Robust control of MIMO systems

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1. Some definitions
   - The $\mathcal{H}_\infty$ norm definition
   - Stability issues

2. What is the $\mathcal{H}_\infty$ performance?
   - $\mathcal{H}_\infty$ norm as a measure of the system gain?
   - How to compute the $\mathcal{H}_\infty$ norm?
   - What is $\mathcal{H}_\infty$ control?

3. Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?
   - Performance analysis and specification using the sensitivity functions: the SISO case
   - Performance analysis and specification using the sensitivity functions: the MIMO case
   - Do not forget some performance limitations

4. How to formulate an $\mathcal{H}_\infty$ control problem?
   - The mixed sensitivity $\mathcal{H}_\infty$ control design
   - How to compute the General Plant $P$?
   - How to extend the control problem with other performance requirements?

5. How to solve an $H_\infty$ control problem?
   - The Static State feedback case
   - The Dynamic Output feedback case
   - The Riccati approach
   - The LMI approach for $\mathcal{H}_\infty$ control design

6. Uncertainty modelling and robustness analysis
   - Introduction
   - Mathematical representation of uncertainties
   - Definition of Robustness analysis
   - Robustness analysis: the unstructured case
   - Robustness analysis: the structured case
   - Robust control design

7. What else in $\mathcal{H}_\infty$ approach?
   - $\mathcal{H}_2$ and multi-objective problems
   - $\mathcal{H}_\infty$ observer design
   - Other interests of the $\mathcal{H}_\infty$ approach
Reference books

To be studied during the course

  https://folk.ntnu.no/skoge/book/, chap 1 to 3 available
  www.ece.lsu.edu/kemin, book slides available
  https://sites.google.com/site/brucefranciscontact/Home/publications
- Carsten Scherer’s courses (MSc Course "Robust Control", MSc Course "Linear Matrix Inequalities in Control")
- + all the MATLAB demo, examples and documentation on the 'Robust Control toolbox’ (https://fr.mathworks.com/products/robust)

Other references (some in french)

  csd.newcastle.edu.au
Robust control in 1 slide?

- Sensitivity function \( S(s) = \frac{1}{1+L(s)} \)
- Complementary Sensitivity function:

\[
y = \frac{G(s)K(s)}{1 + G(s)K(s)} r = \frac{L(s)}{1+L(s)} r = T(s).r
\]
Robust control in 1 slide?

- Sensitivity function $S(s) = \frac{1}{1 + L(s)}$
- Complementary Sensitivity function:

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Why $S$ is the key function in control?

$S$ allows to characterize many things:

- $S = 1 - T \rightarrow S = \frac{r-y}{r}$.
  For performance analysis: $(S(\omega = 0) =$ steady-state error, bandwidth )
- if output disturbance $d_y$ then, $\frac{y}{d_y} = S$.
  $S$ to be minimized!
- distance from -1 to Nyquist plot =
  $$\inf_{\omega} | -1 - L(j\omega) | = \left[ \sup_{\omega} \left| \frac{1}{1 + L(j\omega)} \right| \right]^{-1}$$
- Robustness w.r.t model uncertainties

Robust control: Find $K$ s.t $S$ satisfies all requirements

O. Sename [GIPSA-lab]
Robust control in 1 slide?

- Sensitivity function $S(s) = \frac{1}{1+L(s)}$

- Complementary Sensitivity function:
  \[ y = \frac{G(s)K(s)}{1 + G(s)K(s)} r = \frac{L(s)}{1 + L(s)} r = T(s).r \]

Example: an uncertain mass-spring-damper system controlled by a proportional gain.
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Definition of LTI systems

**Definition (LTI dynamical system)**

Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$ and $D \in \mathbb{R}^{n_z \times n_w}$, a Linear Time Invariant (LTI) dynamical system ($\Sigma_{LTI}$) can be described as:

$$\Sigma_{LTI} : \begin{cases} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{cases} \quad (1)$$

where $x(t)$ is the state which takes values in a state space $X \in \mathbb{R}^n$, $w(t)$ is the input taking values in the input space $W \in \mathbb{R}^{n_w}$ and $z(t)$ is the output that belongs to the output space $Z \in \mathbb{R}^{n_z}$.

The LTI system locally describes the real system under consideration and the linearization procedure allows to treat a linear problem instead of a nonlinear one. For this class of problem, many mathematical and control theory tools can be applied like closed loop stability, controllability, observability, performance, robust analysis, etc. for both SISO and MIMO systems. However, the main restriction is that LTI models only describe the system locally, then, compared to nonlinear models, they lack of information and, as a consequence, are incomplete and may not provide global stabilization.
Signal norms

Reader is also invited to refer to the famous book of Zhou et al., 1996, where all the following definitions and additional information are given. All the following definitions are given assuming signals $x(t) \in \mathbb{C}$, then they will involve the conjugate (denoted as $x^*(t)$). When signals are real (i.e. $x(t) \in \mathbb{R}$), $x^*(t) = x^T(t)$.

**Definition (Norm and Normed vector space)**

- Let $V$ be a finite dimension space. Then $\forall \, p \geq 1$, the application $||.||_p$ is a norm, defined as,

\[ ||v||_p = \left( \sum_i |v_i|^p \right)^{1/p} \]  \hspace{1cm} (2)

- Let $V$ be a vector space over $\mathbb{C}$ (or $\mathbb{R}$) and let $||.||$ be a norm defined on $V$. Then $V$ is a normed space.
1 Some definitions

$L_*$ signal norms

**Definition ($L_1$, $L_2$, $L_\infty$ norms)**

- The 1-Norm of a function $x(t)$ is given by,

$$
\|x(t)\|_1 = \int_0^{+\infty} |x(t)| \, dt
$$

(3)

- The 2-Norm (that introduces the energy norm) is given by,

$$
\|x(t)\|_2 = \sqrt{\int_0^{+\infty} x^*(t)x(t) \, dt} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)X(j\omega) \, d\omega}
$$

(4)

The second equality is obtained by using the Parseval identity.

- The $\infty$-Norm is given by,

$$
\|x(t)\|_\infty = \sup_t |x(t)|
$$

(5)

$$
\|X\|_\infty = \sup_{Re(s)\geq 0} \|X(s)\| = \sup_\omega \|X(j\omega)\|
$$

(6)

if the signals that admit the Laplace transform, analytic in $Re(s) \geq 0$ (i.e. $\in \mathcal{H}_\infty$).
$L_\infty$ and $H_\infty$ spaces

Definition ($L_\infty$ space)

$L_\infty$ is the space of piecewise continuous bounded functions. It is a Banach space of matrix-valued (or scalar-valued) functions on $\mathbb{C}$ and consists of all complex bounded matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, such that,

$$\sup_{\omega \in \mathbb{R}} \sigma[f(j\omega)] < \infty$$

(7)

Definition ($H_\infty$ and $RH_\infty$ spaces)

$H_\infty$ is a (closed) subspace in $L_\infty$ with matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, analytic in $Re(s) > 0$ (open right-half plane). The real rational subspace of $H_\infty$ which consists of all proper and real rational stable transfer matrices, is denoted by $RH_\infty$.

Example

In control theory

$$\frac{s+1}{(s+10)(s+6)} \in RH_\infty$$
$$\frac{s+1}{(s-10)(s+6)} \notin RH_\infty$$
$$\frac{s+1}{(s+10)} \in RH_\infty$$

(8)
The $\mathcal{H}_\infty$ norm of a proper LTI system defined by the state space representation $(A, B, C, D)$ from input $w(t)$ to output $z(t)$ and which belongs to $\mathcal{RH}_\infty$, is the induced energy-to-energy gain (induced $L_2$ norm) defined as,

$$\|G(j\omega)\|_\infty = \sup_{0 < \|u\|_2 < \infty} \frac{\|z\|_2}{\|w\|_2}$$

(9)

Physical interpretations of the $\mathcal{H}_\infty$ norm

- **Frequency-domain interpretation:** the $\mathcal{H}_\infty$ norm represents the maximal gain of the frequency response of the system. For SISO (resp. MIMO) systems, it represents the maximal peak value on the Bode magnitude (resp. singular value) plot of $G(j\omega)$: So it is the maximum steady-state amplification for pure sinusoidal signals. It is also called the worst case attenuation level in the sense that it measures the maximum amplification that the system can deliver on the whole frequency set.

- **Time-domain interpretation:** The $\mathcal{H}_\infty$ norm of an LTI system is equal to the maximum energy amplification of all signals of finite energy.

- Unlike $\mathcal{H}_2$, the $\mathcal{H}_\infty$ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).
Consider
\[ G = -\frac{s - 1}{s + 2}, \quad K = 1 \]

Therefore the control input is non proper:
\[ u = \frac{s + 2}{3} (r - n - d_y) + \frac{s - 1}{3} d_i \]

DEF: A closed-loop system is well-posed if all the transfer functions are proper
\[ \Leftrightarrow \quad I + K(\infty)G(\infty) \text{ is invertible} \]

In the example \( 1 + 1 \times (-1) = 0 \) Note that if \( G \) is strictly proper, this always holds.
**Internal stability**

DEF: A system is internally stable if all the transfer functions of the closed-loop system are stable.

\[
\begin{pmatrix}
    y \\
    u
\end{pmatrix} =
\begin{pmatrix}
    (I + GK)^{-1}GK & (I + GK)^{-1}G \\
    K(I + GK)^{-1} & -K(I + GK)^{-1}G
\end{pmatrix}
\begin{pmatrix}
    r \\
    d_i
\end{pmatrix}
\]

For instance:

\[
G = \frac{1}{s - 1}, \quad K = \frac{s - 1}{s + 1}, \quad \begin{pmatrix}
    y \\
    u
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{s + 2} & \frac{s + 1}{(s - 1)(s + 2)} \\
    \frac{s - 1}{s + 2} & -\frac{1}{s + 2}
\end{pmatrix} \begin{pmatrix}
    r \\
    d_i
\end{pmatrix}
\]

There is one RHP pole (1), which means that this system is not internally stable. This is due here to the pole/zero cancellation (forbidden!!).
# Input-Output Stability

## Definition (BIBO stability)

A system $G (\dot{x} = Ax + Bu; \ y = Cx)$ is **BIBO stable** if a bounded input $u(.) \ (\|u\|_{\infty} < \infty)$ maps a bounded output $y(.) \ (\|y\|_{\infty} < \infty)$.

Now, the quantification (for BIBO stable systems) of the signal amplification (gain) is evaluated as:

$$\gamma_{peak} = \sup_{0 < \|u\|_{\infty} < \infty} \frac{\|y\|_{\infty}}{\|u\|_{\infty}}$$

and is referred to as the **PEAK TO PEAK Gain**.

## Definition ($L_2$ stability)

A system $G (\dot{x} = Ax + Bu; \ y = Cx)$ is **$L_2$ stable** if $\|u\|_2 < \infty$ implies $\|y\|_2 < \infty$.

Now, the quantification of the signal amplification (gain) is evaluated as:

$$\gamma_{energy} = \sup_{0 < \|u\|_2 < \infty} \frac{\|y\|_2}{\|u\|_2}$$

and is referred to as the **ENERGY Gain**, and is such that:

$$\gamma_{energy} = \sup \omega \|G(j\omega)\| := \|G\|_{\infty}$$

For a linear system, these stability definitions are equivalent (but not the quantification criteria).
Small Gain theorem

Consider the so called $M - \Delta$ loop.

![Diagram](image)

**Theorem**

Suppose $M(s)$ in $RH_\infty$ and $\gamma$ a positive scalar. Then the system is well-posed and internally stable for all $\Delta(s)$ in $RH_\infty$ such that $\|\Delta\|_\infty \leq 1/\gamma$ if and only if

$$\|M\|_\infty < \gamma$$
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How to define the system gain?

**SISO systems**

\[ z = Gd, \text{ the gain at a given frequency is simply} \]

\[
\left| \frac{z(\omega)}{d(\omega)} \right| = \left| \frac{G(j\omega)d(\omega)}{d(\omega)} \right| = |G(j\omega)|
\]

The gain depends on the frequency, but since the system is linear it is independent of the input magnitude.

