Further results on global stabilization for multiple integrators with bounded controls

Nicolas Marchand
Laboratoire d’Automatique de Grenoble, INPG-UJF-CNRS UMR 5528, ENSIEG BP 46, 38402 Saint Martin d’Hères Cedex, France
Nicolas.Marchand@inpg.fr

Abstract—In this paper, we propose a class of nonlinear bounded feedbacks for the stabilization of integrators chain that includes the previously existing result of Teel [1]. The main benefits of the proposed method are the significant performance improvement compared to Teel’s work and the global stabilization result obtained with possibly analytic feedbacks compared to the existing results focused on the performance.

1. INTRODUCTION

The stabilization of linear systems with bounded control has been widely studied in the literature and it is not a new idea. For instance optimal control allows this for long [2] even by avoiding bang-bang control [3]. But if one ignores optimality aspects, one may expect more regular or more simple and explicit control laws. The problem of stabilizing a linear system or even multiple integrators is known to be a fully nonlinear problem, namely one can not expect to obtain global stabilization by means of a bounded linear feedback for system of dimension \( n \geq 3 \) [4]. At the origin of the renewed interest in this subject is probably the result proposed by Teel [1] who proposed a class of bounded nonlinear feedbacks that achieves global stabilization of integrator chains. It followed various works that weaken the assumptions and/or extend the result to general controllable linear systems [5, 6] or that try to robustify the feedback with respect to disturbances [6, 7] - with sometimes only a semiglobal stability result. More in the aim of this work, one can find [8, 9, 10] where the objective is to increase the performance of the feedback in term of convergence speed. Indeed, if Teel’s result is nice and founding, its performance is very poor. To disguise this drawback, the above references rely on a one parameter family of saturated linear feedback. Unfortunately, for ‘only ” a semiglobal stability result, these methods are discontinuous and sometimes require optimization - loosing the simplicity that makes one justification to avoid optimal control.

In this paper, we propose a generalization of the class of feedback laws proposed in [1] that allows significantly more efficient bounded feedback laws. The obtained stabilization is global with assumptions as weak as in [4, 6] and the feedback law can be smooth or even analytic.

The paper is organized as follows. In the next section, the main result of this paper is proposed after some preliminary definitions. The last section is devoted to an example that shows the benefit of the proposed control law.

Notations: Let \( \mathbb{R}^* \) denote the set \( \mathbb{R} \setminus \{0\} \). For any real vector \( y \), \( y_i \) will stand for the \( i^{\text{th}} \) coordinates of \( y \). Finally, for any \( \varepsilon > 0 \), \( B(\varepsilon) \) will denote the ball centered at the origin of radius \( \varepsilon \).

2. RESULT

An integrator chain is defined by:

\[
\begin{align*}
\dot{x}_i &= x_{i+1} & \text{for } i \in \{1, \ldots, n-1\} \\
\dot{x}_n &= u
\end{align*}
\]

where \( n \) is some strictly positive integer.

Let us now define what is meant by saturation in the following.

Definition 2.1: A “unitary saturation” will denote any locally lipschitz function \( \sigma : \mathbb{R} \rightarrow [-1, 1] \) such that:

1) for all \( s \in \mathbb{R}^* \), \( \sigma(s) > 0 \),
2) \( \sigma(0) = 0 \),
3) \( \sigma \) is differentiable at 0 with \( \frac{d\sigma}{ds}(0) > 0 \),
4) for all \( s \in \mathbb{R} \), \( |\sigma(s)| \leq 1 \).

Based on the above definition, a saturation function will denote any function \( \sigma^M \) homothetic to a unitary saturation \( \sigma \):

Definition 2.2: For any \( M > 0 \), a saturation function for \( M \) will denote any function \( \sigma^M : \mathbb{R} \rightarrow [-M, M] \) such that \( \sigma : s \mapsto M \sigma^M \left( \frac{s}{M} \right) \) is a unitary saturation function.

This notion of saturation is very general and handles most saturation functions usually used like \( s \mapsto \text{sign}(s) \min(|s|, M) \) or \( s \mapsto M \tanh(\frac{s}{M}) \). Note that the results of this paper remain valid if one assumes only continuity instead of the lipschitz property; all the developments will remain unchanged except that the uniqueness of the closed loop trajectory can no more be assumed and the stability result should be taken as uniform with respect to the different solutions of the differential equation of the closed loop system.

