Average consensus on networks with transmission noise or quantization

Ruggero Carli, Fabio Fagnani, Paolo Frasca, Tom Taylor, and Sandro Zampieri

***Abstract*— In this note we study the average consensus algorithm in a distributed system of agents which are allowed to communicate according to a directed graph. Moreover, the communication between connected agents is not perfect, but affected by some error, which can be either a random additive noise or produced by a quantization. We investigate the effects of these constraints on the performance of the average consensus algorithms.**

I. INTRODUCTION

In recent years much research has been focused on the coordinated control and estimation [1], [2], [4], [5], [6], [8], [9]. In this area, the simplest and more crucial problem is the *coordinated consensus* problem, which has been addressed by many authors [3], [7]. Most of these papers are based on modelling the communication between agents by a (directed) graph and assuming that communications along edges are ideal. In this setting much work has been done and now we are able to design effective consensus algorithms, with mild conditions on the graph structure.

The natural development of these themes is then to look for more realistic communication models and to design algorithms which are effective in this more realistic scenario.

This is what we are trying to do in this note, dealing with communications which are either affected by additive noises, in section III, or constrained to be quantized, in section IV. We will show that these two models have different features, which we study using both mathematical tools and computer simulations. To develop this analysis we need some technical preliminaries, which are collected in section II.

Section V contains the proposal of a algorithm suited for average consensus with quantized communication.

The concluding section, then, summarizes our results and points out some open problems in this emerging field.

II. PRELIMINARIES

In this section we collect some definitions and notations which are used through the paper.

A. Graphs

The communications between agents are modeled by a directed graph $\mathscr{G} = (V,E)$. $V = \{1,\dots,\dots N\}$ is the set of vertices and E is the set of (directed) edges, i.e. a subset of $V \times V$, the set of all ordered couples (h,k) where h and k are in V. We say that the vertices i and j are communicant, or connected, if there exists the couple (j,i) in E. This means that j can transmit information about its state to i. The *adjacency matrix* A is a $\{0,1\}$-valued square matrix indexed by the elements in V defined by letting $A_{ij} = 1$ if and only if $(i,j) \in \mathscr{E}$. Define the *in-degree* of a vertex j as $\sum_i A_{ij}$ and the *out-degree* of a vertex i as $\sum_j A_{ij}$. A graph is called *in-regular* (*out-regular*) of degree k if each vertex has in-degree (out-degree) equal to k. A graph is said a *undirected* (or *symmetric*) graph if $(i,j) \in E$ implies that $(j,i) \in E$. Any $(i,i) \in E$ is called a *self loop*.

A graph is *strongly connected* if for any given pair of vertices (v,v') there exists a path (i.e. an ordered list of edges) which connects v to v'. It is said *fully connected* or *complete* if for any couple of vertices there exists an edge joining them.

Let us now to introduce the concept of *Cayley graph* defined on Abelian groups [16]. Let G be any finite Abelian group (internal operation will always be denoted $+$) of order $|G| = N$, and let S be a subset of G containing zero. The Cayley graph $\mathscr{G}(G,S)$ is the directed graph with vertex set G and arc set

$$\mathscr{E} = \{(g,h) : h - g \in S\}.$$

Notice that a Cayley graph is always in-regular, namely the in-degree of each vertex is equal to $|S|$. If $G = Z_N$ then the graph is said *circulant*.

B. Matrices

A matrix $M \in \mathbb{R}^{N\times N}$ is said *compatible* or *supported* by the graph $\mathscr{G}$ if M_{ij} is positive only if there is an edge from j to i. Conversely, given the matrix M, we can define an induced graph $\mathscr{G}_M$ by taking N nodes and putting an edge (j,i) in E if $M_{ij} > 0$. A matrix is said *nonnegative* [13] if $M_{ij} \geq 0$ for all i and j, and is said *doubly stochastic* if it is nonnegative and the sums along each row and column are equal to 1.

A notion of Cayley structure can also be introduced for matrices. Let G be any finite Abelian group of order $|G| = N$. A matrix $M \in \mathbb{R}^{G\times G}$ is said to be a *Cayley* matrix over the group G if

$$M_{i,j} = M_{i+h,j+h} \qquad \forall\ i,j,h \in G.$$

If $G = Z_N$ then P is said *circulant*.