**How to generalize to MIMO systems?**

we may select:

\[
\frac{\|z(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \|G(j\omega)\|_2?
\]

Which seems to be "independent" of the input magnitude. But this is not a correct definition. Indeed the input direction is of great importance.
The gain of a MIMO system as induced $\mathcal{L}_2$ norm?

Let consider

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

How to define and evaluate its gain?

Consider five different inputs:

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1\ 0)$</td>
<td>$(0\ 1)$</td>
<td>$(0.707\ 0.707)$</td>
<td>$(0.707\ -0.707)$</td>
<td>$(0.6\ -0.8)$</td>
</tr>
</tbody>
</table>

The input magnitudes are:

$$\|d_1\|_2 = \|d_2\|_2 = \|d_3\|_2 = \|d_4\|_2 = \|d_5\|_2 = 1$$

But the corresponding outputs are

<table>
<thead>
<tr>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
<th>$z_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5\ 3)$</td>
<td>$(4\ 2)$</td>
<td>$(6.3630\ 3.5350)$</td>
<td>$(0.7070\ 0.7070)$</td>
<td>$(-0.2\ 0.2)$</td>
</tr>
</tbody>
</table>

and the ratio are $\|z\|_2/\|d\|_2$

<table>
<thead>
<tr>
<th>$|z_1|_2/|d_1|_2$</th>
<th>$|z_2|_2/|d_2|_2$</th>
<th>$|z_3|_2/|d_3|_2$</th>
<th>$|z_4|_2/|d_4|_2$</th>
<th>$|z_5|_2/|d_5|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5.83$</td>
<td>$4.47$</td>
<td>$7.27$</td>
<td>$0.99$</td>
<td>$0.28$</td>
</tr>
</tbody>
</table>

So the gain value differs function of the input vector direction.
How the Singular Value Decomposition can provide such a MIMO gain definition?

Below is represented $\frac{\|z\|_2}{\|d\|_2}$ as a function of $d_{20}/d_{10}$ (where $d = [d_{10}, d_{20}]^T$)

We can see that, depending on the ratio $d_{20}/d_{10}$, the gain varies between 0.27 and 7.34, where $\bar{\sigma}(G') = 7.34$ and $\sigma(G') = 0.27$.

We then have these mathematical definitions:

**MAXIMUM SINGULAR VALUE**

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \bar{\sigma}(G')$$

**MINIMUM SINGULAR VALUE**

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \sigma(G')$$
Characterization of the $\mathcal{H}_\infty$ norm as induced $\mathcal{L}_2$ norm

Finally, in the case of a transfer matrix $G(s)$: (m inputs, p outputs) $u$ vector of inputs, $y$ vector of outputs.

$$\sigma(G(j\omega)) \leq \frac{\|z(\omega)\|_2}{\|d(\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

Example of A two-mass/spring/damper system
2 inputs: $F_1$ and $F_2$ 2 outputs: $x_1$ and $x_2$

$G=ss(A,B,C,D)$: LTI system
$[\text{ninf}, \text{fpeak}] = \text{hinfnorm}(G)$: Compute $H_\infty$ norm and freq
$\text{norm}(G, \text{inf})$: Compute $H_\infty$ norm
$\text{normhinf}(G)$: Compute $H_\infty$ norm
$\text{sigma}(G)$: plot max and min SV
A 7 dof full vertical vehicle model \([z_s \theta \phi z_{usfl} z_{usfr} z_{usrl} z_{usrr}]\):

\[
\begin{align*}
    m_s \ddot{z}_s &= -F_{sfl} - F_{sfr} - F_{srl} - F_{srr} \\
    I_x \ddot{\theta} &= (-F_{sfr} + F_{sfl})t_f + (-F_{srr} + F_{srl})t_r + mha_y \\
    I_y \ddot{\phi} &= (F_{srr} + F_{srl})l_r - (F_{sfr} + F_{sfl})l_f - mha_x \\
    m_{us} \ddot{z}_{usij} &= -F_{si} + F_{tzij}
\end{align*}
\]

Suspension force:

\[F_{si} = k_{ij}(z_{sj} - z_{usij}) + c_{ij}(\dot{z}_{sj} - \dot{z}_{usij}) + u_{ij}\]

Tire force:

\[F_{tzij} = -k_{tij}(z_{usij} - z_{rij})\]
A 7 dof full vertical vehicle model $[z_s \ \dot{\theta} \ \phi \ z_{usfl} \ z_{usfr} \ z_{usrl} \ z_{usrr}]$:

\[
\begin{align*}
    m_s \ddot{z}_s &= -F_{sfl} - F_{sfr} - F_{sr} - F_{sr} \\
    I_x \ddot{\theta} &= (-F_{sf} + F_{sf}) t_f + (-F_{sr} + F_{sr}) t_r + m_h a_y \\
    I_y \ddot{\phi} &= (F_{sr} + F_{sr}) l_r - (F_{sf} + F_{sf}) l_f - m_h a_x \\
    m_{us} \ddot{z}_{us} &= -F_{si} + F_{tz} \\
\end{align*}
\]

Full-car state space model:

$$
\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u
$$

where: 

\[
\begin{align*}
    x &= [z_s \ \dot{\theta} \ \phi \ z_{usfl} \ z_{usfr} \ z_{usrl} \ z_{usrr} \ \dot{z}_s \ \dot{\dot{\theta}} \ \dot{\phi} \ \dot{z}_{usfl} \ \dot{z}_{usfr} \ \dot{z}_{usrl} \ \dot{z}_{usrr}]^T, \\
    w &= [z_{rf} \ z_{rf} \ z_{rrl} \ z_{rrr}]^T, \\
    u &= [u_{fl}, u_{fr}, u_{rl}, u_{rr}]^T \\
    y &= [\ddot{z}_s \ \ddot{\theta} \ \ddot{\phi}] .
\end{align*}
\]

This is a MIMO system with 4 inputs and 3 outputs.
Bode Frequency domain plots: vertical full car model 4 inputs and 3 outputs

12 frequency-domain Bode plots for all the individual transfer functions:
2 What is the $\mathcal{H}_\infty$ performance?

$\mathcal{H}_\infty$ norm

Sigma Frequency domain plots: vertical full car model 4 inputs and 3 outputs

But only 3 singular values plots ($\sigma_1, \sigma_2, \sigma_3$) since the rank of the system transfer matrix is 3!
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How to compute the $\mathcal{H}_\infty$ norm?

As said before, $\mathcal{H}_\infty$ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).

**Method 1:** Since $\| G(j\omega) \|_\infty = \sup_{\omega \in \mathbb{R}} \sigma ( G(j\omega) )$, the intuitive computation is to get the peak on the Bode magnitude plot, which can be estimated using a thin grid of frequency points, $\{ \omega_1, \ldots, \omega_N \}$, and then:

$$\| G(j\omega) \|_\infty \approx \max_{1 \leq k \leq N} \sigma \{ G(j\omega_k) \}$$

**Method 2:** Let the dynamical system $G = (A, B, C, D) \in \mathcal{RH}_\infty$:

$\| G \|_\infty < \gamma$ if and only if $\sigma(D) < \gamma$ and the Hamiltonian $H$ has no eigenvalues on the imaginary axis, where

$$H = \begin{pmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I_n + DR^{-1}D^T)C & -(A + BR^{-1}D^TC) \end{pmatrix} \text{ and } R = \gamma^2 - D^TD$$

Use `norm(sys,inf)` or `hinfnorm(sys,tol)` in Matlab.

**Method 3 (Bounded Real Lemma):** A dynamical system $G = (A, B, C, D)$ is internally stable and with an $\| G \|_\infty < \gamma$ if and only is there exists a positive definite symmetric matrix $P$ (i.e $P = P^T > 0$ s.t

$$\begin{bmatrix} A^T P + P A & P B & C^T \\ B^T P & -\gamma I & D^T \\ C & -\gamma I & -\gamma I \end{bmatrix} < 0, \quad P > 0.$$ (10)
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Problem formulation

Objectives of any control system

to shape the response of the system to a given reference and get (or keep) a stable system in closed-loop, with desired performances, while minimising the effects of disturbances and measurement noises, and avoiding actuators saturation, this despite of modelling uncertainties, parameter changes or change of operating point.

This is formulated as:

Nominal stability (NS): The system is stable with the nominal model (no model uncertainty)

Nominal Performance (NP): The system satisfies the performance specifications with the nominal model (no model uncertainty)

Robust stability (RS): The system is stable for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant)

Robust performance (RP): The system satisfies the performance specifications for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant).

How formulate it in the $H_\infty$ framework?

The overall control objective will be to minimize the $H_\infty$ norm of the closed-loop system from the external variables (references, disturbances, noises..) $w$ to performance output $z$
Towards $\mathcal{H}_\infty$ control design: the General Control Configuration

This approach has been introduced by Doyle (1983). The formulation usually makes use of the general control configuration.

Features

- $P$ is the generalized plant (contains the plant, the weights, the uncertainties if any). It is known.
- $K$ is the controller to be designed.
- The closed-loop transfer matrix from $w$ to $z$ is given by:

$$T_{zw}(s) = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

where $F_l(P, K)$ is referred to as a lower Linear Fractional Transformation.
2 What is the $\mathcal{H}_\infty$ performance?

$\mathcal{H}_\infty$ problem definition

The overall control objective is to minimize some norm of the transfer function from $w$ to $z$, for example, the $\mathcal{H}_\infty$ norm.

Definition ($\mathcal{H}_\infty$ optimal control problem)

$\mathcal{H}_\infty$ control problem: Find a controller $K(s)$ which based on the information in $y$, generates a control signal $u$ which counteracts the influence of $w$ on $z$, thereby minimizing the closed-loop norm from $w$ to $z$.

Definition ($\mathcal{H}_\infty$ suboptimal control problem)

Given $\gamma$ a pre-specified attenuation level, a $\mathcal{H}_\infty$ sub-optimal control problem is to design a stabilizing controller that ensures:

$$\|T_{zw}(s)\|_\infty = \max_{\omega} \bar{\sigma}(T_{zw}(j\omega)) \leq \gamma$$

The optimal problem aims at finding $\gamma_{\text{min}}$ (done using $\text{hinfsyn}$ in MATLAB).

Remarks

- It is worth noting that the $\mathcal{H}_\infty$ control problem is a disturbance attenuation, formulated in the worst-case performance analysis.
- $z$ is then often defined as an "error signal" (to be minimized).
3 Why the $H_\infty$ approach is adapted to control engineering?

1. Some definitions
   - The $H_\infty$ norm definition
   - Stability issues

2. What is the $H_\infty$ performance?
   - $H_\infty$ norm as a measure of the system gain?
   - How to compute the $H_\infty$ norm?
   - What is $H_\infty$ control?

3. Why the $H_\infty$ approach is adapted to control engineering?
   - Performance analysis and specification using the sensitivity functions: the SISO case
   - Performance analysis and specification using the sensitivity functions: the MIMO case
   - Do not forget some performance limitations

4. How to formulate an $H_\infty$ control problem?
   - The mixed sensitivity $H_\infty$ control design
   - How to compute the General Plant $P$?

5. How to solve an $H_\infty$ control problem?
   - The Static State feedback case
   - The Dynamic Output feedback case
   - The Riccati approach
   - The LMI approach for $H_\infty$ control design

6. Uncertainty modelling and robustness analysis
   - Introduction
   - Mathematical representation of uncertainties
   - Definition of Robustness analysis
   - Robustness analysis: the unstructured case
   - Robustness analysis: the structured case
   - Robust control design

7. What else in $H_\infty$ approach?
   - $H_2$ and multi-objective problems
   - $H_\infty$ observer design
   - Other interests of the $H_\infty$ approach
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

Sensitivity functions

The control structure - SISO case

In the SISO case, it leads to:

\[
\begin{align*}
y &= \frac{1}{1+G(s)K(s)} (GKr + dy - GKn + Gd_i) \\
u &= \frac{1}{1+K(s)G(s)} (Kr - Kdy - Kn - KGd_i)
\end{align*}
\]

<table>
<thead>
<tr>
<th>Loop transfer function</th>
<th>$L = G(s)K(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity function</td>
<td>$S(s) = \frac{1}{1+L(s)}$</td>
</tr>
<tr>
<td>Complementary Sensitivity function</td>
<td>$T(s) = \frac{L(s)}{1+L(s)}$</td>
</tr>
</tbody>
</table>

N.B. $S$ is often referred to as the 'Output Sensitivity'.
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

**Stability and robustness margins ...**

Classical definitions:

- **Stability** $\iff |L(j\omega_\pi)| < 1$, where $\omega_\pi$ is the phase crossover frequency defined by $\phi(L(j\omega_\pi)) = -\pi$.

