{k+1} is the set of $x \in \mathbb{R}^n$ for which there exists a selection $i(x) \in \mathbb{N}_q$ such that $A_{i(x)} x \in \Omega_k$. Thus, Ω_k is the set of x that can be driven in Ω in at most k steps and hence $\check{\Omega}$ the set of x that can reach Ω in $\check{N}$ or less steps.

Proposition 1. *The sets Ω_k for all $k \geq 0$ are C^*-sets.*

Algorithm 1 provides a C*-set $\check{\Omega}$ contractive in $\check{N}$ steps, for every initial C*-set $\Omega \in \mathbb{R}^n$, if and only if the switched system (1) is stabilizable, as stated below.

Theorem 1 ([13]). *There exists a Lyapunov function for the switched system (1) if and only if Algorithm 1 ends with finite $\check{N}$.*

Then finite termination of Algorithm 1 is a necessary and sufficient condition for the global asymptotic stabilizability of the switched system (1). An alternative formulation of such a necessary and sufficient condition is presented below.

Theorem 2 ([13]). *There exists a Lyapunov function for the switched system (1) if and only if there exists a C*-set whose gauge function is a Lyapunov function for the system.*

Theorem 2 states that the existence of a C*-set induced Lyapunov function is a necessary and sufficient condition for stabilizability of switched systems. Hence, such functions, nonconvex and homogeneous of order one, form a class of universal Lyapunov functions for the switched systems.

Remark 2. The Algorithm 1 terminates after a finite number of iterations only if the switched system is stabilizable, there is no guarantee of finite termination in general (which means it is a semi-algorithm, to be exact). An analogous, but just sufficient, constructive condition ensuring that there is not a switching law such that the system (1) converges to the origin is given in [13].

Besides a Lyapunov function, Algorithm 1 provides a stabilizing switching control law for system (1), if it terminates in finite time.

Proposition 2 ([13]). *If Algorithm 1 ends with finite $\check{N}$ then $\Psi_{\check{\Omega}} : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the switched system (1) and given the set valued map*

$$\check{\Sigma}(x) = \arg\min_{(i,k)} \{\Psi_{\Omega_k^i}(x) : i \in \mathbb{N}_q, k \in \mathbb{N}_{\check{N}}\} \subseteq \mathbb{N}_q \times \mathbb{N}_{\check{N}}, \tag{4}$$

any switching law defined as $(\check{\sigma}(x), \check{k}(x)) \in \check{\Sigma}(x)$, is a stabilizing switching law. Furthermore, one gets $\Psi_{\check{\Omega}}(x^{\check{\sigma}}_{\check{k}(x)}(x)) \leq \check{\lambda}\Psi_{\check{\Omega}}(x)$ and $\Psi_{\check{\Omega}}(x^{\check{\sigma}}_j(x)) \leq \Psi_{\check{\Omega}}(x)$ for all $j \in \mathbb{N}_{\check{k}(x)}$, with $\check{\lambda} = \min_{\lambda}\{\lambda \geq 0 : \Omega \subseteq \lambda\check{\Omega}\} < 1$.

It could be reasonable, to speed up the convergence of the trajectory of the system to origin, to select among the elements of $\Sigma(x)$, those whose k is minimal.

Remark 3. If the system is stabilizable, then the algorithm ends with finite $\check{N}$ for all initial C*-set Ω. Clearly, the value of $\check{N}$ and the complexity of the set $\check{\Omega}$ depend on the choice of Ω. In particular, if Ω is the Euclidean norm ball (or the union of ellipsoids), the sets Ω_k^i and Ω_k, with $i \in \mathbb{N}_q$ and $k \in \mathbb{N}_{\check{N}}$, are unions of ellipsoids, and $\check{\Omega}$ also. Then, the switching law computation reduces to check the minimum among $x^T P_j x$ with $j \in \check{M}$, where $\{P_j\}_{j \in \check{M}}$ are the $\check{M}$ positive definite matrices defining $\check{\Omega}$, with $\check{M} = q + \cdots + q^{\check{N}} = (q^{\check{N}+1} - q)/(q-1)$, for $q > 1$ and $\check{M} = \check{N}$ for $q = 1$.

2.2 Duality robustness-control of switched systems

In this section, we recall some results from the literature on the stability of a switched linear system with arbitrary switching law $\sigma(\cdot)$ to highlight the analogies with the approaches proposed here for stabilizability.

Consider the linear switched system (1) and assume that the switching law is arbitrary. This would mean that the switching law might be regarded as a parametric uncertainty and the results in [26, 3, 4] on robust stability apply with minor adaptations, see also [24]. The following algorithm provides a polytopic contractive set, and then an induced polyhedral Lyapunov function, for this class of systems, see [5].

Algorithm 2 *Computation of a λ-contractive C-set for (1) with arbitrary switching.*

- ***Initialization***: *given the C-set $\Gamma \subseteq \mathbb{R}^n$ and $\lambda \in [0,1)$, define $\Gamma_0 = \Gamma$ and $k = 0$;*
- ***Iteration*** *for $k \geq 0$: $\Gamma_{k+1} = \Gamma \cap \bigcap_{i \in \mathbb{N}_q} \lambda A_i^{-1} \Gamma_k$;*
- ***Stop*** *if $\Gamma_k \subseteq \Gamma_{k+1}$; denote $\hat{N} = k$ and $\hat{\Gamma} = \Gamma_k$.*

The set $\hat{\Gamma}$ is the maximal λ-contractive set in Γ for the switched system with arbitrary switching law. Provided that Algorithm 2 terminates with finite $\hat{N}$, it can be proved that the system is globally exponentially stable, see [4].

Remark 4. Notice the analogies between Algorithms 1 and 2: they share the same iterative structure and they both generate contractive sets which induce Lyapunov functions if they terminate in a finite time. The main substantial difference resides in the use of intersection/union operators and in the family of sets generated, C*-sets by Algorithm 1 and C-sets by Algorithm 2. Interestingly, the C-sets are closed under the intersection operation whereas C*-sets are closed under the union.

