Robust control of MIMO systems

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O. Sename [GIPSA-lab]
These textbooks are very close to the given course. Some of them are available on internet.


Nonlinear systems

In control theory, dynamical systems are mostly modeled and analyzed thanks to the use of a set of Ordinary Differential Equations (ODE)\(^1\). Nonlinear dynamical system modeling is the "most" representative model of a given system. Generally this model is derived thanks to system knowledge, physical equations etc. Nonlinear dynamical systems are described by nonlinear ODEs.

**Definition (Nonlinear dynamical system)**

For given functions \( f : \mathbb{R}^n \times \mathbb{R}^{nw} \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \times \mathbb{R}^{nw} \rightarrow \mathbb{R}^{nz} \), a nonlinear dynamical system (\(\Sigma_{NL}\)) can be described as:

\[
\Sigma_{NL} : \begin{cases} 
\dot{x} = f(x(t), w(t)) \\
z = g(x(t), w(t))
\end{cases}
\]

(1)

where \(x(t)\) is the state which takes values in a state space \(X \in \mathbb{R}^n\), \(w(t)\) is the input taking values in the input space \(W \in \mathbb{R}^{nw}\) and \(z(t)\) is the output that belongs to the output space \(Z \in \mathbb{R}^{nz}\).

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\(^1\) Partial Differential Equations (PDEs) may be used for some applications such as irrigation channels, traffic flow, etc. but methodologies involved are more complex compared to the ones for ODEs.
The main advantage of nonlinear dynamical modeling is that (if it is correctly described) it catches most of the real system phenomena.

On the other side, the main drawback is that there is a lack of mathematical and methodological tools; e.g. parameter identification, control and observation synthesis and analysis are complex and non systematic (especially for MIMO systems).

In this field, notions of robustness, observability, controllability, closed loop performance etc. are not so obvious. In particular, complex nonlinear problems often need to be reduced in order to become tractable for nonlinear theory, or to apply input-output linearization approaches (e.g. in robot control applications).

As nonlinear modeling seems to lead to complex problems, especially for MIMO systems control, the LTI dynamical modeling is often adopted for control and observation purposes. The LTI dynamical modeling consists in describing the system through linear ODEs. According to the previous nonlinear dynamical system definition, LTI modeling leads to a local description of the nonlinear behavior (e.g. it locally describes, around a linearizing point, the real system behavior).

From now on only Linear Time-Invariant (LTI) systems are considered.
Definition of LTI systems

Definition (LTI dynamical system)

Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$ and $D \in \mathbb{R}^{n_z \times n_w}$, a Linear Time Invariant (LTI) dynamical system ($\Sigma_{LTI}$) can be described as:

$$\Sigma_{LTI} : \begin{cases} 
\dot{x}(t) & = & Ax(t) + Bw(t) \\
z(t) & = & Cx(t) + Dw(t)
\end{cases} \quad (2)$$

where $x(t)$ is the state which takes values in a state space $X \in \mathbb{R}^n$, $w(t)$ is the input taking values in the input space $W \in \mathbb{R}^{n_w}$ and $z(t)$ is the output that belongs to the output space $Z \in \mathbb{R}^{n_z}$.

The LTI system locally describes the real system under consideration and the linearization procedure allows to treat a linear problem instead of a nonlinear one. For this class of problem, many mathematical and control theory tools can be applied like closed loop stability, controllability, observability, performance, robust analysis, etc. for both SISO and MIMO systems. However, the main restriction is that LTI models only describe the system locally, then, compared to nonlinear models, they lack of information and, as a consequence, are incomplete and may not provide global stabilization.
Illustration of the gain computation

For a SISO system, \( y = Gd \), the gain at a given frequency is simply

\[
\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|
\]

The gain depends on the frequency, but since the system is linear it is independent of the input magnitude.

For a MIMO system we may select:

\[
\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \|G(j\omega)\|_2
\]

Which is « independent » of the input magnitude. But this is not a correct definition. Indeed the input direction is of great importance.
A first 'bad' approach

Let consider

\[
G = \begin{bmatrix}
5 & 4 \\
3 & 2
\end{bmatrix}
\]

How to define and evaluate its gain?

Consider five different inputs:

\[
\begin{align*}
&u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & u = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}, & u = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}, & u = \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix}, \\
&y = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, & y = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, & y = \begin{pmatrix} 6.3630 \\ 3.5350 \end{pmatrix}, & y = \begin{pmatrix} 0.7070 \\ 0.7070 \end{pmatrix}, & y = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix}
\end{align*}
\]

Input magnitude:

\[
\|u_1\|_2 = \|u_2\|_2 = \|u_3\|_2 = \|u_4\|_2 = \|u_5\|_2 = 1
\]

But the corresponding outputs are

\[
\begin{align*}
\|y\|_2 / \|u\|_2 &= 5.8310 / 4.4721 = 7.2790 / 0.9998 = 0.2828
\end{align*}
\]
Towards SVD

MAXIMUM SINGULAR VALUE

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \bar{\sigma}(G')$$

MINIMUM SINGULAR VALUE

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \underline{\sigma}(G')$$

We see that, depending on the ratio $d_{20}/d_{10}$, the gain varies between 0.27 and 7.34.
What about eigenvalues?

Eigenvalues are a poor measure of gain. Let

\[ G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix} \]

Eigenvalues are 0 and 0.
But an input vector

\[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

leads to an output vector

\[ \begin{bmatrix} 100 \\ 0 \end{bmatrix} \].

Clearly the gain is not zero.
Now, the maximal singular value is = 100
It means that any signal can be amplified at most 100 times
This is the good gain notion.
The $H_\infty$ norm and related definitions

Towards $H_\infty$ norm

For MIMO systems

Finally, in the case of a transfer matrix $G(s)$ : (m inputs, p outputs) $u$ vector of inputs, $y$ vector of outputs.

$$\sigma(G(j\omega)) \leq \frac{\|y(\omega)\|_2}{\|u(\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

Example of A two-mass/spring/damper system
2 inputs: $F_1$ and $F_2$ 2 outputs: $x_1$ and $x_2$

\[ G=ss(A,B,C,D) : \text{LTI system} \]
\[ \text{normhinf}(G) : \text{Compute Hinf norm} \]
\[ \text{norm}(G, 'inf') : \text{Compute Hinf norm} \]
\[ \text{sigma}(G) : \text{plot max and min SV} \]
Signal norms

Reader is also invited to refer to the famous book of Zhou et al., 1996, where all the following definitions and additional information are given.
All the following definitions are given assuming signals $x(t) \in \mathbb{C}$, then they will involve the conjugate (denoted as $x^*(t)$). When signals are real (i.e. $x(t) \in \mathbb{R}$), $x^*(t) = x^T(t)$.

Definition (Norm and Normed vector space)

Let $V$ be a finite dimension space. Then $\forall p \geq 1$, the application $\|\cdot\|_p$ is a norm, defined as,

$$\|v\|_p = \left( \sum_i |v_i|^p \right)^{1/p}$$

(3)

Let $V$ be a vector space over $\mathbb{C}$ (or $\mathbb{R}$) and let $\|\cdot\|$ be a norm defined on $V$. Then $V$ is a normed space.
\( L_\infty \) norms

**Definition (\( L_1, L_2, L_\infty \) norms)**

The 1-Norm of a function \( x(t) \) is given by,

\[
\|x(t)\|_1 = \int_0^{+\infty} |x(t)| dt
\]

(4)

The 2-Norm (that introduces the energy norm) is given by,

\[
\|x(t)\|_2 = \sqrt{\int_0^{+\infty} x^*(t)x(t)dt} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)X(j\omega)d\omega}
\]

(5)

The second equality is obtained by using the Parseval identity.

The \( \infty \)-Norm is given by,

\[
\|x(t)\|_\infty = \sup_t |x(t)|
\]

(6)

\[
\|X\|_\infty = \sup_{Re(s) \geq 0} \|X(s)\| = \sup_\omega \|X(j\omega)\|
\]

(7)

if the signals that admit the Laplace transform, analytic in \( Re(s) \geq 0 \) (i.e. \( \in H_\infty \)).
About vector spaces

Definition (Banach, Hilbert, Hardy and $L_p$ spaces)

A Banach space is a (real or complex) complete (i.e. all Cauchy sequences, of points in $K$ have a limit that is also in $K$) normed vector space $B$ (with norm $\| \cdot \|_p$).

A Hilbert space is a (real or complex) vector space $H$ with an inner product $\langle \cdot, \cdot \rangle$ that is complete under the norm defined by the inner product. The norm of $f \in H$ is then defined by,

$$\| f \| = \sqrt{\langle f, f \rangle}$$

(8)

Every Hilbert space is a Banach since a Hilbert space is complete with respect to the norm associated with its inner product.

The Hardy spaces ($H_p$) are certain spaces of holomorphic functions (functions defined on an open subset of the complex number plane $\mathbb{C}$ with values in $\mathbb{C}$ that are complex-differentiable at every point) on the unit disk or upper half plane.

The $L_p$ space are spaces of $p$-power integrable functions (function whose integral exists, generally called Lebesgue integral), and corresponding sequence spaces.

For example, $\mathbb{R}^n$ and $\mathbb{C}^n$ with the usual spatial $p$-norm, $\| \cdot \|_p$ for $1 \leq p < \infty$, are Banach spaces. This means that a Banach space is a vector space $B$ over the real or complex numbers with a norm $\| \cdot \|_p$ such that every Cauchy sequence (with respect to the metric $d(x, y) = \| x - y \|$) in $B$ has a limit in $B$. 
**$L_\infty$ and $H_\infty$ spaces**

**Definition ($L_\infty$ space)**

$L_\infty$ is the space of piecewise continuous bounded functions. It is a Banach space of matrix-valued (or scalar-valued) functions on $\mathbb{C}$ and consists of all complex bounded matrix functions $f(j\omega), \forall \omega \in \mathbb{R}$, such that,

$$\sup_{\omega \in \mathbb{R}} \sigma[f(j\omega)] < \infty$$

(9)

**Definition ($H_\infty$ and $RH_\infty$ spaces)**

$H_\infty$ is a (closed) subspace in $L_\infty$ with matrix functions $f(j\omega), \forall \omega \in \mathbb{R}$, analytic in $Re(s) > 0$ (open right-half plane). The real rational subspace of $H_\infty$ which consists of all proper and real rational stable transfer matrices, is denoted by $RH_\infty$. 
The $H_\infty$ norm and related definitions

$L_\infty$ and $H_\infty$ spaces

Example

In control theory

\[
\frac{s+1}{(s+10)(s+6)} \in \mathcal{RH}_\infty
\]

\[
\frac{s+1}{(s-10)(s+6)} \notin \mathcal{RH}_\infty
\]

\[
\frac{s+1}{(s+10)} \in \mathcal{RH}_\infty
\]

(10)
The $H_\infty$ norm and related definitions

**$H_\infty$ norm**

**Definition ($H_\infty$ norm)**

The $H_\infty$ norm of a proper LTI system defined as on (2) from input $w(t)$ to output $z(t)$ and which belongs to $RH_\infty$, is the induced energy-to-energy gain ($L_2$ to $L_2$ norm) defined as,

$$\|G(j\omega)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma (G(j\omega))$$

$$= \sup_{w(s) \in \mathcal{H}_2} \frac{\|z(s)\|_2}{\|w(s)\|_2}$$

$$= \max_{w(t) \in L_2} \frac{\|z\|_2}{\|w\|_2}$$

(11)

**Remark**

**$H_\infty$ physical interpretations**

*This norm represents the maximal gain of the frequency response of the system. It is also called the worst case attenuation level in the sense that it measures the maximum amplification that the system can deliver on the whole frequency set.*

*For SISO (resp. MIMO) systems, it represents the maximal peak value on the Bode magnitude (resp. singular value) plot of $G(j\omega)$, in other words, it is the largest gain if the system is fed by harmonic input signal.*

*Unlike $H_2$, the $H_\infty$ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).*
Recalls on Singular Value Definition

Let $A \in \mathbb{R}^{m \times n}$, There exists unitary matrices:

$$U = [u_1, u_2, \ldots, u_m] \text{ and } V = [v_1, v_2, \ldots, v_n]$$

such that

$$A = U \Sigma V^T, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}, \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$, $p = \min\{m, n\}$.