- **Gain Margin**: indicates the additional gain that would take the closed loop to the critical stability condition ($G_M(dB) = -[|L(j\omega_\pi)|_{dB}$)

- **Phase margin**: quantifies the pure phase delay that should be added to achieve the same critical stability condition ($\Phi_M = 180^\circ + \text{arg}[L(j\omega_c)]$, where $|L(j\omega_c)| = 1(0dB)$)

- **Delay margin**: quantifies the maximal delay that should be added in the loop to achieve the same critical stability condition, $PM/\omega_c$
Robustness margins ...

It is important to consider the module margin that quantifies the minimal distance between the curve and the critical point (-1,0j): this is a robustness margin.

\[
\Delta M = \min_\omega \left| 1 + GK(j\omega) \right|
\]

\[
\Delta M = \frac{1}{M_S}
\]

\[
M_S = \max_\omega |S(j\omega)| = \|S\|_\infty
\]

Good value \( M_S < 2 \) (6dB)

- The MODULE MARGIN is a robustness margin. Indeed, the influence of plant modelling errors on the CL transfer function:

\[
T = \frac{K(s)G(s)}{1 + K(s)G(s)}
\]

is given by:

\[
\frac{\Delta T}{T} = \frac{1}{1 + K(s)G(s)} \Delta G \quad \text{and} \quad S. \frac{\Delta G}{G}
\]

- A good module margin implies good gain and phase margins:

\[
GM \geq \frac{M_S}{M_S - 1} \quad \text{and} \quad PM \geq \frac{1}{M_S}
\]

For \( M_S = 2 \), then \( GM > 2 \) and \( PM > 30^\circ \)

- Last:

\[
M_T = \max_\omega |T(j\omega)|
\]

A good value: \( M_T < 1.5(3.5dB) \)
Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

Sensitivity functions

5 Input/Output performances

Defining two new 'sensitivity functions':

Plant Sensitivity: $SG = S(s).G(s)$ (often referred to as the 'Input Sensitivity', e.g in MATLAB)

Controller Sensitivity: $KS = K(s).S(s)$
% Determination of the sensitivity functions
L=series(G,Khinf) % Loop transfer function L=GK
S=inv(1+L); % S= 1/(1+L)
poleS=pole(S)
T= feedback(L,1)
poleT=pole(T)

%%
SG=S*G;
poleSG=pole(SG)
KS=Khinf*S;
poleKS=pole(KS)

w=logspace(-3,3,500); %% to be adjusted
subplot(2,2,1), sigma(S,w), title('Sensitivity function')
subplot(2,2,2), sigma(T,w), title('Complementary sensitivity function')
subplot(2,2,3), sigma(SG,w), title('Sensitivity*Plant')
subplot(2,2,4), sigma(KS,w), title('Controller*Sensitivity')

Remark: for SISO systems, use bodemag instead of sigma but not for MIMO ones!
Performance analysis and specification using the sensitivity functions: the SISO case- Dynamical behavior

As mentioned in Skogestad & Postlethwaite's book:
The concept of bandwidth is very important in understanding the benefits and trade-offs involved when applying feedback control. Above we considered peaks of closed-loop transfer functions, which are related to the quality of the response. However, for performance we must also consider the speed of the response, and this leads to considering the bandwidth frequency of the system. In general, a large bandwidth corresponds to a faster rise time, since high frequency signals are more easily passed on to the outputs. A high bandwidth also indicates a system which is sensitive to noise and to parameter variations. Conversely, if the bandwidth is small, the time response will generally be slow, and the system will usually be more robust.

Definition

Loosely speaking, bandwidth may be defined as the frequency range \([\omega_1, \omega_2]\) over which control is effective. In most cases we require tight control at steady-state so \(\omega_1 = 0\), and we then simply call \(\omega_2\) the bandwidth. The word "effective" may be interpreted in different ways: globally it means benefit in terms of performance.
3 Why the $\mathcal{H}_{\infty}$ approach is adapted to control engineering? Sensitivity functions

Performance analysis and specification using the sensitivity functions: the SISO case- Bandwidth definitions

**Definition ($\omega_S$)**

The (closed-loop) bandwidth, $\omega_S$, is the frequency where $|S(j\omega)|$ crosses $-3dB \ (1/sqrt2)$ from below.

Remark: $|S| < 0.707$, frequency zone, where $e/r = -S$ is reasonably small

**Definition ($\omega_T$)**

The bandwidth (in term of $T$), $\omega_T$, is the frequency where $|T(j\omega)|$ crosses $-3dB \ (1/sqrt2)$ from above.

**Definition ($\omega_c$)**

The bandwidth (crossover frequency), $\omega_c$, is the frequency where $|L(j\omega)|$ crosses $1 \ (0dB)$, for the first time, from above.
Some remarks

Remark

Usually \( \omega_S < \omega_c < \omega_T \)

Remark

In most cases, the two definitions in terms of \( S \) and \( T \) yield similar values for the bandwidth. In other cases, the situation is generally as follows. Up to the frequency \( \omega_S \), \(|S|\) is less than 0.7, and control is effective in terms of improving performance. In the frequency range \([\omega_S, \omega_T]\) control still affects the response, but does not improve performance. Finally, at frequencies higher than \( \omega_T \), we have \( S \approx 1 \) and control has no significant effect on the response.

Remark

Usually \( \omega_S < \omega_c < \omega_T \)

Finally the following relation is very useful to evaluate the rise time:

\[
    t_r \approx \frac{2.3}{\omega_T}
\]
Performance analysis: answer to ....

Analysis of $S$:
- Steady state error in tracking and output disturbance rejection: $S(\omega = 0) = 0$?
- Maximum peak criterion (Module Margin): $\|S\|_\infty < 2$?
- Bandwidth of $S$

Analysis of $T$:
- Steady state error in tracking: $T(\omega = 0) = 1$?
- Attenuation of measurement noise: $|T(j\omega)|$ small when $\omega \to \infty$?
- Maximum of $T$, $\|T\|_\infty < 1.5$?
- $\omega_T$, bandwidth of $T$ + rise time evaluation $t_r$

Analysis of $KS$:
- Input saturation: $|u(t)| < |u_{max}|$? (where $|u_{max}| < \|KS\|_\infty |r_{max}|$).
- Attenuation of measurement noise: $|KS(j\omega)|$ small when $\omega \to \infty$?

Analysis of $SG$:
- Steady state error in input disturbance rejection: $SG(\omega = 0) = 0$?
- Attenuation of input disturbance effet: $|SG(j\omega)|$ small in the frequency range of interest?
From analysis to specification ... templates

Objective: good performance specifications are important to ensure better control system
Mean: give some templates on the sensitivity functions
For simplicity, presentation for SISO systems first.

Sketch of the method:

1. Robustness and performances in regulation can be specified by imposing frequential templates on the sensitivity functions.
2. If the sensitivity functions stay within these templates, the control objectives are met.
3. These templates can be used for analysis and/or design. In the latter they are considered as weights on the sensitivity functions.
4. The shapes of typical templates on the sensitivity functions are given in the following slides.

Mathematically, these specifications may be captured by an upper bound, on the magnitude of a sensitivity function, given by another transfer function, as for $S$:

$$|S(j\omega)| \leq \frac{1}{|W_\ell(j\omega)|}, \quad \forall \omega \Leftrightarrow \|W_\ell S\|_\infty \leq 1$$

where $W_\ell(s)$ is a WEIGHT selected by the designer.
Template on the sensitivity function $S$ - Weighted sensitivity

Typical specifications in terms of $S$ include:

1. Minimum bandwidth frequency $\omega_S$
2. Maximum tracking error at selected frequencies.
3. System type, or the maximum steady-state tracking error $\epsilon_0$
4. Shape of $S$ over selected frequency ranges.
5. Maximum peak magnitude of $S$, $\|S\|_\infty < M_S$.

The peak specification prevents amplification of noise at high frequencies, and also introduces a margin of robustness; typically we select $M_S = 2$.

How to select the template function

It should:

- be close to the control objectives
- avoid too much under-or over-estimation
- be simple enough to be used later in the control design step
Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

**Sensitivity functions**

Template on the sensitivity function $S$

\[
S(s) = \frac{1}{1 + K(s)G(s)}
\]

\[
\frac{1}{W_e(s)} = \frac{s + \omega_b \varepsilon}{s/M_S + \omega_b}
\]

Generally $\varepsilon \approx 0$ is considered, $M_S < 2$ (6dB) or (3dB - cautious) to ensure sufficient module margin. $\omega_b$ influences the CL bandwidth: $\omega_b \uparrow$

- faster rejection of the disturbance
- faster CL tracking response
- better robustness w.r.t. parametric uncertainties
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

**Sensitivity functions**

**Template on the function $KS$**

$$KS(s) = \frac{K(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_u(s)} = \frac{\varepsilon_1 s + \omega_{bc}}{s + \omega_{bc}/M_u}$$

$M_u$ chosen according to LF behavior of the process (actuator constraints: saturations)

$\omega_{bc}$ influences the CL bandwidth: $\omega_{bc} \downarrow$

- better limitation of measurement noises
- roll-off starting from $\omega_{bc}$ to reduce modeling errors effects
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

**Template on the function $T$**

$$T(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_T(s)} = \frac{\varepsilon_T s + \omega_{bt}}{s + \omega_{bt}/M_T}$$

Generally $\varepsilon_T \simeq 0$ is considered, $M_T < 1.5$ (3dB) to limit the overshoot. $\omega_{bt}$ influences the bandwidth hence the transient behavior of the disturbance rejection properties: $\omega_{bt} \downarrow$

- better noise effects rejection
- better filtering of HF modelling errors
Template on the function $SG$

$$SG(s) = \frac{G(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_{SG}(s)} = \frac{s + \omega_{SG} \varepsilon_{SG}}{s/M_{SG} + \omega_{SG}}$$

$M_{SG}$ allows to limit the overshoot in the response to input disturbances. Generally $\varepsilon_{SG} \approx 0$ is considered, $\omega_{SG}$ influences the CL bandwidth: $\omega_{SG} \uparrow \implies$ faster rejection of the disturbance.
1. Some definitions
   - The $\mathcal{H}_\infty$ norm definition
   - Stability issues

2. What is the $\mathcal{H}_\infty$ performance?
   - $\mathcal{H}_\infty$ norm as a measure of the system gain?
   - How to compute the $\mathcal{H}_\infty$ norm?
   - What is $\mathcal{H}_\infty$ control?

3. Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?
   - Performance analysis and specification using the sensitivity functions: the SISO case
   - Performance analysis and specification using the sensitivity functions: the MIMO case
   - Do not forget some performance limitations

4. How to formulate an $\mathcal{H}_\infty$ control problem?
   - The mixed sensitivity $\mathcal{H}_\infty$ control design
   - How to compute the General Plant $P$?