Before stating our main result, let us define:

Definition 2.3: An “admissible level function”, denotes any function \( \gamma \), locally lipschitz in its arguments, such that for all \( y \in \mathbb{R} \) and \( L, M \in \mathbb{R}^* \):

- \( \gamma(y, L, M) \leq M \) for all \( y \) such that \( |y| > L \)
- \( \gamma(y, L, M) \leq M + L - |y| \) for all \( y \in [-L, L] \)

With this definition, one has:

Theorem 2.1: There exists a linear change of coordinates \( y = Tx \) such that for any set unitary saturation functions \( \{\sigma_i\}_{i \in \{1, \ldots, n\}} \), there is \( \{L_i\}_{i \in \{2, \ldots, n\}} \) such that for any set of admissible level functions \( \{\gamma_i\}_{i \in \{1, \ldots, n-1\}} \) and any set of strictly positive real pairs \( \{(M_i, L_i)\}_{i \in \{1, \ldots, n\}} \) with for
\( i \in \{1, \ldots, n - 1\}, M_i < L_{i+1} \leq L_{i+1}, \) the feedback law
\[
    u = -\sigma_n^{M_n} (y_n + \sigma_n^{M_n-1} (y_{n-1} + \ldots \right)
\]
\[
+ \sigma_2^{\gamma_2(y_2, L_2, M_2)} (y_2) + \sigma_1^{\gamma_1(y_1)} (y_1) \ldots)
\]
globally asymptotically stabilizes system (1).

Theorem 2.1 is proved in appendix A.

The above result is a generalization of Theorem 2.1 of [1] in the sense that, taking, for all \( i \in \{1, \ldots, n - 1\}, \gamma_i(y_{i+1}, L_{i+1}, M_i) \equiv M_i \) as admissible level functions and \( M_i < \frac{1}{2} L_{i+1} \) and imposing upon the \( \sigma_i^{M_i} \)’s to be linear in some neighborhood \([-L_i, L_i]\) of the origin, one retrieves the well known result of [1] - with lighter assumptions on the \( \sigma_i \)’s also later proposed in [5][1] for different control laws.

Note that when the control bounds \( M_i \)’s go to the infinite, the proposed feedback law as well as the one proposed in [1] behaves like a linear state feedback \( u_{uc} \). The notable change in the above result holds in the fact that the “level” \( \gamma_i(y_{i+1}, L_{i+1}, M_i) \)'s of the saturation functions \( \sigma_i^{\gamma_i(y_{i+1}, L_{i+1}, M_i)} \)'s are allowed to change according to the value of \( y_{i+1} \). As one can see in the proof below, the \( L_i \)’s define a neighborhood \( \mathcal{N} = \{z \text{ s.t. } z_i \in [-L_i, L_i]\} \) of the origin where the unconstrained feedback \( u_{uc} \) is sufficient to stabilize the states of the system to the origin. In particular, if the \( \sigma_i \)'s are linear in some neighborhood \( \mathcal{N}' \) of the origin, the result will hold for all \( L_i \)’s such that \( \mathcal{N} \subset \mathcal{N}' \). The potential benefit of the proposed extension is shown on simulations in the next section.

3. Example

We consider here a chain of five integrators and impose the constraint \( M_n = 20 \), that is \(|u| \leq 20\). The initial conditions for all simulations where \( x_0 = (1 \ 1 \ 1 \ 1 \ 1) \) and the trajectories are all represented in the \( x \) state space.

We first applied the control law proposed in [1]. We took saturation functions of the form \( \sigma_i^{M_i} : s \rightarrow \text{sign}(s) \min(|s|, M_i) \), that is taking \( M_i = L_{i+1} \) for all \( i \in \{1, \ldots, n - 1\} \). To fulfill the condition \( M_i < \frac{1}{2} L_{i+1} \) of [1], we took \( M_i = \frac{1}{2} \min L_{i+1} \) for \( i = 1, \ldots, n - 1 \). With these assumptions, the derivative of the saturation functions at the origin is unitary and the coordinate transformation is given by (see [1] for further details):
\[
    y_{n-i} = \sum_{j=0}^{i} \frac{j!}{(i-j)!} x_{n-j}
\]
The control law is then given by:
\[
    u = -\sigma_n^{M_n} (y_n + \sigma_n^{M_n-1} (y_{n-1} + \ldots \right)
\]
\[
+ \sigma_2^{\gamma_2(y_2, L_2, M_2)} (y_2) + \sigma_1^{\gamma_1(y_1)} (y_1) \ldots)
\]

