Computing control invariant sets in high dimension is easy

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Abstract

In this paper we consider the problem of computing control invariant sets for linear controlled high-dimensional systems with constraints on the input and on the states. Set inclusions conditions for control invariance are presented that involve the N-step sets and are posed in form of linear programming problems. Such conditions allow to overcome the complexity limitation inherent to the set addition and vertices enumeration and can be applied also to high dimensional systems. The efficiency and scalability of the method are illustrated by computing approximations of the maximal control invariant set, based on the 10-step operator, for a system whose state and input dimensions are 30 and 15, respectively.

Key words: Invariance, computational methods, convex analysis

1 Introduction

Invariance and contractivity of sets are central properties in modern control theory. Although the first important results on invariance date back to the beginning of the seventies [3], this topic gained considerable interest in the recent years, see in particular the works by Blanchini and coauthors [5,7], mainly due to its relation with constrained control and popular optimization-based control techniques as Model Predictive Control, see [21].

Iterative procedures are given for the computation of control invariant sets that permit their practical implementation. Most of those procedures are substantially based on the one-step backward operator that associates to any set the states that can be steered into by an admissible input. Different algorithms based on the one-step operator exist for computing control invariants, that substantially differs from the initial set. For instance, if the algorithms are initialized with the state constraints set, [5,17,27], the one-step operator generates a sequence of outer approximations of the maximal control invariant that converges to it under compactness assumptions, see [3]. If, instead, the procedure is initialized with a control invariant set, a non-decreasing sequence of control invariant sets are obtained that converges from the inside to the maximal control invariant set, see the considerations on minimum-time ultimate boundedness problem in [4,7]. A particular case, suggests to initialize the procedure with the set containing the origin only (which is a control invariant in the general framework), obtaining the sequence of *i*-step null-controllable sets, that are control invariant and converges to the maximal control invariant set, see [14,16,22,10].

Thus, although the abstract iterative procedures for obtaining control invariant sets apply also for nonlinear systems, [12,13], the practical computation of the one-step set, that is the basis for them, is often prohibitively complex for their application in high dimension even in the linear context. Some constructive approaches are based on Minkowski sum and projection procedure, as in [16,4,6], which are hardly applicable in high dimension due to their numerical complexity. Other methods are based on conditions involving the vertices of the sets under analysis, [14,19,22,24], but the vertices number may grow combinatorially with the space dimension and the vertices computation is hardly manageable in high dimension. The numerical complexity has also been addressed by considering linear feedback and ellipsoidal control invariant sets, see the monograph [9], or by fixing the polyhedral set complexity [8,1,28].

In this paper we address the main problem related to the complexity of the N-step operator, for discrete-time deterministic controlled systems, with polyhedral constraints on the input and on the state. Considering polyhedral sets, such operator can be expressed in terms of Minkowski sum of polyhedra and then as an NP-complete problem [29], hardly manageable in high dimension. An algorithm is presented for determining control invariant sets that is based on a set inclusion condition involving the N-step set of a polyhedron but does not require to explicitly compute the Minkowski sum nor to have the vertices representation of the sets. Such

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condition is posed as an LP feasibility problem, hence solvable even in high dimension. Examples that show the low conservatism and the high scalability of the approach are provided.

Notations Denote with $\mathbb{R}_+$ the set of nonnegative real numbers. Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. The i-th element of a finite set of matrices or vectors is denoted as A_i. Using the notation from [26], given a mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, its inverse mapping is denoted $M^{-1} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. If M is a single-valued linear mapping, we also denote, with slight abuse of notation, the related matrices $M \in \mathbb{R}^{n\times m}$ and, if M is invertible, $M^{-1} \in \mathbb{R}^{m\times n}$. Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ we use the notation $(a,b) = [a^T \ b^T]^T \in \mathbb{R}^{n+m}$. The symbol 0 denotes, besides the zero, also the matrices of appropriate dimensions whose entries are zeros and the origin of a vectorial space, its meaning being determined by the context. The symbol $\mathbf{1}$ denotes the vector of entries 1 and I the identity matrix, their dimension is determined by the context. The subset of $\mathbb{R}^n$ containing the origin only is $\{0\}$. The symbol $\oplus$ denotes the Minkowski set addition, i.e. given $C, D \subseteq \mathbb{R}^n$ then $C \oplus D = \{x + y \in \mathbb{R}^n : \ x \in C, \ y \in D\}$. To simplify the notation, the propositions involving the existential quantifier in the definition of sets are left implicit, e.g. $\{x \in A : f(x,y) \leq 0, \ y \in B\}$ means $\{x \in A : \ \exists y \in B \text{ s.t.} f(x,y) \leq 0\}$. The unit box in $\mathbb{R}^n$ is denoted $\mathscr{B}^n$.

2 Problem formulation and preliminary results

The objective of this paper is to provide a constructive method to compute a control invariant set for controlled linear systems with constraints on the input and on the state. We would like to obtain a polytopic invariant set that could be computed through convex optimization problems. The main aim is to provide a method to obtain admissible control invariant sets for high-dimensional systems, thus no complex computational operations are supposed to be allowed.

The system is given by

$$x^+ = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input, with constraints

$$x \in X = \{y \in \mathbb{R}^n : \ Fy \leq f\}, \quad u \in U = \{v \in \mathbb{R}^m : \ Gv \leq g\}. \tag{2}$$

Assumption 1 *The matrix A is nonsingular.*

Assumption 1, not necessary but imposed here to easy the presentation, is not very restrictive. Recall for instance that every discretized linear system with no delay satisfies it. Anyway, the case of nonsingular A is developed in [11].