5. How to extend the control problem with other performance requirements?

6. Uncertainty modelling and robustness analysis
   - Introduction
   - Mathematical representation of uncertainties
   - Definition of Robustness analysis
   - Robustness analysis: the unstructured case
   - Robustness analysis: the structured case
   - Robust control design

7. What else in $\mathcal{H}_\infty$ approach?
   - $\mathcal{H}_2$ and multi-objective problems
   - $\mathcal{H}_\infty$ observer design
   - Other interests of the $\mathcal{H}_\infty$ approach
Performance analysis and specification using the sensitivity functions: the MIMO case

The output & the control input satisfy the following equations:

\[(I_p + G(s)K(s))y(s) = (GKr + dy - GKn + Gd_i)\]
\[(I_m + K(s)G(s))u(s) = (Kr - Kdy - Kn - KGd_i)\]

BUT : \(K(s)G(s) \neq G(s)K(s)\) !!
Sensitivity functions - The MIMO case

Definitions

Output and Output complementary sensitivity functions:

\[ S_y = (I_p + GK)^{-1}, \quad T_y = (I_p + GK)^{-1}GK, \quad S_y + T_y = I_p \]

Input and Input complementary sensitivity functions:

\[ S_u = (I_m + KG)^{-1}, \quad T_u = KG(I_m + KG)^{-1}, \quad S_u + T_u = I_m \]

Properties

\[ T_y = GK(I_p + GK)^{-1} \]
\[ T_u = (I_m + KG)^{-1}KG \]
\[ S_u K = KS_y \]
MIMO Input/Output performances

Defining two new 'sensitivity functions':

Plant Sensitivity: $S_y G = S_y(s) \cdot G(s)$
Controller Sensitivity: $K S_y = K(s) \cdot S_y(s)$
Performance analysis and specification using the sensitivity functions: the MIMO case - Some classical analysis criteria (1)

- The transfer function $KS_y(s)$ should be upper bounded so that $u(t)$ does not reach the physical constraints, even for a large reference $r(t)$
- The effect of the measurement noise $n(t)$ on the plant input $u(t)$ can be made « small » by making the sensitivity function $KS_u(s)$ small (in High Frequencies)
- The effect of the input disturbance $d_i(t)$ on the plant input $u(t) + d_i(t)$ (actuator) can be made « small » by making the sensitivity function $S_u(s)$ small (take care to not trying to minimize $T_u$ which is not possible)
Performance analysis and specification using the sensitivity functions: the MIMO case - Some classical analysis criteria (2)

- The plant output $y(t)$ can track the reference $r(t)$ by making the complementary sensitivity function $T_y(s)$ equal to 1. (servo pb)

- The effect of the output disturbance $d_y(t)$ (resp. input disturbance $d_i(t)$) on the plant output $y(t)$ can be made « small » by making the sensitivity function $S_y(s)$ (resp. $S_yG(s)$) « small »

- The effect of the measurement noise $n(t)$ on the plant output $y(t)$ can be made « small » by making the complementary sensitivity function $T_y(s)$ « small »
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

**Sensitivity functions**

### Trade-offs

But

$$S_\star + T_\star = I_\star$$

Some trade-offs are to be looked for...

These trade-offs can be reached if one aims:

- to reject the disturbance effects in low frequencies
- to minimize the noise effects in high frequencies

It remains to require:

- $S_y$ and $S_yG$ to be small in low frequencies to reduce the load (output and input) disturbance effects on the controlled output
- $T_y$ and $KS_y$ to be small in high frequencies to reduce the effects of measurement noises on the controlled output and on the control input (actuator efforts)
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

## Synthesis

Provide a clear and detailed *frequency-domain* performance analysis using the sensitivity functions in order to explain the trade-off performance/robustness/actuator constraints.

### Qualitative analysis

Use of $S_y$, $T_y$, $K S_y$, $S_y G$ to:

- predict the behavior of the output w.r.t different external inputs (reference, disturbance, noise)
- predict the behavior of the control input w.r.t different external inputs (reference, disturbance, noise)

### Quantitative analysis

Use of $S_y$, $T_y$, $K S_y$, $S_y G$ to:

- Stability analysis and margins.
- Compute the steady-state errors in tracking, output and input disturbance attenuations.
- Give the maximum of the input/output gains to analyze the transient behaviors of the output and control input (incl. saturation).
- Give all the bandwidths of the sensitivity functions
- Evaluate the rise time in tracking
- Evaluate the robustness margins
The direct extension of the performances objectives to MIMO systems could be formulated as follows:

1. **Disturbance attenuation/closed-loop performances:**
   \[
   \bar{\sigma}(S_y(j\omega)) < \frac{1}{|W_1(j\omega)|}
   \]
   with \(|W_1(j\omega)| > 1\) for \(\omega < \omega_b\)

2. **Actuator constraints:**
   \[
   \bar{\sigma}(KS_y(j\omega)) < \frac{1}{|W_2(j\omega)|}
   \]
   with \(|W_2(j\omega)| > 1\) for \(\omega > \omega_h\)

3. **Robustness to multiplicative uncertainties:**
   \[
   \bar{\sigma}(T_y(j\omega)) < \frac{1}{|W_3(j\omega)|}
   \]
   with \(|W_3(j\omega)| > 1\) for \(\omega > \omega_l\)

However these objectives do not consider the specific MIMO structure of the system, i.e. the input-output relationship between actuators and sensors. It is then better to define the objectives accordingly with the system inputs and outputs.
Towards MIMO systems

Let us consider a system with 2 inputs and 1 output and define:

\[ G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \]

Therefore

\[ GK = G_1 K_1 + G_2 K_2, \quad KG = \begin{pmatrix} K_1 G_1 & K_1 G_2 \\ K_2 G_1 & K_2 G_2 \end{pmatrix} \]

and the sensitivity functions are:

\[ S_y = \frac{1}{1 + G_1 K_1 + G_2 K_2}, \quad KS_y = \begin{pmatrix} \frac{K_1}{1+G_1 K_1+G_2 K_2} \\ \frac{K_2}{1+G_1 K_1+G_2 K_2} \end{pmatrix} \]

While a single template \( W_e \) is convenient for \( S_y \) it is straightforward that the following diagonal template should be used for \( KS_y \):

\[ W_u(s) = \begin{pmatrix} W_u^1(s) & 0 \\ 0 & W_u^2(s) \end{pmatrix} \]

where \( W_u^1 \) and \( W_u^2 \) are chosen in order to account for each actuator specificity (constraint).
3 Why the $\mathcal{H}_\infty$ approach is adapted to control engineering?

Sensitivity functions

The MIMO general case

Let us consider $G$ with $m$ inputs and $p$ outputs.

- In the MIMO case the simplest way is to defined the templates as diagonal transfer matrices, i.e. using $(M_{S_i}, \omega_{b_i}, \varepsilon_i)$
- In that case, a weighting function should be dedicated for each input, and for each output.
- These weighting functions may of course be different if the specifications on each actuator (e.g. saturation), and on each sensor (e.g. noise), are different.

In addition, during the performance analysis step, take care to plot, in addition to the MIMO sensitivity functions, the individual ones related to each input/output to check if the individual specification is met. Hence, in the simplest case:

1. If the specifications are identical then it is sufficient to plot:
   - $\tilde{\sigma}(S_y(j\omega))$ and $\frac{1}{|W_e(j\omega)|}$, for all $\omega$
   - $\tilde{\sigma}(K S_y(j\omega))$ and $\frac{1}{|W_u(j\omega)|}$, for all $\omega$

2. If the specifications are different, one should plot
   - $\tilde{\sigma}(S_y(i,:))$ and $\frac{1}{|W_e^i|}$, for all $\omega, i = 1, \ldots, p$
   - $\tilde{\sigma}(K S_y(k,:))$ and $\frac{1}{|W_u^k|}$, for all $\omega, k = 1, \ldots, m$.

   i.e. $p$ plots for all output behaviors and $m$ plots for the input ones.

3. In a very general case, plot $\tilde{\sigma}(S_y)$ with $\tilde{\sigma}(1/W_e)$
More on weighting functions

When tighter (harder) objectives are to be met ..... the templates can be defined more accurately by transfer functions of order greater than 1, as

$$W_e(s) = \left( \frac{s/M_S + \omega_b}{s + \omega_b \varepsilon} \right)^k,$$

if a roll-off of $-20 \times k$ dB per decade is required.

Take care to the choice of the parameters ($M_S$, $\omega_b$, $\varepsilon$) to avoid incoherent objectives!
Final objectives

In terms of control synthesis, all these specifications can be tackled in the following problem: find $K(s)$ s.t.

$$
\left\| \begin{array}{c}
W_e S_y \\
W_u K S_y \\
W_T T_y \\
W_{SG} S_y G \\
\end{array} \right\|_\infty \leq 1 \Rightarrow \begin{array}{c}
\|W_e S_y\|_\infty \leq 1 \\
\|W_u K S_y\|_\infty \leq 1 \\
\|W_{SG} S_y G\|_\infty \leq 1 \\
\|W_T T_y\|_\infty \leq 1 \\
\end{array}
$$

Often, the simpler following one (referred to as the mixed sensitivity problem) is studied:

Find $K$ s.t. $\left\| \begin{array}{c}
W_e S_y \\
W_u K S_y \\
\end{array} \right\|_\infty \leq 1$

since the latter allows to consider the closed-loop output performance as well as the actuator constraints.
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   - $\mathcal{H}_\infty$ observer design
   - Other interests of the $\mathcal{H}_\infty$ approach
Introduction

Framework

Main extracts of this part: see Goodwin et al 2001. "Performance limitations in control are not only inherently interesting, but also have a major impact on real world problems."

Objective: take into account the limitations inherent to the system or due to actuators constraints, before designing the controller..... Understanding what is not possible is as important as understanding what is possible!

Example of structural constraints

\[ S + T = 1, \forall \omega \]

We then cannot have, for any frequency \( \omega_0 \), \(|S(j\omega_0)| < 1 \) and \(|T(j\omega_0)| < 1 \). This implies that, disturbance and noise rejection cannot be achieved in the same frequency range.

Bode’s Sensitivity Integral for open-loop stable systems

It is known that, for an open loop stable plant:

\[
\int_{0}^{\infty} \log|S(j\omega)| d\omega = 0
\]

Then the frequency range where \(|S(j\omega)| < 1 \) is balanced by the frequencies where \(|S(j\omega)| > 1 \)
Bode sensitivity

Nice interpretation of the balance between reduction and magnification of the sensitivity.

For open-loop unstable systems we have a stronger constraint:

\[
\int_{0}^{\infty} \log|\det(S(j\omega))|d\omega = \pi \sum_{i=1}^{N_p} \text{Re}(p_i),
\]

where \( p_i \) design the \( N_p \) RHP poles. Therefore, in the presence of RHP poles, the control effort necessary to stabilize the system is paid in terms of amplification of the sensitivity magnitude.
The interesting case of systems with RHP zeros

**Theorem**

Let $G(s)$ a MIMO plant with one RHP zero at $s = z$, and $W_p(s)$ be a scalar weight. Then, closed-loop stability is ensured only if:

$$\| W_p(s)S(s) \| \geq |W_p(s = z)|$$

To illustrate the use of that theorem, if $W_p$ is chosen as:

$$W_p(s) = \left( \frac{s/M_S + \omega b}{s + \omega b \varepsilon} \right),$$

and, if the controller meets the requirements, then

$$\| W_p(s)S(s) \|_\infty \leq 1$$

Therefore a necessary condition is:

$$\left| \frac{z/M_S + \omega b}{z + \omega b \varepsilon} \right| \leq 1$$

To conclude, if $z$ is real, and if the performance specifications are such that $M_S = 2$ and $\varepsilon = 0$, then a necessary condition to meet the performance requirements is:

$$\omega_b \leq \frac{z}{2}$$
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The mixed sensitivity $H_\infty$ control design

How to consider performance specification in $H_\infty$ control?

Illustration on the $H_\infty$ SISO problem: 
\[ \|T_{ew}(s)\|_\infty = \begin{bmatrix} W_e S \\ W_u KS \end{bmatrix} \leq \gamma \]

In that case the closed-loop system $T_{ew}(s)$ must have 1 input and 2 outputs. Since $S = \frac{r-y}{r}$ and $KS = \frac{u}{r}$, the control scheme needs only one external input $r$.