Finally, for linear parametric uncertain systems, the existence of a polyhedral Lyapunov function is a necessary and sufficient condition for asymptotic stability.

Theorem 3 ([26, 4]). *There is a Lyapunov function for a linear parametric uncertain system if and only if there is a polyhedral Lyapunov function for the system.*

The result in Theorem 3 holds for general parametric uncertainty and applies also for switched systems with arbitrary switching law, as remarked in [24].

Remark 5. As for the duality of Algorithms 1 and 2 highlighted in Remark 4, evident conceptual analogies hold between Theorems 2 and 3. Then the class of gauge functions induced by C*-sets is universal for linear switched systems with switching control law, in analogy with the class of polyhedral functions (i.e. induced by C-sets) for the case of arbitrary switching law, [3, 4].

3 Novel conditions for stabilizability and comparisons

As seen above, system (1) is stabilizable if and only if there exists $N \in \mathbb{N}$ such that

$$\Omega \subseteq \mathrm{int}\Big(\bigcup_{i \in \mathscr{I}^{[1:N]}} \Omega_i \Big) \quad \text{with} \quad \Omega_i = \Omega_i(\Omega) = \{x \in \mathbb{R}^n : \ \mathbb{A}_i x \in \Omega\}. \tag{5}$$

Since the stabilizability property does not depend on the choice of the initial C*-set Ω, even if N does, focusing on the case $\Omega = \mathbf{B}$ and ellipsoidal pre-images entails no loss of generality, see [13]. Then condition (3) can be replaced by

$$\mathbf{B} \subseteq \mathrm{int}\Big(\bigcup_{i \in \mathscr{I}^{[1:N]}} \mathbf{B}_i \Big) \quad \text{with} \quad \mathbf{B}_i = \{x \in \mathbb{R}^n :\ x^T \mathbb{A}_i^T \mathbb{A}_i x \leq 1\}, \tag{6}$$

for what concerns stabilizability, although the value N might depend on the choice of Ω. The set inclusions (5) or (6) are the stopping conditions of the algorithm and then must be numerically checked at every step. The main computational issue consists in determining if a C*-set Ω is included into the interior of the union of some C*-sets. This problem is very complex in general, also in the case of ellipsoidal sets where it relates to quantifier elimination over real closed fields [7]. On the other hand, the condition given by Theorem 1 provides an exact characterization of the complexity inherent to the problem of stabilizing a switched linear system.

The objective here is to consider alternative conditions for stabilizability to provide geometrical and numerical insights and analyze their conservatism by comparison with the necessary and sufficient one given in Theorem 1.

3.1 Lyapunov-Metzler BMI conditions

The condition we are considering first is related to the Lyapunov-Metzler inequalities that is sufficient and given by a set of BMI inequalities involving Metzler matrices.

Theorem 4 ([17]). *If there exist $P_i > 0$, with $i \in \mathscr{I}$, and $\pi \in \mathscr{M}_q$ such that*

$$A_i^T \left(\sum_{j=1}^{q} \pi_{ji} P_j \right) A_i - P_i < 0, \qquad \forall i \in \mathscr{I}, \tag{7}$$

holds, then the switched system (1) is stabilizable.

As proved in [17], the satisfaction of (7) implies that the homogeneous function induced by the set $\bigcup_{i \in \mathscr{I}} \mathscr{E}(P_i)$ is a control Lyapunov function. A first relation between the Lyapunov-Metzler condition (7) and the geometric one (5) is provided below. We prove that the satisfaction of (7) implies that the condition given by Theorem 1 holds for the particular case of $\Omega = \bigcup_{i \in \mathscr{I}} A_i \mathscr{E}(P_i)$ and $N = 1$.

Proposition 3 ([14]). *If the Lyapunov-Metzler condition (7) holds then (5) holds with $N = 1$ and $\Omega = \bigcup_{i \in \mathscr{I}} A_i \mathscr{E}(P_i)$.*

Proposition 3 provides a geometrical meaning of the Lyapunov-Metzler condition and a first relation with the necessary and sufficient condition given in Theorem 1. In fact, for the general case of $q \in \mathbb{N}$ the Lyapunov-Metzler condition is just sufficient for $\bigcup_{i \in \mathscr{I}} A_i \mathscr{E}(P_i) \subseteq \mathrm{int}(\bigcup_{i \in \mathscr{I}} \mathscr{E}(P_i))$ to hold. Moreover, it is proved in [14] that the condition is also necessary for $q = 2$.

3.2 Generalized Lyapunov-Metzler conditions

Two generalizations of the Lyapunov-Metzler condition can be given, by relaxing the intuitive but unnecessary constraint stating that the number of ellipsoids and the number of modes are equal.

Proposition 4 ([14]). *If there exist $M \in \mathbb{N}$ and $P_i > 0$, with $i \in \mathscr{I}^{[1:M]}$, and $\pi \in \mathscr{M}_{M_{\mathscr{I}}}$ such that*

$$\mathbb{A}_i^T \left(\sum_{j \in \mathscr{I}^{[1:M]}} \pi_{ji} P_j \right) \mathbb{A}_i - P_i < 0, \qquad \forall i \in \mathscr{I}^{[1:M]},$$

holds, then the switched system (1) is stabilizable.

Proposition 4 extends the Lyapunov-Metzler condition (7), which is recovered for $M = 1$. Another possible extension is obtained by maintaining the sequence length in 1 but increasing the number of ellipsoids involved.