Some basic properties and methods, well assessed in the literature, concerning control invariant sets are recalled hereafter. Consider the set Ω containing the origin, i.e. $0 \in \Omega$, and $Q_k(\Omega, U)$ defined as

$$\begin{aligned} Q_k(\Omega,U) = \{x \in \mathbb{R}^n : \ & A^k x + \sum_{i=0}^{k-1} A^{k-1-i} B u_{k-i} \in \Omega, \\ & u_i \in U \ \forall i \in \mathbb{N}_k\}. \end{aligned} \tag{3}$$

The basic algorithm for obtaining a control invariant set consists in searching, given Ω, for the minimal N such that

$$\Omega \subseteq \operatorname{co}\left(\bigcup_{k=1}^{N} Q_k(\Omega,U)\right). \tag{4}$$

As a matter of fact, all the N for which (4) holds, lead to a control invariant set. Moreover, if (4) is satisfied, then it is satisfied for every $K \geq N$, leading to a non-decreasing sequence of nested control invariant sets.

Thus, the algorithm computes the preimages of Ω until the stop condition (4) holds and then all the states in

$$\bar{Q}_N(\Omega,U) = \operatorname{co}\left(\bigcup_{k=1}^{N} Q_k(\Omega,U)\right) \tag{5}$$

can be steered in Ω, thus in $\bar{Q}_N(\Omega,U)$, in N steps at most.

Given the initial set Ω, a condition characterizing an invariant set, alternative to (4), is the following

$$\Omega \subseteq Q_N(\Omega,U), \tag{6}$$

which is equivalent to the fact that every state in $Q_N(\Omega,U)$ can be steered in Ω in exactly N steps. This means that (6) implies, but is not equivalent to (4) and the resulting invariant set would be $\bar{Q}_N(\Omega,U)$ as in (5). Condition (6), which will be referred to as N-step condition in what follows, is just sufficient for (4) to hold but it does not require the computation of the convex hull of several sets at every iteration.

Now, to obtain estimations of the maximal control invariant set contained in X, consider

$$\begin{aligned} Q_k^x(\Omega,U,X) = \{x \in X : \ & A^k x + \sum_{i=0}^{k-1} A^{k-1-i} B u_{k-i} \in \Omega, \\ & A^j x + \sum_{i=0}^{j-1} A^{j-1-i} B u_{j-i} \in X \ \ \forall j \in \mathbb{N}_k, \\ & u_i \in U \ \ \forall i \in \mathbb{N}_k\}, \end{aligned} \tag{7}$$

that is the set of states $x \in X$ for which an admissible sequence of input of length k exists driving the state in Ω in k

steps by maintaining the trajectory in X. The resulting control invariant set would then be given by

$$\bar{Q}_N^x(\Omega,U,X) = \mathrm{co}\left(\bigcup_{k=1}^{N} Q_k^x(\Omega,U,X)\right), \qquad (8)$$

provided that condition

$$\Omega \subseteq Q_N^x(\Omega,U,X) \qquad (9)$$

holds. Note that (9) is just sufficient but, in general, less complex to be checked than $\Omega \subseteq \bar{Q}_N^x(\Omega,U,X)$.

Remark 1 *The value of N for which invariance condition (6) and (9) hold depends on the choice of Ω. Clearly, if Ω is a control invariant set, then the conditions hold for all $N \geq 1$. Moreover, for every Ω there exists $\alpha > 0$ and $N \geq 1$ such that $\alpha\Omega$ satisfies (6) or (9), under mild stabilizability conditions.*

2.1 Algorithm

The main issue which impedes the application of the algorithm in high dimension is the fact that computing the Minkowski set addition is a complex operation, as it is an NP-complete problem, see [15,29]. Moreover the addition leads to sets whose representation complexity increases. Considering, in fact, two polytopic sets Ω and Δ, their sum has in general more facets and vertices than Ω and Δ. Thus, the algorithm given above requires the computation of the Minkowski sum, hardly manageable in high dimension, and generates polytopes with an increasing number of facets and vertices. Another source of complexity is the convex hull in (4), (5) or (8), as the explicit computation of the convex hull is a non-convex operation whose complexity grows exponentially with the dimension, see [2]. Furthermore, also the vertices representation of the sets is a potential limitation for high dimensional systems, since the number of vertices may grow combinatorially with the dimension. Finally, approaches are provided, for instance in [16,6], that require the computation of the projection of polytopes, operation whose complexity is equivalent to the one of Minkowski sum. As can be seen from the comparison, provided in [15], between different projection algorithms, polytope projections are not suitable when projecting on high dimensions. This can be also heuristically checked by computing the projection over an n-dimensional subspace of a randomly generated $2n$-dimensional polytope. Using the MPT toolbox [18], for instance, we needed more than 40 seconds to project a polytope from $\mathbb{R}^{10}$ into a 5-dimensional subspace, more than 15 minutes to project from $\mathbb{R}^{12}$ to $\mathbb{R}^{6}$.

The main objective of this paper is to design a method for testing conditions (6) and (9) and for having a, potentially implicit, representation of sets (5) and (8) by means of convex optimization problems, then applicable also to relatively high dimensional systems, to obtain control invariant sets, avoiding the vertices representation of the sets and Minkowski sum or polytope projections computation.