Control objectives:
\[
\begin{align*}
y &= Gu = GK(r-y) \Rightarrow \text{tracking error} : \varepsilon = Sr \\
u &= K(r-y) = K(r-Gu) \Rightarrow \text{actuator force} : u = KSr
\end{align*}
\]

To cope with that control objectives the following control scheme is considered:

Objective w.r.t sensitivity functions: 
\[ \|W_e S\|_\infty \leq 1, \quad \|W_u KS\|_\infty \leq 1. \]

Idea: define 2 new virtual controlled outputs:
\[
\begin{align*}
e_1 &= W_e Sr \\
e_2 &= W_u KSr
\end{align*}
\]
The mixed sensitivity $\mathcal{H}_\infty$ control design - Problem definition

The performance specifications on the tracking error & on the actuator, given as some weights on the controlled output, then leads to the new control scheme:

The associated general control configuration is:

External inputs $w = r$

Control Inputs $u$

Measured outputs $r - y$

Controlled Outputs $e = (e_1, e_2)^T$

$$ e = \begin{bmatrix} W_e & -W_e G \\ 0 & W_u \\ I & -G \end{bmatrix} $$

$$ K $$

$r(t) \xrightarrow{K(s)} e_1(t) \xrightarrow{W_e(s)} \xrightarrow{W_u(s)} e_2(t) \xrightarrow{y(t)}$
The mixed sensitivity $\mathcal{H}_\infty$ control problem

The corresponding $\mathcal{H}_\infty$ suboptimal control problem is therefore to find a controller $K(s)$ such that:

$$\|T_{ew}(s)\|_\infty = \left\| \begin{bmatrix} W_e S \\ W_u K S \end{bmatrix} \right\|_\infty \leq \gamma$$

The mixed sensitivity $\mathcal{H}_\infty$ control design - The closed-loop system

Using the definition of the lower Linear Fractional Transformation, we get

$$T_{ew}(s) = F_l(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

$$= \begin{bmatrix} W_e \\ 0 \end{bmatrix} + \begin{bmatrix} -W_e G \\ W_u \end{bmatrix} K (I + GK)^{-1} I$$

$$= \begin{bmatrix} W_e S \\ W_u K S \end{bmatrix}$$
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How to compute the General Plant $P$?

Generation of $P$ using **sysic**

For the previous mixed sensitivity $H_\infty$ control problem the Matlab code to get the generalized plant $P$ is as follows:

```matlab
% Generalized plant P is found with function sysic
systemnames = 'G We Wu';
inputvar = '[ r(1); u(1)];
outputvar = '[We; Wu; r-G];
input_to_G = '[u];
input_to_We = '[r-G];
input_to_Wu = '[u];
sysoutname = 'P';
cleanupsysic = 'yes';
sysic;
% Find H-infinity optimal controller
nmeas=1; nu=1;
[K,CL,GAM,INFO] = hinfsyn(P,nmeas,nu,'DISPLAY','ON');
gopt
```
How to formulate an $H_{\infty}$ control problem?

Generation of $P$ using **Simulink**

A convenient solution for more complex control structures is to build the $H_{\infty}$ General Control configuration (without the controller), in order to formalize the generalized Plant, with its input/output vectors. Then the `linmod` command allows to get the state space representation of $P$.

Matlab code for linear model extraction

```matlab
[A,B,C,D]=linmod('model')
P=ss(A,B,C,D)
```

*model*: Name of the Simulink® system from which the linear model is extracted.

It is worth noting that this method works also for a MIMO system $G$, with MIMO weighting functions $W_e$ and $W_u$, and vectors of inputs/outputs.
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What about disturbance attenuation?

Extension of the $H_\infty$ control problem: to account for input disturbance rejection, the control scheme must include $d_i$:

$$T_{ew} = \begin{bmatrix} W_e S_y & W_e S_y G \\ W_u K S_y & W_u T_u \end{bmatrix}$$

When $W_d = 1$ this corresponds to the closed-loop system:

Remarks: Note that $W_u T_u$ is an additional constraint that may lead to an increase of the attenuation level $\gamma$ since it is not part of the objectives. Hopefully $T_u$ is low pass, and $W_u$ as well. The input weight has to be on $u$ not $u + d_i$ which would lead to an unsolvable problem.
4 How to formulate an $H_{\infty}$ control problem?

How to extend the control problem with other performance requirements?

How to improve the disturbance attenuation using $W_d$?

The previous problem, allows to ensure the input disturbance rejection, but does not provide any additional d-o-f to improve it (without impacting the tracking performance). In order to 'decouple' both performance objectives, the idea is to add a disturbance model that indeed changes the disturbance rejection properties.

Let then consider: $d_i(t) = W_d.d$. In that case the closed-loop system is This corresponds to the closed-loop system:

$$T_e w = \begin{bmatrix} W_e S_y & W_e S_y G W_d \\ W_u K S_y & W_u T_u W_d \end{bmatrix}$$

and the template expected for $S_y G$ is now $\frac{1}{W_d.W_e}$.

First interest: improve the disturbance weight as $W_d = 100$... but this has a price (see Fig. below for an example).
More generally...

To include multiple objectives in a SINGLE $\mathcal{H}_\infty$ control problem, there are 2 ways:

1. add some external inputs (reference, noise, disturbance, uncertainties ...)
2. add new controlled outputs

Of course both ways increase the dimension of the problem to be solved....thus the complexity as well. Moreover additional constraints appear that are not part of the objectives ....

General rule: first think simple !!
Some extensions: the 2-DOF case

In some cases it is interesting to decouple the transient response in tracking from the stabilization loop (as in RST controllers). This is the case of 2 dof control structure.

Pay attention when building $P$ since:

- External inputs: $r$, $d_i$, $d_y$ and $n$
- Control Input: $u$
- Controlled outputs $z_1$ and $z_2$
- Measurements: $r$ and $y + n$

The controller solution will be such as

$$u = \begin{bmatrix} K_r & K_y \end{bmatrix} \begin{bmatrix} r \\ -y - n \end{bmatrix}$$
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The Generalized Plant

General Control Configuration

The first step of any $H_\infty$ control problem is to define the considered control configuration according to the choice of:

- the exogeneous and control inputs, the controlled and measured output variables,
- the structure of the controller (1dof, 2dof ....)
- the performance specifications (weighting functions)

Generalized Plant $P$

From the General Control Configuration, we can define/compute the Generalized Plant $P$. The outcome of the previous step is the state space representation of $P$, as:

$$
P \begin{cases} 
\dot{x} = Ax + B_1 w + B_2 u \\
z = C_1 x + D_{11} w + D_{12} u \\
y = C_2 x + D_{21} w + D_{22} u 
\end{cases}
$$
5 How to solve an $H_\infty$ control problem?  

The Static State feedback case

A first case: the state feedback control problem

Let consider the system:

$$\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t)
\end{align*}$$  \hspace{1cm} (11)

Formulation

In this case, the measured output vector is the state vector, and $K$ is a constant gain. The objective is to find a state feedback control law $u = -Kx$ such that:

$$\|Tzw(s)\|_\infty \leq \gamma$$

Solution

The method consists in applying the Bounded Real Lemma to the closed-loop system, and then try to obtain some convex solutions (LMI formulation). This is achieved if and only if there exists a positive definite symmetric matrix $P$ (i.e $P = P^T > 0$) s.t.

$$\begin{bmatrix}
(A - B_2 K)^T & P & (A - B_2 K) \\
* & -\gamma I & D_{12}^T K \\
* & * & -\gamma I
\end{bmatrix} < 0, \quad P > 0.$$  \hspace{1cm} (12)
Solution of the state feedback control problem

Use of change of variables

First, left and right multiplication by $diag(P^{-1}, I_n, I_n)$, and use $Q = P^{-1}$ and $Y = -KP^{-1}$. It leads to

$$
\begin{bmatrix}
AQ + B_2Y + QA^T + YT B_2^T \\
* \\
* \\
\end{bmatrix}
\begin{bmatrix}
B_1 \\
-\gamma I \\
D_{11}^T \\
\end{bmatrix}
\begin{bmatrix}
QC_1^T + Y^TD_{12}^T \\
\end{bmatrix}
\begin{bmatrix}
* \\
-\gamma I \\
\end{bmatrix}
< 0, \quad Q > 0. \quad (13)
$$

The state feedback controller is then:

$$K = -YQ^{-1}$$

Comments

The state feedback control design could be formulated considering other objectives as:

- Pole placement
- minimization of $H_2$ norm of the closed-loop system
- Solve a mixed objective $H_2/H_\infty$
- Take into account some uncertainties

This will lead to other LMI formulations
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The Dynamic Output feedback case

It will be shown how to formulate such a control problem using "classical" control tools. The procedure will be 2-steps:

**Get $P$:** Build the General Control Configuration scheme s.t. the closed-loop system matrix does correspond to the tackled $H_\infty$ problem (for instance the mixed sensitivity problem). Use of Matlab, sysic tool. 
A state space representation of $P$, the generalized plant, is needed.

**Compute $K$:** Use an optimisation algorithm that finds the controller $K$ solution of the considered problem. 
The calculation of the controller, solution of the $H_\infty$ control problem, can then be done using the Riccati approach or the LMI approach of the $H_\infty$ control problem [Zhou et al.(1996)Zhou, Doyle, and Glover] [Skogestad and Postlethwaite(1996)].

Notations:

$$P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

- $x \in \mathbb{R}^n$: plant state variables \cup state variables of weights
- $w \in \mathbb{R}^{nw}$: external inputs
- $z \in \mathbb{R}^{nz}$: controlled outputs
- $u \in \mathbb{R}^{nu}$ control inputs
- $y \in \mathbb{R}^{ny}$ measured outputs (inputs of the controller)
Problem formulation

Let $K(s)$ be a dynamic output feedback LTI controller defined as

$$K(s) : \begin{cases} \dot{x}_K(t) & = & A_K x_K(t) + B_K y(t), \\ u(t) & = & C_K x_K(t) + D_K y(t). \end{cases}$$

where $x_K \in \mathbb{R}^n$, and $A_K$, $B_K$, $C_K$ and $D_K$ are matrices of appropriate dimensions.

**Remark.** The controller will be considered here of the same order (same number of state variables) $n$ than the generalized plant, which here, in the $H_{\infty}$ framework, the order of the optimal controller.

With $P(s)$ and $K(s)$, the closed-loop system $N(s)$ is:

$$N(s) : \begin{cases} \dot{x}_{cl}(t) & = & A_{CL} x_{cl}(t) + B_{CL} w(t), \\ z(t) & = & C_{CL} x_{cl}(t) + D_{CL} w(t), \end{cases} \quad (14)$$

where $x_{cl}^T(t) = [x^T(t) \ x_K^T(t)]$ and

$$\begin{align*}
A_{CL} & = \begin{pmatrix} A + B_2 \ D_K & C_2 & \ B_2 \ C_K \\ B_K & C_2 & \ A_K \end{pmatrix}, \\
B_{CL} & = \begin{pmatrix} B_1 + B_2 \ D_K & D_{21} \\ B_K & D_{21} \end{pmatrix}, \\
C_{CL} & = \begin{pmatrix} C_1 + D_{12} \ D_K & C_2, \ D_{12} \ C_K \end{pmatrix}, \\
D_{CL} & = B_1 + B_2 \ D_K & D_{21}.
\end{align*}$$

The aim is of course to find matrices $A_K$, $B_K$, $C_K$ and $D_K$ s.t. the $H_{\infty}$ norm of the closed-loop system (14) is as small as possible, i.e. $\gamma_{opt} = \min \gamma$ s.t. $\|N(s)\|_{\infty} < \gamma$. 

O. Sename [GIPSA-lab]
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Assumptions for the Riccati method (step 1/3)

A1: \((A, B_2)\) stabilizable and \((C_2, A)\) detectable: necessary for the existence of stabilizing controllers

A2: \(\text{rank}(D_{12}) = n_u\) and \(\text{rank}(D_{21}) = n_y\): Sufficient to ensure the controllers are proper, hence realizable

A3: \(\forall \omega \in \mathbb{R}, \text{rank} \left( \begin{pmatrix} A - j\omega I_n & B_2 \\ C_1 & D_{12} \end{pmatrix} \right) = n + n_u\)

A4: \(\forall \omega \in \mathbb{R}, \text{rank} \left( \begin{pmatrix} A - j\omega I_n & B_1 \\ C_2 & D_{21} \end{pmatrix} \right) = n + n_y\) Both ensure that the optimal controller does not try to cancel poles or zeros on the imaginary axis which would result in CL instability

A5:
\[
D_{11} = 0, \quad D_{22} = 0, \quad D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix},
\]
\[
\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 & I_{n_y} \end{bmatrix}
\]

not necessary but simplify the solution (does correspond to the given theorem next but can be easily relaxed)
The problem solvability (step 2/3)

The first step is to check whether a solution does exist or not, to the optimal control problem.

Theorem (1)

Under the assumptions A1 to A5, there exists a dynamic output feedback controller
\[ u(t) = K(.) y(t) \]
such that the closed-loop system is internally stable and the \( H_\infty \) norm of the closed-loop system from the exogenous inputs \( w(t) \) to the controlled outputs \( z(t) \) is less than \( \gamma \), if and only if

1. the Hamiltonian \( H = \begin{pmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T & C_1 \\ -A^T & \end{pmatrix} \) has no eigenvalues on the imaginary axis.