Now we give some notational conventions. Given a matrix $M \in \mathbb{R}^{N\times N}$, $\mathrm{diag}(M)$ means a diagonal matrix with the same diagonal elements of the matrix M and $\mathrm{out}(M)$ means the matrix obtained from M by putting equal to 0 all its diagonal elements, i.e. $\mathrm{out}(M) = M - \mathrm{diag}(M)$.

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III. Additive Communication Noise

In this section we study the performance of some distributed consensus algorithms when communications are affected by a transmission additive noise.

A. Noiseless communication algorithm

In the standard noiseless consensus algorithm we have that the agent i has a state $x_i(t)$ and the algorithm updates the state according to the formula

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij} x_j(t), \tag{1}$$

More compactly we can write

$$x(t+1) = Px(t), \tag{2}$$

where $x(t)$ is the column vector with entries $x_i(t)$ and P is the matrix with entries P_{ij}. It is well known in the literature [3], [11] that, if P is a doubly stochastic matrix with positive diagonal and with $\mathscr{G}_P$ strongly connected, then the previous algorithm solves the *average consensus problem*, namely

$$\lim_{t\to+\infty} x(t) = \frac{1}{N}\left(\sum_{i=1}^{N} x_i(0)\right)\mathbf{1}, \tag{3}$$

where $\mathbf{1}$ is the (column) vector of all ones. This can be shown because the previous conditions imply that

(A) 1 is the only eigenvalue of P on the unit circle centered in 0;
(B) the eigenvalue 1 has algebraic multiplicity one and $\mathbf{1}$ is its eigenvector;
(C) all the other eigenvalues are strictly inside the unit disk centered in 0.

B. Communication noise. Mean square analysis

Let us consider now a noisy version of the previous algorithm, namely the following systems

$$x(t+1) = Px(t) + n(t), \tag{4}$$

where $n(t)$ is a N-dimensional white noise with with zero mean and covariance $\mathbb{E}[n(t)n(t)^T] = Z$. This model can describe different ways in which the noise enters in the consensus algorithm.

For instance, if the agent i can access only to an approximation $\hat{x}_{j,i}(t)$ of $x_j(t)$, then the algorithm (1) becomes

$$x_i(t+1) = P_{ii}x_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} P_{ij}\hat{x}_{j,i}(t), \tag{5}$$

If $\hat{x}_{j,i}(t)$ is a noisy version of $x_j(t)$, namely $\hat{x}_{j,i}(t) = x_j(t) + \nu_{ij}(t)$, where $\nu_{ij}(t)$ are i.i.d. random variables of zero mean and variance σ^2, then we have that (5) can be written in the form of (4), taking

$$n_i(t) = \sum_{\substack{j=1\\ j\neq i}}^{N} P_{ij}\nu_{ij}(t). \tag{6}$$

In this case we have that $Z = \sigma^2\{\,\mathrm{diag}(PP^T) - [\mathrm{diag}(P)]^2\}$.

If now we consider the average of the positions $x_a(t) = \frac{1}{N}\mathbf{1}^T x(t)$ we can define the vector $\Delta(t) = x(t) - x_a(0)\mathbf{1}$ together with the *disagreement* $\tilde{\Delta}(t) = x(t) - x_a(t)\mathbf{1}$ and the *bias* $\beta(t) = x_a(t) - x_a(0)$ and their mean squared 2-norms,

$$w(t) = \frac{1}{N}\mathbb{E}[||\Delta(t)||^2],$$

$$v(t) = \frac{1}{N}\mathbb{E}[||\tilde{\Delta}(t)||^2]$$

$$\eta(t) = \mathbb{E}[|\beta(t)|^2].$$

Since with a communication noise there is no hope to reach an average consensus in the usual sense, we are interested in describing the performances of these systems with respect to their capacity to lead the agents *near* to consensus at the average of the initial conditions. The following theorem shows that this is not possible.