Proposition 5 ([14]). *If for every $i \in \mathscr{I}$ there exist a set of indices $\mathscr{K}_i = \mathbb{N}_{h_i}$, with $h_i \in \mathbb{N}$; a set of matrices $P_k^{(i)} > 0$, with $k \in \mathscr{K}_i$, and there are $\pi_{m,k}^{(p,i)} \in [0,1]$, satisfying $\sum_{p \in \mathscr{I}} \sum_{m \in \mathscr{K}_p} \pi_{m,k}^{(p,i)} = 1$ for all $k \in \mathscr{K}_i$, such that*

$$A_i^T \left(\sum_{p \in \mathscr{I}} \sum_{m \in \mathscr{K}_p} \pi_{m,k}^{(p,i)} P_m^{(p)} \right) A_i - P_k^{(i)} < 0, \quad \forall i \in \mathscr{I}, \, \forall k \in \mathscr{K}_i,$$

holds, then the switched system (1) is stabilizable.

Geometrically, Proposition 5 provides a condition under which there exists a C*-set composed by a finite number of ellipsoids that is contractive.

3.3 LMI sufficient condition

The main drawback of the necessary and sufficient condition for stabilizability is its inherent complexity. The Lyapunov-Metzler-based approach leads to a more practical BMI sufficient condition. Nevertheless, the complexity could be still computationally prohibitive, see [33]. Our next aim is to formulate an alternative condition that could be checked by convex optimization algorithms.

Theorem 5 ([14]). *The switched system (1) is stabilizable if there exist $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{N_{\mathscr{I}}}$ such that $\eta \geq 0$, $\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i = 1$ and*

$$\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I. \tag{8}$$

We wonder now if the sufficient condition given in Theorem 5 is also necessary. The answer is negative, in general, as proved by the following counter-example.

Example 1. The aim of this illustrative example is to show a case for which the inclusion condition (6) is satisfied with $N = 1$, but there is not a finite value of $\hat{N} \in \mathbb{N}$ for which condition (8) holds. Consider the three modes given by the matrices

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} R\left(\frac{2\pi}{3}\right), \quad A_3 = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} R\left(\frac{-2\pi}{3}\right),$$

with $a = 0.6$. Set $\Omega = \mathbf{B}$. By geometric inspection, condition (6) holds at the first step, i.e. for $N = 1$, see [14]. On the other hand, A_i are such that $\det(A_i^T A_i) = a^2 a^{-2} = 1$ and $\mathrm{tr}(A_i^T A_i) = a^2 + a^{-2} = 3.1378$ while the determinant and trace of the matrix defining $\mathbf{B}$ are 1 and 2, respectively. Notice that $a^2 + a^{-2} > 2$ for every a different from 1 or -1 and $a^2 + a^{-2} = 2$ otherwise.

For every N and every $\mathbf{B}_i$ with $i \in \mathscr{I}^{[1:N]}$, the related $\mathbb{A}_i$ is such that $\det(\mathbb{A}_i^T \mathbb{A}_i) = 1$ and $\mathrm{tr}(\mathbb{A}_i^T \mathbb{A}_i) \geq 2$. Notice that, for all the matrices $Q > 0$ in $\mathbb{R}^{2\times 2}$ such that $\det(Q) = 1$, then $\mathrm{tr}(Q) \geq 2$ and $\mathrm{tr}(Q) = 2$ if and only if $Q = I$, since the determinant is the product of the eigenvalues and the trace its sum. Thus, for every subset of the ellipsoids $\mathbf{B}_i$, determined by a subset of indices $K \subseteq \mathscr{I}^{[1:N]}$, we have that $\sum_{i\in K} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I$, cannot hold, since either $\mathrm{tr}(\mathbb{A}_i^T \mathbb{A}_i) > 2$ or $\mathbb{A}_i^T \mathbb{A}_i = I$. Thus the LMI condition (8) is sufficient but not necessary.

Another interesting implication that follows from Example 1 concerns the stabilizability through periodic switching sequences.

Proposition 6 ([14]). *The existence of a stabilizing periodic switching law is sufficient but not necessary for the stabilizability of the system (1).*

In the proof of Proposition 6 we used the fact that the existence of a stabilizing periodic switching law implies the satisfaction of the LMI condition, see [14]. One might wonder if there exists an equivalence relation between periodic stabilizability and condition (8). The answer is provided below.

Theorem 6 ([14]). *A stabilizing periodic switching law for the system (1) exists if and only if condition (8) holds.*

Note that, although periodic stabilizability and condition (8) are equivalent from the stabilizability point of view, the computational aspects and the resulting controls are different. Indeed, checking periodic stabilizability consists of an eigenvalue test for a number of matrices exponential in M, see Lemma 1, while condition (8) is an LMI that grows exponentially with N. On the other hand, M is always greater or equal than N, much greater in general. Finally, notice that the periodic law is in open loop whereas (8) leads to a state-dependent switching law.

The LMI condition (8) can be used to derive the controller synthesis techniques. If (8) holds, then there is $\mu \in [0,1)$ such that

$$\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i \leq \mu^2 I. \tag{9}$$

A stabilizing controller does not necessarily select at each time step $k \in \mathbb{N}$ the input to be applied. This can be done only at $\{k_p\}_{p\in\mathbb{N}}$ with $k_0 = 0$, and $k_p < k_{p+1} \leq k_p + N$, for all $p \in \mathbb{N}$. At time k_p, the controller selects the sequence of inputs to be applied up to step $k_{p+1}-1$. The instant k_{p+1} is also determined by the controller at time k_p. More precisely, the controller acts as follows for all $p \in \mathbb{N}$, let

$$i_p = \arg \min_{i \in \mathscr{I}^{[1:N]}} (x_{k_p}^T \mathbb{A}_i^T \mathbb{A}_i x_{k_p}). \tag{10}$$

Then, the next instant k_{p+1} is given by

$$k_{p+1} = k_p + l(i_p), \tag{11}$$

with $l(i_p)$ length of i_p, and the controller applies the sequence of inputs

$$\sigma_{k_p+j-1} = i_{p,j}, \quad \forall j \in \{1, \dots, l(i_p)\}. \tag{12}$$

Theorem 7 ([14]). *Assume that (8) holds, and consider the control given by (10), (11), (12). For all $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we have $\|x_k\| \leq \mu^{k/N-1} L^{N-1} \|x_0\|$, where $L \geq \|A_i\|$, for all $i \in \mathscr{I}$ and $L \geq 1$, and the controlled system is globally exponentially stable.*

From Theorem 7, the LMI condition (8) implies that the switched system with the switching rule given by (10), (11), (12) is globally exponentially stable. Nevertheless, neither the Euclidean norm of x nor the function $\min_{i \in \mathscr{I}^{[1:N]}} (x^T \mathbb{A}_i^T \mathbb{A}_i x)$ are monotonically decreasing along the trajectories. On the other hand a positive definite homogeneous nonconvex function decreasing at every step can be inferred for a different switching rule.

