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GENERICITY AND RANK DEFICIENCY OF HIGH ORDER SYMMETRIC TENSORS

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ABSTRACT

Blind Identification of Under-Determined Mixtures (UDM) is involved in numerous applications, including Multi-Way factor Analysis (MWA) and Signal Processing. In the latter case, the use of High-Order Statistics (HOS) like Cumulants leads to the decomposition of symmetric tensors. Yet, little has been published about rank-revealing decompositions of symmetric tensors. Definitions of rank are discussed, and useful results on Generic Rank are proved, with the help of tools borrowed from Algebraic Geometry.

1. INTRODUCTION

Several extensions of the Singular Value Decomposition (SVD) to K-way arrays are possible, and we are interested in the Canonical Decomposition (CAND), which allows us to define a *Tensor Rank*. CAND is essential in the process of Blind Identification of Under-Determined Mixtures (UDM), i.e., linear mixtures with more inputs than observable outputs. Despite its interest, this subject has not been much addressed in the general literature, and even less in Signal Processing. For instance, for several years, the so-called Parafac algorithm is used to fit data arrays to a multilinear model [1, 2]. Yet, the minimization of this matching error is an ill-posed problem in general, since the set of tensors of rank smaller than r is not closed, unless $r = 1$.

See [2, 3, 4, 5] and references therein for a list of application areas, including Speech, Mobile Communications, Machine Learning, Factor Analysis with K-way arrays (MWA), Biomedical Engineering, Psychometrics and Chemometrics.

2. NOTATION

2.1. Arrays

Arrays with more than one index will be denoted in bold uppercase; vectors are one-way arrays, and will be denoted in bold lowercase. Plain uppercases will be mainly used to denote dimensions. For our purpose, only a few notations related to arrays [4, 5] are necessary. In this paper, the outer product of two arrays of order M and N is denoted $\boldsymbol{C} = \boldsymbol{A} \circ \boldsymbol{B}$ and is an array of order $M + N$:

$$C_{ij\cdots\ell\, ab\cdots d} = A_{ij\cdots\ell} B_{ab\cdots d}. \tag{1}$$

For instance, the outer product of two vectors, $\boldsymbol{u} \circ \boldsymbol{v}$, is a matrix. Conversely, the mode-p inner product between two arrays having the same pth dimension is denoted $\boldsymbol{A} \bullet_p \boldsymbol{B}$, and is obtained by summing over the pth index. More precisely, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are of orders M and N respectively, this yields for $p = 1$ the array of order $M + N - 2$:

$$\{\boldsymbol{A} \bullet_1 \boldsymbol{B}\}_{i_2\cdots i_M, j_2\cdots j_N} = \sum_k A_{k i_2\cdots i_M} B_{k j_2\cdots j_N}.$$

For instance, the standard matrix-vector product can be written as $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}^{\mathrm{T}} \bullet_1 \boldsymbol{u}$. Note that some authors [6, 4] denote this contraction product as $\boldsymbol{A} \times_p \boldsymbol{B}$, which we find much less convenient.

We shall say that a d-way array is *square* if all its d dimensions are identical. A square array will be called *symmetric* if its entries do not change by any permutation of its d indices. The linear space of square d-way arrays of size N is of dimension N^d, whereas the space of symmetric d-way arrays of same size is of dimension $\binom{N+d-1}{d}$.

In this framework, only d-way arrays that enjoy the *multi-linearity property* by linear change of coordinates will be considered; they will be referred to as *tensors* [7]. To illustrate this property, let $\boldsymbol{T}$ be a tensor of third order of dimensions $P_1 \times P_2 \times P_3$, and let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be three matrices of size $K_1 \times P_1$, $K_2 \times P_2$, and $K_3 \times P_3$, respectively (in general $K_i = P_i$ and matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are invertible, but this is actually not mandatory in most of our discussion). Then tensor $\boldsymbol{T}$ is transformed by the multi-linear map $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ into a tensor $\boldsymbol{T}'$ defined by:

$$T'_{ijk} = \sum_{abc} A_{ia} B_{jb} C_{kc} T_{abc}, \tag{2}$$

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which may be written in compact form as $\boldsymbol{T}' = \boldsymbol{T} \bullet_1 \boldsymbol{A} \bullet_2 \boldsymbol{B} \bullet_3 \boldsymbol{C}$.