Singular values are good measures of the size of a matrix. Singular vectors are good indications of strong/weak input or output directions. Note that:

$Av_i = \sigma_i u_i$ and $A^T u_i = \sigma_i v_i$. Then

$$A A^T u_i = \sigma_i^2 u_i$$

$\bar{\sigma} = \sigma_{max}(A) = \sigma_1 = $ the largest singular value of $A$.

and

$\underline{\sigma} = \sigma_{min}(A) = \sigma_p = $ the smallest singular value of $A$. 
**L₂ and H₂ spaces**

**Definition (L₂ space)**

L₂ is the space of piecewise continuous square integrable functions. It is a Hilbert space of matrix-valued (or scalar-valued) functions on \( \mathbb{C} \) and consists of all complex matrix functions \( f(j\omega) \), \( \forall \omega \in \mathbb{R} \), such that,

\[
\|f\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}[f^*(j\omega)f(j\omega)]d\omega} < \infty
\]  

(12)

The inner product for this Hilbert space is defined as (for \( f, g \in L₂ \))

\[
<f, g> = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}[f^*(j\omega)g(j\omega)]d\omega}
\]  

(13)

**Definition (H₂ and RH₂ spaces)**

H₂ is a subspace (Hardy space) of L₂ with matrix functions \( f(j\omega) \), \( \forall \omega \in \mathbb{R} \), analytic in \( Re(s) > 0 \) (functions that are locally given by a convergent power series and differentiable on each point of its definition set). In particular, the real rational subspace of H₂, which consists of all strictly proper and real rational stable transfer matrices, is denoted by RH₂.
$\mathcal{L}_2$ and $\mathcal{H}_2$ spaces

Example

In control theory

\[
\frac{s+1}{(s+10)(s+6)} \in \mathcal{RH}_2
\]

\[
\frac{s+1}{(s-10)(s+6)} \not\in \mathcal{RH}_2
\]

\[
\frac{s+1}{s+10} \not\in \mathcal{RH}_2
\]  \hfill (14)
The \( H_{\infty} \) norm and related definitions

\( LTI \) systems and signals norms

\( \mathcal{H}_2 \) norm

Definition (\( \mathcal{H}_2 \) norm)

The \( \mathcal{H}_2 \) norm of a strictly proper LTI system defined as on (2) from input \( w(t) \) to output \( z(t) \) and which belongs to \( \mathcal{RH}_2 \), is the energy (\( L_2 \) norm) of the impulse response \( g(t) \) defined as,

\[
\| G(j\omega) \|_2 = \sqrt{\int_{-\infty}^{+\infty} g^*(t)g(t)dt} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}[G^*(j\omega)G(j\omega)] d\omega}
\]

\[= \sup_{w(s) \in \mathcal{H}_2} \frac{||z(s)||_{\infty}}{||u(s)||_2} \tag{15}\]

The norm \( \mathcal{H}_2 \) is finite if and only if \( G(s) \) is strictly proper (i.e. \( G(s) \in \mathcal{RH}_2 \)).

Remark

\( \mathcal{H}_2 \) physical interpretations and remarks

For SISO systems, it represents the area located below the so called Bode diagram.

For MIMO systems, the \( \mathcal{H}_2 \) norm is the impulse-to-energy gain of \( z(t) \) in response to a white noise input \( w(t) \) (satisfying \( W^*(j\omega)W(j\omega) = I \), i.e. uniform spectral density).

The \( \mathcal{H}_2 \) norm can be computed analytically (through the use of the controllability and observability Grammians) or numerically (through LMI s).
Outline

1. The $H_\infty$ norm and related definitions
   - $H_\infty$ norm as a measure of the system gain
   - LTI systems and signals norms

2. Performance analysis
   - Definition of the sensitivity functions
   - Frequency-domain analysis
   - Stability issues

3. Performance specifications for analysis and/or design
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   - MIMO case
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   - Some useful lemmas

7. Introduction to LPV systems and control
   - LPV modelling
   - LPV Control
Objectives of any control system [Skogestad and Postlethwaite(1996)]
shape the response of the system to a given reference and get (or keep) a stable system in closed-loop, with desired performances, while minimising the effects of disturbances and measurement noises, and avoiding actuators saturation, this despite of modelling uncertainties, parameter changes or change of operating point.
This is formulated as:

**Nominal stability (NS):** The system is stable with the nominal model (no model uncertainty)

**Nominal Performance (NP):** The system satisfies the performance specifications with the nominal model (no model uncertainty)

**Robust stability (RS):** The system is stable for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant)

**Robust performance (RP):** The system satisfies the performance specifications for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant).
The control structure

![Complete control scheme diagram]

**Figure**: Complete control scheme

In the SISO case, it leads to:

\[
\begin{align*}
y &= \frac{1}{1+G(s)K(s)} (GKr + dy - GKn + Gdi) \\
u &= \frac{1}{1+K(s)G(s)} (Kr - Kdy - Kn - KGdi)
\end{align*}
\]

<table>
<thead>
<tr>
<th>Loop transfer function</th>
<th>( L = G(s)K(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity function</td>
<td>( S(s) = \frac{1}{1+L(s)} )</td>
</tr>
<tr>
<td>Complementary Sensitivity function</td>
<td>( T(s) = \frac{L(s)}{1+L(s)} )</td>
</tr>
</tbody>
</table>

N.B. \( S \) is often referred to as the 'Output Sensitivity'.

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Defining two new 'sensitivity functions':

**Plant Sensitivity:** \( SG = S(s) \cdot G(s) \) (often referred to as the 'Input Sensitivity', e.g. in **Matlab**)

**Controller Sensitivity:** \( KS = K(s) \cdot S(s) \)
Definition of the sensitivity functions- The MIMO case

The output & the control input satisfy the following equations:

\[
(I_p + G(s)K(s))y(s) = (GKr + dy - GKn + Gdi) \\
(I_m + K(s)G(s))u(s) = (Kr - Kdy - Kn - KGdi)
\]

BUT : \(K(s)G(s) \neq G(s)K(s)\) !!
Definition of the sensitivity functions - The MIMO case

Definitions

Output and Output complementary sensitivity functions:

\[ S_y = (I_p + GK)^{-1}, \quad T_y = (I_p + GK)^{-1}GK, \quad S_y + T_y = I_p \]

Input and Input complementary sensitivity functions:

\[ S_u = (I_m + KG)^{-1}, \quad T_u = KG(I_m + KG)^{-1}, \quad S_u + T_u = I_m \]

Properties

\[ T_y = GK(I_p + GK)^{-1} \]
\[ T_u = (I_m + KG)^{-1}KG \]
\[ S_u K = KS_y \]
MIMO Input/Output performances

Defining two new 'sensitivity functions':

Plant Sensitivity:  \( S_y G = S_y(s) \cdot G(s) \)

Controller Sensitivity:  \( K S_y = K(s) \cdot S_y(s) \)

Input performance  
Output performance
The transfer function \( KS_y(s) \) should be upper bounded so that \( u(t) \) does not reach the physical constraints, even for a large reference \( r(t) \).

The effect of the measurement noise \( n(t) \) on the plant input \( u(t) \) can be made « small » by making the sensitivity function \( KS_u(s) \) small (in High Frequencies).

The effect of the input disturbance \( d_i(t) \) on the plant input \( u(t) + d_i(t) \) (actuator) can be made « small » by making the sensitivity function \( S_u(s) \) small (take care to not trying to minimize \( T_u \) which is not possible).
The plant output $y(t)$ can track the reference $r(t)$ by making the complementary sensitivity function $T_y(s)$ equal to 1. (servo pb)

The effect of the output disturbance $d_y(t)$ (resp. input disturbance $d_i(t)$) on the plant output $y(t)$ can be made « small » by making the sensitivity function $S_y(s)$ (resp. $S_yG(s)$) « small »

The effect of the measurement noise $n(t)$ on the plant output $y(t)$ can be made « small » by making the complementary sensitivity function $T_y(s)$ « small »
Trade-offs

But

\[ S_\star + T_\star = I_\star \]

Some trade-offs are to be looked for...

These trade-offs can be reached if one aims:

- to reject the disturbance effects in low frequencies
- to minimize the noise effects in high frequencies

It remains to require:

- \( S_y \) and \( S_yG \) to be small in low frequencies to reduce the load (output and input) disturbance effects on the controlled output
- \( T_y \) and \( KS_y \) to be small in high frequencies to reduce the effects of measurement noises on the controlled output and on the control input (actuator efforts)
Stability and robustness margins ...

Classical definitions:

**Gain Margin**: indicates the additional gain that would take the closed loop to the critical stability condition

**Phase margin**: quantifies the pure phase delay that should be added to achieve the same critical stability condition

**Delay margin**: quantifies the maximal delay that should be added in the loop to achieve the same critical stability condition
Robustness margins ...

It is important to consider the module margin that quantifies the minimal distance between the curve and the critical point (-1,0j): this is a robustness margin.

$$\Delta M = \min_{\omega} \|1 + G(s)K(s)\|$$

A good module margin implies good gain and phase margins:

$$GM \geq \frac{M_S}{M_S - 1} \text{ and } PM \geq \frac{1}{M_S}$$

For $M_S = 2$, then $GM > 2$ and $PM > 30^\circ$

Last:

$$MT = \max_{\omega} \|T(j\omega)\|$$

A good value: $MT < 1.5(3.5dB)$
Performance analysis

Frequency-domain analysis

Frequency-domain analysis - Dynamical behavior

As mentioned in [Skogestad and Postlethwaite(1996)]:
The concept of bandwidth is very important in understanding the benefits and trade-offs involved when applying feedback control. Above we considered peaks of closed-loop transfer functions, which are related to the quality of the response. However, for performance we must also consider the speed of the response, and this leads to considering the bandwidth frequency of the system. In general, a large bandwidth corresponds to a faster rise time, since high frequency signals are more easily passed on to the outputs. A high bandwidth also indicates a system which is sensitive to noise and to parameter variations. Conversely, if the bandwidth is small, the time response will generally be slow, and the system will usually be more robust.

Definition

Loosely speaking, bandwidth may be defined as the frequency range \([\omega_1, \omega_2]\) over which control is effective. In most cases we require tight control at steady-state so \(\omega_1 = 0\), and we then simply call \(\omega_2\) the bandwidth. The word "effective" may be interpreted in different ways: globally it means benefit in terms of performance.
Definition ($\omega_S$)

The (closed-loop) bandwidth, $\omega_S$, is the frequency where $|S(j\omega)|$ crosses $-3dB$ ($1/\sqrt{2}$) from below.

Remark: $|S| < 0.707$, frequency zone, where $e/r = -S$ is reasonably small

Definition ($\omega_T$)

The bandwidth (in term of $T$), $\omega_T$, is the frequency where $|T(j\omega)|$ crosses $-3dB$ ($1/\sqrt{2}$) from above.

Definition ($\omega_c$)

The bandwidth (crossover frequency), $\omega_c$, is the frequency where $|L(j\omega)|$ crosses $1$ ($0dB$), for the first time, from above.
Some remarks

Remark

Usually \( \omega_S < \omega_c < \omega_T \)

Remark

In most cases, the two definitions in terms of \( S \) and \( T \) yield similar values for the bandwidth. In other cases, the situation is generally as follows. Up to the frequency \( \omega_S \), \(|S| \) is less than 0.7, and control is effective in terms of improving performance. In the frequency range \([\omega_S, \omega_T]\) control still affects the response, but does not improve performance. Finally, at frequencies higher than \( \omega_T \), we have \( S \approx 1 \) and control has no significant effect on the response.

Remark

Usually \( \omega_S < \omega_c < \omega_T \)

Finally the following relation is very useful to evaluate the rise time:

\[
t_R = \frac{2.3}{\omega_T}
\]
Examples
Well-posedness

Consider

\[ G = -\frac{s - 1}{s + 2}, \quad K = 1 \]

Therefore the control input is non proper:

\[ u = \frac{s + 2}{3} (r - n - d_y) + \frac{s - 1}{3} d_i \]

DEF: A closed-loop system is well-posed if all the transfer functions are proper

\[ \Leftrightarrow \quad I + K(\infty)G(\infty) \text{ is invertible} \]

In the example \( 1 + 1 \times (-1) = 0 \) Note that if \( G \) is strictly proper, this always holds.
DEF: A system is internally stable if all the transfer functions of the closed-loop system are stable.

\[
(y \quad u) = \begin{pmatrix} (I + GK)^{-1}GK \\ K(I + GK)^{-1} \\ -K(I + GK)^{-1}G \end{pmatrix} \begin{pmatrix} r \\ d_i \end{pmatrix}
\]

For instance:

\[
G = \frac{1}{s - 1}, \quad K = \frac{s - 1}{s + 1}, \quad \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} \\ \frac{s+1}{s-1} \\ \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} r \\ d_i \end{pmatrix}
\]

There is one RHP pole (1), which means that this system is not internally stable. This is due here to the pole/zero cancellation (forbidden!!).
Small Gain theorem

Consider the so called $M - \Delta$ loop.

![Diagram of $M - \Delta$ loop]

**Figure: $M - \Delta$ form**

**Theorem**

Suppose $M(s)$ in $RH_\infty$ and $\gamma$ a positive scalar. Then the system is well-posed and internally stable for all $\Delta(s)$ in $RH_\infty$ such that $\|\Delta\|_\infty \leq 1/\gamma$ if and only if

$$\|M\|_\infty < \gamma$$
Input-Output Stability

Definition (BIBO stability)

A system $G (\dot{x} = Ax + Bu; y = Cx)$ is **BIBO stable** if a bounded input $u(.) (\|u\|_{\infty} < \infty)$ maps a bounded output $y(.) (\|y\|_{\infty} < \infty)$.