2. there exists \( X_\infty \geq 0 \) t.q. \( A^T X_\infty + X_\infty A + X_\infty (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty + C_1^T C_1 = 0 \),

3. the Hamiltonian \( J = \begin{pmatrix} A^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -B_1 & B_1^T \\ -A & \end{pmatrix} \) has no eigenvalues on the imaginary axis.

4. there exists \( Y_\infty \geq 0 \) t.q. \( A Y_\infty + Y_\infty A^T + Y_\infty (\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty + B_1 B_1^T = 0 \),

5. the spectral radius \( \rho(X_\infty Y_\infty) \leq \gamma^2 \).
Controller reconstruction (step 3/3)

Theorem (2)

If the necessary and sufficient conditions of the Theorem 1 are satisfied, then the so-called central controller is given by the state space representation

\[
K_{sub}(s) = \begin{bmatrix}
\hat{A}_\infty & -Z_\infty L_\infty \\
F_\infty & 0
\end{bmatrix}
\]

with

\[
\hat{A}_\infty = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \\
F_\infty = -B_2^T X_\infty, \quad L_\infty = -Y_\infty C_2^T \\
Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}
\]

The Controller structure is indeed an observer-based state feedback control law, with

\[
u_2(t) = -B_2^T X_\infty \hat{x}(t),
\]

where \(\hat{x}(t)\) is the observer state vector

\[
\dot{\hat{x}}(t) = A \hat{x}(t) + B_1 \hat{w}(t) + B_2 u(t) + Z_\infty L_\infty \left( C_2 \hat{x}(t) - y(t) \right).
\]

(15)

and \(\hat{w}(t)\) is defined as

\[
\hat{w}(t) = \gamma^{-2} B_1^T X_\infty \hat{x}(t).
\]

Remark. \(\hat{w}(t)\) is an estimation of the worst case disturbance. \(Z_\infty L_\infty\) is the filter gain for the OE problem of estimating \(\hat{x}(t)\) in the presence of the worst case disturbance.
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In this case only $A_1$ is necessary. The solution is based on the use of the Bounded Real Lemma, and some relaxations that lead to an LMI problem to be solved [Scherer(1990)].

When we refer to the $H_\infty$ control problem, we mean: Find a controller $K$ for system $P$ such that, given $\gamma_\infty$,

$$\|F_l(P, K)\|_\infty < \gamma_\infty$$

(16)

The minimum of this norm is denoted as $\gamma^*_\infty$ and is called the optimal $H_\infty$ gain. Hence, it comes,

$$\gamma^*_\infty = \min_{(A_K, B_K, C_K, D_K) s.t. \sigma A_{CL} \subset \mathbb{C}^-} \|T_{zw}(s)\|_\infty$$

(17)

As presented in the previous sections, this condition is fulfilled thanks to the BRL. As a matter of fact, the system is internally stable and meets the quadratic $H_\infty$ performances iff. $\exists P = P^T \succ 0$ such that,

$$
\begin{bmatrix}
A_{CL}^T P + P A_{CL} & P B_{CL} & C_{CL}^T D_{CL} \\
B_{CL}^T P & -\gamma_\infty I \\
C_{CL} & D_{CL} & -I
\end{bmatrix} < 0
$$

(18)

where $A_{CL}, B_{CL}, C_{CL}, D_{CL}$ are given in (14). Since this inequality is not an LMI and not tractable for SDP solver, relaxations have to be performed (indeed it is a BMI), as proposed in [Scherer et al.(1997)Scherer, Gahinet, and Chilali].
The LMI approach for $\mathcal{H}_\infty$ control design - Problem solution

Theorem (LTI/$\mathcal{H}_\infty$ solution [Scherer et al.(1997)Scherer, Gahinet, and Chilali])

A dynamical output feedback controller of the form $K(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ that solves the $\mathcal{H}_\infty$ control problem, is obtained by solving the following LMIs in $(X, Y, \tilde{A}, \tilde{B}, \tilde{C} \text{ and } \tilde{D})$, while minimizing $\gamma_\infty$,

$$
\begin{bmatrix}
M_{11} & (\ast)^T & (\ast)^T & (\ast)^T \\
M_{21} & M_{22} & (\ast)^T & (\ast)^T \\
M_{31} & M_{32} & M_{33} & (\ast)^T \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{bmatrix} \prec 0
$$

$$
\begin{bmatrix}
X & I_n \\
I_n & Y
\end{bmatrix} \succ 0
$$

where,

$$
M_{11} = AX + XA^T + B_2\tilde{C} + \tilde{C}^TB_2^T \\
M_{22} = YA + A^TY + \tilde{B}C_2 + C_2^T\tilde{B}^T \\
M_{32} = B_1^TY + D_{21}\tilde{B}^T \\
M_{41} = C_1X + D_{12}\tilde{C} \\
M_{43} = D_{11} + D_{12}\tilde{D}D_{21} \\
M_{21} = \tilde{A} + A^T + C_2^T\tilde{D}^TB_2^T \\
M_{31} = B_1^T + D_{21}\tilde{D}^TB_2^T \\
M_{33} = -\gamma_\infty I_{n_u} \\
M_{42} = C_1 + D_{12}\tilde{D}C_2 \\
M_{44} = -\gamma_\infty I_{n_y}
$$
Controller reconstruction

Once $A, B, C, D, X$ and $Y$ have been obtained, the reconstruction procedure consists in finding non singular matrices $M$ and $N$ s.t. $MN^T = I - XY$ and the controller $K$ is obtained as follows

$$
\begin{align*}
D_K &= \tilde{D} \\
C_K &= (\tilde{C} - D_c C_2 X)M^{-T} \\
B_K &= N^{-1}(\tilde{B} - YB_2 D_c) \\
A_K &= N^{-1}(\tilde{A} - YAX - YB_2 D_c C_2 X - NB_c C_2 X - YB_2 C_c M^T)M^{-T}
\end{align*}
$$

where $M$ and $N$ are defined such that $MN^T = I_n - XY$ (that can be solved through a singular value decomposition plus a Cholesky factorization).

**Remark.** Note that other relaxation methods can be used to solve this problem, as suggested by [Gahinet(1994)].
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   - What is $\mathcal{H}_\infty$ control?

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   - Performance analysis and specification using the sensitivity functions: the MIMO case
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   - Other interests of the $\mathcal{H}_\infty$ approach
Introduction

- A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller.
- How to take into account the difference between the actual system and the model?
- A solution: using a model set BUT: very large problem and not exact yet.

**A method:** these differences are referred as model uncertainty.

The approach:

1. **determine the uncertainty set:** mathematical representation
2. **check Robust Stability**
3. **check Robust Performance**

Lots of forms can be derived according to both our knowledge of the physical mechanism that cause the uncertainties and our ability to represent these mechanisms in a way that facilitates convenient manipulation.

Several origins:

- Approximate knowledge and variations of some parameters
- Measurement imperfections (due to sensor)
- At high frequencies, even the structure and the model order is unknown (100)
- Choice of simpler models for control synthesis
- Controller implementation

Two classes: parametric uncertainties / neglected or unmodelled dynamics.
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Example 1: uncertainties

Let consider the example from (Sokestad & Postlewaite, 1996).

\[ \tilde{G}(s) = \frac{k}{1 + \tau s} e^{-sh}, \quad 2 \leq k, h, \tau \leq 3 \]

Let us choose the nominal parameters as, \( k = h = \tau = 2.5 \) and \( G \) the according nominal model. We can define the 'relative' uncertainty, which is actually referred as a MULTIPLICATIVE UNCERTAINTY, as

\[ \tilde{G}(s) = G(s)(I + W_m(s)\Delta(s)) \]

with \( W_m(s) = \frac{3.5s + 0.25}{s + 1} \)

and \( \|\Delta\|_{\infty} \leq 1 \)

![Graph showing relative uncertainties](Relative uncertainties (Gp-G)/G)
Example 2: unmodelled dynamcis

Let us consider the system:

\[ \tilde{G}(s) = G_0(s) \frac{1}{1 + \tau s}, \quad \tau \leq \tau_{\text{max}} \]

This can be modelled as:

\[ \tilde{G}(s) = G_0(s)(I + W_m(s)\Delta(s)) \]

with \( W_m(s) = \frac{\tau_{\text{max}} j\omega}{1 + \tau_{\text{max}} j\omega} \) and \( ||\Delta||_\infty \leq 1 \)

This can be represented as:

![Block diagram of the system](image)
Example 3: parametric uncertainties

Consider the first order system:

\[ G(s) = \frac{1}{s + a}, \quad a_0 - b < a < a_0 + b \]

Define now:

\[ a = a_0 + \delta.b \quad \text{with} \quad |\delta| < 1 \]

Then it leads:

\[ \frac{1}{s + a} = \frac{1}{s + a_0 + \delta.b} = \frac{1}{s + a_0} (1 + \frac{\delta.b}{s + a_0})^{-1} \]

This can then be represented as a Multiplicative Inverse Uncertainty:

\[ z = y_\Delta = \frac{1}{s + a_0} (w - bu_\Delta) \]

\[ N = \begin{pmatrix} -\frac{b}{s + a_0} & 1 \\ -\frac{b}{s + a_0} & \frac{1}{s + a_0} \end{pmatrix} \]

\[ \Delta = \delta \]
Example 3 (cont.) same example with state space formulation

Let us first the transfer function $G(s) = \frac{1}{s+a}$ as

\[
G : \begin{cases}
\dot{x} &= (-a_0 - \delta b)x + w \\
z &= x
\end{cases}
\]  

(22)

In order to use an LFT, let us define the uncertain input:

\[u_{\Delta} = \delta x,\]

Then the previous system can be rewritten in the following LFR:

where $\Delta$ and $y_{\Delta}$ are given as:

\[\Delta = [\delta], \quad y_{\Delta} = (x)\]

and $N$ given by the state space representation:

\[
N : \begin{cases}
\dot{x} &= -a_0 x - bu_{\Delta} + w \\
y_{\Delta} &= x \\
z &= x
\end{cases}
\]  

(23)
Example 4: parametric uncertainties in state space equations

Let us consider the following uncertain system:

\[
G : \begin{cases}
    \dot{x}_1 &= (\ -2 + \delta_1 \) x_1 + (-3 + \delta_2) x_2 \\
    \dot{x}_2 &= (\ -1 + \delta_3 \) x_2 + u \\
    y &= x_1
\end{cases}
\] (24)

In order to use an LFT, let us define the uncertain inputs:

\[ u_{\Delta_1} = \delta_1 x_1, \quad u_{\Delta_2} = \delta_2 x_2, \quad u_{\Delta_3} = \delta_3 x_2 \]

Then the previous system can be rewritten in the following LFR:

where \( \Delta \) and \( y_\Delta \) are given as:

\[
\Delta = \begin{bmatrix}
\delta_1 & 0 & 0 \\
0 & \delta_2 & 0 \\
0 & 0 & \delta_3
\end{bmatrix}, \quad y_\Delta = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}
\]

and \( N \) given by the state space representation:

\[
N : \begin{cases}
    \dot{x}_1 &= -2x_1 - 3x_2 + u_{\Delta_1} + u_{\Delta_2} \\
    \dot{x}_2 &= -x_2 + u + u_{\Delta_3} \\
    y &= x_1
\end{cases}
\]
Towards LFR (LFT)

The previous computations are in fact the first step towards an unified representation of the uncertainties: the Linear Fractional Representation (LFR).

Indeed the previous schemes can be rewritten in the following general representation as:

This LFR gives then the transfer matrix from \( w \) to \( z \), and is referred to as the upper Linear Fractional Transformation (LFT):

\[
F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}
\]

This LFT exists and is well-posed if \((I - N_{11}\Delta)^{-1}\) is invertible.
In this representation $N$ is known and $\Delta(s)$ collects all the uncertainties taken into account for the stability analysis of the uncertain closed-loop system. $\Delta(s)$ shall have the following structure:

$$\Delta(s) = \text{diag}\{\Delta_1(s), \cdots, \Delta_q(s), \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \epsilon_1 I_{c_1}, \cdots, \epsilon_c I_{c_c}\}$$

with $\Delta_i(s) \in \mathcal{RH}_{\infty}^{k_i \times k_i}$, $\delta_i \in \mathbb{R}$ and $\epsilon_i \in \mathbb{C}$.