3 N-step condition for control invariance

As noticed above, a first main issue is related to checking whether the sum of several polytopes contains a polytope, see the N-step stop condition (6) and (9).

Consider first the N-step condition (6), characterized by the Minkowski sum of several sets. The explicit definition of the Minkowski sum of sets could be avoided by employing its implicit representation. Indeed, given two polyhedral sets $\Gamma = \{x \in \mathbb{R}^m : Hx \in h\}$ and $\Delta = \{y \in \mathbb{R}^p : Gy \leq g\}$ and $P \in \mathbb{R}^{n\times m}$ and $T \in \mathbb{R}^{n\times p}$ we have that $P\Gamma \oplus T\Delta = \{x \in \mathbb{R}^n : x = Py + Tz,\ Hy \leq h,\ Gz \leq g\}$. Thus, the explicit hyperplane or vertex representation of the sum can be replaced by the implicit one, given by the projection of a polyhedron in higher dimension. On the other hand, one might wonder if the stop condition $\Omega \subseteq Q_N(\Omega,U)$ could be checked without the explicit representation of $Q_N(\Omega,U)$.

The first hint to do is that the inclusion condition is testable through a set of LP problems provided the vertices of Ω are available. Such an assumption is not very restrictive, since Ω is a design parameter that could be determined such that both the hyperplane and vertices representation should be available, a box for instance. Nevertheless, and since we are aiming at invariant sets for high dimensional systems, the use of vertices should be avoided if possible. Consider for instance, in fact, a system with $n = 20$. The unit box in $\mathbb{R}^{20}$ is characterized by 40 hyperplanes, but it has $2^{20} \simeq 10^6$ vertices. Then checking if it is contained in a set could require to solve more than a million of LP problems.

We consider then the possibility of testing whether a polyhedron is included in the sum of polyhedra by employing only their hyperplane representations and without the explicit representation of the sum of sets. The following result, based on the Farkas lemma and widely used on set theory and invariant methods for control, is useful for this purpose.

Lemma 1 *Two polyhedral sets $\Gamma = \{x \in \mathbb{R}^n : Hx \leq h\}$, with $H \in \mathbb{R}^{p\times n}$, and $\Delta = \{x \in \mathbb{R}^n : Gx \leq g\}$, with $G \in \mathbb{R}^{q\times n}$, satisfy $\Gamma \subseteq \Delta$ if and only if there exists a non-negative matrix $T \in \mathbb{R}^{q\times p}$ such that $TH = G$ and $Th \leq g$.*

Consider now the stop condition (6), which is suitable for applying the Lemma 1, as illustrated below.

The main issue for applying Lemma 1 is the fact that obtaining the explicit hyperplane representation of the set at right-hand side of (6) is numerically hardly affordable, mainly in relatively high dimension. In fact, given two polyhedra $\Gamma \subseteq \mathbb{R}^m$ and $\Delta \subseteq \mathbb{R}^p$, to determine L and l such that $P\Gamma \oplus Q\Delta = \{x \in \mathbb{R}^n :\ Lx \leq l\}$ is an NP-complete problem, see [29]. Nevertheless, a sufficient condition in form of LP feasibility problem is given below.

Theorem 1 *Consider $\Omega = \{x \in \mathbb{R}^n : Hx \leq h\}$ and U as in (2), with $H \in \mathbb{R}^{n_h \times n}$ and $G \in \mathbb{R}^{n_g \times m}$, and suppose that $0 \in \Omega$ and $0 \in U$. Then the set $\bar{Q}_N(\Omega, U)$ as in (5) is a control invariant set if there exist $T \in \mathbb{R}^{n_{\bar{g}} \times n_h}$ and $M \in \mathbb{R}^{\bar{n} \times \bar{n}}$, with $n_{\bar{g}} = n_h + N n_g$ and $\bar{n} = n + Nm$, such that*

$$\begin{cases} T\bar{H} = \bar{G}M \\ Th \leq \bar{g} \\ \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} M \end{cases} \tag{10}$$

hold with

$$\bar{G} = \begin{bmatrix} HA^N & HB & HAB & \dots & HA^{N-1}B \\ 0 & G & 0 & \dots & 0 \\ 0 & 0 & G & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & G \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} h \\ g \\ g \\ \dots \\ g \end{bmatrix} \tag{11}$$
$$\bar{H} = \begin{bmatrix} H & 0 & 0 & \dots & 0 \end{bmatrix}$$

where $\bar{G} \in \mathbb{R}^{n_{\bar{g}} \times \bar{n}}$, $\bar{g} \in \mathbb{R}^{n_{\bar{g}}}$, and $\bar{H} \in \mathbb{R}^{n_h \times \bar{n}}$.