Now, the quantification (for BIBO stable systems) of the signal amplification (gain) is evaluated as:

$$\gamma_{peak} = \sup_{0 < \|u\|_{\infty} < \infty} \frac{\|y\|_{\infty}}{\|u\|_{\infty}}$$

and is referred to as the **PEAK TO PEAK Gain**.

Definition ($L_2$ stability)

A system $G (\dot{x} = Ax + Bu; y = Cx)$ is **$L_2$ stable** if $\|u\|_2 < \infty$ implies $\|y\|_2 < \infty$.

Now, the quantification of the signal amplification (gain) is evaluated as:

$$\gamma_{energy} = \sup_{0 < \|u\|_2 < \infty} \frac{\|y\|_2}{\|u\|_2}$$

and is referred to as the **ENERGY Gain**, and is such that:

$$\gamma_{energy} = \sup \omega \|G(j\omega)\| := \|G\|_{\infty}$$

For a linear system, these stability definitions are equivalent (but not the quantification criteria).

O. Sename [GIPSA-lab]
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1. The $H_\infty$ norm and related definitions
   - $H_\infty$ norm as a measure of the system gain?
   - LTI systems and signals norms

2. Performance analysis
   - Definition of the sensitivity functions
   - Frequency-domain analysis
   - Stability issues

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   - SISO case
   - MIMO case
   - Some performance limitations

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   - LMI in control
   - The LMI approach for $H_\infty$ control design
   - Some useful lemmas

7. Introduction to LPV systems and control
   - LPV modelling
   - LPV Control
Objective: good performance specifications are important to ensure better control system
mean: give some templates on the sensitivity functions
For simplicity, presentation for SISO systems first.
Sketch of the method:

Robustness and performances in regulation can be specified by imposing frequential
templates on the sensitivity functions.

If the sensitivity functions stay within these templates, the control objectives are met.
These templates can be used for analysis and/or design. In the latter they are considered as
weights on the sensitivity functions

The shapes of typical templates on the sensitivity functions are given in the following slides

Mathematically, these specifications may be captured by an upper bound, on the magnitude of a
sensitivity function, given by another transfer function, as for $S$:

$$|S(j\omega)| \leq \frac{1}{|W_e(j\omega)|}, \quad \forall \omega \Leftrightarrow \|W_eS\|_{\infty} \leq 1$$

where $W_e(s)$ is a WEIGHT selected by the designer.
Performance specifications
SISO case

Template on the sensitivity function - Weighted sensitivity

Typical specifications in terms of $S$ include:

- Minimum bandwidth frequency $\omega_S$
- Maximum tracking error at selected frequencies.
- System type, or the maximum steady-state tracking error $\epsilon_0$
- Shape of $S$ over selected frequency ranges.
- Maximum peak magnitude of $S$, $\|S\|_\infty < M_S$.

The peak specification prevents amplification of noise at high frequencies, and also introduces a margin of robustness; typically we select $M_S = 2$. 
Template on the sensitivity function $S$

$$S(s) = \frac{1}{1 + K(s)G(s)}$$

$$\frac{1}{W_e(s)} = \frac{s + \omega_b \varepsilon}{s/M_S + \omega_b}$$

Generally $\varepsilon \approx 0$ is considered, $M_S < 2$ (6dB) or (3dB - cautious) to ensure sufficient module margin.

$\omega_b$ influences the CL bandwidth: $\omega_b \uparrow$
- faster rejection of the disturbance
- faster CL tracking response
- better robustness w.r.t. parametric uncertainties
Template on the function $KS$

$$KS(s) = \frac{K(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_u(s)} = \frac{\varepsilon_1 s + \omega_{bc}}{s + \omega_{bc}/M_u}$$

$M_u$ chosen according to LF behavior of the process (actuator constraints: saturations)

$\omega_{bc}$ influences the CL bandwidth: $\omega_{bc} \downarrow$

better limitation of measurement noises

roll-off starting from $\omega_{bc}$ to reduce modeling errors effects
Template on the function $T$

$$T(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_T(s)} = \frac{\varepsilon_T s + \omega_{bt}}{s + \omega_{bt}/M_T}$$

Generally $\varepsilon_T \simeq 0$ is considered, $M_T < 1.5$ (3dB) to limit the overshoot. $\omega_{bt}$ influences the bandwidth hence the transient behavior of the disturbance rejection properties: $\omega_{bt} \downarrow$

- better noise effects rejection
- better filtering of HF modelling errors
Template on the function $SG$

\[
SG(s) = \frac{G(s)}{1 + K(s)G(s)}
\]

\[
\frac{1}{W_{SG}(s)} = \frac{s + \omega_{SG} \varepsilon_{SG}}{s/M_{SG} + \omega_{SG}}
\]

$M_{SG}$ allows to limit the overshoot in the response to input disturbances. Generally $\varepsilon_{SG} \approx 0$ is considered, $\omega_{SG}$ influences the CL bandwidth: $\omega_{SG} \uparrow \implies$ faster rejection of the disturbance.
A first insight into the MIMO case

The direct extension of the performances objectives to MIMO systems could be formulated as follows:

1. **Disturbance attenuation/closed-loop performances:**
   \[
   \bar{\sigma}(S_y) < \frac{1}{W_1(j\omega)}
   \]
   with \( W_1(j\omega) > 1 \) for \( \omega < \omega_b \)

2. **Actuator constraints:**
   \[
   \bar{\sigma}(KS_y) < \frac{1}{W_2(j\omega)}
   \]
   with \( W_2(j\omega) > 1 \) for \( \omega > \omega_n \)

3. **Robustness to multiplicative uncertainties:**
   \[
   \bar{\sigma}(T_y) < \frac{1}{W_3(j\omega)}
   \]
   with \( W_3(j\omega) > 1 \) for \( \omega > \omega_k \)

However these objectives do not consider the specific MIMO structure of the system, i.e. the input-output relationship between actuators and sensors. It is then better to define the objectives accordingly with the system inputs and outputs.
Towards MIMO systems

Let us consider a system with 2 inputs and 1 output and define:

\[ G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \]

Therefore

\[ GK = G_1 K_1 + G_2 K_2, \quad KG = \begin{pmatrix} K_1 G_1 & K_1 G_2 \\ K_2 G_1 & K_2 G_2 \end{pmatrix} \]

and the sensitivity functions are:

\[ S_y = \frac{1}{1 + G_1 K_1 + G_2 K_2}, \quad KS_y = \begin{pmatrix} \frac{K_1}{1+G_1 K_1+G_2 K_2} \\ \frac{K_2}{1+G_1 K_1+G_2 K_2} \end{pmatrix} \]

While a single template \( W_e \) is convenient for \( S_y \) it is straightforward that the following diagonal template should be used for \( KS_y \):

\[ W_u(s) = \begin{pmatrix} W_u^1(s) & 0 \\ 0 & W_u^2(s) \end{pmatrix} \]

where \( W_u^1 \) and \( W_u^2 \) are chosen in order to account for each actuator specificity (constraint).
Let us consider $G$ with $m$ inputs and $p$ outputs.

In the MIMO case the simplest way is to defined the templates as diagonal transfer matrices, i.e. using $(M_{Si}, \omega_{bi}, \varepsilon_i)$

In that case, a weighting function should be dedicated for each input, and for each output. These weighting functions may of course be different if the specifications on each actuator (e.g. saturation), and on each sensor (e.g. noise), are different.

In addition, during the performance analysis step, take care to plot, in addition to the MIMO sensitivity functions, the individual ones related to each input/output to check if the individual specification is met. Hence, in the simplest case:

If the specifications are identical then it is sufficient to plot:
\[
\begin{align*}
\bar{\sigma}(S_y(j\omega)) & \text{ and } \frac{1}{|W_e(j\omega)|}, \text{ for all } \omega \\
\bar{\sigma}(KS_y(j\omega)) & \text{ and } \frac{1}{|W_u(j\omega)|}, \text{ for all } \omega
\end{align*}
\]

If the specifications are different, one should plot
\[
\begin{align*}
\bar{\sigma}(S_y(i,:)) & \text{ and } \frac{1}{|W_i^e|}, \text{ for all } \omega, i = 1, \ldots, p \\
\bar{\sigma}(KS_y(k,:)) & \text{ and } \frac{1}{|W_i^u|}, \text{ for all } \omega, k = 1, \ldots, m.
\end{align*}
\]
i.e. $p$ plots for all output behaviors and $m$ plots for the input ones.

In a very general case, plot $\bar{\sigma}(S_y)$ with $\bar{\sigma}(1/W_e)$
More on weighting functions

When tighter (harder) objectives are to be met ..... the templates can be defined more accurately by transfer functions of order greater than 1, as

$$W_e(s) = \left( \frac{s/MS + \omega b}{s + \omega b \varepsilon} \right)^k,$$

if a roll-off of $-20 \times k$ dB per decade is required. Take care to the choice of the parameters $(M_S, \omega_b, \varepsilon)$ to avoid incoherent objectives!
Final objectives

In terms of control synthesis, all these specifications can be tackled in the following problem: find $K(s)$ s.t.

\[
\begin{bmatrix}
W_e S \\
W_u K S \\
W_T T \\
W_{SG} S G
\end{bmatrix}
\leq 1 \Rightarrow \|W_e S\|_{\infty} \leq 1 \quad \|W_u K S\|_{\infty} \leq 1 \quad \|W_{SG} S G\|_{\infty} \leq 1
\]

Often, the simpler following one (referred to as the mixed sensitivity problem) is studied:

Find $K$ s.t. \[
\begin{bmatrix}
W_e S \\
W_u K S
\end{bmatrix}
\leq 1
\]

since the latter allows to consider the closed-loop output performance as well as the actuator constraints.
Performance specifications

Introduction

Framework

Main extracts of this part: see Goodwin et al 2001. "Performance limitations in control are not only inherently interesting, but also have a major impact on real world problems."
Objective: take into account the limitations inherent to the system or due to actuators constraints, before designing the controller..... Understanding what is not possible is as important as understanding what is possible!

Example of structural constraints

\[ S + T = 1, \forall \omega \]

We then cannot have, for any frequency \( \omega_0 \), \( |S(j\omega_0)| < 1 \) and \( |T(j\omega_0)| < 1 \). This implies that, disturbance and noise rejection cannot be achieved in the same frequency range.

Bode’s Sensitivity Integral for open-loop stable systems

It is known that, for open loop stable plant:

\[ \int_{0}^{\infty} \log|S(j\omega)|d\omega = 0 \]

Then the frequency range where \( |S(j\omega)| < 1 \) is balanced by the frequencies where \( |S(j\omega)| > 1 \)
Bode sensitivity

Nice interpretation of the balance between reduction and magnification of the sensitivity.

For unstable system we have stronger constraints:

\[
\int_0^\infty \log|\text{det}(S(j\omega))|d\omega = \pi \sum_{i=1}^{N_p} \text{Re}(p_i),
\]

where \(p_i\) design the \(N_p\) RHP poles. Therefore, in the presence of RHP poles, the control effort necessary to stabilize the system is paid in terms of amplification of the sensitivity magnitude.
The interesting case of systems with RHP zeros

Theorem

Let $G(s)$ a MIMO plant with one RHP zero at $s = z$, and $W_p(s)$ be a scalar weight. Then, closed-loop stability is ensured only if:

$$\| W_p(s) S(s) \| \geq |W_p(s = z)|$$

To illustrate the use of that theorem, if $W_p$ is chosen as:

$$W_p(s) = \left( \frac{s/M_S + \omega b}{s + \omega_b \epsilon} \right),$$

and, if the controller meets the requirements, then

$$\| W_p(s) S(s) \|_\infty \leq 1$$

Therefore a necessary condition is:

$$| \frac{z/M_S + \omega b}{z + \omega_b \epsilon} | \leq 1$$

To conclude, if $z$ is real, and if the performance specifications are such that: $M_S = 2$ and $\epsilon = 0$, then a necessary condition to meet the performance requirements is:

$$\omega_b \leq \frac{z}{2}$$
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The General Control Configuration

This approach has been introduced by Doyle (1983). The formulation makes use of the general control configuration.

\[ P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \]

where \( P \) is the generalized plant (contains the plant, the weights, the uncertainties if any) ; \( K \) is the controller. The closed-loop transfer function is given by:

\[ T_{ew}(s) = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \]

where \( F_l(P, K) \) is referred to as a lower Linear Fractional Transformation.
The overall control objective is to minimize some norm of the transfer function from $w$ to $e$, for example, the $\mathcal{H}_\infty$ norm.

**Definition ($\mathcal{H}_\infty$ optimal control problem)**

$\mathcal{H}_\infty$ control problem: Find a controller $K(s)$ which based on the information in $y$, generates a control signal $u$ which counteracts the influence of $w$ on $e$, thereby minimizing the closed-loop norm from $w$ to $e$.