Remark: $\Delta(s)$ includes

- $q$ full block transfer matrices,
- $r$ real diagonal blocks referred to as ’repeated scalars’ (indeed each block includes a real parameter $\delta_i$ repeated $r_i$ times),
- $c$ complex scalars $\epsilon_i$ repeated $c_i$ times.

Constraints: The uncertainties must be normalized, i.e such that:

$$\|\Delta\|_{\infty} \leq 1, \quad |\delta_i| \leq 1, \quad |\epsilon_i| \leq 1$$
Uncertainty types

We have seen in the previous examples the two important classes of uncertainties, namely:

- **UNSTRUCTURED UNCERTAINTIES**: we ignore the structure of $\Delta$, considered as a full complex perturbation matrix, such that $\|\Delta\|_\infty \leq 1$.
  We then look at the maximal admissible norm for $\Delta$, to get Robust Stability and Performance. This will give a global sufficient condition on the robustness of the control scheme.
  This may lead to conservative results since all uncertainties are collected into a single matrix ignoring the specific role of each uncertain parameter/block.

- **STRUCTURED UNCERTAINTIES**: we take into account the structure of $\Delta$, (always such that $\|\Delta\|_\infty \leq 1$).
  The robust analysis will then be carried out for each uncertain parameter/block.
  This needs to introduce a new tool: the Structured Singular Value. We then can obtain more fine results but using more complex tools.

The analysis is provided in what follows for both cases.
In Matlab this analysis is provided in the tools `robuststab` and `robustperf`. 
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Robustness analysis: problem formulation

Since the analysis will be carried you for a closed-loop system, $N$ should be defined as the connection of the plant and the controller. Therefore, in the framework of the $H_\infty$ control, the following extended General Control Configuration is considered:

![Diagram of $P - K - \Delta$ structure]

$N$ is such that

$$N = F_l(P, K)$$
Robust analysis: problem definition

In the global $P - K - \Delta$ General Control Configuration, the transfer matrix from $w$ to $z$ (i.e. the closed-loop uncertain system) is given by:

$$z = F_u(N, \Delta)w,$$

with $F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$.

and the objectives are then formulated as follows:

**Nominal stability (NS):** $N$ is internally stable

**Nominal Performance (NP):** $\|N_{22}\|_\infty < 1$ and NS

**Robust stability (RS):** $F_u(N, \Delta)$ is stable $\forall \Delta$, $\|\Delta\|_\infty < 1$ and NS

**Robust performance (RP):** $\|F_u(N, \Delta)\|_\infty < 1$ $\forall \Delta$, $\|\Delta\|_\infty < 1$ and NS
Summary of the methodology

Uncertainty definition
- Parameters, neglected dynamics ...
  - Unstructured case
    - Model type (additive, multiplicative ...)
      - Small Gain theorem
        - Sensitivity function template
          - RS \(|S_*| < 1/|W_*|\)
          - RP \(\sim NP + RS < 1\)
  - Structured case
    - Structured Small Gain theorem
      - RS \(\mu_{\Delta r}(N_{11}) < 1\)
      - RP \(\mu_{\Delta}(N) < 1\)
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Towards Robust stability analysis

Robust Stability: with a given controller $K$, we determine whether the system remains stable for all plants in the uncertainty set. According to the definition of the previous upper LFT, when $N$ is stable, the instability may only come from $(I - N_{11}\Delta)$. Then it is equivalent to study the $M - \Delta$ structure, given as:

![Diagram](image)

This leads to the definition of the Small Gain Theorem

**Theorem (Small Gain Theorem)**

Suppose $M \in RH_\infty$. Then the closed-loop system in Fig. 7 is well-posed and internally stable for all $\Delta \in RH_\infty$ such that:

$$||\Delta||_\infty \leq \delta (\text{resp. } < \delta) \quad \text{if and only if} \quad ||M(s)||_\infty < 1/\delta (\text{resp. } ||M(s)||_\infty \leq 1)$$
Definition of the uncertainty types

**Additive**

\[ W_A(s) \xrightarrow{\Delta_A(s)} y \xrightarrow{\Delta_A(s)} u \xrightarrow{+} y \]

**Additive inverse**

\[ u \xrightarrow{-} \Delta_{iA}(s) \xrightarrow{W_{iA}(s)} y \]

**Output Multiplicative**

\[ u \xrightarrow{+} W_0(s) \xrightarrow{\Delta_0(s)} y \xrightarrow{+} u \]

**Input Multiplicative**

\[ W_1(s) \xrightarrow{\Delta_1(s)} u \xrightarrow{+} y \]

**Output Inverse Multiplicative**

\[ u \xrightarrow{-} \Delta_{i0}(s) \xrightarrow{W_{i0}(s)} y \]

**Input Inverse Multiplicative**

\[ u \xrightarrow{+} \Delta_{i1}(s) \xrightarrow{W_{i1}(s)} y \]

**Figure:** 6 uncertainty representations
Robust stability analysis: additive case

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

\[ \tilde{G}(s) = G(s) + W_A(s) \Delta_A(s). \]

Computing the \( N - \Delta \) form gives

\[ N(s) = \begin{pmatrix} -W_A K S_y & W_A K S_y \\ S_y & T_y \end{pmatrix} \]

Output Multiplicative uncertainties:

\[ \tilde{G}(s) = (I + W_O(s) \Delta_O(s)) G(s). \]

Then it leads

\[ N(s) = \begin{pmatrix} -W_O T_y & W_O T_y \\ S_y & T_y \end{pmatrix} \]
General results

Theorem (Small Gain Theorem)

Consider the different uncertainty types, and assume that NS is achieved, i.e. $M \in RH_{\infty}$ for each type. Then the closed-loop system is robustly stable, i.e. internally stable for all $\Delta_k \in RH_{\infty}$ (for $k = A, 0, I, iO, ii$) such that:

- **Additive**: $\| W_A KS_y \|_{\infty} \leq 1$
- **Additive Inverse**: $\| W_{iA} S_y \|_{\infty} \leq 1$
- **Output Multiplicative**: $\| W_O Ty \|_{\infty} \leq 1$
- **Input Multiplicative**: $\| W_I Tu \|_{\infty} \leq 1$
- **Output Inverse Multiplicative**: $\| W_{iO} Sy \|_{\infty} \leq 1$
- **Input Inverse Multiplicative**: $\| W_{iI} Su \|_{\infty} \leq 1$

This gives some robustness templates for the sensitivity functions. However this may be conservative.
Here Robust Stability is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e

\[ \tilde{G} = G(I + W_m \Delta_m) \]

Then the loop transfer function is given as:

\[ \tilde{L} = \tilde{G}K = GK(I + W_m \Delta_m) = L + W_m L \Delta_m; \]

According to the Nyquist theorem, RS is achieved the the closed-loop system is stable for any \( \tilde{L} \) should not encircle, i.e \( \tilde{L} \) should not encircle -1 for all uncertainties. According to the figure, a sufficient condition is then:

\[ |W_m L| < |1 + L|, \quad \forall \omega \]

\[ \Leftrightarrow \left| \frac{W_m L}{1+L} \right| < 1, \quad \forall \omega \]

\[ \Leftrightarrow |W_m T| < 1 \quad \forall \omega \]
A first insight in Robust Performance

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

![Control scheme diagram]

Case of **Output Multiplicative** uncertainties:
\[ \tilde{G}(s) = (I + W_O(s)\Delta_O(s))G(s). \]
Computing the \( N - \Delta \) form gives

\[
N(s) = \begin{bmatrix}
N_{11}(s) & N_{12}(s) \\
N_{21}(s) & N_{22}(s)
\end{bmatrix} = \begin{bmatrix}
-W_OT_y & W_OT_y \\
-W_eS_y & W_eS_y
\end{bmatrix}
\]

The objectives are then formulated as follows:

- **NS**: \( N \) is internally stable
- **NP**: \( \|W_eS_y\|_\infty < 1 \) and NS
- **RS**: \( \|W_OT_y\|_\infty < 1 \) and NS
- **RP**: \( \|F_u(N, \Delta)\|_\infty < 1 \forall \Delta, \|\Delta\|_\infty < 1 \)

Sufficient condition: NS and
\[
\bar{\sigma}(W_OT_y) + \bar{\sigma}(W_eS_y) < 1, \forall \omega
\]
Illustration on the SISO case

Here Robust Performance is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e

\[ \tilde{G} = G(I + W_m \Delta_m) \]

Then the loop transfer function is given as:

\[ \tilde{L} = \tilde{G}K = GK(I + W_m \Delta_m) = L + W_m L \Delta_m \]

First NP is achieved when:

\[ |W_e S| < 1, \quad \forall \omega, \quad \Leftrightarrow \quad |W_e| < |1 + L|, \quad \forall \omega. \]

Therefore RP is achieved if

\[ |W_e \tilde{S}| < 1, \quad \forall \tilde{S}, \forall \omega \]

\[ \Leftrightarrow \quad |W_e| < |1 + \tilde{L}|, \quad \forall \tilde{L}, \forall \omega \]

Since \[ |1 + \tilde{L}| \geq |1 + L| - |W_m L \Delta_m| \], a sufficient condition is actually:

\[ |W_e| + |W_m L| < |1 + L|, \quad \forall \omega \]

\[ \Leftrightarrow \quad |W_e S| + |W_m T| < 1, \quad \forall \omega \]
### 6 Robust analysis

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The structured case

\[ \Delta = \{ \text{diag}\{\Delta_1, \cdots, \Delta_q, \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \epsilon_1 I_{c_1}, \cdots, \epsilon_c I_{c_c} \} \in \mathbb{C}^{k \times k} \} \] (26)

with \( \Delta_i \in \mathbb{C}^{k_i \times k_i} \), \( \delta_i \in \mathbb{R} \), \( \epsilon_i \in \mathbb{C} \),

where \( \Delta_i(s), i = 1, \ldots, q \), represent full block complex uncertainties, \( \delta_i(s), i = 1, \ldots, r \), real parametric uncertainties, and \( \epsilon_i(s), i = 1, \ldots, c \), complex parametric uncertainties.

Taking into account the uncertainties leads to the following General Control Configuration,

\[ \Delta(s) \]

\[ K(s) \]

\[ P(s) \]

where \( \Delta \in \Delta \).
The structured singular value

Let consider the $M - \Delta$ structure with structured uncertainties. We look for the smallest structured $\Delta$ which makes $(I - M\Delta)$ singular. We need to introduce $\mu$, the structured singular value, defined as:

**Definition ($\mu$)**

$$
\mu_\Delta(M) := \frac{1}{\min\{\sigma(\Delta) : \Delta \in \Delta, \det(I - \Delta M) = 0\}}, \quad \text{for } M \in \mathbb{C}^{n \times n}
$$

**Theorem (The structured Small Gain Theorem)**

Let $M(s)$ be a MIMO LTI stable system and $\Delta(s)$ a LTI uncertain stable matrix, (i.e. $\in \mathcal{RH}_\infty$). The $M - \Delta$ structure is stable for all $\Delta(s) \Delta \in \Delta$ with $\sigma(\Delta) < 1$ if and only if:

$$
\forall \omega \in \mathbb{R} \quad \mu_\Delta(M(j\omega)) \leq 1, \quad \text{with } M(s) := N_{11}(s)
$$

More generally both following statements are equivalent

- **For $\bar{\mu} \in \mathbb{R}$, $M(s)$ and $\Delta(s)$ belong to $\mathcal{RH}_\infty$, and**
  $$
  \forall \omega \in \mathbb{R}, \quad \mu_\Delta(M(j\omega)) \leq \bar{\mu}
  $$

- **the $M - \Delta$ structure is stable for any uncertainty $\Delta(s)$ of the form (26) such that**:
  $$
  \|\Delta(s)\|_\infty < 1/\bar{\mu}
  $$
Build the whole control scheme

**Fictive uncertainties**: full complex matrix representing the $H_{\infty}$ norm specifications

**Real uncertainties**: block diagonal matrix

Disturbances & references $w$ → $W_i$ → $G$ → $W_0$ → $e$

Control input $u$ → $K$ → $P$ → $N$ → $\gamma$ → Measured output
Introduction of a fictive block

Usually only real parametric uncertainties (given in $\Delta_r$) are considered for RS analysis. RP analysis also needs a fictive full block complex uncertainty, as below,

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

and the closed-loop transfer matrix is:

$$T_{ew}(s) = N_{22}(s) + N_{21}(s)\Delta(s)(I - N_{11}(s))^{-1}N_{12}(s)$$ (27)
Robust analysis theorem

For RS, we shall determine how large $\Delta$ (in the sense of $H_\infty$) can be without destabilizing the feedback system. From (27), the feedback system becomes unstable if $\det(I - N_{11}(s)) = 0$ for some $s \in \mathbb{C}$, $\Re(s) \geq 0$. The result is then the following.