Then, we applied the control law of Theorem 2.1 with, here too, \( \sigma_i^{M_i} : s \rightarrow \text{sign}(s) \min(|s|, M_i) \). As already mentioned, the \( L_i \)’s can be taken equal to the \( M_i \)’s since the saturation function \( \sigma_i^{M_i} \) are linear in \([-M_i, M_i]\). \( L_i \) was imposed to be equal to \( L_i \), that is: \( L_i = L_i = M_i \) for all \( i \in \{2, \ldots, n\} \). The admissible level function \( \gamma_i \) where taken such that, for all \( i \in \{1, \ldots, n - 1\}, \gamma_i(y_{i+1}, L_{i+1}, M_i) = M_i \) if \(|y_{i+1}| > L_{i+1}\) and \( \gamma_i(y_{i+1}, L_{i+1}, M_i) = M_i + L_{i+1} - |y_{i+1}| \) if \(|y_{i+1}| \leq L_{i+1}\). Finally to insure that for \( i \in \{1, \ldots, n - 1\}, M_i < L_{i+1} \leq L_{i+1}, \) we took \( M_i = \frac{1}{10000} L_{i+1}. \) With these choices, the transformation is the same as in [1].

Beside these two nonlinear control laws, we have applied the unconstrained feedback \( u_{uc} = \sum_{i=1}^{n} y_i \) that corresponds for the two above control laws to the case where the \( M_i \)'s go to the infinite.

The result shown on Figure I clearly comes to light that the extension proposed in the present paper significantly increases the performances of the control law proposed by Teel [1] by increasing the convergence speed as well as reducing the transitory excursions of the states. The performances are close to the unconstrained case, however, to be more complete, the work needs to be compared with time optimal control.

4. Conclusion

In this paper, we propose a general class of stabilizing saturated feedback laws for linear integrator chain. It extends the class proposed by Teel [1], allowing a significant increase of the performances in terms of convergence speed and transitory excursions. Contrary to some existing work with the same aim, one achieves global stability with possibly smooth control laws.

APPENDIX

A. Proof of Theorem 2.1

Let us start the proof with an preliminary lemma that avoids to use a global lipschitz argument as in [1] to justify that the trajectory of the closed loop system can not diverge in finite time.

Lemma 1.1: Any closed loop trajectory of any linear system can not diverge in finite time under bounded control.

Lemma 1.1 is proved in appendix B.

Let \( \alpha_i \) denote the derivative of \( \sigma_i \) at the origin and \( \beta_i := \prod_{j=1}^{i} \alpha_i \). Since the \( \alpha_i \)'s are assumed to be non zero, one can make the transformation:
\[
    y_{n-i} = \beta_{n-i} \sum_{j=0}^{i} \frac{j!}{(i-j)!} x_{n-j}
\]
the inverse transformation being given by:
\[
    x_{n-i} = \sum_{j=0}^{i} (-1)^{i+j} \frac{j!}{(i-j)!} \beta_{n-j} y_{n-j}
\]
System (1) then becomes:
\[
\begin{cases}
    \dot{y}_i = \sum_{k=i+1}^{n} \beta_k y_k + u & \text{for } i = 1, \ldots, n - 1 \\
    \dot{y}_n = u
\end{cases}
\]
We shall first prove that the feedback law of Theorem 2.1 locally asymptotically stabilizes (2). First note that since the $\sigma_i$’s are assumed to be differentiable at the origin, one has:

$$\forall \varepsilon_i > 0, \exists \eta_i > 0 \text{ s.t. } \forall y \in [-\eta_i, \eta_i],$$

$$\sigma_i(y) = \alpha_i y + r'_i \quad \text{with} \quad |r'_i| \leq \varepsilon_i |y|$$