Proof: Consider condition (6), sufficient for $\bar{Q}_N(\Omega, U)$ to be a control invariant set. The right-hand side term of (6) is given by

$$\begin{aligned} Q_N(\Omega, U) = \{x \in \mathbb{R}^n : \ & HA^N x + HBu_1 + HA^1 Bu_2 + \dots \\ & + HA^{N-1} Bu_N \leq h, \ Gu_i \leq g, \ \forall i \in \mathbb{N}_N\} \end{aligned}$$

and then it is the projection on $\mathbb{R}^n$ of the set

$$\begin{aligned} \bar{\Omega}_N = \{(x, u_1, u_2, \dots, u_N) \in \mathbb{R}^{\bar{n}} : \ & HA^N x + HBu_1 + HABu_2 \dots \\ + HA^{N-1} Bu_N \leq h, \ Gu_i \leq g \ \forall i \in \mathbb{N}_N\} = \{ & \bar{x} \in \mathbb{R}^{\bar{n}} : \ \bar{G}\bar{x} \leq \bar{g}\}, \end{aligned}$$

with $\bar{x} = (x, u_1, u_2, \dots, u_N) \in \mathbb{R}^{\bar{n}}$ and $\bar{G}$ and $\bar{g}$ as in (11). The set Ω is the projection on $\mathbb{R}^n$ of the set

$$\begin{aligned} \bar{\Omega} &= \{(x, u_1, u_2, \dots, u_N) \in \mathbb{R}^{\bar{n}} : \ Hx \leq h\} \\ &= \{\bar{x} \in \mathbb{R}^{\bar{n}} : \ \bar{H}\bar{x} \leq h\} \subseteq \mathbb{R}^{\bar{n}} \end{aligned}$$

with $\bar{H}$ as in (11). Thus, condition (6) is equivalent to

$$\text{proj}_x \bar{\Omega} \subseteq \text{proj}_x \bar{\Omega}_N, \tag{12}$$

since $\Omega = \text{proj}_x \bar{\Omega}$ and $Q_N = \text{proj}_x \bar{\Omega}_N$. Thus, to prove that $\Omega \subseteq Q_N(\Omega, U)$ is equivalent to check whether the projection of $\bar{\Omega}_N$ on $\mathbb{R}^n$ contains the projection of $\bar{\Omega}$. Unfortunately, condition (12) is not suitable for using Lemma 1 and then we search for a sufficient condition for (12) to hold such that the lemma can be applied directly.

Consider any linear single-valued mapping $M : \mathbb{R}^{\bar{n}} \rightrightarrows \mathbb{R}^{\bar{n}}$, characterized by a, possibly non-invertible, matrix $M \in \mathbb{R}^{\bar{n} \times \bar{n}}$, such that the value of x through M is preserved, i.e. $\text{proj}_x M((x, u_1, \dots, u_N)) = x$ for all $(x, u_1, \dots, u_N) \in \mathbb{R}^{\bar{n}}$. Clearly, the value of x is preserved also through the inverse mapping of M, that is $\text{proj}_x M^{-1}((x, u_1, \dots, u_N)) = x$ for all $(x, u_1, \dots, u_N) \in \mathbb{R}^{\bar{n}}$. This means that $\text{proj}_x \bar{\Omega}_N = \text{proj}_x M^{-1} \bar{\Omega}_N$ and then (12) is equivalent to

$$\text{proj}_x \bar{\Omega} \subseteq \text{proj}_x M^{-1} \bar{\Omega}_N. \tag{13}$$

Then, the existence of M preserving the x and such that

$$\bar{\Omega} \subseteq M^{-1} \bar{\Omega}_N \tag{14}$$

holds, is a sufficient condition for (13), and thus also for (12), to be satisfied. Notice that necessity of (14) for (13) is not straightforward, since $\text{proj}_x \Gamma \subseteq \text{proj}_x \Delta$ does not imply $\Gamma \subseteq \Delta$, in general.

The condition on the matrix M such that $\text{proj}_x M((x, u_1, \dots, u_N)) = x$ for all $(x, u_1, \dots, u_N) \in \mathbb{R}^{\bar{n}}$ is

$$\begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix} M \tag{15}$$

and then, from Lemma 1, it follows that conditions (14) and (15) are equivalent to the existence of $T \in \mathbb{R}^{n_{\bar{g}} \times n_{\bar{h}}}$ and $M \in \mathbb{R}^{(\bar{n}) \times (\bar{n})}$ satisfying (10). Then (10) is a sufficient condition for $\Omega \subseteq Q_N(\Omega, U)$. ■

The result given above can be directly extended to the problem in presence of constraints on the state.

Theorem 2 *Consider $\Omega = \{x \in \mathbb{R}^n : Hx \leq h\}$ and X and U as in (2), with $H \in \mathbb{R}^{n_h \times n}$, $F \in \mathbb{R}^{n_f \times n}$ $G \in \mathbb{R}^{n_g \times m}$, and suppose that $0 \in \Omega$, $0 \in X$ and $0 \in U$. Then the set $\bar{Q}_N^x(\Omega, U, X)$ as in (8) is a control invariant set contained in X if there exist $T \in \mathbb{R}^{n_{\bar{g}} \times n_h}$ and $M \in \mathbb{R}^{\bar{n} \times \bar{n}}$, with $n_{\bar{g}} = n_h + N n_g + N n_f$ and $\bar{n} = n + Nm$, such that (10) holds with*

$$\bar{G} = \begin{bmatrix} HA^N & HB & HAB & \dots & HA^{N-1}B \\ 0 & G & 0 & \dots & 0 \\ 0 & 0 & G & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & G \\ \hline FA^N & FB & FAB & \dots & FA^{N-1}B \\ FA^{N-1} & 0 & FB & \dots & FA^{N-2}B \\ \dots & \dots & \dots & \dots & \dots \\ FA & 0 & 0 & \dots & FB \\ F & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} h \\ g \\ g \\ \dots \\ g \\ \hline f \\ f \\ \dots \\ f \\ f \end{bmatrix}$$
$$\bar{H} = \begin{bmatrix} H & 0 & 0 & \dots & 0 \end{bmatrix} \tag{16}$$

where $\bar{G} \in \mathbb{R}^{n_{\bar{g}} \times \bar{n}}$, $\bar{g} \in \mathbb{R}^{n_{\bar{g}}}$, and $\bar{H} \in \mathbb{R}^{n_h \times \bar{n}}$.