Proposition 7 ([14]). *Given the switched system (1), suppose there exist $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{N_{\mathscr{I}}}$ such that $\eta \geq 0$, $\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i = 1$ and (8) hold. Then there is $\lambda \in [0,1)$ such that the function*

$$V(x) = \min_{i \in \mathscr{I}^{[1:N]}} (x^T \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i x), \tag{13}$$

where n_i is the length of $i \in \mathscr{I}^{[1:N]}$, satisfies $V(A_{\sigma(x)} x) \leq \lambda V(x)$ for all $x \in \mathbb{R}^n$, with

$$i^*(x) = \arg \min_{i \in \mathscr{I}^{[1:N]}} (x^T \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i x), \tag{14}$$

and $\sigma(x) = i_1^*(x)$.

Remark 6. If the LMI (8) has a solution, then there exists a scalar $\mu \in [0,1)$, such that (9) is verified. The value of μ induces the rate of convergence λ for the Lyapunov function (13). Thus one might solve the optimization problem $\min_{\mu^2, \eta} \mu^2$ subject to (9), to get higher convergence rate.

3.4 Stabilizability conditions relations

The implications between the stabilizability conditions, whose proofs can be found in [14], are summarized in Figure 1. Remark that, compared to the Lyapunov-Metzler inequalities (7), the LMI condition (8) concerns a convex problem and it is less conservative. On the other hand, the dimension of the LMI problem might be consistently higher than the BMI one. The direct extension to the case of output-based switching design is not straightforward and requires further research. Nevertheless, since the LMI condition and the periodic stabilizability are equivalent, if (8) has a solution then an open-loop stabilizing switching sequence can be designed, and no output is necessary to stabilize the system.

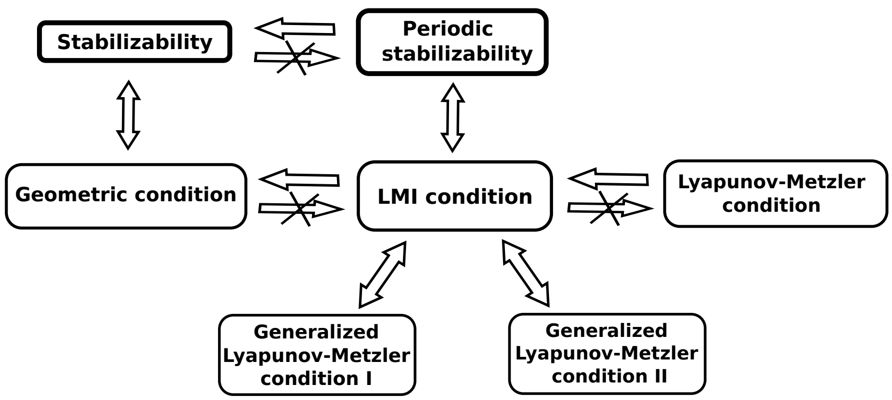

Fig. 1 Implications diagram of stabilizability conditions.

4 Control co-design for discrete-time switched linear systems

Consider now the discrete-time controlled switched linear system

$$x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k, \tag{15}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and the control input at time $k \in \mathbb{N}$, respectively; $\sigma : \mathbb{N} \to \mathscr{I}$ is the switching law and $\{A_i\}_{i\in\mathscr{I}}$ and $\{B_i\}_{i\in\mathscr{I}}$, with $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ for all $i \in \mathscr{I}$. A time-varying control policy $\nu : \mathbb{R}^n \times \mathbb{N} \to \mathscr{I} \times \mathbb{R}^{m\times n}$, is such that $\nu(x,k) = \big(\sigma(x,k),\, K(x,k)\big) \in \mathscr{I} \times \mathbb{R}^{m\times n}$, where $K(x,k)$ is the state feedback gain that may change at every instant, i.e. such that $u_k(x_k) = K(x_k,k)x_k$.

Remark 7. As proved in [35], see Theorems 5 and 7 in particular, the attention can be restricted without loss of generality to static control policies of the form

$$\nu(x) = \big(\sigma(x),\, K(x)\big) \in \mathscr{I} \times \mathbb{R}^{m\times n}, \tag{16}$$

such that $\nu(ax) = \nu(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, and to piecewise quadratic Lyapunov functions. Moreover $K(x)$ belongs to a finite set, i.e. $K(x) \in \mathscr{K} = \{\kappa_i\}_{i\in\mathbb{N}_M}$.

The switched system in closed loop with (16) reads

$$x_{k+1} = \left(A_{\sigma(x_k)} + B_{\sigma(x_k)} K(x_k)\right) x_k, \tag{17}$$

where $\sigma(x_k) = \sigma_k$. We denote with $x_k^\nu(x_0) \in \mathbb{R}^n$ the state of the system (15) at time k starting from $x(0) = x_0$ by applying the control policy ν. Given $\sigma \in \mathscr{I}^D$ we denote with $x_k^\sigma(x_0)$ the state of (17) at time $k \le D$ starting at x_0 under the switching sequence σ. The dependence of x_k^ν and x_k^σ on the initial conditions will be dropped.

Definition 5. The system (15) is globally exponentially stabilizable if there are a control policy $\nu(x)$ as in (16), $c \ge 0$ and $\lambda \in [0,1)$ such that $\|x_k^\nu(x_0)\| \le c\lambda^k \|x_0\|$, for all $x_0 \in \mathbb{R}^n$, with x_k state of (17).