Moreover, since for all $\gamma > 0$, $\sigma_{\gamma}$ is homothetic to $\sigma_i$ with ratio $\gamma$, one also has that:

$$\forall y \in [-\eta_i, \eta_i], \quad \sigma_i(y) = \alpha_i y + r'_i$$

with $|r'_i| \leq \varepsilon_i |y|$

it follows, thanks to the definition of saturation functions with variable level, that the linearization of $\sigma_{\gamma}$ at the origin is independent of $\gamma$. It Hence, for all $\varepsilon_1 > 0$ and all $y_1 \in [-\eta_1, \eta_1]$, one has:

$$u = -\sigma_n^M(y_n + \sigma_{n-1}^\gamma(y_{n-1}, L_n, M_{n-1}) (y_{n-1} + \cdots + \sigma_2^\gamma(y_2, L_3, M_2) (y_2 + \alpha_1 y_1 + r'_1) \cdots))$$

with $r'_1 \leq \varepsilon_1 |y_1|$ and therefore taking

$$\eta'_1 := (|\alpha_1| + \varepsilon_1) \eta_1$$

one has:

$$|\alpha_1 y_1 + r'_1| \leq \eta'_1$$

Hence, for all $\varepsilon_2$ and all $y_2 \in [-\eta_2 + \eta'_1, \eta_2 - \eta'_1]$ (with $\eta_1$ chosen sufficiently small so that $\eta'_1 < \eta_2$), one has $y_2 + \alpha_1 y_1 + r'_1 \in [-\eta_2, \eta_2]$ and consequently:

$$u = -\sigma_n^M(y_n + \sigma_{n-1}^\gamma(y_{n-1}, L_n, M_{n-1}) (y_{n-1} + \cdots + \sigma_3^\gamma(y_3, L_4, M_3) (\alpha_2 (y_2 + \alpha_1 y_1 + r'_1) + r'_2) \cdots))$$

with $r'_2 \leq \varepsilon_2 |y_2 + \alpha_1 y_1 + r'_1|$ and therefore taking

$$\eta'_2 := (|\alpha_2| + \varepsilon_2) \eta_2$$

one has:

$$|\alpha_2 (y_2 + \alpha_1 y_1 + r'_1) + r'_2| \leq \eta'_2$$

Going on further for all $y_i$, it follows that for all $\{\varepsilon_i > 0\}_{i \in \{1, \ldots, n\}}$ there is some $\{\eta_i > 0\}_{i \in \{1, \ldots, n\}}$ such that
for all $y \in \{z \text{ s.t. } |z_i| \leq \eta_i\}$, one has:

$$u = - (\alpha_n y_n + \alpha_n (\alpha_{n-1} y_{n-1} + \alpha_{n-1} (\cdots + 
 \alpha_2 (\alpha_1 y_1 + r_1') \cdots + r_{n-1}') + r_n')$$

$$= - \sum_{i=1}^{n} \beta_i y_i + r'$$

where $r' := - r'_n - \beta_n r'_{n-1} - \cdots - \beta_2 r'_1$ encompasses all higher order terms and is such that:

$$|r'_1| \leq \varepsilon_1 |y_1|$$

$$|r'_2| \leq \varepsilon_2 (|y_2| + |\alpha_1| |y_1| + |r'_1|)$$

$$\vdots$$

$$|r'_n| \leq \varepsilon_n \left( |y_n| + \sum_{j=1}^{n-1} \prod_{k=j}^{n-1} |\alpha_k| |y_j| + \sum_{j=1}^{n-2} \prod_{k=j+1}^{n-2} |\alpha_k| |r'_{n-1}| + \sum_{j=1}^{n-3} \prod_{k=j+1}^{n-3} |\alpha_k| |r'_{n-2}| \right)$$

The closed loop system then follows with equation (2):

$$\dot{y}_i = - \sum_{k=1}^{i} \beta_k y_k + r' \quad \text{for } i \in \{1, \ldots, n\}$$

and is obviously locally asymptotically stable for any sufficiently small $\varepsilon_i$’s. The key point in the above calculations is that the errors can be bounded independently of the choice made for the $\gamma_i$’s and hence of the $L_i$’s and the $M_i$’s.

Let us now prove that the closed loop system necessarily joins any predefined neighborhood of the origin with an appropriate choice of the $N_i$’s and the $M_i$’s.