**Definition ($\mathcal{H}_\infty$ suboptimal control problem)**

Given $\gamma$ a pre-specified attenuation level, a $\mathcal{H}_\infty$ sub-optimal control problem is to design a stabilizing controller that ensures:

$$\|T_{ew}(s)\|_\infty = \max_{\omega} \bar{\sigma}(T_{ew}(j\omega)) \leq \gamma$$

The optimal problem aims at finding $\gamma_{min}$. 
$\mathcal{H}_\infty$ control design - Problem formulation

It will be shown how to formulate such a control problem using "classical" control tools. The procedure will be 2-steps:

**Get $P$:** Build a control scheme s.t. the closed-loop system matrix does correspond to the tackled $H_\infty$ problem (for instance the mixed sensitivity problem). Use of *Matlab, sysic* tool.

**Compute $K$:** Use an optimisation algorithm that finds the controller $K$ solution of the considered problem.

**Illustration on**

\[
\left\| \begin{array}{c}
W_e S \\
W_u K S
\end{array} \right\|_\infty \leq \gamma
\]

In that case the closed-loop system must be
\[
\left\| T_{ew}(s) \right\|_\infty = \left\| \begin{array}{c}
W_e S \\
W_u K S
\end{array} \right\|_\infty \leq \gamma
\]
i.e a system with 1 input and 2 outputs. Since $S = \frac{r - y}{r}$ and $KS = \frac{u}{r}$, the control scheme needs only one external input $r$. 
How to consider performance specification in $\mathcal{H}_\infty$ control?

Note that: in practice the performance specification concerns at least two sensitivity functions ($S$ and $KS$) in order to take into account the tracking objective as well as the actuator constraints.

**Control objectives:**

\[
y = Gu = GK(r - y) \Rightarrow \text{tracking error: } \varepsilon = Sr \\
u = K(r - y) = K(r - Gu) \Rightarrow \text{actuator force: } u = KSr
\]

To cope with that control objectives the following control scheme is considered:

**Objective w.r.t sensitivity functions:**

\[
\|WeS\|_\infty \leq 1, \quad \|WuKS\|_\infty \leq 1.
\]

**Idea:** define 2 new virtual controlled outputs:

\[
e_1 = WeSr \\
e_2 = WuKSr
\]
About the control structure - Problem definition

The performance specifications on the tracking error & on the actuator, given as some weights on the controlled output, then leads to the new control scheme:

The associated general control configuration is:

\[
\begin{bmatrix}
W_e & -W_e G \\
0 & W_u \\
I & -G
\end{bmatrix}
\]

\[e=(e_1, e_2)^T\]
About the control structure- Problem definition

The corresponding $\mathcal{H}_\infty$ suboptimal control problem is therefore to find a controller $K(s)$ such that

$$\|T_{ew}(s)\|_\infty = \left\| \begin{bmatrix} W_e S \\ W_u K S \end{bmatrix} \right\|_\infty \leq \gamma$$

with

$$T_{ew}(s) = F_l(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

$$= \begin{bmatrix} W_e \\ 0 \end{bmatrix} + \begin{bmatrix} -W_e G \\ W_u \end{bmatrix} K (I + G K)^{-1} I$$

$$= \begin{bmatrix} W_e S \\ W_u K S \end{bmatrix}$$

in Matlab

```matlab
% Generalized plant P is found with function sysic
systemnames = 'G We Wu';
inputvar = '[r(1);u(1)]';
outputvar = '[We; Wu; r-G]';
input_to_G = '[u]';
input_to_We = '[r-G]';
input_to_Wu = '[u]';
sysoutname = 'P';
cleanupsysic = 'yes';
sysic;
% Find H-infinity optimal controller
nmeas=1; nu=1;
[K,CL,GAM,INFO] = hinfsyn(P,nmeas,nu,'DISPLAY','ON');
gopt
```

O. Sename [GIPSA-lab]
What about disturbance attenuation?

To account for input disturbance rejection, the control scheme must include $d_i$:

This corresponds to the closed-loop system.

$$T_{ew} = \begin{bmatrix} W_e S_y & W_e S_y G \\ W_u K S_y & W_u T_u \end{bmatrix}$$

The new $\mathcal{H}_\infty$ control problem therefore includes the input disturbance rejection objective, thanks to $S_y G$ that should satisfy the same template as $S$, i.e., an high-pass filter!

Remarks: Note that $W_u T_u$ is an additional constraint that may lead to an increase of the attenuation level $\gamma$ since it is not part of the objectives. Hopefully $T_u$ is low pass, and $W_u$ as well. The input weight has to be on $u$ not $u + d_i$ which would lead to an unsolvable problem.
Improve the disturbance attenuation

The previous problem, allows to ensure the input disturbance rejection, but does not provide any additional d-o-f to improve it (without impacting the tracking performance). In order to 'decouple' both performance objectives, the idea is to add a disturbance model that indeed changes the disturbance rejection properties.

Let then consider: \( d_i(t) = W_d.d \). In that case the closed-loop system is This corresponds to the closed-loop system.

\[
T_{ew} = \begin{bmatrix}
W e S_y & W e S_y G W_d \\
W_u K S_y & W_u T_u W_d
\end{bmatrix}
\]

and the template expected for \( S_y G \) is now \( \frac{1}{W_d . W_e} \).

First interest: improve the disturbance weight as \( W_d = 100 \)... but this has a price (see Fig. below for an example).
More generally...

To include multiple objectives in a SINGLE $\mathcal{H}_\infty$ control problem, there are 2 ways:

1. add some external inputs (reference, noise, disturbance, uncertainties ...)
2. add new controlled outputs

Of course both ways increase the dimension of the problem to be solved....thus the complexity as well. Moreover additional constraints appear that are not part of the objectives ....

General rule: first think simple !!
Solving the $H_\infty$ control problem

The solution of the $H_\infty$ control problem is based on a state space representation of $P$, the generalized plant, that includes the plant model and the performance weights. The calculation of the controller, solution of the $H_\infty$ control problem, can then be done using the Riccati approach or the LMI approach of the $H_\infty$ control problem [Zhou et al. (1996) Zhou, Doyle, and Glover] [Skogestad and Postlethwaite (1996)].

Notations:

\[ P = \begin{bmatrix}
\dot{x} = Ax + B_1 w + B_2 u \\
e = C_1 x + D_{11} w + D_{12} u \\
y = C_2 x + D_{21} w + D_{22} u
\end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\]

with $x \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ et $y \in \mathbb{R}^n$. 

O. Sename [GIPSA-lab]
Problem formulation

Let $K(s)$ be a dynamic output feedback LTI controller defined as

$$K(s) : \begin{cases} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t), \\ u(t) &= C_K x_K(t) + D_K y(t). \end{cases}$$

where $x_K \in \mathbb{R}^n$, and $A_K$, $B_K$, $C_K$ and $D_K$ are matrices of appropriate dimensions.

**Remark.** The controller will be considered here of the same order (same number of state variables) $n$ than the generalized plant, which here, in the $\mathcal{H}_\infty$ framework, the order of the optimal controller.

With $P(s)$ and $K(s)$, the closed-loop system $N(s)$ is:

$$N(s) : \begin{cases} \dot{x}_{cl}(t) &= A x_{cl}(t) + B w(t), \\ z(t) &= C x_{cl}(t) + D w(t), \end{cases} \quad (16)$$

where $x_{cl}^T(t) = [x^T(t) \ x_K^T(t)]$ and

$$\begin{align*}
A &= \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\
B_K C_2 & A_K \end{pmatrix}, \\
B &= \begin{pmatrix} B_1 + B_2 D_K D_{21} \\
B_K D_{21} \end{pmatrix}, \\
C &= \begin{pmatrix} C_1 + D_{12} D_K C_2, & D_{12} C_K \end{pmatrix}, \\
D &= B_1 + B_2 D_K D_{21}.
\end{align*}$$

The aim is of course to find matrices $A_K$, $B_K$, $C_K$ and $D_K$ s.t. the $\mathcal{H}_\infty$ norm of the closed-loop system (16) is as small as possible, i.e. $\gamma_{opt} = \min \gamma$ s.t. $\|N(s)\|_\infty < \gamma$. 
Assumptions for the Riccati method

A1: $(A, B_2)$ stabilizable and $(C_2, A)$ detectable: necessary for the existence of stabilizing controllers

A2: $\text{rank}(D_{12}) = n_u$ and $\text{rank}(D_{21}) = n_y$: Sufficient to ensure the controllers are proper, hence realizable

A3: $\forall \omega \in \mathbb{R}, \text{rank} \left( \begin{array}{cc} A - j\omega I & B_2 \\ C_1 & D_{12} \end{array} \right) = n + n_u$

A4: $\forall \omega \in \mathbb{R}, \text{rank} \left( \begin{array}{cc} A - j\omega I & B_1 \\ C_2 & D_{21} \end{array} \right) = n + n_y$ Both ensure that the optimal controller does not try to cancel poles or zeros on the imaginary axis which would result in CL instability

A5: $D_{11} = 0, \quad D_{22} = 0, \quad D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix}$, 
\[
\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}
\]
not necessary but simplify the solution (does correspond to the given theorem next but can be easily relaxed)
The problem solvability

The first step is to check whether a solution does exist or not, to the optimal control problem.

Theorem (1)

Under the assumptions A1 to A5, there exists a dynamic output feedback controller $u(t) = K(.) y(t)$ such that the closed-loop system is internally stable and the $\mathcal{H}_\infty$ norm of the closed-loop system from the exogenous inputs $w(t)$ to the controlled outputs $z(t)$ is less than $\gamma$, if and only if

i. the Hamiltonian $H = \begin{pmatrix} A & \gamma^{-2} B_1 \gamma^{-1} B_1^T - B_2 B_2^T \\ -C_1 & C_1^T \end{pmatrix}$ has no eigenvalues on the imaginary axis.

ii. there exists $X_\infty \succeq 0$ t.q. $A^T X_\infty + X_\infty A + X_\infty \left( \gamma^{-2} B_1 \gamma^{-1} B_1^T - B_2 B_2^T \right) X_\infty + C_1^T C_1 = 0$,

iii. the Hamiltonian $J = \begin{pmatrix} A^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{pmatrix}$ has no eigenvalues on the imaginary axis.

iv. there exists $Y_\infty \succeq 0$ t.q. $A Y_\infty + Y_\infty A^T + Y_\infty \left( \gamma^{-2} C_1^T C_1 - C_2^T C_2 \right) Y_\infty + B_1 B_1^T = 0$,

v. the spectral radius $\rho(X_\infty Y_\infty) \leq \gamma^2$. 
Controller reconstruction

Theorem (2)

If the necessary and sufficient conditions of the Theorem 1 are satisfied, then the so-called central controller is given by the state space representation

\[
K_{sub}(s) = \begin{bmatrix} \hat{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix}
\]

with

\[
\hat{A}_\infty = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \\
F_\infty = -B_2^T X_\infty, \quad L_\infty = -Y_\infty C_2^T \\
Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}
\]

The Controller structure is indeed an observer-based state feedback control law, with

\[
u_2(t) = -B_2^T X_\infty \hat{x}(t),
\]

where \(\hat{x}(t)\) is the observer state vector

\[
\dot{x}(t) = A \hat{x}(t) + B_1 \hat{w}(t) + B_2 u(t) + Z_\infty L_\infty \left( C_2 \hat{x}(t) - y(t) \right).
\]  

(17)

and \(\hat{w}(t)\) is defined as

\[
\hat{w}(t) = \gamma^{-2} B_1^T X_\infty \hat{x}(t).
\]

Remark. \(\hat{w}(t)\) is an estimation of the worst case disturbance. \(Z_\infty L_\infty\) is the filter gain for the OE problem of estimating \(\hat{x}(t)\) in the presence of the worst case disturbance.
Outline

1. The $\mathcal{H}_\infty$ norm and related definitions
   - $\mathcal{H}_\infty$ norm as a measure of the system gain
   - LTI systems and signals norms

2. Performance analysis
   - Definition of the sensitivity functions
   - Frequency-domain analysis
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   - Representation of uncertainties
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   - LPV modelling
   - LPV Control
Introduction

A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller

How to take into account the difference between the actual system and the model?
A solution: using a model set BUT : very large problem and not exact yet

**A method:** these differences are referred as model uncertainty.

The approach

- determine the uncertainty set: mathematical representation
- check Robust Stability
- check Robust Performance

Lots of forms can be derived according to both our knowledge of the physical mechanism that cause the uncertainties and our ability to represent these mechanisms in a way that facilitates convenient manipulation.