Proof: Condition (10) with (16) can be proved to imply the constrained invariant condition (9) by reasonings analogous to those of Theorem 1. ■

Given the sets Ω, U and X, to obtain the greatest multiple of Ω such that (9) holds, that is the greatest $\alpha \in \mathbb{R}$ such that

$$\alpha\Omega \subseteq Q_N^x(\alpha\Omega, U, X), \tag{17}$$

is equivalent to compute the smallest nonnegative β, with $\beta = \alpha^{-1}$, such that

$$\Omega \subseteq Q_N^x(\Omega, \beta U, \beta X).$$

This consists in replacing g with βg in (16) and leads to the following LP problem in T, M and β

$$\begin{aligned} \alpha^{-1} = \beta = \min_{\beta \in \mathbb{R}_+} \ & \gamma \\ \text{s.t.} \ & T\bar{H} = \bar{G}M \\ & Th \le \gamma\hat{g} + \tilde{g} \\ & \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} M \end{aligned} \tag{18}$$

with $\hat{g} = (0, g, g, \dots, g, f, \dots, f, f)$ and $\tilde{g} = (h, 0, 0, \dots, 0)$.

Remark 2 *Clearly, if Ω is a control invariant set, then the greatest α satisfying (17) is not smaller than* 1.

Note that directly maximizing α would yield to replace h by αh in (10) and (16) and then to a nonlinear optimization problem. Analogous computational considerations hold for the case of absence of state constraints, as in Theorem 1, which is a particular case of Theorem 2 with $X = \mathbb{R}^n$.

4 State inclusion test

In the previous section, a condition for (9) to hold is given that does not require the computation of the preimage sets $Q_N^x(\Omega, U, X)$, then avoiding the computation of Minkowski addition, see Theorem 2. Once $\alpha\Omega$ is computed by solving (18), one possible choice to obtain a control invariant set is given by

$$\bar{\Omega}^x = \mathrm{co}\left(\bigcup_{k=1}^{N} \Omega_k^x\right) \quad \text{with} \quad \Omega_k^x = Q_k^x(\alpha\Omega, U, X). \tag{19}$$

To have an explicit representation of $\bar{\Omega}^x$ requires to compute the convex hull of the union of several sets, each one given by the Minkowski sum of sets, but the convex hull operation is numerically demanding. Hereafter we provide a convex condition to check if a given $x \in \mathbb{R}^n$ belongs to the invariant set $\bar{\Omega}^x$ without computing it explicitly.

The theorem below provides a representation of the control invariant $\bar{\Omega}^x$ in terms of linear equalities and inequalities.

Theorem 3 *Let Assumption 1 hold. Consider $\Omega = \{x \in \mathbb{R}^n : Hx \le h\}$ bounded, X and U as in (2), with $H \in \mathbb{R}^{n_h \times n}$, $F \in \mathbb{R}^{n_f \times n}$, $G \in \mathbb{R}^{n_g \times m}$, and suppose that $0 \in \Omega$, $0 \in X$, $0 \in U$ and U is bounded. Given α solution of (18) then the set $\bar{\Omega}^x$ defined by (19) can be written as follows*

$$\begin{aligned} \bar{\Omega}^x = \{ & x \in \mathbb{R}^n : \ x = \sum_{k=1}^{N} z_k; \\ & HA^k z_k + \sum_{i=0}^{k-1} HA^{k-1-i} B v_{k,k-i} \le \alpha\lambda_k h, \ \forall k \in \mathbb{N}_N; \\ & FA^j z_k + \sum_{i=0}^{j-1} FA^{j-1-i} B v_{k,j-i} \le \lambda_k f, \ \forall j \in \mathbb{N}_k, \ \forall k \in \mathbb{N}_N; \\ & F z_k \le \lambda_k f, \ \forall k \in \mathbb{N}_N; \quad G v_{k,i} \le \lambda_k g \ \forall i \in \mathbb{N}_k, \ \forall k \in \mathbb{N}_N; \\ & \lambda \ge 0, \ \sum_{k=1}^{N} \lambda_k = 1 \}. \end{aligned} \tag{20}$$

Proof: Given an arbitrary collection of non-empty convex sets $\Gamma_i \subseteq \mathbb{R}^n$ with $I \in \mathbb{N}$ and $i \in \mathbb{N}_I$, note first that

$$\mathrm{co}\left(\bigcup_{i \in \mathbb{N}_I} \Gamma_i\right) = \bigcup_{\substack{\lambda \ge 0 \\ \mathbf{1}^T\lambda = 1}} \left(\bigoplus_{i \in \mathbb{N}_I} \lambda_i \Gamma_i\right)$$

see Chapter 3 in [25]. Provided condition (17) is satisfied and from Lemma 2 in Appendix A, the control invariant set is given by