Theorem 1: Let

$$Y = I - \frac{1}{N}\mathbf{1}\mathbf{1}^*$$

Then the evolution of the mean squared norms w, v, and η is given by the following formulas.

$$\begin{aligned} v(t) &= \frac{1}{N}\,\mathrm{tr}\left[(YP)^t\mathbb{E}[x(0)x(0)^T](P^TY)^t\right] + \\ &\quad \frac{1}{N}\,\mathrm{tr}\left[\sum_{i=0}^{t-1}(YP)^i Z (P^TY)^i\right] && (7)\\ \eta(t) &= \frac{t}{N^2}(\mathbf{1}^T Z\mathbf{1}) && (8)\\ w(t) &= v(t) + \eta(t) && (9)\end{aligned}$$

Remark 1: Equation (8) implies that instead of being average preserving, as hoped, when communications are noisy these algorithms are affected, in mean, by a drift on the (estimated) average position, which is linear in time. On the other hand, since YP is a asymptotically stable matrix, then the first term in sum in (7) tends to zero and so

$$v(\infty) = \frac{1}{N}\,\mathrm{tr}\left[\sum_{i=0}^{\infty}(YP)^i Z (P^TY)^i\right]$$

and this allows us to see that $v(t)$ is bounded as t goes to infinity, while $w(t)$ is not.

C. Examples

The above analysis can be specialized in some special cases.

Complete Graph

Consider $P = N^{-1}\mathbf{1}\mathbf{1}^T$. In case we consider the noise structure given in (6), we have that

$$\eta(t) = \sigma^2\frac{N-1}{N^3}t$$

$$v(t) = \sigma^2\frac{N-1}{N^2} \quad \text{for } t \geq 1$$

Cayley Graph

In a Cayley graph one can compute $w(t), v(t), \eta(t)$ as function of its spectrum, showing that also on these graphs, the performances are corrupted by noise. Here we will regard the initial condition $x(0)$ as the realization of a random variable of zero mean and variance τ^2, independent from the noises.

In case we consider the noise structure given in (6), we have that Z is Cayley and diagonal and so all its eigenvalues coincides with $\mu = \sigma^2 \sum_{j=2}^{N} P_{1j}^2$.

Denote $\sigma(P) = \{\lambda_0, \lambda_1, \ldots, \lambda_{N-1}\}$ the spectrum of P, where by the properties of P we have that $\lambda_0 = 1$ and $|\lambda_i| < 1$, $\forall 1 \leq i \leq N-1$. Then, we are able to compute

$$v(t) = \frac{\tau^2}{N} \sum_{j=1}^{N-1} |\lambda_j|^{2t} + \frac{\mu}{N} \sum_{j=1}^{N-1} \frac{1-|\lambda_j|^{2t}}{1-|\lambda_j|^2}$$
$$\eta(t) = \frac{\mu}{N} t.$$

Remark that $w(t)$ is the sum of three terms: a summation of decreasing exponentials, a term saturating to a constant and a linear drift. Indeed,

$$v(\infty) = \frac{\mu}{N} \sum_{j=1}^{N-1} \frac{1}{1-|\lambda_j|^2}.$$

A similar bound is presented in [14] where the authors study the system (4) taking n_i to be a process noise, rather than coming from communication.

Remark 2: An interesting result can be found if we assume that $G = Z_M^k$ and $N = M^k$. Let $e_h \in \mathbb{Z}_M^k, h = 1, \ldots, k$ be the column vector with all zeros except a 1 at position h. Let $e^{(0)} = (0, \ldots, 0)$. Assume now that P has the following structure

$$P_{ij} = \begin{cases} \frac{1}{k+1} & \text{if } i - j = e_h \ \exists h: \ 0 \leq h \leq M \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Then, it is possible to prove that

$$v(\infty) \approx \begin{cases} C_k N & \text{if } k = 1 \\ C_k \log N & \text{if } k = 2 \\ C_k & \text{if } k \geq 3 \end{cases} \tag{11}$$

where C_k is a suitable constant independent of N. This result is presented in [15].

D. Sensible solutions against drift

Since the drift problem appears because the agents get and use information which is corrupted by noise, an idea to overcome can be by selecting in some way the information that is worth to use. This can be done if the agents are "aware of the problem", namely if they know something about the probability distribution of the noise.

In this case, a criterion can be to discard any information about agents whose position is too near. Of course, doing this, the agents renounce the idea of seeking a perfect agreement.