Several origins:

- Approximate knowledge and variations of some parameters
- Measurement imperfections (due to sensor)
- At high frequencies, even the structure and the model order is unknown (100
- Choice of simpler models for control synthesis
- Controller implementation

Two classes: parametric uncertainties / neglected or unmodelled dynamics
Robust analysis

Representation of uncertainties

Example 1: uncertainties [Skogestad and Postlethwaite(2005)]

\[ \tilde{G}(s) = \frac{k}{1 + \tau s} e^{-s h}, \quad 2 \leq k, h, \tau \leq 3 \]

Let us choose the nominal parameters as, \( k = h = \tau = 2.5 \) and \( G \) the according nominal model. We can define the 'relative' uncertainty, which is actually referred as a MULTIPLICATIVE UNCERTAINTY, as

\[ \tilde{G}(s) = G(s)(I + W_m(s)\Delta(s)) \]

with \( W_m(s) = \frac{3.5s + 0.25}{s + 1} \)

and \( \|\Delta\|_\infty \leq 1 \)
Example 2: unmodelled dynamics

Let us consider the system:

\[
\tilde{G}(s) = G(s) \frac{1}{1 + \tau s}, \quad \tau \leq \tau_{max}
\]

This can be modelled as:

\[
\tilde{G}(s) = G_0(s)(I + W_m(s)\Delta(s)), \quad W_m(s) = \frac{\tau_{max}j\omega}{1 + \tau_{max}j\omega}
\]

with \(\|\Delta\|_{\infty} \leq 1\)

which can be represented as

\[
N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} = \begin{pmatrix} 0 & I \\ G_0W_m(s) & G_0(s) \end{pmatrix}
\]
Example 3: parametric uncertainties

Consider the first order system:

\[ G(s) = \frac{1}{s + a} , \quad a_0 - b < a < a_0 + b \]

Define now:

\[ a = a_0 + \delta b \]

with \(|\delta| < 1\). Then it leads:

\[ \frac{1}{s + a} = \frac{1}{s + a_0 + \delta b} = \frac{1}{s + a_0} \left(1 + \frac{\delta b}{s + a_0}\right)^{-1} \]

This can then be represented as a Multiplicative Inverse Uncertainty:

\[ z = y_{\Delta} = \frac{1}{s + a_0} (w - b u_{\Delta}) \]
Example 4: parametric uncertainties in state space equations

Let us consider the following uncertain system:

\[
G : \begin{cases}
\dot{x}_1 &= (-2 + \delta_1)x_1 + (-3 + \delta_2)x_2 \\
\dot{x}_2 &= (-1 + \delta_3)x_2 + u \\
y &= x_1
\end{cases}
\]

(18)

In order to use an LFT, let us define the uncertain inputs:

\[
u_\Delta_1 = \delta_1 x_1, \quad u_\Delta_2 = \delta_2 x_2, \quad u_\Delta_3 = \delta_3 x_2
\]

Then the previous system can be rewritten in the following LFR:

where \( \Delta \) and \( y_\Delta \) are given as:

\[
\Delta = \begin{bmatrix}
\delta_1 & 0 & 0 \\
0 & \delta_2 & 0 \\
0 & 0 & \delta_3
\end{bmatrix}, \quad y_\Delta = \begin{pmatrix}
x_1 \\
x_2 \\
2 x_1 + u + u_\Delta
\end{pmatrix}
\]

and \( N \) given by the state space representation:

\[
N : \begin{cases}
\dot{x}_1 &= -2x_1 - 3x_2 + u_\Delta_1 + u_\Delta_2 \\
\dot{x}_2 &= -x_2 + u + u_\Delta_3 \\
y &= x_1
\end{cases}
\]
Towards LFR (LFT)

The previous computations are in fact the first step towards an unified representation of the uncertainties: the Linear Fractional Representation (LFR). Indeed the previous schemes can be rewritten in the following general representation as:

\[
F_u(N, \Delta) = N_{22} + N_{21} \Delta (I - N_{11} \Delta)^{-1} N_{12}
\]

This LFT exists and is well-posed if \((I - N_{11} \Delta)^{-1}\) is invertible.
LFT definition

In this representation $N$ is known and $\Delta(s)$ collects all the uncertainties taken into account for the stability analysis of the uncertain closed-loop system. $\Delta(s)$ shall have the following structure:

$$\Delta(s) = diag \{ \Delta_1(s), \cdots, \Delta_q(s), \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \epsilon_1 I_{c_1}, \cdots, \epsilon_c I_{c_c} \}$$

with $\Delta_i(s) \in \mathcal{RH}_\infty^{k_i \times k_i}$, $\delta_i \in \mathbb{R}$ and $\epsilon_i \in \mathbb{C}$.

**Remark:** $\Delta(s)$ includes

- $q$ full block transfer matrices,
- $r$ real diagonal blocks referred to as ‘repeated scalars’ (indeed each block includes a real parameter $\delta_i$ repeated $r_i$ times),
- $c$ complex scalars $\epsilon_i$ repeated $c_i$ times.

**Constraints:** The uncertainties must be normalized, i.e such that:

$$\|\Delta\|_\infty \leq 1, \ |\delta_i| \leq 1, \ |\epsilon_i| \leq 1$$
Uncertainty types

We have seen in the previous examples the two important classes of uncertainties, namely:

**UNSTRUCTURED UNCERTAINTIES:** we ignore the structure of $\Delta$, considered as a full complex perturbation matrix, such that $\|\Delta\|_\infty \leq 1$. We then look at the maximal admissible norm for $\Delta$, to get Robust Stability and Performance. This will give a global sufficient condition on the robustness of the control scheme. This may lead to conservative results since all uncertainties are collected into a single matrix ignoring the specific role of each uncertain parameter/block.

**STRUCTURED UNCERTAINTIES:** we take into account the structure of $\Delta$, (always such that $\|\Delta\|_\infty \leq 1$). The robust analysis will then be carried out for each uncertain parameter/block. This needs to introduce a new tool: the **Structured Singular Value**. We then can obtain more fine results but using more complex tools.

The analysis is provided in what follows for both cases. In *Matlab* this analysis is provided in the tools `robuststab` and `robustperf`. 
Robustness analysis: problem formulation

Since the analysis will be carried you for a closed-loop system, $N$ should be defined as the connection of the plant and the controller. Therefore, in the framework of the $H_\infty$ control, the following extended General Control Configuration is considered:

$$N = F_l(P, K)$$
Robust analysis: problem definition

In the global $P - K - \Delta$ General Control Configuration, the transfer matrix from $w$ to $z$ (i.e. the closed-loop uncertain system) is given by:

$$z = F_u(N, \Delta)w,$$

with $F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$.

and the objectives are then formulated as follows:

**Nominal stability (NS):** $N$ is internally stable

**Nominal Performance (NP):** $\|N_{22}\|_\infty < 1$ and NS

**Robust stability (RS):** $F_u(N, \Delta)$ is stable $\forall \Delta$, $\|\Delta\|_\infty < 1$ and NS

**Robust performance (RP):** $\|F_u(N, \Delta)\|_\infty < 1$ $\forall \Delta$, $\|\Delta\|_\infty < 1$ and NS
Towards Robust stability analysis

Robust Stability= with a given controller $K$, we determine wether the system remains stable for all plants in the uncertainty set.

According to the definition of the previous upper LFT, when $N$ is stable, the instability may only come from $(I - N_{11}\Delta)$. Then it is equivalent to study the $M - \Delta$ structure, given as:

![Diagram](image)

Figure: $M - \Delta$ structure

This leads to the definition of the Small Gain Theorem

**Theorem (Small Gain Theorem)**

Suppose $M \in RH_{\infty}$. Then the closed-loop system in Fig. 7 is well-posed and internally stable for all $\Delta \in RH_{\infty}$ such that:

$$\|\Delta\|_{\infty} \leq \delta (resp. < 1) \text{ if and only if } \|M(s)\|_{\infty} < 1/\delta (resp. \|M(s)\|_{\infty} \leq 1)$$
Definition of the uncertainty types

**Additive**

\[
\begin{align*}
&G(s) \quad + \\
&\Delta_A(s) \quad + \\
&W_A(s) \quad + \\
&y \\
&u \\
&u_\Delta
\end{align*}
\]

**Additive inverse**

\[
\begin{align*}
&G(s) \quad + \\
&\Delta_{iA}(s) \quad + \\
&W_{iA}(s) \quad + \\
&y \\
&u \\
&u_\Delta
\end{align*}
\]

**Output Multiplicative**

\[
\begin{align*}
&G(s) \quad + \\
&W_0(s) \quad + \\
&\Delta_0(s) \quad + \\
&y \\
&u \\
&u_\Delta
\end{align*}
\]

**Input Multiplicative**

\[
\begin{align*}
&G(s) \quad + \\
&W_1(s) \quad + \\
&\Delta_I(s) \quad + \\
&y \\
&u \\
&u_\Delta
\end{align*}
\]

**Output Inverse Multiplicative**

\[
\begin{align*}
&G(s) \quad + \\
&\Delta_{iO}(s) \quad + \\
&W_{iO}(s) \quad + \\
&y \\
&u \\
&u_\Delta
\end{align*}
\]

**Input Inverse Multiplicative**

\[
\begin{align*}
&G(s) \quad + \\
&\Delta_{ii}(s) \quad + \\
&W_{ii}(s) \quad + \\
&y \\
&u \\
&u_\Delta
\end{align*}
\]

Figure: 6 uncertainty representations
Robust analysis

Robust stability analysis: additive case

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

\[ r(t) \quad + \quad \varepsilon(t) \quad u(t) \quad \tilde{G}(s) \quad y(t) \]

\[ r(t) + \varepsilon(t) \rightarrow K(s) \rightarrow \tilde{G}(s) \rightarrow y(t) \]

**Figure: Control scheme**

Additive case:
\[ \tilde{G}(s) = G(s) + W_A(s) \Delta_A(s). \]
Computing the \( N - \Delta \) form gives

\[ N(s) = \begin{pmatrix} -W_A K S_y & W_A K S_y \\ S_y & T_y \end{pmatrix} \]

The objective is to obtain:

Output Multiplicative uncertainties:
\[ \tilde{G}(s) = (I + W_O(s) \Delta_O(s))G(s). \]
Then it leads

\[ N(s) = \begin{pmatrix} -W_O T_y & W_O T_y \\ S_y & T_y \end{pmatrix} \]
General results

Theorem (Small Gain Theorem)

Consider the different uncertainty types, and assume that NS is achieved, i.e. $M \in RH_\infty$ for each type. Then the closed-loop system is robustly stable, i.e. internally stable for all $\Delta_k \in RH_\infty$ (for $k = A, 0, I, iO, il$) with $\|\Delta\|_{\infty} \leq 1$, if and only if:

- **Additive**: $\|W_A KS_y\|_{\infty} \leq 1$
- **Additive Inverse**: $\|W_iA S_y G\|_{\infty} \leq 1$
- **Output Multiplicative**: $\|W_O T_y\|_{\infty} \leq 1$
- **Input Multiplicative**: $\|W_I T_u\|_{\infty} \leq 1$
- **Output Inverse Multiplicative**: $\|W_iO S_y\|_{\infty} \leq 1$
- **Input Inverse Multiplicative**: $\|W_iI S_u\|_{\infty} \leq 1$

This gives some robustness templates for the sensitivity functions. However this may be conservative.
Illustration on the SISO case

Here Robust Stability is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e

\[ \tilde{G} = G(I + W_m \Delta_m) \]

Then the loop transfer function is given as:

\[ \tilde{L} = \tilde{G}K = GK(I + W_m \Delta_m) = L + W_m L \Delta_m; \]

According to the Nyquist theorem, RS is achieved the the closed-loop system is stable for any \( \tilde{L} \) should not encircle, i.e \( \tilde{L} \) should not encircle -1 for all uncertainties. According to the figure, a sufficient condition is then:

\[ |W_m L| < |1 + L|, \quad \forall \omega \]

\[ \iff |W_m L| \frac{1}{1 + L} < 1, \quad \forall \omega \]

\[ \iff |W_m T| < 1 \quad \forall \omega \]
A first insight in Robust Performance

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

Case of **Output Multiplicative** uncertainties:
\[ \tilde{G}(s) = (I + W_O(s)\Delta_O(s))G(s). \]
Computing the \( N - \Delta \) form gives
\[
N(s) = \begin{bmatrix}
N_{11}(s) & N_{12}(s) \\
N_{21}(s) & N_{22}(s)
\end{bmatrix}
= \begin{pmatrix}
-W_OT_y & W_OT_y \\
-W_eS_y & W_eS_y
\end{pmatrix}
\]

The objectives are then formulated as follows:

**NS:** \( N \) is internally stable

**NP:** \( \|W_eS_y\|_\infty < 1 \) and NS

**RS:** \( \|W_OT_y\|_\infty < 1 \) and NS

**RP:** \( \|F_u(N, \Delta)\|_\infty < 1 \) \( \forall \Delta, \|\Delta\|_\infty < 1 \),

Sufficient condition: NS and
\[
\bar{\sigma}(W_OT_y) + \bar{\sigma}(W_eS_y) < 1, \forall \omega
\]
Illustration on the SISO case

Here Robust Performance is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e.

\[
\tilde{G} = G(I + W_m \Delta_m)
\]

Then the loop transfer function is given as:

\[
\tilde{L} = \tilde{G}K = GK(I + W_m \Delta_m) = L + W_m L \Delta_m;
\]

First NP is achieved when:

\[
|W_e S| < 1 \quad \forall \omega, \quad \Leftrightarrow \quad |W_e| < |1 + L|, \quad \forall \omega.
\]

Therefore RP is achieved if

\[
|W_e \tilde{S}| < 1, \quad \forall \tilde{S}, \forall \omega
\]
\[
\Leftrightarrow \quad |W_e| < |1 + \tilde{L}|, \quad \forall \tilde{L}, \forall \omega
\]

Since \( |1 + \tilde{L}| \geq |1 + L| - |W_m L \Delta_m| \), a sufficient condition is actually:

\[
|W_e| + |W_m L| < |1 + L|, \quad \forall \omega
\]
\[
\Leftrightarrow \quad |W_e S| + |W_m T| < 1, \quad \forall \omega
\]
The structured case

\[ \Delta = \{ \text{diag}\{ \Delta_1, \cdots, \Delta_q, \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \epsilon_1 I_{c_1}, \cdots, \epsilon_c I_{c_c} \} \in \mathbb{C}^{k \times k} \} \]  \hspace{1cm} (20)

with \( \Delta_i \in \mathbb{C}^{k_i \times k_i} \), \( \delta_i \in \mathbb{R} \), \( \epsilon_i \in \mathbb{C} \),

where \( \Delta_i(s), i = 1, \ldots, q \), represent full block complex uncertainties, \( \delta_i(s), i = 1, \ldots, r \), real parametric uncertainties, and \( \epsilon_i(s), i = 1, \ldots, c \), complex parametric uncertainties.

