Average consensus on networks with quantized communication

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Several agents in a network have to communicate in order to achieve an agreement about the average of their states.
Average consensus problem

Linear dynamical system on $\mathbb{R}^N$

$$x(t + 1) = P x(t),$$

componentwise $x_i(t + 1) = \sum_{i=1}^{N} P_{ij} x_j(t), \quad i = 1 \ldots N$

The matrix $P$ has to depend on the communication network:
if $j$ does not communicate with $i$, then $P_{ij} = 0$

**Goal:** design $K$ so that all agents tend to share the same state

$$\lim_{t \to +\infty} x_i(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(0) \quad \forall i.$$
In distributed control and information theory.

- Data fusion in sensor networks
- Coordination and rendezvous of UAV and robots
- Load balancing between processors
Our problem

We suppose that the agents can exchange information through a
- time-invariant
- strongly connected
- digital
communication network.

Information has to be symbolic, i.e. quantized:

Normalized to 1 uniform quantization: \( q[x(t)] = \text{round} [x(t)] \).

\[ \Rightarrow \text{the classical consensus can not be reached.} \]

Perfect links & naive quantization

- If links are **not digital**, it is known that we can choose $P$ such that
  - $P_{ii} > 0 \ \forall i$
  - $P$ is non negative
  - $P$ is doubly stochastic, i.e. $\sum_{j=1}^{N} P_{ij} = \sum_{i=1}^{N} P_{ij} = 1$.

- With **digital** links, the *naive* approach
  
  $$x_i(t+1) = P_{ii} x_i(t) + \sum_{j \neq i} P_{ij} q[x_j(t)],$$

  fails because this *non linear* map
  - does **not** drive the agents to equal states,
  - does **not** preserve their average.
Our proposal

Take a slightly different evolution map:

\[ x_i(t + 1) = x_i(t) - (1 - P_{ii}) q[x_i(t)] + \sum_{j \neq i} P_{ij} q[x_j(t)], \]

\[ x(t + 1) = x(t) + (P - I) q[x(t)], \]

This map preserves the average of states and drives them nearer to consensus.

Example: random geometric graph.
Remarks:

- we are interested in the asymptotic of average consensus disagreement
  \[ \Delta(t) := x(t) - \frac{1}{N} \sum_{i=1}^{N} x_i(0) 1. \]

- The quantization errors \( e(t) := x(t) - q[x(t)] \) are bounded

Three approaches can be useful:

- Working on the actual quantized map. \( d_\infty(P) := \limsup_{t \to \infty} \frac{1}{\sqrt{N}} ||\Delta(t)||_2. \)

- Worst case analysis. \( d^w := \lim_{t \to \infty} \sup_{||e||_\infty \leq 1/2} \frac{1}{N} ||\Delta(t)||_2 \)

- Probabilistic analysis: consider \( e \) as a random variable.
  \( d^r_\infty(P) := \sqrt{\lim_{t \to \infty} \frac{1}{N} \mathbb{E}[||\Delta(t)||^2]} \)
we study the actual system

- very difficult: results only in special cases.

Complete graphs with uniform weights, \( P = \frac{1}{N} \mathbf{1} \mathbf{1}^T \).
\[ \Rightarrow d_\infty(P) \leq 1 \]

Directed circuits with uniform weights, \( k = 1/2 \).
Thanks to a symbolic dynamics underlying the system, \( \Rightarrow \)
\[ d_\infty(P) \leq 1/2 \]

\[ x_i(t + 1) = (1 - k)x_i(t) + kx_i(t) \]
Worst case

+ it’s easier
+ it gives upper bounds on the actual system, since $d_\infty(P) \leq d^w_\infty(P)$
- results are often very conservative.

We find two main bounds

1. $d^w_\infty(P) \leq \frac{C_P}{1 - \rho_{ess}(P)}$

2. if $P$ is symmetric $\Rightarrow$ $d_\infty(P) \leq \frac{1}{2} \sum_{s=0}^{\infty} \rho(P^s(I - P))$  $\rho$ spectral radius
Consequences on the dependence on $N$.

- If there is a uniform lower bound $G$ on the spectral gaps, the performance does not worsen in $N$.

- If $P$ is symmetric Cayley matrix with bounded degree, (its spectral gap is infinitesimal in $N$) $\Rightarrow$ $d_{\infty}(P_N) = O(\log N)$.

- This bound is tight for the hypercube graphs: $d_{\infty}^w(P) = \frac{\log_2 N}{2}$.

Is this divergence intrinsic in the system?
it’s easier

- there is little *a priori* justification, since the original system is not random
- it gives no upper bound

+ results are near to typical simulated results (*a posteriori* justification).

e is supposed to be a random variable acting as an additive noise, having
- zero mean,
- variance $\sigma^2$.

Then

$$d_{\infty}^r(P) = \sqrt{\frac{1}{N} \sigma^2 \sum_{i=1}^{N-1} \frac{|1 - \lambda_i|^2}{1 - |\lambda_i|^2}}, \quad \lambda_i \text{ are the eigenvalues of } P.$$
In general, if \( P_{ii} \geq \varepsilon \forall i, \) \( \Rightarrow \) \( d^{r}_{\infty}(P) \leq \frac{1-\varepsilon}{\varepsilon}. \)

This applies to sequences of Cayley graphs.

Examples:

**Hypercube** \( d^{r}_{\infty}(P) = \sqrt{\frac{N-1}{N}} \sigma \)

**Directed circuit** \( d^{r}_{\infty}(P) = \sqrt{\frac{N-1}{N}} \frac{k}{1-k} \sigma^2 \)

**Undirected circuit** \( \lim_{N \to \infty} d^{r}_{\infty}(P) = \left( \frac{1}{\sqrt{1-2k}} - 1 \right)^{1/2} \sigma \)

- In most cases, \( d^{r}_{\infty}(P) \) can be bounded uniformly in \( N \) \( \Rightarrow \) good scalability
- \( d^{r}_{\infty}(P) \) can not be bounded uniformly on other parameters (e.g, \( k \) in directed circuit)
Simulations: hypercube graph

The method scales very well in $N$.

- No logarithmic divergence.
- Well compatible with the probabilistic (uniform noise) result.

Remark that the $x$-axis is logarithmic.
In the directed circuit, the performance depends on the parameter $k$.

- No logarithmic divergence in $N$
- Qualitatively compatible with probabilistic results:
  - In $k$
  - In $N$ for small $k$
Discussion and open problems

The algorithm works very well.

- it preserves the average of initial conditions,
- it drives the agents near to the consensus in typical cases (e.g. uniform weights),

But:

- which is the right theoretical approach?
- are there trade-offs between asymptotical vicinity to consensus and speed of convergence?
- are there better algorithms? E.g., we know that using uniform quantization in an encoder-decoder scheme with memory (Zooming in-zooming out), average consensus is reachable.