2.2. Polynomials

Any symmetric tensor of order d and dimension K can be associated with a unique homogeneous polynomial of degree d in K variables via the expression [9, 5]:

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{j}} T_{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{f}(\boldsymbol{j})} \tag{3}$$

where for any integer vector $\boldsymbol{j}$ of dimension d, one associates bijectively the integer vector $\boldsymbol{f}(\boldsymbol{j})$ of dimension K, each entry f_k of $\boldsymbol{f}(\boldsymbol{j})$ being equal to the number of times index k appears in $\boldsymbol{j}$. We have in particular $|\boldsymbol{f}(\boldsymbol{j}))| = d$. We also assume the following conventions (rather standard in algebraic geometry): $\boldsymbol{x}^{\boldsymbol{j}} \stackrel{\text{def}}{=} \prod_{k=1}^{K} x_k^{j_k}$ and $|\boldsymbol{f}| \stackrel{\text{def}}{=} \sum_{k=1}^{K} f_k$, where $\boldsymbol{j}$ and $\boldsymbol{f}$ are integer vectors. The converse is true as well, and the correspondence between symmetric tensors and homogeneous polynomials is obviously bijective.

Now for asymmetric tensors, the same association is not possible. In order to connect tensor spaces with algebraic geometry, tensors are associated with multilinear maps [8]. This justifies the use of the Zariski topology.

3. DEFINITION OF RANKS

Let $\boldsymbol{T}$ be a symmetric tensor as defined in section 2.1.

CAND and rank. Any tensor can always be decomposed (possibly non-uniquely) as:

$$\boldsymbol{T} = \sum_{i=1}^{r} \boldsymbol{u}(i) \circ \boldsymbol{v}(i) \circ \cdots \circ \boldsymbol{w}(i). \tag{4}$$

The *Tensor Rank* is defined as the smallest integer $r(\boldsymbol{T})$ such that this decomposition holds exactly. Among other properties, note that this Canonical Decomposition (CAND) holds valid in a ring, and that the CAND of a multilinear transform of $\boldsymbol{T}$ equals the multilinear transform of the CAND of $\boldsymbol{T}$. In other words, if $(\boldsymbol{u}, \boldsymbol{v}, \dots, \boldsymbol{w})$ is the CAND of $\boldsymbol{T}$, then $(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{B}\boldsymbol{v}, \dots, \boldsymbol{C}\boldsymbol{w})$ is the CAND of $\boldsymbol{T} \bullet_1 \boldsymbol{A} \bullet_2 \boldsymbol{B} \bullet_3 \boldsymbol{C}$.

If in (4), we have $\boldsymbol{u}(i) = \boldsymbol{v}(i) = \cdots = \boldsymbol{w}(i)$ for every i, then we may call it a *symmetric* CAND, yielding a *symmetric rank*, $r_s(\boldsymbol{T})$.

Genericity. A property is referred to as *typical* if it is true on a non zero volume set. On the other hand, a property is said to be *generic* if is true almost everywhere. More formal definitions will be given in section 4. It is important to distinguish between typical and generic properties; for instance, as will be subsequently seen, there can be several *typical ranks*, but only a single *generic rank*.

Through the bijection (3), the CAND (4) of symmetric tensors can be transposed to homogeneous polynomials (also called *quantics*), as pointed out in [9]. This allows to talk indifferently either about CAND of tensors or quantics.