Namely, we introduce a family of threshold functions

$$f(z) = \begin{cases} z & \text{if } |z| > R \\ 0 & \text{if } |z| \leq R \end{cases}$$

and we change the evolution equation (5) into

$$x_i(t+1) = x_i(t) + \sum_{j=1}^{N} P_{ij} f(\hat{x}_{j,i}(t) - x_i(t)). \tag{12}$$

In spite of its simplicity, simulations (Figure 1) show this method as effective in avoiding drift.

Namely, if we suppose that the noise probability distribution has bounded support, i.e. that $\exists B$ such that $P(|n_{ij}| > B) = 0$, we have that

- if $B > R$ we still have a linear drift in the average position, but slower[1];
- if $B \leq R$ we avoid the drift, and asymptotically in time the agents will have states such that both v_∞ and w_∞ are finite and depending on N and R, so that the best choice is $R = B$.

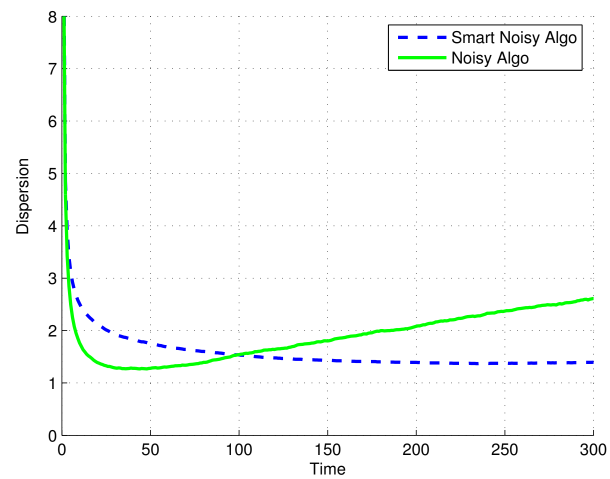

Fig. 1. Time evolution of $w(t)$ on a circulant graph of degree 1, where $x_i(t+1) = \frac{1}{2}x_i(t) + \frac{1}{2}x_{i+1}(t)$ and $N = 20$ (solid line). The improvement obtained discarding some information (dashed) is clear.

IV. Quantized Communication

In this section we deal again with a set of agents which communicate in order to estimate the average value of their initial conditions. Differently from most of the literature we assume that the communication network is constituted only of digital links. This implies the exchange of perfect information between the systems is not allowed. In fact, through a digital channel, the j-th agent can only send to the i-th agent symbolic data. Hence beside the aspects induced by the choice of the structure of the communication network, one have also to face the effects of the quantization constraints due to the digital links.

This problem is now starting to attract the interest of the scientific community, as in [12] and [10]. In particular in the latter the authors consider an interesting randomized communication scheme on a connected undirected graph which allows agents which have integer states to converge to the average of initial conditions.[2]

[1] Of course, this is also the case when the noise probability distribution has unbounded support.

[2] Actually, the consensus is reached in the sense that the initial sum of the states is preserved and the asymptotic states of the agents can only differ by 1.

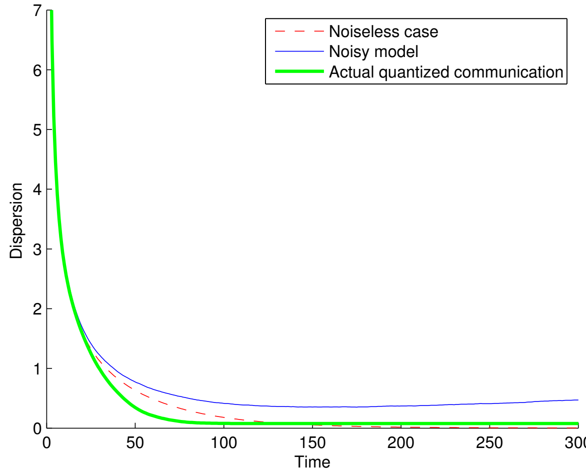

Fig. 2. The plot compares the evolution of $w(t)$ in the actual quantized case (13) with its noisy approximation and the perfect communication case, on a circulant graph where $x_i(t+1) = \frac{1}{2}x_i(t) + \frac{1}{2}x_{i+1}(t)$ and $N = 20$.