In Sections 3.3 and 3.4 we proved that, for autonomous systems as (1), periodic stabilizability is more conservative than generic stabilizability. On the other hand, the equivalent condition is much more computationally tractable. Indeed, the condition in case of periodic stabilizability is an LMI in the parameter N that might by much smaller than the periodic cycle length. Hereafter we focus on a condition analogous to the LMI one (8) for the controlled switched system (15) to determine a stabilizing control policy (16) for periodic stabilizable systems.

From Remark 7, the problem of co-design is equivalent to determine a stabilizing static control policy as in (16), with finite number of feedback gains, and a piecewise quadratic Lyapunov function for the system (17). Applying Theorem 6, the objective is to search for sequences of modes and feedback gains, fulfilling the LMI condition (8) in the context of co-design. That is, given a sequence $\vartheta \in \mathscr{I}$, of length J, and a time instant $j \in \mathbb{N}_J$, a gain among the finite set $\mathscr{K}$ can be applied, denoted as K_j^ϑ and whose value has to be designed. Then, with a slight abuse of notation, given $J \in \mathbb{N}$ and a sequence $\vartheta \in \mathscr{I}^J$, we denote

$$\mathbb{F}_\vartheta = \prod_{j=1}^{J} F_{\vartheta_j} = F_{\vartheta_J} \dots F_{\vartheta_1} = (A_{\vartheta_J} + B_{\vartheta_J} K_J^\vartheta) \dots (A_{\vartheta_1} + B_{\vartheta_1} K_1^\vartheta). \tag{18}$$

Thus a set of $N_{\mathscr{I}} = \sum_{k=1}^N q^k$ matrices $\mathbb{F}_\vartheta$, one for every $\vartheta \in \mathscr{I}^{[1:N]}$, can be defined as in (18) that are parameterized in the gains $\{K_j^\vartheta\}_{j\in\mathbb{N}_{|\vartheta|}}$. We focus on the control policy for (15) of the form (16) where $K(x)$ belongs to one of the elements of a sequence associated to a mode in $\mathscr{I}^{[1:N]}$. Then, $K(x)$ is a gain among the $\sum_{k=1}^N kq^k$ possible, i.e. $K(x) \in \mathscr{K}$ where

$$\mathscr{K} = \{\kappa_i\}_{i\in\mathbb{N}_M} = \{K_j^\vartheta \in \mathbb{R}^{m\times n} : \ \vartheta \in \mathscr{I}^{[1:N]},\ j \in \mathbb{N}_{|\vartheta|}\}, \tag{19}$$

with $M = \sum_{k=1}^N kq^k$. Given a switching law $\vartheta : \mathbb{N} \to \mathscr{I}$ and a sequence of feedback gains $K^\vartheta : \mathbb{N} \to \mathbb{R}^{m\times n}$, we denote with $x_k^\vartheta(x)$ the state at time k starting at x if the control $\nu_k = (\vartheta_k, K_k^\vartheta)$ is applied at k for all $k \in \mathbb{N}$. As for the case without control input, the concept of periodic ϑ-stabilizability can be given for the system (15).

Definition 6. The system (15) is periodic ϑ-stabilizable if there exist: a periodic switching law $\vartheta : \mathbb{N} \to \mathscr{I}$ and a periodic sequence $K^{\vartheta} : \mathbb{N} \to \mathbb{R}^{m\times n}$, both of cycle length $D \in \mathbb{N}$; $c \geq 0$ and $\lambda \in [0,1)$ such that $\|x_k^{\vartheta}(x)\| \leq c\lambda^k \|x\|$ holds for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

Clearly periodic ϑ-stabilizability is sufficient for exponential stabilizability of (15) as in Definition 5. From Definition 6 and Theorem 6, the conditions

$$\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i = 1, \tag{20}$$

$$\sum_{j \in \mathscr{I}^{[1:N]}} \eta_j \mathbb{F}_j^T \mathbb{F}_j < I, \tag{21}$$

are necessary and sufficient for periodic ϑ-stabilizability of system (15). Thus, condition (21) is an LMI that provides the exact characterization of ϑ-stabilizability, together with (20). Below we give a convex condition equivalent to (21).

Proposition 8 ([15]). *Given $N \in \mathbb{N}$, $\eta \in \mathbb{R}^{N_{\mathscr{I}}}$ with $\eta > 0$, and the set of gains (19), condition (21) holds if and only if for every $j \in \mathscr{I}^{[1:N]}$ there exist $|j|-1$ nonsingular matrices $G_{j,k} \in \mathbb{R}^{n\times n}$ with $k \in \mathbb{N}_{|j|-1}$ and $R_j \in \mathbb{R}^{n\times n}$ such that $R_j = R_j^T > 0$ and*

$$\begin{bmatrix} \eta_j I & X_{j,|j|} & 0 & \dots & 0 & 0 & 0 \\ X_{j,|j|}^T & Y_{j,|j|-1} & X_{j,|j|-1} & \dots & 0 & 0 & 0 \\ 0 & X_{j,|j|-1}^T & Y_{j,|j|-1} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & Y_{j,2} & X_{j,2} & 0 \\ 0 & 0 & 0 & \dots & X_{j,2}^T & Y_{j,1} & X_{j,1} \\ 0 & 0 & 0 & \dots & 0 & X_{j,1}^T & R_j \end{bmatrix} > 0 \tag{22}$$

for every $j \in \mathscr{I}^{[1:N]}$ with $X_{j,1} = \eta_j \mathbb{F}_{j_1}$ and $X_{j,k+1} = \mathbb{F}_{j_{k+1}} G_{j,k}$ and $Y_{j,k} = G_{j,k} + G_{j,k}^T$ for all $k \in \mathbb{N}_{|j|-1}$ and

$$\sum_{j \in \mathscr{I}^{[1:N]}} R_j < I. \tag{23}$$

The following theorems, based on Proposition 8, provide a necessary and sufficient LMI condition for periodic ϑ-stabilizability of the controlled system (15), see their proofs in [15]. Moreover, the explicit form of the control law (16) is given.