On the other hand, given a symmetric tensor $\boldsymbol{S}$, one can compute its CAND either in $\mathcal{T}_s$ or in $\mathcal{T}$. Since the CAND in $\mathcal{T}_s$ is constrained, we obviously also have the inequality between rank and symmetric rank:

$$\forall \boldsymbol{S} \in \mathcal{T}_s,\ r(\boldsymbol{S}) \leq r_s(\boldsymbol{S}). \tag{5}$$

4. GENERIC AND TYPICAL RANKS

Define the set of tensors $\mathcal{Y}_r = \{\boldsymbol{T} \in \mathcal{T} \mid r(\boldsymbol{T}) \leq r\}$ with values in $\mathbb{C}$. Also denote $\bar{\mathcal{Y}}_r$ its Zariski closure, which is the smallest variety containing $\mathcal{Y}_r$ [10], and $\mathcal{Z}_r = \{\boldsymbol{T} \in \mathcal{T} : r(\boldsymbol{T}) = r\}$. Then we have obviously, for every r, $\mathcal{Y}_{r-1} \sqcup \mathcal{Z}_r = \mathcal{Y}_r$ and the sum $\mathcal{Y}_1 + \cdots + \mathcal{Y}_1$ of r times $\mathcal{Y}_1$ is $\mathcal{Y}_r$.

Let us define the following ranks:

- $\bar{R} = \min\{r \mid \bar{\mathcal{Y}}_r = \mathcal{T}\}$,
- $R = \min\{r \mid \mathcal{Y}_r = \mathcal{T}\}$.

By definition, we have $\bar{R} \leq R$. We shall prove in this section that the generic rank exists in $\mathbb{C}$ and that it is equal to $\bar{R}$.

Definition of Typical ranks. An integer r is not a typical rank if $\mathcal{Z}_r$ has a zero volume, which means that $\mathcal{Z}_r$ is contained in a non trivial closed set. Alternatively, r is a typical rank if $\mathcal{Z}_r$ is dense with the Zariski topology, which means that $\overline{\mathcal{Z}}_r = \mathcal{T}$.

Definition of the Generic rank. When a typical rank is unique, it may be called generic.

Next, even if we know that $\mathcal{Y}_1$ is closed as a determinantal variety, $\mathcal{Y}_r$ are generally not closed for $r > 1$. This is another major difference with matrices, for which all $\mathcal{Y}_r$ are closed. One can actually prove a simple lemma, that yields a more accurate statement.

Lemma 1. *The sets $\bar{\mathcal{Y}}_k$, $k \geq 1$, are irreducible.*

Proof. For $r \geq 1$, the variety $\bar{\mathcal{Y}}_r$ is the closure of the image $\mathcal{Y}_r$ of the map:

$$\begin{aligned} \phi_r : \mathbb{C}^{r \times n \times d} &\to \mathcal{T} \\ (\boldsymbol{u}, \dots, \boldsymbol{w}) &\mapsto \textstyle\sum_{i=1}^{r} \boldsymbol{u}(i) \circ \boldsymbol{v}(i) \circ \cdots \circ \boldsymbol{w}(i) \end{aligned}$$

where $\boldsymbol{u}(i), \dots, \boldsymbol{w}(i)$ are d vectors $\mathbb{C}^n$ for each $i = 1, \dots, r$. Consider now two polynomials f, g such that $fg \equiv 0$ on $\bar{\mathcal{Y}}_r$. As $\bar{\mathcal{Y}}_r$ is the Zariski closure of $\mathcal{Y}_r$, this is equivalent to $fg \equiv 0$ on $\mathcal{Y}_r$ or

$$(fg) \circ \phi_r = (f \circ \phi_r)(g \circ \phi_r) \equiv 0.$$

Thus either $f \equiv 0$ or $g \equiv 0$ on $\mathcal{Y}_r$ or equivalently on $\bar{\mathcal{Y}}_r$, which proves that $\bar{\mathcal{Y}}_r$ is an irreducible variety. For more details on properties of parameterized varieties, see [10]. See also the proof [8] for 3^{rd} order tensors. $\square$

Lemma 2. *We have* $\bar{R} = \min\{r \mid \bar{\mathcal{Y}}_r = \bar{\mathcal{Y}}_{r+1}\}$.