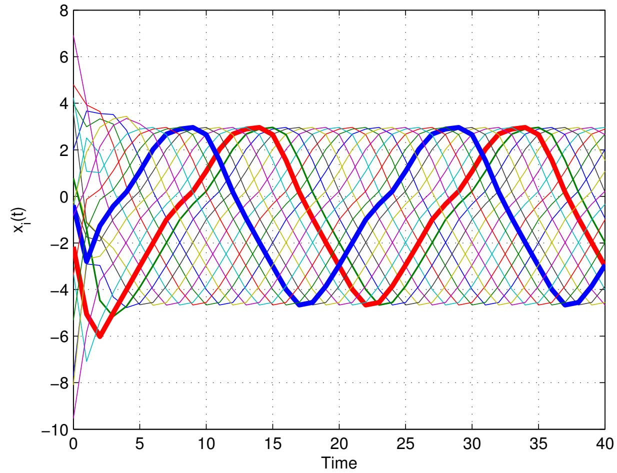

Fig. 3. Periodic dynamic of $x_i(t)$ following (13) on a directed circulant graph.

Our approach is different because we suppose that agents can only exchange integer numbers between them, but their states are real numbers. Moreover, we do not assume the graph to be undirected.

With these hypotheses, the agents evolve following

$$x(t+1) = \operatorname{diag}(P)\,x(t) + \operatorname{out}(P)\,q(x(t)) \tag{13}$$

where with $q : \mathbb{R}^N \to \mathbb{Z}^N$ we denote the quantizer, which maps each component of x into the nearest integer, namely, if $x = (x_1, \dots, x_N)^T$ and $n = (n_1, \dots, n_N)^T$ we have that $, \forall 1 \leq i \leq N$,

$$q(x) = n \in \mathbb{Z}^N \quad \Leftrightarrow \quad \begin{array}{ll} x_i \in [n_i - 1/2, n_i + 1/2[, & \text{if } x_i \geq 0 \\ x_i \in]n_i - 1/2, n_i + 1/2], & \text{if } x_i < 0. \end{array}$$

Remark 3: The system (13) does not conserve the average of the agents' states, i.e., $x_a(t+1) \neq x_a(t)$.

Then, in general this algorithm will not be able to drive the states to an *average* consensus, but to a weaker condition of consensus at some other value.

A. Additive white noise model

A first approach to the analysis of the system (13) could be trying to model the quantization error as an additive noise, following (4)

$$\begin{aligned} x(t+1) = & Px(t) + \operatorname{out}(P)(q(x) - x) = && (14) \\ & Px(t) + \operatorname{out}(P)\nu(t), && (15) \end{aligned}$$

Here $\nu(t)$ is a vector of noises which are uniformly distributed in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. These are intended to model, in the average, the behavior of the rounding error.

Even if sensible, this model is not a good description of the behavior of the system with quantized communication. This is visualized in Figure 2.

Its more evident drawback is that such a noisy model can be proved to be affected by a drift in time of its average of states, while this does not occur in the actual quantized system. In fact, if we denote $\Delta(t) = x(t) - x_a(0)\mathbf{1}$ and $u(t) = \frac{1}{N}||\Delta(t)||^2$, it is possible to prove, by convexity arguments, that $u(t)$ is bounded in time for the system (13).

Thus the only way to study the quantization seems to be by dealing with the actual evolution law (13). Solving exactly its dynamics is in general not simple, and can be tackled only in some special cases.

In the next section will be presented a possible way to overcome in part the drawbacks of this naive approach to quantization.

V. Quantized Communication II

In the above section we have seen how the naive approach to quantization leads to a system difficult to study and with poor performances (Figure 3). Then it is natural to look for some clever quantization algorithm.

The starting point is the constraint that allows each agent to access only the integer approximation of its neighbors' states. Then, the designer can only look for a smart way for the agents to use their own information.

Then, we propose the evolution scheme

$$x_i(t+1) = x_i(t) - q(x_i(t)) + \sum_{j=1}^{N} P_{ij}\,q(x_j(t)) \tag{16}$$

or, more concisely,

$$x(t+1) = x(t) + (P - I)\,q(x(t)). \tag{17}$$

This scheme avoids the main shortcoming of (13), noticed in Remark 3.