$$\begin{aligned} \bar{\Omega}^x &= \mathrm{co}\left(\bigcup_{k=1}^{N} \Omega_k^x\right) = \bigcup_{\substack{\lambda \ge 0 \\ \mathbf{1}^T\lambda = 1}} \left(\bigoplus_{k=1}^{N} \lambda_k \Omega_k^x\right) \\ &= \bigcup_{K \subseteq \mathbb{N}_N} \bigcup_{\substack{\lambda \in \Lambda(K) \\ \mathbf{1}^T\lambda = 1}} \left(\bigoplus_{k \in K} \lambda_k \Omega_k^x\right) = \{x \in \mathbb{R}^n : \ x = \sum_{k \in K} \lambda_k y_k; \\ & \quad y_k \in \Omega_k^x, \ \lambda_k > 0, \ \ \forall k \in K; \ \sum_{k \in K} \lambda_k = 1, \ \forall K \subseteq \mathbb{N}_N \}, \end{aligned} \tag{21}$$

since $0 \cdot \Omega_k^x = \{0\}$ for all $k \in \mathbb{N}_n$. Then, from $\lambda_k > 0$ for every $k \in K$ and defining $z_k = \lambda_k y_k$ for all $k \in K$, it follows

$$\begin{aligned} \bar{\Omega}^x = \{ & x \in \mathbb{R}^n : \ x = \sum_{k \in K} z_k; \ z_k/\lambda_k \in \Omega_k^x, \ \lambda_k > 0, \ \ \forall k \in K; \\ & \sum_{k \in K} \lambda_k = 1, \ \forall K \subseteq \mathbb{N}_N \} = \{x \in \mathbb{R}^n : \ x = \sum_{k \in K} z_k; \\ & HA^k z_k/\lambda_k + \sum_{i=0}^{k-1} HA^{k-1-i} B u_{k,k-i} \le \alpha h, \ \forall k \in K; \\ & FA^j z_k/\lambda_k + \sum_{i=0}^{j-1} FA^{j-1-i} B u_{k,j-i} \le f, \ \forall j \in \mathbb{N}_k, \ \forall k \in K; \\ & F z_k/\lambda_k \le f, \ \forall k \in K; \quad G u_{k,i} \le g \ \forall i \in \mathbb{N}_k, \ \forall k \in K; \\ & \lambda_k > 0, \ \ \forall k \in K; \quad \sum_{k \in K} \lambda_k = 1, \ \forall K \subseteq \mathbb{N}_N \} \end{aligned} \tag{22}$$

from (7). By introducing $v_{k,i} = \lambda_k u_{k,i}$ for all $i \in \mathbb{N}_k$, $k \in K$ and $K \subseteq \mathbb{N}_N$, then

$$\begin{aligned}\bar{\Omega}^x = \{x \in \mathbb{R}^n :\ & x = \sum_{k\in K} z_k; \\ & HA^k z_k + \sum_{i=0}^{k-1} HA^{k-1-i} B v_{k,k-i} \le \alpha\lambda_k h,\ \forall k \in K; \\ & FA^j z_k + \sum_{i=0}^{j-1} FA^{j-1-i} B v_{k,j-i} \le \lambda_k f,\ \forall j \in \mathbb{N}_k,\ \forall k \in K; \\ & F z_k \le \lambda_k f,\ \forall k \in K; \quad G v_{k,i} \le \lambda_k g\ \forall i \in \mathbb{N}_k,\ \forall k \in K; \\ & \lambda_k > 0,\ \ \forall k \in K; \quad \sum_{k\in K} \lambda_k = 1,\ \forall K \subseteq \mathbb{N}_N\}.\end{aligned} \tag{23}$$

If $\lambda_k = 0$, as for all $k \notin K$ and every $K \subseteq \mathbb{N}_N$, then $G v_{k,i} \le \lambda_k g$ implies $v_{k,i} = 0$ for all $i \in \mathbb{N}_k$, from the boundedness of U. Thus $\lambda_k = 0$ implies also $z_k = 0$, from the boundedness of Ω and Assumption 1. Hence the expression (20) can be recovered by posing $\lambda_k = 0, v_{k,i} = 0$ and $z_k = 0$ in (23) for all $k \notin K$ and every $K \subseteq \mathbb{N}_N$. ■

Theorem 3 implies that checking if $x \in \bar{\Omega}^x$ resorts to solve an LP feasibility problem in the variables $x, z_k, v_{k,i}, \lambda_k$ for all $i \in \mathbb{N}_k$ and $k \in \mathbb{N}_N$, then in a space of dimension $n + Nn + 0.5N(N+1)m + N$. Such a representation is particularly suitable to be used in optimization-based control, as model predictive control for instance, since it reduces to enforcing the linear constraints characterizing $\bar{\Omega}^x$.

5 Numerical examples

The different results presented in this paper are illustrated through numerical examples. The optimization problems are solved using YALMIP interface [20] and Mosek optimizer [23] on an Intel® Core™ i7-6600U CPU @ 2.60GHz × 4 processor laptop with 16GB of RAM.

5.1 Example 1

Here we compare the computational burden required to check the invariant condition (17) by solving (16)-(18) with an alternative approach based on known properties of computational geometry. First we describe this approach. Suppose that both the hyperplanes and the vertices representation of Ω are available. This assumption, not needed for our method that only requires the H-representation, could be reasonably posed since Ω can be arbitrary chosen, and then fixed to be the unitary box, i.e. $\Omega = \mathscr{B}^n$. Then, the 2^n vertices can be easily obtained, which is not the case for general polytopes. Thus, the exact maximal α such that (17) is satisfied is given by

$$\begin{aligned}\alpha^* = \max_{\alpha,\ u_{i,j}} \ & \alpha \\ \text{s.t.}\ \ & \bar{G} \cdot (\alpha v_j, u_{1,j}, \ldots, u_{N,j},) \le \alpha\tilde{g} + \hat{g}, \quad \forall j \in \mathbb{N}_{2^n}\end{aligned} \tag{24}$$

where v_j is the j-th vertex, with $j \in \mathbb{N}_{2^n}$, and $\bar{G}, \tilde{g}$ and $\hat{g}$ are defined in and below Theorem 2. The constraints in the LP problem (24) impose that every vertex of $\alpha\Omega$ is contained in $Q_N^x(\alpha\Omega, U, X)$ and their number is equal to the number of vertices of Ω, hence exponentially growing with the system dimension.