Proof. Suppose that there exist $r < \bar{R}$ such that $\bar{\mathcal{Y}}_r = \bar{\mathcal{Y}}_{r+1}$. Then $\bar{\mathcal{Y}}_r + \mathcal{Y}_1 = \bar{\mathcal{Y}}_{r+1} = \bar{\mathcal{Y}}_r$ so that we also have

$$\bar{\mathcal{Y}}_r + \mathcal{Y}_1 + \mathcal{Y}_1 = \bar{\mathcal{Y}}_r + \mathcal{Y}_1 + \cdots + \mathcal{Y}_1 = \bar{\mathcal{Y}}_r.$$

As the sum of R times $\mathcal{Y}_1$ is $\mathcal{T}$, we deduce that $\bar{\mathcal{Y}}_r = \mathcal{T}$ and that $r \geq \bar{R}$, which contradicts our hypothesis. By definitions $\bar{\mathcal{Y}}_{\bar{R}} = \bar{\mathcal{Y}}_{\bar{R}+1} = \mathcal{T}$, which proves the lemma. See also the proof of [8]. □

Theorem 3. *The varieties $\bar{\mathcal{Z}}_r$ can be ordered by inclusion as follows:*

$$\text{if } r_1 < r_2 < \bar{R} < r_3 \leq R, \text{ then } \bar{\mathcal{Z}}_{r_1} \subsetneq \bar{\mathcal{Z}}_{r_2} \subsetneq \bar{\mathcal{Z}}_{\bar{R}} \supsetneq \bar{\mathcal{Z}}_{r_3}.$$

Proof. By lemma 2, we deduce that for $r < \bar{R}$

$$\bar{\mathcal{Y}}_r \neq \bar{\mathcal{Y}}_{r+1}.$$

As $\bar{\mathcal{Y}}_r$ is an irreducible variety, we have $\dim(\bar{\mathcal{Y}}_r) < \dim(\bar{\mathcal{Y}}_{r+1})$. As $\mathcal{Y}_r \cup \mathcal{Z}_{r+1} = \mathcal{Y}_{r+1}$, we deduce that

$$\bar{\mathcal{Y}}_r \cup \bar{\mathcal{Z}}_{r+1} = \bar{\mathcal{Y}}_{r+1},$$

which implies by the irreducibility of $\bar{\mathcal{Y}}_{r+1}$, that $\bar{\mathcal{Z}}_{r+1} = \bar{\mathcal{Y}}_{r+1}$. Consequently, for $r_1 < r_2 < \bar{R}$, we have

$$\bar{\mathcal{Z}}_{r_1} = \bar{\mathcal{Y}}_{r_1} \subsetneq \bar{\mathcal{Z}}_{r_2} = \bar{\mathcal{Y}}_{r_2} \subsetneq \bar{\mathcal{Z}}_{r_2} = \mathcal{T}.$$

Let us prove now that if $\bar{R} < r_3$, we have $\bar{\mathcal{Z}}_{r_3} \subsetneq \mathcal{T}$. Suppose that $\bar{\mathcal{Z}}_{r_3} = \mathcal{T}$, then $\mathcal{Z}_{r_3}$ is dense in $\mathcal{T}$ as well as $\mathcal{Z}_{\bar{R}}$ for the Zariski topology. This implies that $\mathcal{Z}_{r_3} \cap \mathcal{Z}_{\bar{R}} \neq \emptyset$, which is false because a tensor cannot have two different ranks. Consequently, we have $\bar{\mathcal{Z}}_{r_3} \subsetneq \mathcal{T}$. □

Since the $\mathcal{Y}_r$'s are parameterized and thus are algebraic constructible sets [11], and since the closure of an algebraic constructible set for the Euclidean topology and the Zariski topology are the same, the above reasoning holds true for many other topologies on $\mathbb{R}$ or $\mathbb{C}$, and we have in particular:

Corollary 4. *Let μ be a measure on Borel subsets of $\mathcal{T}$ with respect to the Euclidean topology on $\mathcal{T}$. Let $\bar{R}$ be the generic rank in $\mathcal{T}$. If μ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{T}$, then*

$$\mu(\{\boldsymbol{T} \in \mathcal{T} \mid r(\boldsymbol{T}) \neq \bar{R}\}) = 0.$$

In particular, this corollary tells us that $\mathcal{Z}_{\bar{R}}$ is also dense in $\mathcal{T}$ with respect to the Euclidean topology, and holds if μ is the Gaussian measure on $\mathcal{T}$. In other words, the rank of a tensor whose entries are drawn randomly according to an absolutely continuous distribution (e.g. Gaussian) is $\bar{R}$ with probability 1