Proposition 2: The algorithm (17) conserves the average of the initial conditions.

Proof: Since P is doubly stochastic we have that

$$\begin{aligned} & x_a(t+1) = N^{-1}\mathbf{1}^T x(t+1) = \\ & N^{-1}\mathbf{1}^T x(t) + N^{-1}\mathbf{1}^T (P - I)\,q(x(t)) = N^{-1}\mathbf{1}^T x(t) = x_a(t), \end{aligned}$$

for all t. ■

Moreover, one can prove that this is the only local choice[3] which satisfies the constraint and allows to preserve the average of states.

Then, we want to study how the proposed algorithm leads the agents near to the consensus.

A. Communication between two agents

As a first example, we show that the quantized system can behave rather poorly even in the trivial case when the system is made of only two communicating agents.

Proposition 3: Let the evolution law be

$$x_i(t+1) = x_i(t) + k\left[q(x_{i+1}(t)) - q(x_i(t))\right] \quad i = 1,2,$$

with $0 < k < 1$, and let $\delta(t) = |x_1(t) - x_2(t)|$. Then

$$\delta(\infty) \leq \begin{cases} \dfrac{k}{1-k} & \text{if } k \geq 1/2 \\ 1 & \text{if } k \leq 1/2. \end{cases} \tag{18}$$

Remark that the limit behavior depends heavily on the parameter k, which represents the weight each agent assigns to the value it receives from its neighbor.

B. The pursuit graph case

We consider a circulant graph (pursuit graph) where each agent i communicates with only one neighbor and evolves following

$$x_i(t+1) = x_i(t) + \frac{1}{2}\left[q(x_{i+1}(t)) - q(x_i(t))\right] \quad i = 1,\ldots,N. \tag{19}$$

The dynamic (19) can be studied by mean of a symbolic dynamic approach. We define, first, $n_i = \lfloor 2x_i \rfloor$, mapping the real dynamics into an equivalent dynamics between integers $n_i(t+1) = g(n_i(t), n_{i+1}(t))$ with

$$g(n_i(t), n_{i+1}(t)) = \left\lfloor \frac{n_i(t)}{2} \right\rfloor + \left\lceil \frac{n_{i+1}(t)}{2} \right\rceil.$$

For this map g, it is possible to study the evolution and the limit behavior (fixed and periodic points), and then to recover information on the dynamics of x_i.

Theorem 4: The limit state of system (19) is reached in finite time, can be either a fixed or a periodic point, and is such that

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| \leq 1. \tag{20}$$

This theorem tells that, in this special family of graphs, the agents get close to the average agreement and this does not depend on the number of the agents.

[3]We call *local* choice, a choice which is a local rule, that is which can be done with no global knowledge on the graph topology or the population of agents.

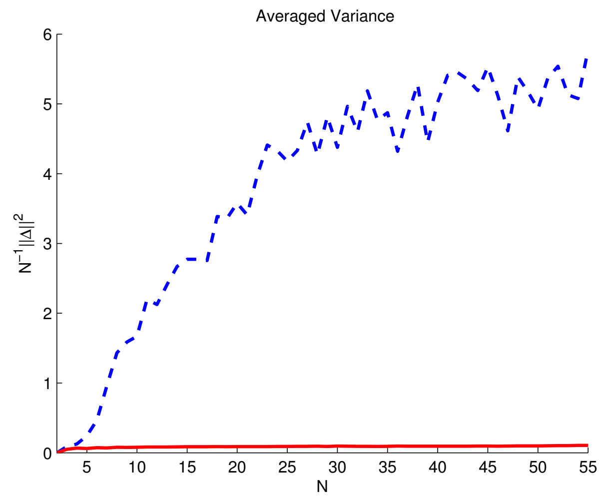

Fig. 4. Asymptotic variance of the states for the scheme (13), dashed, and the proposed (17), solid line, applied on symmetric circulant graphs.