The exact maximal α^* solution of (24) and the α obtained by solving (18) with (16) are computed for randomly generated controllable systems with real eigenvalues with increasing n and $m = \lceil n/2 \rceil$. The sets are $\Omega = \mathscr{B}^n, U = 10\mathscr{B}^m$ and $X = 100\mathscr{B}^n$ and $N = 2$. The computation times are given in Figure 1 in function of the state dimension n.

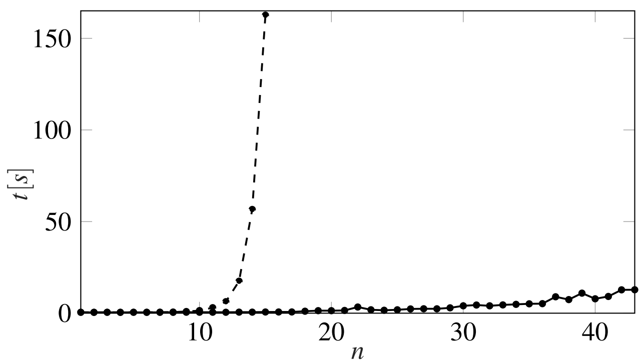

Fig. 1. Computation times in seconds to solve (24), in dashed line, and to solve (18) with (16), in solid line, in function of the state dimension n.

Note that the proposed method permits to check the invariance condition up to a 40 dimensional system with 20 inputs in less than 12 seconds, while the alternative approach needs more than $160s$ for $n = 15$.

In Figure 2 we report the values of α obtained by solving (18) for 1000 randomly generated systems with n between 1 and 12 and also the normalized mismatch with respect to α^* given by (24), i.e. $|\alpha - \alpha^*|/\alpha^*$, in logarithmic scale. It can be noticed that the approximation error is several order of magnitude smaller than the values of α, in most of the cases included between 10^{-4} and 10^{-12}, which might be due to the numerical precision rather than to real inaccuracy.

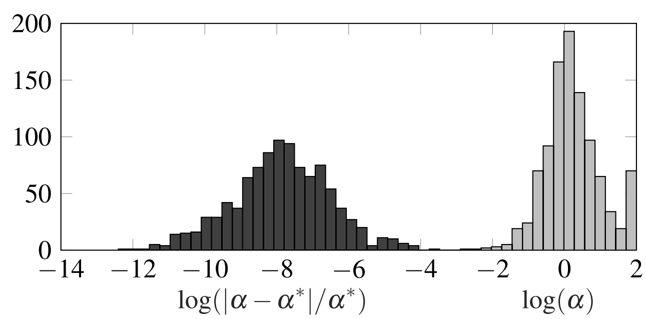

Fig. 2. Histograms of the values of $\log(\alpha)$, in light gray, and $\log(|\alpha - \alpha^*|/\alpha^*)$, in dark gray, over 1000 tests.

Finally, the Minkowski sum has been employed to compute $Q_N^x(\alpha\Omega, U, X)$ a posteriori, for evaluating its computational cost, but we could not go further than $n = 4$.

5.2 Example 2

We apply now the proposed method to an high dimensional system, in particular with $n = 30$ and $m = 15$ with horizons $N = 5, 10$. To provide some hints on the conservatism of the control invariant obtained with respect to the maximal control invariant set, we build a system for which the latter can be computed. Indeed the classical algorithms for computing the maximal control invariant set are too computationally demanding to be applied to high dimensional systems in general. Then, a specific structure has to be imposed to the system dynamics for computing the maximal control invariant set to be compared with our results. In particular, we consider system (1) with

$$A = P^{-1}\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{15} \end{bmatrix} P, \quad B = P^{-1}\begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_{15} \end{bmatrix} \tag{25}$$

where $A_i \in \mathbb{R}^{2\times 2}$ and $B_i \in \mathbb{R}^2$, for $i \in \mathbb{N}_{15}$, are matrices whose entries are randomly generated such that all A_i have instable poles and the pairs (A_i, B_i) are controllable and the maximal control invariant is obtained, as illustrated below, after 5 iterations at most. The latter requirement has been introduced for sets convergence reasons. The matrix $P \in \mathbb{R}^{30\times 30}$ is a randomly generated nonsingular matrix. Figure 3 provides a graphical representation of A and B, for which the maximal values (15.303 for A and 49.0516 for B) are depicted in white, the minimal ones (-13.4866 for A and -60.4621 for B) are drawn in black, the other values are proportional degree of gray. The matrices are not sparse, not a single null entry is present either in A or B, and are available under request.