Let $V_n$ be the Lyapunov function defined by $V_n := y_n^2$. Clearly, one has:

$$\dot{V}_n = - 2 y_n \sigma_n M_n (y_n + \gamma_{n-1} (y_{n-1}, M_{n-1}) (y_{n-1} + \cdots + \gamma_2 (y_2, L_2, M_2) (y_2 + \gamma_1 (y_1, L_1, M_1) (y_1)))$$

By definition of admissible level functions, if $|y_n| > L_n$, one has:

$$\gamma_{n-1} (y_n, L_n, M_{n-1}) \leq M_{n-1} < L_n$$

and hence $y_n$ and $y_n + \gamma_{n-1} (y_{n-1}, M_{n-1}) (y_{n-1} + \cdots + \gamma_2 (y_2, L_2, M_2) (y_2 + \gamma_1 (y_1, L_1, M_1) (y_1)))$ have same sign. By definitions 2.1 and 2.2 it follows that $V_n$ is strictly decreasing and $y_n$ necessarily joins in finite time $[-L_n, L_n]$. During that time, the remaining states $y_1, \ldots, y_{n-1}$ remain bounded thanks to lemma 1.1.

Let us now define $V_{n-1} := y_{n-1}^2$. Using the differentiability of $\sigma_n$ at the origin, one knows that for any $\varepsilon > 0$, there is $L_n > 0$ such that for all $L_n \leq L_n$ and all $y \in [-2 L_n, 2 L_n]$, $\sigma_n M_n (y) = \alpha_n y + r'_n$ with $|r'_n| \leq \varepsilon_n |y|$. Since in $[-L_n, L_n]$, one has:

$$|y_n + \sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots) | \leq |y_n| + \gamma_{n-1} (y_n, L_n, M_{n-1})$$

$$\leq M_{n-1} + L_n < 2 L_n$$

it follows that for all $y_n$ in $[-L_n, L_n]$:

$$u = - [\alpha_n (y_n + \sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots)) + r'_n]$$

with:

$$|r'_n| \leq \varepsilon_n |y_n + \sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots)| < 2 \varepsilon_n L_n$$

Hence, the evolution of $V_{n-1}$ is given by:

$$\dot{V}_{n-1} = 2 y_{n-1} (\beta_n y_n - u)$$

$$= - 2 y_{n-1} (\sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots) + r'_n)$$

If $y_{n-1}$ is outside $[-L_{n-1}, L_{n-1}]$, then

$$|y_{n-1} + \sigma_{n-2}^\gamma (y_{n-1}, L_{n-1}, M_{n-2}) (\cdots) | \geq L_{n-1} - M_{n-2} > 0$$

Hence, choosing a sufficiently small $\varepsilon_n$, one can insure that $r'_n$ is sufficiently small so that $\sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots)$ and $\sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots) + r'_n$ will be of same sign and consequently that $y_{n-1}$ and $\sigma_{n-1}^\gamma (y_{n-1}, L_{n-1}, M_{n-1}) (\cdots) + r'_n$ also at least outside $[-L_{n-1}, L_{n-1}]$. Hence, $V_{n-1}$ will be strictly decreasing and $y_{n-1}$ reaches $[-L_{n-1}, L_{n-1}]$ in finite time. Here again, invoking lemma 1.1 the remaining states $y_1, \ldots, y_{n-2}$ remain bounded.

Repeating the same reasoning for all $y_i$ it follows that taking sufficiently small $\varepsilon_i$’s, there exists $L_i$’s such that for any initial condition and any $L_i \leq L_i$, the state $y$ of the system rejoins the neighborhood $\mathcal{N} := \{z \text{ s.t. } |z_i| \leq [-L_i, L_i] \}$ in finite time. Taking $L_i$ so that $\mathcal{N}$ is sufficiently small to insure the asymptotic stability ends the proof.

**B. Proof of Lemma 1.1**

Let $\dot{x} = Ax + Bu$ where $u$ is such that $\|u\| \leq M$. Then, one has:

$$\frac{d}{dt} \|x(t)\|^2 = 2 x^T Ax + 2 x^T Bu \leq 2 \lambda_{\max}(A) \|x\|^2 + 2 \lambda_{\max}(B) M \|x\|^2$$

recognizing a Bernoulli ordinary differential equation, it follows:

$$\|x(t)\|^2 \leq \left[ \frac{\lambda_{\max}(B) M}{\lambda_{\max}(A)} \left( e^{\lambda_{\max}(A)t} - 1 \right) + \|x(0)\| e^{\lambda_{\max}(A)t} \right]^2$$

which ends the proof. Note that the result does only assumes the existence of a closed loop trajectory.
REFERENCES