Taking into account the uncertainties leads to the following General Control Configuration,

Figure: General control configuration with uncertainties

where \( \Delta \in \Delta \).
The structured singular value

To handle parametric uncertainties, we need to introduce $\mu$, the structured singular value, defined as:

**Definition ($\mu$)**

For $M \in \mathbb{C}^{n \times n}$, the structure singular value is defined as:

$$
\mu_\Delta(M) := \frac{1}{\min\{\sigma(\Delta) : \Delta \in \Delta, \det(I - \Delta M) \neq 0\}}
$$

In other words, it allows to find the smallest structured $\Delta$ which makes $\det(I - M \Delta) = 0$.

**Theorem (The structured Small Gain Theorem)**

Let $M(s)$ be a MIMO LTI stable system and $\Delta(s)$ a LTI uncertain stable matrix, (i.e. $\in \mathcal{RH}_\infty$). The system in Fig. 7 is stable for all $\Delta(s)$ in (20) if and only if:

$$
\forall \omega \in \mathbb{R} \quad \mu_\Delta(M(j \omega)) \leq 1, \text{ with } M(s) := N_{zv}(s)
$$

More generally both following statements are equivalent

For $\bar{\mu} \in \mathbb{R}$, $N(s)$ and $\Delta(s)$ belong to $\mathcal{RH}_\infty$, and

$$
\forall \omega \in \mathbb{R}, \quad \mu_\Delta(M(j \omega)) \leq \bar{\mu}
$$

the system represented in figure 7 is stable for any uncertainty $\Delta(s)$ of the form (20) such that:

$$
||\Delta(s)||_{\infty} < 1/\bar{\mu}
$$

O. Sename [GIPSA-lab]
Build the whole control scheme

Fictive uncertainties: full complex matrix representing the $H_{\infty}$ norm specifications

Real uncertainties: block diagonal matrix

Disturbances & references

Control input

Controlled outputs

Measured output
Robust analysis
Robustness analysis: the structured case

Introduction of a fictive block

Usually only real parametric uncertainties (given in $\Delta_r$) are considered for RS analysis. RP analysis also needs a fictive full block complex uncertainty, as below,

$$\Delta(s)$$

![Diagram]

**Figure: $N\Delta$**

where

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

and the closed-loop transfer matrix is:

$$T_{ew}(s) = N_{22}(s) + N_{21}(s) \Delta(s)(I - N_{11}(s))^{-1} N_{12}(s)$$

(21)
Robust analysis theorem

For RS, we shall determine how large $\Delta$ (in the sense of $H_\infty$) can be without destabilizing the feedback system. From (21), the feedback system becomes unstable if $\det(I - N_{11}(s)) = 0$ for some $s \in \mathbb{C}$, $\Re(s) \geq 0$. The result is then the following.

**Theorem**

Assume that the nominal system $N_{ew}$ and the perturbations $\Delta$ are stable. Then the feedback system is stable for all allowed perturbations $\Delta$ such that $||\Delta(s)||_\infty < 1/\beta$ if and only if $\forall \omega \in \mathbb{R}, \mu_\Delta(N_{11}(j\omega)) \leq \beta$.

Assuming nominal stability, RS and RP analysis for structured uncertainties are therefore such that:

$$\begin{align*}
\text{NP} &\iff \overline{\sigma}(N_{22}) = \mu_{\Delta_f}(N_{22}) \leq 1, \ \forall \omega \\
\text{RS} &\iff \mu_{\Delta_r}(N_{11}) < 1, \ \forall \omega \\
\text{RP} &\iff \mu_\Delta(N) < 1, \ \forall \omega, \ \Delta = \begin{bmatrix} \Delta_f & 0 \\ 0 & \Delta_r \end{bmatrix}
\end{align*}$$

Finally, let us remark that the structured singular value cannot be explicitly determined, so that the method consists in calculating an upper bound and a lower bound, as closed as possible to $\mu$. 
Summary

The steps to be followed in the RS/RP analysis for structured uncertainties are then:

- Definition of the real uncertainties $\Delta_r$ and of the weighting functions
- Evaluation of $\mu(N_{22})\Delta_f, \mu(N_{11})\Delta_r$, and $\mu(N)\Delta$
- Computation of the admissible intervals for each parameter

Remark: The Robust Performance analysis is quite conservative and requires a tight definition of the weighting functions that do represent the performance objectives to be satisfied by the uncertain closed-loop system. Therefore it is necessary to distinguish the weighting functions used for the nominal design from the ones used for RP analysis.
Brief overview on robust control design

In order to design a robust control, i.e. a controller for which the synthesis actually accounts for uncertainties, some of the methods are:

**Unstructured uncertainties:** Consider an uncertainty weight (unstructured form), and include the Small Gain Condition through a new controlled output. For example, robustness face to Output Multiplicative Uncertainties can be considered into the design procedure adding the controlled output $e_y = W_O y$, which, when tracking performance is expected, leads to the condition $\| W_O T_y \|_\infty \leq 1$.

**Structured uncertainties:** the design of a robust controller in the presence of such uncertainties is the $\mu$ — synthesis. It is handled through an interactive procedure, referred to as the $DK$ iteration. This procedure is much more involved than a "simple" $H_\infty$ control design and often leads to an increase of the order of the controller (which is already the sum of the order of the plant and of the weighting functions).

Use other mathematical representation of parametric uncertainties, [Scherer and Wieland(2004)], as for instance the **polytopic model**. In that case the set of uncertain parameters is assumed to be a polytope (i.e. a convex) set. The stability issue in that framework is referred to as the 'Quadratic stability' i.e find a single Lyapunov function for the uncertainty set. While in the general case this is an unbounded problem, in the polytopic case (or in the affine case), the stability is to be analyzed only at the vertices of the polytope, which is a finite dimensional problem.

This approach can then be applied to find a single controller, valid over the polytopic set. Note that this approach gives rise to the LPV design for polytopic systems, as described next.
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Brief on optimisation

Definition (Convex function)

A function \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) is convex if and only if for all \( x, y \in \mathbb{R}^m \) and \( \lambda \in [0, 1] \),

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

(22)

Equivalently, \( f \) is convex if and only if its epigraph,

\[
\text{epi}(f) = \{(x, \lambda)|f(x) \leq \lambda\}
\]

(23)

is convex.

Definition ((Strict) LMI constraint)

A Linear Matrix Inequality constraint on a vector \( x \in \mathbb{R}^m \) is defined as,

\[
F(x) = F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0(\succ 0)
\]

(24)

where \( F_0 = F_0^T \) and \( F_i = F_i^T \in \mathbb{R}^{n \times n} \) are given, and symbol \( F \succeq 0(\succ 0) \) means that \( F \) is symmetric and positive semi-definite (\( \succeq 0 \)) or positive definite (\( \succ 0 \)), i.e. \( \{\forall u | u^TFu(\succ) \geq 0\} \).
Convex to LMIs

Example

Lyapunov equation. A very famous LMI constraint is the Lyapunov inequality of an autonomous system $\dot{x} = Ax$. Then the stability LMI associated is given by,

$$
\begin{align*}
    x^T P x & > 0 \\
    x^T (A^T P + PA) x & < 0
\end{align*}
$$

which is equivalent to,

$$
F(P) = \begin{bmatrix} -P & 0 \\ 0 & A^T P + PA \end{bmatrix} < 0
$$

where $P = P^T$ is the decision variable. Then, the inequality $F(P) < 0$ is linear in $P$.

LMI constraints $F(x) \succeq 0$ are convex in $x$, i.e. the set $\{x | F(x) \succeq 0\}$ is convex. Then LMI based optimization falls in the convex optimization. This property is fundamental because it guarantees that the global (or optimal) solution $x^*$ of the the minimization problem under LMI constraints can be found efficiently, in a polynomial time (by optimization algorithms like e.g. Ellipsoid, Interior Point methods).
LMI problem

Two kind of problems can be handled

**Feasibility:** The question whether or not there exist elements $x \in X$ such that $F(x) < 0$ is called a feasibility problem. The LMI $F(x) < 0$ is called feasible if such $x$ exists, otherwise it is said to be infeasible.

**Optimization:** Let an objective function $f : S \rightarrow R$ where $S = \{x | F(x) < 0\}$. The problem to determine

$$V_{opt} = \inf_{x \in S} f(x)$$

is called an optimization problem with an LMI constraint. This problem involves the determination of $V_{opt}$, the calculation of an almost optimal solution $x$ (i.e., for arbitrary $\epsilon > 0$ the calculation of an $x \in S$ such that $V_{opt} \leq f(x) \leq V_{opt} + \epsilon$, or the calculation of a optimal solutions $x_{opt}$ (elements $x_{opt} \in S$ such that $V_{opt} = f(x_{opt})$).
Examples of LMI problem

Stability analysis is a *feasibility* problem.
LQ control is an optimization problem, formulated as:

**LQ control**

Consider a controllable system \( \dot{x} = Ax + Bu \). Find a state feedback \( u(t) = -Kx(t) \) s.t
\[
J = \int_{0}^{\infty} (x^T Q x + u^T R u) dt
\]
is minimum (given \( Q > 0 \) and \( R > 0 \)) is an optimization problem whose solution is obtained solving the Riccati equation:

\[
\text{Find } P > 0, \text{ s.t. } A^T P + PA - PBR^{-1}B^T P + Q = 0
\]

and then the state feedback is given by:

\[
u(t) = -R^{-1}B^T Px(t)\]

which is equivalent to: find \( P > 0 \) s.t
\[
\begin{bmatrix}
    A^T P + PA + Q & PB \\
    B^T P & R
\end{bmatrix} > 0
\]
Semi-Definite Programming (SDP) Problem

LMI programming is a generalization of the Linear Programming (LP) to cone positive semi-definite matrices, which is defined as the set of all symmetric positive semi-definite matrices of particular dimension.

Definition (SDP problem)

A SDP problem is defined as,

\[
\begin{align*}
\min & \quad c^T x \\
\text{under constraint} & \quad F(x) \succeq 0
\end{align*}
\]

(27)

where \( F(x) \) is an affine symmetric matrix function of \( x \in \mathbb{R}^m \) (e.g. LMI) and \( c \in \mathbb{R}^m \) is a given real vector, that defines the problem objective.

SDP problems are theoretically tractable and practically:

They have a polynomial complexity, i.e. there exists an algorithm able to find the global minimum (for a given a priori fixed precision) in a time polynomial in the size of the problem (given by \( m \), the number of variables and \( n \), the size of the LMI).

SDP can be practically and efficiently solved for LMIs of size up to \( 100 \times 100 \) and \( m \leq 1000 \) see ElGhaoui, 97. Note that today, due to extensive developments in this area, it may be even larger.
The state feedback design problem

Stabilisation
Let us consider a controllable system $\dot{x} = Ax + Bu$. The problem is to find a state feedback $u(t) = -Kx(t)$ s.t the closed-loop system is stable.

Using the Lyapunov theorem, this amounts at finding $P = P^T > 0$ s.t:

$$
(A - BK)^T P + P (A - BK) < 0
\Leftrightarrow
A^T P + PA - K^T B^T P - PBK < 0
$$

which is obviously not linear...

Solution; use of change of variables

First, left and right multiplication by $P^{-1}$ leads to

$$
P^{-1} A^T + AP^{-1} - P^{-1} K^T B^T - BK P^{-1} < 0
\Leftrightarrow
Q A^T + AQ + Y^T B^T + BY < 0
$$

with $Q = P^{-1}$ and $Y = -KP^{-1}$.

The problem to be solved is therefore formulated as an LMI ! and without any conservatism !
The Bounded Real Lemma

The $\mathcal{L}_2$-norm of the output $z$ of a system $\Sigma_{LTI}$ is uniformly bounded by $\gamma$ times the $\mathcal{L}_2$-norm of the input $w$ (initial condition $x(0) = 0$).