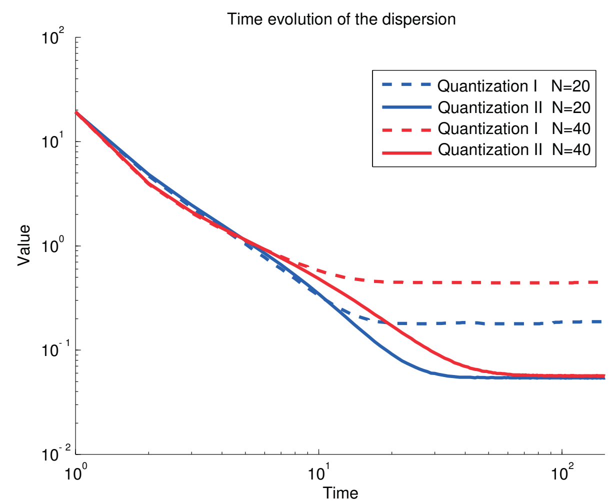

Fig. 5. Time evolution of the variance of states for the scheme (13), dashed, and our algorithm (17), solid line, applied on two random geometric graphs with different N.

C. Simulations on general graphs

The analysis is difficult for general graphs, but simulations allow some remarks. While on some graphs, e.g. the complete graph and the pursuit graph considered in the above subsection, the performance of the algorithms are similar, on some others the asymptotical performance of the algorithm (17) scales better with N than the naive (13). A sample of these simulations is shown in Figures 4 and 5.

D. General results

Inspired by the good performances obtained in simulations, some general results can be obtained about (17). In the following we only assume that the matrix P satisfies the conditions $(A),(B),(C)$ stated in Section III. We denote by $\rho(P)$ its essential spectral radius [11], which is the largest in modulus eigenvalue, different from 1, of the matrix P, namely

$$\rho(P) = \max\{|\lambda| : \lambda \in \sigma(P) \setminus 1\}.$$

Then the following result can be proved.

Theorem 5: Let $\Delta(t) = (I - N^{-1}\mathbf{1}\mathbf{1}^T)x(t)$ and let $d(t) = \frac{1}{\sqrt{N}}||\Delta(t)||_2$. Consider the evolution equation (17) and let ρ be the essential spectral radius of P. Then the following facts hold.

i) $d(t) \leq \frac{||I-P||_2}{2}\frac{1-\rho^t}{1-\rho} + \rho^t \frac{||\Delta(0)||_2}{\sqrt{N}}$

ii) $d_\infty = \limsup_{t\to\infty} d(t) \leq \frac{1}{1-\rho}$

Remark 4: The claim *ii*) gives a bound which is finite in time, but which can depend on N. An interesting case is given by the Cayley matrices. Indeed, it is known [11] that if P is a Cayley matrix and ν is the in-degree of $\mathcal{G}_P$ then $\rho(P) \geq 1 - \frac{C}{N^{2/\nu}}$ where $C > 0$ is a constant independent of the graph and of N. Hence, if we consider ν fixed and N tends to infinity, we have that $1-\rho(P)$ tends to 0 as $N^{-\frac{2}{\nu}}$ implying that the bound *ii*) diverges polynomially in N.

However, about this family of matrices, something more refined than this remark is given by the following result.

Theorem 6: Let P be any symmetric Cayley matrix. Denote by ν the in-degree of $\mathcal{G}_P$. Then

$$d_\infty \leq C_\nu \log N, \tag{21}$$

where $C_\nu > 0$ is a constant depending on ν.

This means that the worst case behavior of our scheme scales, for the family of Cayley matrices, better than polynomially with N.

VI. Concluding remarks

This paper attempted to look for some insight into the behavior of linear averaging algorithms for the consensus problem on a graph with additional constraints in the communication. This has been done in two cases, when communication is affected by an additive noise and when communication is quantized. In the former case we showed that such a system will fail to find an average consensus, due to a drift from the initial average. We also proposed some heuristic techniques to avoid this problem.

In the quantized communication several questions remain open. Here we tried to begin a study of the heavy effects of a constraint of quantized communication on the well-known diffusion agreement schemes [11]. Then we proposed a new agreement algorithm of this class, which is able to preserve the average of the initial data, and which gives good performances and a good scalability with N. Then,the next step is to make a deeper mathematical study of this algorithm, in order to understand the origin of its features and design further improvements.

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