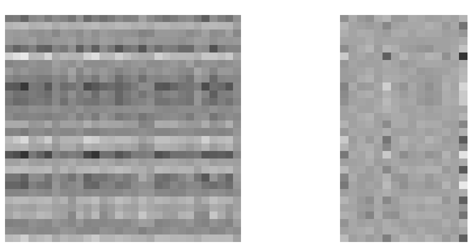

Fig. 3. Graphical representation of matrices A and B.

Thus, the dynamics of system with state $y = Px$, is controllable and it is, in practice, composed by 15 decoupled two-dimensional subsystems with one control input each. Hence, the maximal control invariant set in the space for the overall system in y, denoted Σ, is given by the Cartesian product of the maximal control invariant sets of the 15 subsystems. That

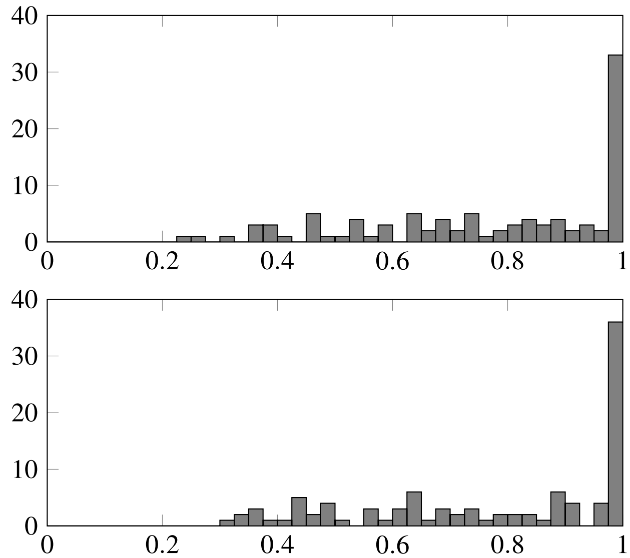

Fig. 4. Histograms of the values r_Ω/r_Σ for $N = 5, 10$ for a systems with $n = 30$ and $m = 15$.

is $\Sigma = \prod_{i=1}^{15}\Sigma_i$ where Σ_i are the maximal control invariant set in $10\mathscr{B}^2$ for the i-th subsystem with input bound $10\mathscr{B}^2$. Hence Σ can be computed by computing Σ_i, being (A_i, B_i) a two-dimensional controllable system, for all $i \in \mathbb{N}_{15}$. Therefore, $P^{-1}\Sigma \subseteq \mathbb{R}^{30}$ is the maximal control invariant set for the system (1) in x with (25). After computing $P^{-1}\Sigma$, the linear problem (18) has been solved to obtain $\bar{\Omega}^x$ with $N = 5, 10$ and sets $\Omega = P^{-1}\mathscr{B}^{30}$, $U = 10\mathscr{B}^{15}$ and $X = 10P^{-1}\mathscr{B}^{30}$.

To quantify the difference between the maximal control invariant set $P^{-1}\Sigma$ and the set $\bar{\Omega}^x$, 100 vectors $v \in \mathbb{R}^n$ are generated randomly. Then, (a lower approximation of) the maximal values of r_Σ and r_Ω are computed such that $r_\Sigma v \in P^{-1}\Sigma$ and $r_\Omega v \in \bar{\Omega}^x$, through dichotomy method. In practice, we search for (approximations of) the intersections between the ray $v_r = \{rv \in \mathbb{R}^n : \; r \geq 0\}$ and the boundaries of the sets $P^{-1}\Sigma$ and $\bar{\Omega}^x$. The ratio between r_Ω/r_Σ is an indicator of the mismatch between the maximal control invariant set $P^{-1}\Sigma$ and $\bar{\Omega}^x$, the closer to one, the closer are the intersections between the ray v_r and the two sets.

Figure 4 shows the histograms of the ratio r_Ω/r_Σ for $N = 5, 10$. As expected, the higher is the horizon N, the closer are the sets Σ and $\bar{\Omega}^x$.

6 Conclusions

In this paper we addressed the problem of computing control invariant sets for linear systems with state and input polyhedral constraints. Invariance conditions are given, that are set inclusions involving the N-step sets, which are posed in form of LP optimization problems, instead of Minkowski sum of polyhedra. Then the procedures based on those conditions are applicable even for high dimensional systems.

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A Appendix

Lemma 2 *Given $K \subseteq \mathbb{N}_N$ and defined $\bar{K} = \mathbb{N}_n / K$ and*

$$\Lambda(K) = \{\lambda \in \mathbb{R}^n : \ \lambda_k > 0 \ \forall k \in K, \quad \lambda_k = 0 \ \forall k \in \bar{K}\} \tag{A.1}$$

one has

$$\begin{aligned} &\{\lambda \in \mathbb{R}^n : \ \lambda_k \geq 0 \ \forall k \in \mathbb{N}_n, \ \mathbf{1}^T \lambda = 1\} \\ &\quad = \bigcup_{K \subseteq \mathbb{N}_N} \{\lambda \in \Lambda(K) : \mathbf{1}^T \lambda = 1\} \end{aligned} \tag{A.2}$$

Proof: Note that for every $\lambda \in \Lambda(K)$, λ_k is strictly positive if and only if $k \in K$, i.e. K denotes the set of indices such that λ_k is not zero, in practice. For every λ in the l.h.s. of (A.2), there exists a K, that is the set of indices for which $\lambda_k > 0$, such that $\lambda \in \Lambda(K)$. Analogously, every λ in the r.h.s. of (A.2), also satisfies $\lambda \geq 0$ and then it is contained in the l.h.s. set. ∎