A dynamical system $G = (A, B, C, D)$ is internally stable and with an $||G||_\infty < \gamma$ if and only is there exists a positive definite symmetric matrix $P$ (i.e $P = P^T > 0$) s.t

$$
\begin{bmatrix}
A^T P + P A & P B & C^T D \\
B^T P & -\gamma I & 0 \\
C & D & -\gamma I
\end{bmatrix} < 0, \quad P > 0.
$$

(28)

The Bounded Real Lemma (BRL), can also be written as follows (see Scherer)

$$
\begin{bmatrix}
I & 0 & 0 & 0 \\
A & B & 0 & 0 \\
0 & I & 0 & 0 \\
C & D & 0 & 0
\end{bmatrix}^T
\begin{bmatrix}
0 & P & 0 & 0 \\
P & 0 & 0 & 0 \\
0 & 0 & -\gamma^2 I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 & 0 \\
A & B & 0 & 0 \\
0 & I & 0 & 0 \\
C & D & 0 & 0
\end{bmatrix} < 0
$$

(29)

Note that the BRL is an LMI if the only unknown (decision variables) are $P$ and $\gamma$ (or $\gamma^2$).
Quadratic stability

This concept is very useful for the stability analysis of uncertain systems. Let us consider an uncertain system

\[ \dot{x} = A(\delta)x \]

where \( \delta \) is a parameter vector that belongs to an uncertainty set \( \Delta \).

**Theorem**

The considered system is said to be quadratically stable for all uncertainties \( \delta \in \Delta \) if there exists a (single) "Lyapunov function" \( P = P^T > 0 \) s.t

\[ A(\delta)^T P + PA(\delta) < 0, \text{ for all } \delta \in \Delta \] (30)

This is a sufficient condition for ROBUST Stability which is obtained when \( A(\delta) \) is stable for all \( \delta \in \Delta \).
- Solvability

In this case only $A1$ is necessary. The solution is based on the use of the Bounded Real Lemma, and some relaxations that leads to an LMI problem to be solved [Scherer(1990)].

when we refer to the $\mathcal{H}_\infty$ control problem, we mean: Find a controller $C$ for system $M$ such that, given $\gamma_\infty$,

$$\|\mathcal{F}_l(P, K)\|_\infty < \gamma_\infty$$  \hspace{1cm} (31)

The minimum of this norm is denoted as $\gamma_\infty^*$ and is called the optimal $\mathcal{H}_\infty$ gain. Hence, it comes,

$$\gamma_\infty^* = \min_{(A_K, B_K, C_K, D_K) \text{s.t. } \sigma A \subset C^-} \|Tzw(s)\|_\infty$$  \hspace{1cm} (32)

As presented in the previous sections, this condition is fulfilled thanks to the BRL. As a matter of fact, the system is internally stable and meets the quadratic $\mathcal{H}_\infty$ performances iff. $\exists \mathcal{P} = \mathcal{P}^T \succ 0$ such that,

$$
\begin{bmatrix}
  A^T \mathcal{P} + \mathcal{P} A & \mathcal{P} B & C^T \\
  B^T \mathcal{P} & -\gamma_2^2 I & D^T \\
  C & D & -I
\end{bmatrix} < 0
$$  \hspace{1cm} (33)

where $A, B, C, D$ are given in (16). Since this inequality is not an LMI and not tractable for SDP solver, relaxations have to be performed (indeed it is a BMI), as proposed in [Scherer et al.(1997b)Scherer, Gahinet, and Chilali].
Theorem (LTI/$\mathcal{H}_\infty$ solution [Scherer et al.(1997a)Scherer, Gahinet, and Chilali])

A dynamical output feedback controller of the form $C(s) = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$ that solves the $\mathcal{H}_\infty$ control problem, is obtained by solving the following LMIs in $(X, Y, \bar{A}, \bar{B}, \bar{C}$ and $\bar{D})$, while minimizing $\gamma_\infty$,

$$
\begin{bmatrix}
M_{11} & (*)^T & (*)^T & (*)^T \\
M_{21} & M_{22} & (*)^T & (*)^T \\
M_{31} & M_{32} & M_{33} & (*)^T \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{bmatrix}
\prec 0
$$

(34)

where,

$$
\begin{align*}
M_{11} &= AX + XA^T + B_2\bar{C} + \bar{C}^T B_2^T \\
M_{22} &= YA + A^T Y + \bar{B}C_2 + C_2^T \bar{B}^T \\
M_{32} &= B_1^T Y + D_{21}^T \bar{B}^T \\
M_{41} &= C_1 X + D_{12} \bar{C} \\
M_{43} &= D_{11} + D_{12} \bar{D}D_{21} \\
M_{21} &= \bar{A} + A^T + C_2^T \bar{D}^T B_2^T \\
M_{31} &= B_1^T + D_{21}^T \bar{D}^T B_2^T \\
M_{33} &= -\gamma_\infty I_{n_u} \\
M_{42} &= C_1 + D_{12} \bar{D}C_2 \\
M_{44} &= -\gamma_\infty I_{n_y}
\end{align*}
$$

(35)
Controller reconstruction

Once $A$, $B$, $C$, $D$, $X$ and $Y$ have been obtained, the reconstruction procedure consists in finding non singular matrices $M$ and $N$ s.t. $M N^T = I - X Y$ and the controller $K$ is obtained as follows:

$$
\begin{align*}
D_c &= \tilde{D} \\
C_c &= (\tilde{C} - D_c C_2 X) M^{-T} \\
B_c &= N^{-1} (\tilde{B} - Y B_2 D_c) \\
A_c &= N^{-1} (\tilde{A} - Y A X - Y B_2 D_c C_2 X - N B_c C_2 X - Y B_2 C_c M^T) M^{-T}
\end{align*}
$$

(36)

where $M$ and $N$ are defined such that $M N^T = I_n - X Y$ (that can be solved through a singular value decomposition plus a Cholesky factorization).

**Remark.** Note that other relaxation methods can be used to solve this problem, as suggested by [Gahinet(1994)].
Schur lemma

Lemma

Let $Q = Q^T$ and $R = R^T$ be affine matrices of compatible size, then the condition

\[
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix} \succeq 0
\]  

(37)

is equivalent to

\[
\begin{align*}
R & \succ 0 \\
Q - SR^{-1}S^T & \preceq 0
\end{align*}
\]  

(38)

The Schur lemma allows to a convert a quadratic constraint (ellipsoidal constraint) into an LMI constraint.
Kalman-Yakubovich-Popov lemma

Lemma

For any triple of matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $M \in \mathbb{R}^{(n+m) \times (n+m)} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, the following assessments are equivalent:

1. There exists a symmetric $K = K^T \succ 0$ s.t.

\[
\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + M < 0
\]

2. $M_{22} < 0$ and for all $\omega \in \mathbb{R}$ and complex vectors $\text{col}(x, w) \neq 0$

\[
\begin{bmatrix} A - j\omega I & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \implies \begin{bmatrix} x \\ w \end{bmatrix}^T M \begin{bmatrix} x \\ w \end{bmatrix} < 0
\]

If $M = -\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$ then the second statement is equivalent to the condition that, for all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$,

\[
\begin{bmatrix} I \\ C(j\omega I - A)^{-1} B + D \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} C(j\omega I - A)^{-1} B + D \\ I \end{bmatrix} > 0
\]

This lemma is used to convert frequency inequalities into Linear Matrix Inequalities.
Projection Lemma

Lemma

For given matrices $W = W^T$, $M$ and $N$, of appropriate size, there exists a real matrix $K = K^T$ such that,

$$ W + MKN^T + NK^T M^T \prec 0 $$  \hspace{1cm} (39)

if and only if there exist matrices $U$ and $V$ such that,

$$ W + MU + U^T M^T \prec 0 $$

$$ W + NV + V^T N^T \prec 0 $$  \hspace{1cm} (40)

or, equivalently, if and only if,

$$ M_\perp^T W M_\perp \prec 0 $$

$$ N_\perp^T W N_\perp \prec 0 $$  \hspace{1cm} (41)

where $M_\perp$ and $N_\perp$ are the orthogonal complements of $M$, $N$ respectively (i.e. $M_\perp^T M = 0$).

The projection lemma is also widely used in control theory. It allows to eliminate variable by a change of basis (projection in the kernel basis). It is involved in one of the $\mathcal{H}_\infty$ solutions [Doyle et al.(1989)Doyle, Glover, Khargonekar, and Francis]see e.g.
**Completion Lemma**

**Lemma**

Let $X = X^T, Y = Y^T \in \mathbb{R}^{n \times n}$ such that $X > 0$ and $Y > 0$. The three following statements are equivalent:

There exist matrices $X_2, Y_2 \in \mathbb{R}^{n \times r}$ and $X_3, Y_3 \in \mathbb{R}^{r \times r}$ such that,

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} \succ 0 \quad \text{and} \quad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \quad (42)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \quad \text{and} \quad \text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + r$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \quad \text{and} \quad \text{rank}[XY - I] \leq r$$

This lemma is useful for solving LMIs. It allows to simplify the number of variables when a matrix and its inverse enter in a LMI.
Finsler’s lemma

This Lemma allows the elimination of matrix variables.

Lemma

The following statement are equivalent

\[ x^T A x < 0 \text{ for all } x \neq 0 \text{ s.t: } B x = 0 \]
\[ \tilde{B}^T A \tilde{B} < 0 \text{ where } B \tilde{B} = 0 \]
\[ A + \lambda B^T B < 0 \text{ for some scalar } \lambda \]
\[ A + X B + B^T X^T < 0 \text{ for some matrix } X \]
\[ B^\perp^T A B^\perp < 0 \]
Outline

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   - LTI systems and signals norms

2. Performance analysis
   - Definition of the sensitivity functions
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   - Introduction
   - SISO case
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5. Uncertainty modelling and robustness analysis
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6. Introduction to Linear Matrix Inequalities
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7. Introduction to LPV systems and control
   - LPV modelling
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The LPV approach

Definition of a Linear Parameter Varying system

\[
\Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}
\]

\( x(t) \in \mathbb{R}^n, \ldots, \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in \Omega \), is a vector of time-varying parameters (\( \Omega \) convex set)

(Scherer, ACC Tutorial 2012)

Dampened mass-spring system:

\[
\ddot{p} + c \dot{p} + k(t) p = u, \quad y = x
\]

First-order state-space representation:

\[
\frac{d}{dt} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k(t) & -c \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix}
\]
Different Models

According to the dependency on the parameter set, we may have several classes of models:

Affine parameter dependency: \( A(\rho) = A_0 + A_1 \rho_1 + \ldots + A_N \rho_N \)

Polynomial dependency: \( A(\rho) = A_0 + A_1 \rho + A_2 \rho^2 + \ldots + A_S \rho^S \)

Rational dependency: \( A(\rho) = [A_{n0} + A_{n1} \rho_{n1} + \ldots + A_{nN} \rho_{nN}] [I + A_{d1} \rho_{d1} + \ldots + A_{dN} \rho_{dN}]^{-1} \)

If \( \rho = \rho(x(t), t) \) then the system is referred to as quasi-LPV.