Theorem 8 ([15]). *The system (15) is periodically ϑ-stabilizable if and only if there exist $N \in \mathbb{N}$; $\eta \in \mathbb{R}^{N_{\mathscr{I}}}$ such that $\eta > 0$ and (20) holds; and for every $j \in \mathscr{I}^{[1:N]}$ there are:*

- *$|j|-1$ nonsingular matrices $G_{j,k} \in \mathbb{R}^{n\times n}$, with $k \in \mathbb{N}_{|j|-1}$;*
- *$|j|$ matrices $Z_{j,k} \in \mathbb{R}^{m\times n}$ with $k \in \mathbb{N}_{|j|}$;*
- *a symmetric positive definite matrix $R_j \in \mathbb{R}^{n\times n}$;*

such that (22) and (23) hold with

$$\begin{aligned} X_{j,1} &= \eta_j A_{j_1} + B_{j_1} Z_{j,1}, & \\ X_{j,k+1} &= A_{j_{k+1}} G_{j,k} + B_{j_{k+1}} Z_{j,k+1}, & \forall k \in \mathbb{N}_{|j|-1}, \\ Y_{j,k} &= G_{j,k} + G_{j,k}^T, & \forall k \in \mathbb{N}_{|j|-1}, \end{aligned} \tag{24}$$

and gains

$$\begin{aligned} K_1^j &= \eta_j^{-1} Z_{j,1}, & \\ K_{k+1}^j &= Z_{j,k+1} G_{j,k}^{-1}, & \forall k \in \mathbb{N}_{|j|-1}, \end{aligned} \tag{25}$$

for all $j \in \mathscr{I}^{[1:N]}$.

The following theorem provides a ϑ-stabilizability condition, a control policy and a bound on the decreasing of the Euclidean norm every N steps at most.

Theorem 9 ([15]). *Suppose there exist* $\alpha > 1$ *and* $N \in \mathbb{N}$*;* $\eta \in \mathbb{R}^{N_{\mathscr{I}}}$ *such that* $\eta > 0$*; matrices* $G_{j,k} \in \mathbb{R}^{n \times n}$ *with* $k \in \mathbb{N}_{|j|-1}$*,* $Z_{j,k} \in \mathbb{R}^{m \times n}$ *with* $k \in \mathbb{N}_{|j|}$ *and* $R_j \in \mathbb{R}^{n \times n}$ *as defined in Theorem 8 such that (22)-(23) and (24) hold and*

$$\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i = \alpha. \tag{26}$$

Then system (15) is periodically ϑ*-stabilizable and* $\|\mathbb{F}_{\vartheta(x)} x\|_2 < \lambda \|x\|_2$ *holds for all* $x \in \mathbb{R}^n$*, with*

$$\vartheta = \vartheta(x) = \arg \min_{j \in \mathscr{I}^{[1:N]}} (x^T \mathbb{F}_j^T \mathbb{F}_j x),$$

and $\lambda = \alpha^{-1/2}$*. Given* $x(t) = x$*, the stabilizing control policy is defined from (25) within an horizon of length* $|\vartheta|$ *as*

$$\nu(x,k) = (\sigma(x,k), K(x,k)) = \left(\vartheta_k, K_k^{\vartheta}\right) \tag{27}$$

to be applied at time $t+k-1$*, for all* $k \in \mathbb{N}_{|\vartheta|}$.

From Theorem 9, the value α is related to λ and then could serve for obtaining the fastest decreasing rate, for a given N, by solving the following LMI problem

$$\begin{aligned} \alpha = \sup_{\alpha,\eta,G_{j,k},Z_{j,k},R_j} & \sum_{j \in \mathscr{I}^{[1:N]}} \eta_j \\ \text{s.t. } & (22)-(23)-(24), \end{aligned} \tag{28}$$

with $\eta, G_{j,k}, Z_{j,k}, R_j$ as defined in Theorem 8.

Remark 8. A nonconvex control Lyapunov function $V(x)$, decreasing at every step, analogous to (13), and a state-dependent control policy $\nu(x)$ as in (16) can be defined by solving on-line an LMI problem, see [15].

The interested reader is referred to [15] for a detailed comparison analysis, in terms of conservatism and complexity, of this approach with respect to methods from the literature, such as those presented in [35, 36] and in [17, 12].

5 Numerical examples

Some illustrative examples, taken from [13, 14, 15], follow.

Example 2. Consider the system (1) with $q = 4$, $n = 2$ and

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.8 \end{bmatrix}, \quad A_2 = 1.1\,R(\frac{2\pi}{5}) \quad A_3 = 1.05\,R(\frac{2\pi}{5} - 1), \quad A_4 = \begin{bmatrix} -1.2 & 0 \\ 1 & 1.3 \end{bmatrix}.$$

The matrices A_i, with $i \in \mathbb{N}_4$, are not Schur, which implies that the system (1) is not stabilizable by any constant switching law. We apply Algorithm 1 with $\Omega = \mathbb{B}^2$. The

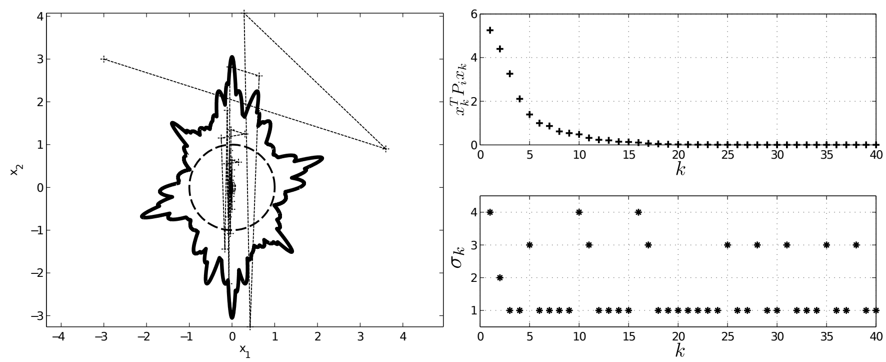

Fig. 2 Left: ball $\mathbb{B}^2$ in dashed and $\bigcup_{k \in \mathbb{N}_5} \Omega_k$ in solid line. Trajectory starting from $x_0 = (-3,3)^T$ in dotted line. Right: Lyapunov function and switching control law in time.

algorithm stops at the fifth iteration. Figure 2 left, emphasizes that $\mathbb{B}^2$ is included in $\bigcup_{k \in \mathbb{N}_5} \Omega_k$. A stabilizing switching law and the related Lyapunov function are given in Figure 2 right, for the initial condition $x_0 = (-3,3)^T$.