**Theorem ([Skogestad and Postlethwaite(1996)])**

Assume that the nominal system $N_{22}$ and the perturbations $\Delta$ are stable. Then the feedback system is stable for all allowed perturbations $\Delta$ such that $||\Delta(s)||_\infty < 1/\beta$ if and only if

$$\forall \omega \in \mathbb{R}, \quad \mu_{\Delta} (N_{11}(j\omega)) \leq \beta.$$ 

Assuming nominal stability, RS and RP analysis for structured uncertainties are therefore such that:

$$\text{NP} \Leftrightarrow \sigma(N_{22}) = \mu_{\Delta_f}(N_{22}) \leq 1, \ \forall \omega$$

$$\text{RS} \Leftrightarrow \mu_{\Delta_r}(N_{11}) < 1, \ \forall \omega$$

$$\text{RP} \Leftrightarrow \mu_{\Delta}(N) < 1, \ \forall \omega, \ \Delta = \begin{bmatrix} \Delta_f & 0 \\ 0 & \Delta_r \end{bmatrix}$$

Finally, let us remark that the structured singular value cannot be explicitly determined, so that the method consists in calculating an upper bound and a lower bound, as closed as possible to $\mu$. 
Summary

The steps to be followed in the RS/RP analysis for structured uncertainties are then:

- Definition of the real uncertainties $\Delta_r$ and of the weighting functions
- Evaluation of $\mu(N_{22})\Delta_f$, $\mu(N_{11})\Delta_r$, and $\mu(N)\Delta$
- Computation of the admissible intervals for each parameter

Remark: The Robust Performance analysis is quite conservative and requires a tight definition of the weighting functions that do represent the performance objectives to be satisfied by the uncertain closed-loop system. Therefore it is necessary to distinguish the weighting functions used for the nominal design from the ones used for RP analysis.
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   - What is $\mathcal{H}_\infty$ control?

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   - Performance analysis and specification using the sensitivity functions: the SISO case
   - Performance analysis and specification using the sensitivity functions: the MIMO case
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   - $\mathcal{H}_\infty$ observer design
   - Other interests of the $\mathcal{H}_\infty$ approach
Brief overview

In order to design a robust control, i.e. a controller for which the synthesis actually accounts for uncertainties, some of the methods are:

- **Unstructured uncertainties**: Consider an uncertainty weight (unstructured form), and include the Small Gain Condition through a new controlled output. For example, robustness face to Output Multiplicative Uncertainties can be considered into the design procedure adding the controlled output $e_y = W_O y$, which, when tracking performance is expected, leads to the condition $\| W_O T_y \|_\infty \leq 1$.

- **Structured uncertainties**: The design of a robust controller in the presence of such uncertainties is the $\mu$ – synthesis. It is handled through an interactive procedure, referred to as the $DK$ iteration. This procedure is much more involved than a "simple" $H_\infty$ control design and often leads to an increase of the order of the controller (which is already the sum of the order of the plant and of the weighting functions).

- Use other mathematical representation of parametric uncertainties, [Scherer and Wieland(2004)], as for instance the **polytopic model**. In that case the set of uncertain parameters is assumed to be a polytope (i.e. a convex) set. The stability issue in that framework is referred to as the 'Quadratic stability' i.e find a single Lyapunov function for the uncertainty set. While in the general case this is an unbounded problem, in the polytopic case (or in the affine case), the stability is to be analyzed only at the vertices of the polytope, which is a finite dimensional problem.

This approach can then be applied to find a single controller, valid over the polytopic set. Note that this approach gives rise to the LPV design for polytopic systems, as described next.
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$\mathcal{H}_2$ design

The $\mathcal{H}_\infty$ norm considered above gives the system gain when input and output are measured using the $L_2$ norm. Rather than bounding the output energy, it may be desirable to keep the peak amplitude of the controlled output below a certain level, e.g. to avoid actuator saturations. Now, when we refer to the $\mathcal{H}_2$ control problem, we mean: Find a controller $C$ for system $M$ such that, given $\gamma_\infty$,

$$||F_l(M, C)||_2 < \gamma_2$$

(28)

The $\mathcal{H}_2$ problem can be expressed as follow

$$\begin{bmatrix}
    A^T K + K A & K B \\
    B^T K & -I
\end{bmatrix} < 0\quad \begin{bmatrix}
    K & C^T \\
    C & Z
\end{bmatrix} > 0, \quad \text{Trace}(Z) < \gamma_2$$

where $A_{CL}, B_{CL}, C_{CL}, D_{CL}$ are given in (14).
**H∞/H₂ problem formulation**

Useful to deal with different objectives functions of the external signal types (noise, disturbance..).

The resulting LMI based problem formulation consists in solving the following problem subject to $K = K^T \succ 0$ (note that to obtain LMIs, the same change of variable as introduced in the $H_\infty$ and $H_2$ problems can be applied).

Objectives:

\[
T_\infty = \| \frac{z_\infty}{w_\infty} \|_\infty < \gamma_\infty
\]

\[
T_2 = \| \frac{z_2}{w_2} \|_2 < \gamma_2
\]

\[
\begin{bmatrix}
\begin{bmatrix}
A_{CL}^T K + KA_{CL} & KB_\infty & C_T^T \\
B_{\infty}^T K & -\gamma_2^2 I & D_{T1}\n
\end{bmatrix}
& \begin{bmatrix}
K B_2 \\
\bar{C} \n
\end{bmatrix}
& \begin{bmatrix}
C_2^T \\
-D_{T1}^\infty \n
\end{bmatrix}
\end{bmatrix}
< 0
\]

\[
\begin{bmatrix}
\begin{bmatrix}
A_{CL}^T K + KA_{CL} & KB_2 \\
B_2^T K & -I \n
\end{bmatrix}
\end{bmatrix}
< 0
\]

\[
\begin{bmatrix}
K \\
C_2 \n
\end{bmatrix}
> 0 , \text{Trace}(Z) < \gamma_2^2
\]

Even after the change of basis, it is impossible (non convex problem) to minimize simultaneously the $H_\infty$ and $H_2$ criteria. As a consequence, the problem is usually reformulated as one of the problems below:

- A linear combination of $\gamma_\infty$ and $\gamma_2$, e.g.:

\[
\gamma_{mix} = \alpha \gamma_\infty + (1 - \alpha) \gamma_2 , \text{where } \alpha \in [0 1]
\]  

- Minimize $\gamma_\infty$ (resp. $\gamma_2$) while fixing $\gamma_2$ (resp. $\gamma_\infty$).
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\( H_\infty \) observer definition

Let consider the system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Ew(t) + Bu(t) \\
y(t) &= Cx(t) + Fw(t)
\end{align*}
\]  

(30)

The objective is

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))
\]

(31)

where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimated state of \( x(t) \) and \( L \) is the \( n \times p \) constant observer gain matrix to be designed.

The estimated error, \( e(t) := x(t) - \hat{x}(t) \), satisfies:

\[
\dot{e}(t) = (A - LC)e(t) + (E - LF)w(t)
\]

(32)

Problem definition

System (31) is said to be a \( H_\infty \) observer for the above system if:

\[
\lim_{t \to \infty} e(t) \to 0 \quad \text{for} \quad w(t) \equiv 0
\]

(33)

\[
\|T_{ew}(s)\|_\infty \leq \gamma \quad \text{under} \quad \hat{e}(t = 0) = 0
\]

(34)
\( H_{\infty} \) observer design

The method consists (as for state feedback design) to apply the BRL to the error equation and use some change of variables to get some LMIs. This is achieved if and only if there exists a positive definite symmetric matrix \( P \) (i.e. \( P = P^T > 0 \)) s.t

\[
\begin{bmatrix}
(A - LC)^T P + P (A - LC) & P (E - LF) & I_n \\
* & -\gamma I & 0 \\
* & * & -\gamma I
\end{bmatrix} < 0, \quad P > 0. \tag{35}
\]

Use of change of variables

Let define \( Y = -KP_L \). It leads to

\[
\begin{bmatrix}
AP + YC + PA^T + CTY^T & PE + YF & I_n \\
* & -\gamma I & 0 \\
* & * & -\gamma I
\end{bmatrix} < 0, \quad Q > 0. \tag{36}
\]

The observer gain is then:

\[ L = -P^{-1}Y \]
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Other interests of the $\mathcal{H}_\infty$ approach

**Control**

Using LMIs the previous methods can be designed to take into account

- **Pole placement constraints**: useful to avoid fast dynamics and high frequencies in the controller (to facilitate digital implementation).

- **Input and state constraints**: some results allow to included together with $\mathcal{H}_\infty$ performance, the saturation constraints on the input (to provide an anti-windup scheme) (and state constraints)

- **Passivity performance**: used to enforce dissipative properties of the closed loop (this property is widely used in e.g. electrical systems, robotic applications). This property ensures that the introduced energy is dissipated into the system. This approach is linked with the passivity theory.

**Observer design, Fault Diagnosis and Fault Tolerant Control**

- Design of $\mathcal{H}_\infty$ observer and robust observers.

- Design $\mathcal{H}_\infty$ observers for Fault Detection and Isolation (FDI) (sometimes using a bank of observers) and for Fault Estimation as well.

- Reconfiguration of (state or dynamic output) feedback control: The controller changes according to detected faults

Last be not least: **all what has been seen in the course does exist for discrete-time systems.**
Some PhD students on robust and/or LPV control

- Waleed Nwesaty, "LPV/$H_\infty$ control design of on-board energy management systems for electrical vehicles", PhD GIPSA-lab, Université Grenoble Alpes, 2015.
- Soheib Fergani, "$H_\infty$/LPV robust MIMO control of vehicle dynamics", PhD, GIPSA-lab, Université Grenoble Alpes, 2014.
- Maria Rivas, "Modeling and Control of a Spark Ignited Engine for Euro 6 European Normative", PhD, GIPSA-lab / RENAULT, Grenoble INP, 2012.
- David Hernandez, "Robust control of hybrid electro-chemical generators", PhD, GIPSA-lab / G2Elab, Grenoble INP, 2011.
- Emilie Roche, "Commande Linéaire à Paramètres Variants discrète à échantillonnage variable : application à un sous-marin autonome", PhD, GIPSA-lab, Grenoble INP, 2011.
- Corentin Briat, "Robust control and observation of LPV time-delay systems", PhD, GIPSA-lab, INP Grenoble, 2008.
- Christophe Gauthier, "Commande multivariable de la pression d’injection dans un moteur Diesel Common Rail", PhD, LAG / DELPHI, Grenoble INP, 2007.
- Julien Brely, "Régulation multivariable de filières de production de fibre de verre", PhD, LAG / ST Gobain Vetrotex, Grenoble INP, 2003.
- Giampaolo Filardi, "Robust Control design strategies applied to a DVD-video player", PhD, LAG / ST Microelectronics, Grenoble INP, 2003.
Further studies

P. Gahinet.
A linear matrix inequality approach to $\mathcal{H}_\infty$ control.

C. Scherer.
The riccati inequality and state-space $\mathcal{H}_\infty$-optimal control.

C. Scherer and S. Wieland.
*LMI in control (lecture support, DELFT University)*.
2004.

C. Scherer, P. Gahinet, and M. Chilali.
Multiobjective output-feedback control via LMI optimization.

S. Skogestad and I. Postlethwaite.
*Multivariable Feedback Control. Analysis and Design*.
John Wiley and Sons, Chichester, 1996.

*Robust and Optimal Control*.
New Jersey, 1996.