For instance:

\[
\dot{x}(t) = x^2(t) = \rho(t)x(t)
\]

with \( \rho = x \).
Polytopic models

Let us denote $N$ the number of parameters. The parameter are assumed to be bounded: $\rho_i \in [\underline{\rho}_i, \bar{\rho}_i]$. The vector of parameters evolves inside a polytope represented by $Z = 2^N$ vertices $\omega_i$, as

$$\rho \in \text{Co}\{\omega_1, \ldots, \omega_Z\}$$

(43)

It is then written as the convex combination:

$$\rho = \sum_{i=1}^{Z} \alpha_i \omega_i, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{Z} \alpha_i = 1$$

(44)

where the vertices are defined by a vector $\omega_i = [\nu_{i1}, \ldots, \nu_{iN}]$ where $\nu_{ij}$ equals $\underline{\rho}_j$ or $\bar{\rho}_j$. A system is then represented as

$$\Sigma(\rho) = \sum_{k=1}^{Z} \alpha_k(\rho) \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \quad \text{with } \sum_{k=1}^{2^N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

where $\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ is a LTI system corresponding to a vertex $k$. 
Poytopic models

For a LPV system with 2 parameters, boundend \([\rho_{1,2}, \bar{\rho}_{1,2}]\), the corresponding polytope owns 4 vertices as:

\[ \mathcal{P}_\rho = \left\{ (\rho_1, \rho_2), (\rho_1, \bar{\rho}_2), (\bar{\rho}_1, \rho_2), (\bar{\rho}_1, \bar{\rho}_2) \right\} \]  \hspace{1cm} (45)

The polytopic coordinates are \((\alpha_i)\) are obtained as:

\[
\begin{align*}
\omega_1 &= (\rho_1, \rho_2), \quad \alpha_1 = \left( \frac{\bar{\rho}_1 - \rho_1}{\rho_1 - \rho_2} \right) \times \left( \frac{\rho_2 - \rho_2}{\bar{\rho}_2 - \rho_2} \right) \\
\omega_2 &= (\rho_1, \bar{\rho}_2), \quad \alpha_2 = \left( \frac{\rho_1 - \rho_1}{\rho_1 - \rho_2} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right) \\
\omega_3 &= (\bar{\rho}_1, \rho_2), \quad \alpha_3 = \left( \frac{\rho_1 - \rho_1}{\bar{\rho}_1 - \rho_2} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right) \\
\omega_4 &= (\bar{\rho}_1, \bar{\rho}_2), \quad \alpha_4 = \left( \frac{\rho_1 - \rho_1}{\bar{\rho}_1 - \rho_2} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right)
\end{align*}
\] \hspace{1cm} (46)

where \(\rho_1\) and \(\rho_2\) are the instantaneous values of the parameters \(\rho_{i}^{(k)}\) in the implementation step. The LPV system is then rewritten under the polytopic representation:

\[
\begin{pmatrix}
A(\rho_{1,2}) & B(\rho_{1,2}) \\
C(\rho_{1,2}) & D(\rho_{1,2})
\end{pmatrix}
= \alpha_1 \begin{pmatrix}
A(\omega_1) & B(\omega_1) \\
C(\omega_1) & D(\omega_1)
\end{pmatrix} + \alpha_2 \begin{pmatrix}
A(\omega_2) & B(\omega_2) \\
C(\omega_2) & D(\omega_2)
\end{pmatrix} \\
+ \alpha_3 \begin{pmatrix}
A(\omega_3) & B(\omega_3) \\
C(\omega_3) & D(\omega_3)
\end{pmatrix} + \alpha_4 \begin{pmatrix}
A(\omega_4) & B(\omega_4) \\
C(\omega_4) & D(\omega_4)
\end{pmatrix}
\] \hspace{1cm} (47)
LFT models

The equation of the system under LFR representation is as follows:

\[
\begin{bmatrix}
\dot{x} \\
\Delta z \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A & B_\Delta & B_1 & B_2 \\
C_\Delta & D_{\Delta\Delta} & D_{\Delta1} & D_{\Delta2} \\
C_1 & D_{1\Delta} & D_{11} & D_{12} \\
C_2 & D_{2\Delta} & D_{21} & D_{22}
\end{bmatrix} \begin{bmatrix}
x \\
\Delta w \\
z \\
u
\end{bmatrix}
\]

**Figure:** System under LFT form
Towards a LFT/LPV model of an AUV

Considered the NL model with 2 state variables (the pitch angle $\theta$ and velocity $q$):

$$\dot{\theta} = \cos(\phi)q - \sin(\phi)r$$
$$M\dot{q} = -p \times r(I_x - I_z) - m[Z_g(q \times w - r \times v)] - (Z_q m - Z_f \mu V)gsin(\theta)$$
$$-(X_q m - X_f \mu V)gcos(\theta)cos(\phi) + Mwq w|q| + Mqq q|q| + F_{fins}$$

(48)

where $M$ interia matrix, $m$ AUV mass, $V$ volume and $\mu$ mass density. The other parameters $(I_x, I_z, Z_g, Z_q, Z_f, X_q, X_f, M_{wq})$ appear in the dynamical and hydrodynamical functions. $F_{fins}$: forces and moments due to the fins.

Tangent linearization around $X_{eq} = [0 \ u_{eq} \ 0 \ 0 \ 0 \ 0 \ 0 \ \theta_{eq} \ 0 \ 0]$: 

$$\dot{\tilde{\theta}} = \tilde{q}$$
$$M\dot{\tilde{q}} = [-(Z_g m - Z_f \mu V)gcos(\theta_{eq}) + (X_g m - X_f \mu V)gsin(\theta_{eq})]\tilde{\theta} + F_{fins_{eq}}$$

where $\tilde{\theta}$ and $\tilde{q}$ are the variations of $\theta$ and $q$. 
A LFR model

An LFR model of the vehicle described in the form:

\[ \Delta(\rho) \]

The \( \Delta \) block contains the varying part of the model, which depends on the linearization point \( (\theta_{eq}) \). The LFR form is defined by:

\[
\begin{cases}
    x_{k+1} = Ax_k + [B_\Delta \ B_1] \begin{bmatrix} u_\Delta \\ u \end{bmatrix} \\
    (y_\Delta) = \begin{bmatrix} C_\Delta \\ C_1 \end{bmatrix} x_k + \begin{bmatrix} D_\Delta & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} u_\Delta \\ u \end{bmatrix}
\end{cases}
\]

(49)
A LFR model

with

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_\Delta = \begin{bmatrix} 0 \\ -(Z_m - Z_f \rho V)g \\ X_m - X_f \rho V \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0 \\ F_{fins} \end{bmatrix}, \quad C_\Delta = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
D_\Delta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\] (50)

\[
z_\Delta = \begin{bmatrix} \Delta \theta \\ \Delta \theta \end{bmatrix}; \quad u_\Delta = \Delta z_\Delta; \quad \Delta = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}
\] (51)

with:

\[
\left\{ \begin{array}{l}
\rho_1 = \cos(\theta_{eq}) \\
\rho_2 = \sin(\theta_{eq})
\end{array} \right.
\] (52)
Towards LPV control

The "gain scheduling" approach

Some references

- Modelling, identification: (Bruzelius, Bamieh, Lovera, Toth)
- Control: (Shamma, Apkarian & Gahinet, Adams, Packard, Beker ...)
- Stability, stabilization: (Scherer, Wu, Blanchini ...)
- Geometric analysis: (Bokor & Balas)
The $H_\infty/LPV$ control problem

**Definition**

Find a LPV controller $C(\rho)$ s.t the closed-loop system is stable and for $\gamma_\infty > 0$, $\sup \|z\|_2^2 < \gamma_\infty$,

Unbounded set of LMIs (Linear Matrix Inequalities) to be solved ($\rho \in \Omega$)

**Some approaches:** polytopic, LFT, gridding. See Arzelier [HDR, 2005], Bruzelius [Thesis, 2004], Apkarian et al. [TAC, 1995]...

**A solution:** The "polytopic" approach [C. Scherer et al. 1997]

Problem solved off line for each vertex of a polytope (convex optimisation) (using here a single Lyapunov function i.e. quadratic stabilization).

On-line the controller is computed as the convex combination of local linear controllers

$$C(\rho) = \sum_{k=1}^{2N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}, \sum_{k=1}^{2N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

Easy implementation !!
LPV control design

Dynamical LPV generalized plant:

\[ \Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \] (53)

LPV controller structure:

\[ S(\rho) : \begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \] (54)

LPV closed-loop system:

\[ \mathcal{CL}(\rho) : \begin{bmatrix} \dot{\xi} \\ z \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \] (55)
LPV control design

$\mathcal{H}_\infty$ criteria Apkarian et al. [TAC, 1995]

Stabilize system $CL(\rho)$ (find $K > 0$) while minimizing $\gamma_\infty$.

\[
\begin{bmatrix}
A(\rho)^T K + KA(\rho) & KB_{\infty}(\rho) & C_{\infty}(\rho)^T \\
B_{\infty}(\rho)^T K & -\gamma^2 I & D_{\infty}(\rho)^T \\
C_{\infty}(\rho) & D_{\infty}(\rho) & -I
\end{bmatrix} < 0
\]

Infinite set of LMIs to solve ($\rho \in \Omega$) ($\Omega$ is convex)


LFT, Gridding, Polytopic
LPV control design

Polytopic approach

Solve the LMIs at each vertex of the polytope formed by the extremum values of each varying parameter, with a common $K$ Lyapunov function.

$$C'(\rho) = \sum_{k=1}^{2^N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}$$

where,

$$\alpha_k(\rho) = \frac{\prod_{j=1}^{N} |\rho_j - C^c(\omega_k)_j|}{\prod_{j=1}^{N} (\bar{\rho}_j - \rho_j)}$$

where $C^c(\omega_k)_j = \{\bar{\rho}_j$ if $(\omega_k)_j = \rho_j$ or $\rho_j\}$ otherwise.

$$\sum_{k=1}^{2^N} \alpha_k(\rho) = 1 \ , \ \alpha_k(\rho) > 0$$
LPV/\( \mathcal{H}_\infty \) control synthesis

Proposition - feasibility (brief) Scherer et al. (1997)

Solve the following problem at each vertices of the parametrized points (illustration with 2 parameters):

\[
\gamma^* = \min \gamma \\
\text{s.t. (57)} |\rho_1, \rho_2| \\
\text{s.t. (57)} |\rho_1, \rho_2| \\
\text{s.t. (57)} |\rho_1, \rho_2| \\
\text{s.t. (57)} |\rho_1, \rho_2|
\]

\[
\begin{bmatrix}
AX + B_2 \tilde{C}(\rho_1, \rho_2) + (\star)^T \\
\tilde{A}(\rho_1, \rho_2) + AT \\
B_1^T \\
C_1X + D_{12} \tilde{C}(\rho_1, \rho_2)
\end{bmatrix} \prec 0
\]

\[
\begin{bmatrix}
Y A + \tilde{B}(\rho_1, \rho_2)C_2 + (\star)^T \\
B_1^TY + D_{21}^T \tilde{B}(\rho_1, \rho_2)^T \\
C_1 \\
D_{11} - \gamma I
\end{bmatrix} \prec 0
\]

O. Sename [GIPSA-lab]
Proposition - reconstruction (brief) Scherer et al. (1997)

Reconstruct the controllers as,

\[
\text{solve } (59) \begin{bmatrix} \rho_1, \rho_2 \\ \rho_1, \bar{\rho}_2 \\ \bar{\rho}_1, \rho_2 \\ \bar{\rho}_1, \bar{\rho}_2 \end{bmatrix}
\]

\[
C_c(\rho_1, \rho_2) = \tilde{C}(\rho_1, \rho_2)M^{-T}
\]
\[
B_c(\rho_1, \rho_2) = N^{-1}\tilde{B}(\rho_1, \rho_2)
\]
\[
A_c(\rho_1, \rho_2) = N^{-1}(\tilde{A}(\rho_1, \rho_2) - YAX - NB_c(\rho_1, \rho_2)C_2X - YB_2C_c(\rho_1, \rho_2)M^T)M^{-T}
\]

where $M$ and $N$ are defined such that $MN^T = I - XY$ which may be chosen by applying a singular value decomposition and a Cholesky factorization.
Interest of the LPV approach

LPV is a key tool to the control of complex systems.

*Some examples*:

Modelling of complex systems (non linear)

- Use of a quasi-LPV representation to include *non linearities* in a linear state space model (even delays)
- Transformation of *constraints* (e.g. saturation) into an 'external' parameter
- Modelling of LTV, hybrid (e.g. switching control)

*BUT*:

A q-LPV system is not equivalent to the non linear one:

- **stability**: $\rho = \rho(x(t), t)$ is assumed to be bounded... so are the state trajectories
- **controllability**: some non controllable modes of a non linear system may vanish according to the LPV representation
Interest of the LPV approach

Some of works using LPV approaches - former PhD students

Gain-scheduled control

- Account for various operating conditions using a variable "equilibrium point": (Gauthier 2007)
- Control with real-time performance adaptation using parameter dependent weighting functions from endogenous or exogenous parameters (Poussot 2008, Do 2011)
- Control under computation constraints: $H_\infty$ variable sampling rate controller with sampling dependent performances (Robert 2007, Roche 2011, Robert et al., IEEE TCST 2010))

Coordination of several actuators for MIMO systems

- An LPV structure for control allocation Poussot et al. (CEP 2011)
- Selection of a specific parameter for the control activation Poussot et al. (VSD 2011)
Some PhD students on robust and/or LPV control

Maria Rivas, "Modeling and Control of a Spark Ignited Engine for Euro 6 European Normative", PhD, GIPSA-lab / RENAULT, Grenoble INP, 2012.


David Hernandez, "Robust control of hybrid electro-chemical generators", PhD, GIPSA-lab / G2Elab, Grenoble INP, 2011.

Emilie Roche, "Commande Linéaire à Paramètres Variants discrète à échantillonnage variable : application à un sous-marin autonome", PhD, GIPSA-lab, Grenoble INP, 2011.


Corentin Briat, "Robust control and observation of LPV time-delay systems", PhD, GIPSA-lab, INP Grenoble, 2008.

Christophe Gauthier, "Commande multivariable de la pression d'injection dans un moteur Diesel Common Rail", PhD, LAG / DELPHI, Grenoble INP, 2007.


Alessandro ZIN, "Sur la commande robuste de suspensions automobiles en vue du contrôle global de châssis", PhD, LAG / Grenoble INP, 2005.

Julien Brely, "Régulation multivariable de filières de production de fibre de verre", PhD, LAG / ST Gobain Vetrotex, Grenoble INP, 2003.

Giampaolo Filardi, "Robust Control design strategies applied to a DVD-video player", PhD, LAG / ST Microelectronics, Grenoble INP, 2003.


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