Example 3. Consider the system (1) with $q = 2$, $n = 2$ and

$$A_1 = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.9 \end{bmatrix} R(\theta), \quad A_2 = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix},$$

non-Schur. From Figure 3, left, one can infer that the system is not stabilizable if $\theta = 0$. Nevertheless, taking $\theta = \frac{\pi}{5}$, Algorithm 1 stops after seven steps implying the stabilizability of the system, see Figure 3, right.

Example 4. Consider (1) with $q = 2, n = 2, x_0 = [-3,3]^T$ and the non-Schur matrices

$$A_1 = 1.01R\left(\frac{\pi}{5}\right), \quad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix}.$$

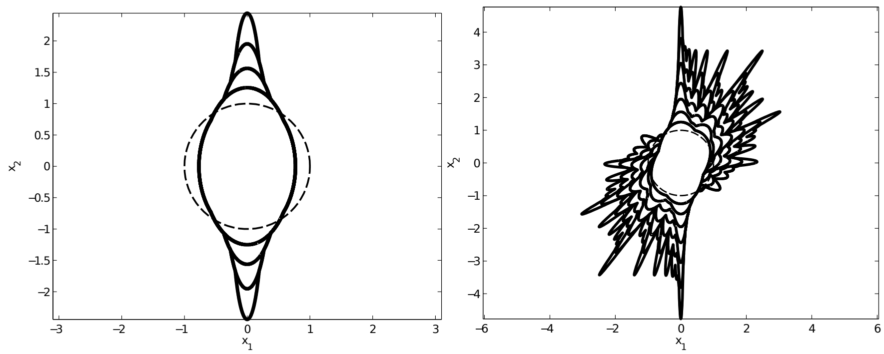

Fig. 3 Ball $\mathbb{B}^2$ in dashed and $\cup_{k \in \mathbb{N}_i} \Omega_k$, for $i \in \mathbb{N}_i$ in solid line for Example 3, with $\theta = 0$ (left) and $\theta = \frac{\pi}{5}$ (right).

Four switching laws are designed and compared: the geometric condition given in Theorem 1, proving the stabilizability of the system; the min-switching strategy (10)-(12) related to the LMI condition (8); the switching control law given in Proposition 7 and the periodic switching law, that exists from Theorem 6.

As noticed in [17, 13], for systems with $q = 2$ the Lyapunov-Metzler inequalities become two linear matrix inequalities once two parameters, both contained in $[0, 1]$, are fixed. Such LMIs have been checked for this example to be infeasible on a grid of these two parameters, with step of 0.01. It is then reasonable to conclude that the Lyapunov-Metzler inequalities are infeasible for this numerical example.

Then, an iterative procedure is applied to determine $N \in \mathbb{N}$ such that (6) is satisfied. The result is that (6) holds with $N = 5$ and then the homogeneous function induced by the obtained set is a control Lyapunov function and the related min-switching rule is a stabilizing law. The state evolution and the switching law are depicted in Figure 4, left.

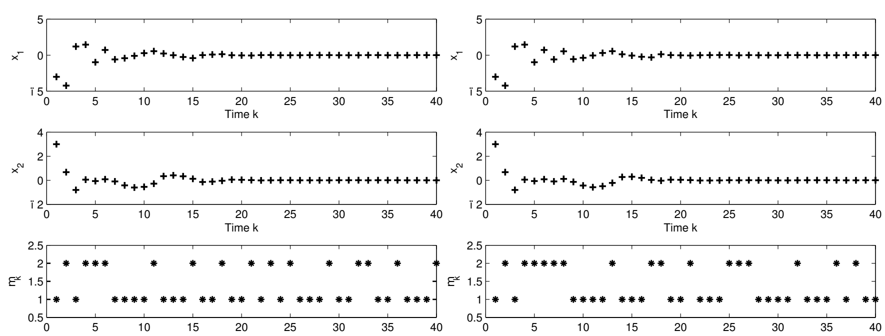

Fig. 4 State evolution and switching control induced by the geometric condition (6) (left), and min-switching control (10)-(12) (right).

The LMI condition (8) is solved with $N = 7$ and the min-switching law (10)-(12) is applied to the system at first. The control results in the concatenation of elements of $\mathscr{I}^{[1,7]}$, respectively of lengths $\{7,6,5,7,7,\dots\}$. The time-varying length of the switching subsequences is a consequence of the state dependence of the min-switching strategy. The resulting behavior is depicted in Figure 4, right. Then, the control law defined in Proposition 7, namely (14) with $\lambda = 0.9661$, is applied and the result is shown in Figure 5, left. The value of λ is obtained by solving the optimization problem described in Remark 6.

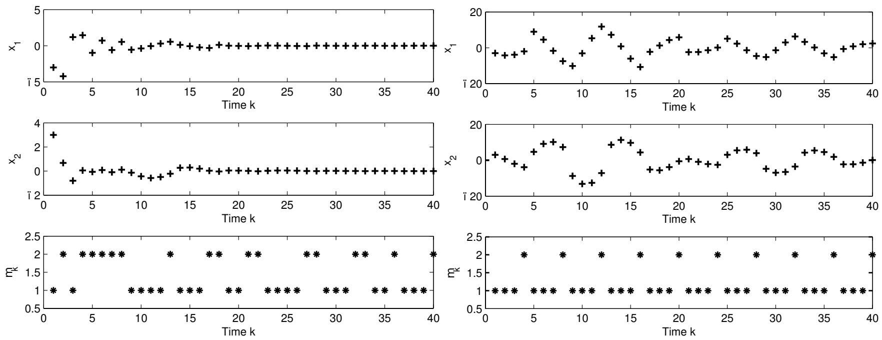

Fig. 5 State evolution and min-switching control (14) (left) and periodic switching control with $M = 4$ (right).

The periodic switching law of length $M = 4$ is then obtained, by searching the shorter sequence of switching modes which yields a Schur matrix $\mathbb{A}_i$. The resulting evolution is represented in Figure 5, right.

Finally a comparison between the switching laws is provided in Figure 6, where the time-evolution of the Euclidean distance of the state from the origin is depicted.

Example 5. Consider Example 2 in [35], that is a 4-dimensional system with 4 modes whose matrices are

$$A_1 = \begin{bmatrix} 0.5 & -1 & 2 & 3 \\ 0 & -0.5 & 2 & 4 \\ 0 & -1 & 2.5 & 2 \\ 0 & 0 & 0 & 1.5 \end{bmatrix}, A_2 = \begin{bmatrix} -0.5 & -1 & 2 & 1 \\ 0 & 1.5 & -2 & 0 \\ 0 & 0 & 0.5 & 0 \\ -2 & -1 & 2 & 2.5 \end{bmatrix}, A_3 = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & -0.5 \\ 0 & 0.5 & 1 & -0.5 \\ 1 & 0 & 0 & 0.5 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 2 & -2 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \tag{29}$$

The conditions of Theorem 9 are satisfied with horizon $N = 3$. Besides the inherent computational benefit of having a stabilization condition in form of LMI with respect to the algorithmic method presented in [35], also the control obtained is

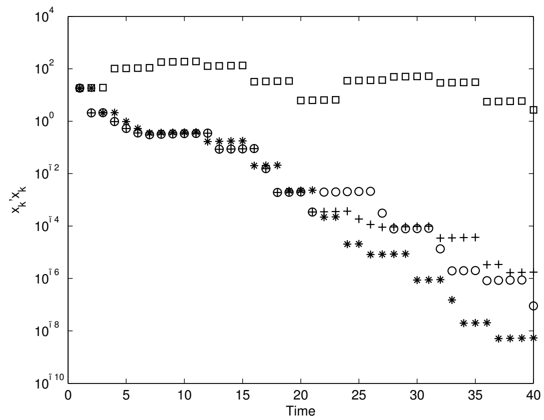

Fig. 6 Comparison between the evolution of the Euclidean norm of the state for the different switching laws: induced by geometric condition (6) (star); min-switching law (10)-(12) (cross); min-switching control (14) (circle) and periodic rule (square).

substantially simpler and more efficient. Actually, in [35] stabilizability is proved by means of an algorithm which inspects control horizons of length 7 resulting in a piecewise quadratic function determined by 13 matrices. Moreover, a much faster convergence rate is obtained by solving the LMI problem (28), if compared with the results in [35], see Figure 7 where $x_0 = [1\ 1\ 0\ -1]^T$ and Figure 4 in [35], top.

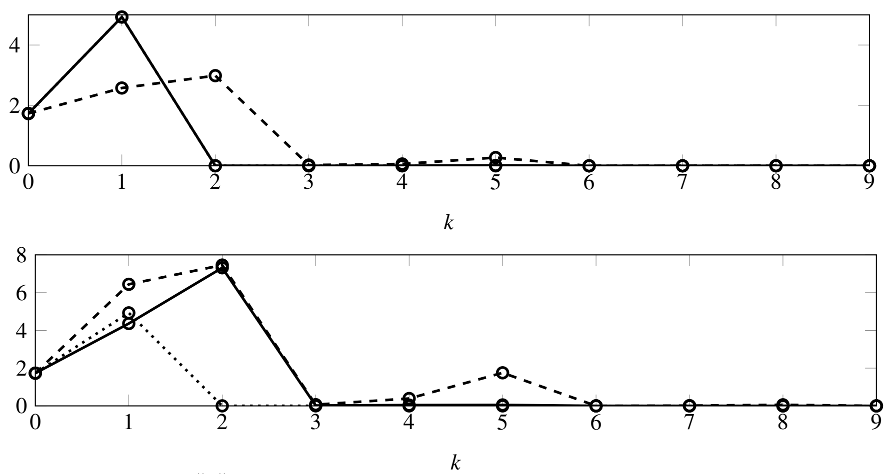

Fig. 7 Evolutions of $\|x\|_2$ with control (27), in solid, with min-switching of Remark 8 in dashed and with the periodic control in dotted with (29) (top) and with A_4 multiplied by 2.5, (bottom). In the top figure the solid and dotted lines are overlapped.

Finally, A_4 being Schur, with 4 eigenvalues equal to 0.5, a trivial stabilizing solution exists. Then we define a new A_4 by multiplying the one in (29) by 2.5. All the eigenvalues of the new A_4 are then equal to 1.25. The evolutions of the Euclidean norm of the state, for $x_0 = [1\ 1\ 0\ -1]^T$, under the obtained controls are depicted in Figure 7, bottom.

6 Conclusions

We considered the problems of stabilizability and control co-design for switched linear systems. Via a set-theory approach, a geometric necessary and sufficient condition for the stabilizability have been provided, proving that the family of nonconvex, homogeneous functions induced by a C*-set is a universal class of Lyapunov functions. Then, a novel LMI condition has been presented that overcomes the computational issues related to the geometric condition. Such a condition, together with others from the literature, have been analyzed and compared in terms of conservatism and computational complexity. Finally, an LMI condition is given for control co-design that is proved to be necessary and sufficient for the stabilizability of switched systems that admit periodic control policies.

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