Other inequalities. A lower bound can be derived:

$$\bar{R}_s \geq \left\lceil \frac{\binom{K+d-1}{d}}{K} \right\rceil. \tag{6}$$

An upper bound has also been derived for real [12] or complex [13] tensors:

$$\bar{R}_s \leq \binom{K+d-2}{d-1}. \tag{7}$$

Because the space of symmetric tensors, $\mathcal{T}_s$, is included in the subspace of $\mathcal{T}$ of square tensors, maximal and generic ranks are related for every fixed order d and dimension K by:

$$\bar{R} \geq \bar{R}_s, \quad \text{and} \quad R \geq R_s \tag{8}$$

Note that (8) and (5) are in reverse order, but there is no contradiction: the spaces are not the same.

It is then legitimate to ask oneself whether the symmetric rank and the rank are the same. A partial answer to this question is provided by the two results below.

Theorem 5. *Let $\boldsymbol{A}$ be a complex symmetric tensor of dimension n and order k. If $r_s(\boldsymbol{A}) \leq n$, then $r_s(\boldsymbol{A}) = r(\boldsymbol{A})$ generically.*

Theorem 6. *Let $\boldsymbol{A}$ be a complex symmetric tensor of dimension n and order k. If $r_s(\boldsymbol{A}) = 1$ or 2, then $r_s(\boldsymbol{A}) = r(\boldsymbol{A})$.*

For reasons of space, the proof will be reported in a full-length version of this paper. A general proof showing that $r_s(\boldsymbol{A}) = r(\boldsymbol{A})$ generically, even for $r_s(\boldsymbol{A}) > n$, is being completed.

5. TOPOLOGY

These statements extend previous results [14], and prove that there can be only *one* subset $\mathcal{Z}_r$ of non empty interior, and the latter is dense in $\mathcal{T}$; this result needs however an algebraically closed field (e.g. the field $\mathbb{C}$ of complex numbers).

The results of section 4 are indeed not valid in the real field. In fact, the conjecture of Kruskal [17] according to which there could be several typical ranks for given order and dimensions, has been proved recently by Ten Berge [18]. See the example in section 6.

6. EXAMPLES

Lack of closeness. Let's give now a few examples. It has been shown [9, 15] that symmetric tensors of order 3 and dimension 3 have a generic rank $\bar{R}_s = 4$ but a maximal rank $R_s = 5$. This means that only $\mathcal{Z}_4$ is dense in $\bar{\mathcal{Y}}_4 = \bar{\mathcal{Y}}_5$, and $\mathcal{Z}_3$ and $\mathcal{Z}_5$ are not closed and of empty interior. On the other hand, $\mathcal{Z}_1$ is closed.

In order to make this statement even more explicit, let's now define a sequence of rank-2 tensors converging to a rank-3 one. This will be a simple proof of the lack of closure of $\mathcal{Y}_r$ for $r > 1$. For this purpose, let $\{\boldsymbol{x}_i, \boldsymbol{y}_i\}$ be linearly independent vectors. Then the following tensor is of rank 2 for any real $\varepsilon > 0$:

$$\boldsymbol{T}_\varepsilon = \boldsymbol{x}_1 \circ \boldsymbol{x}_2 \circ (\boldsymbol{x}_3 - \varepsilon^{-1}\boldsymbol{y}_3) + (\boldsymbol{x}_1 + \varepsilon\boldsymbol{y}_1) \circ (\boldsymbol{x}_2 + \varepsilon\boldsymbol{y}_2) \circ \varepsilon^{-1}\boldsymbol{y}_3.$$

But it converges towards a rank-3 tensor when ε tends to zero. This becomes clear by rewriting $\boldsymbol{T}_\varepsilon$ as:

$$\boldsymbol{T}_\varepsilon = [\boldsymbol{x}_1 \circ \boldsymbol{x}_2 \circ \boldsymbol{x}_3 + \boldsymbol{y}_1 \circ \boldsymbol{x}_2 \circ \boldsymbol{y}_3 + \boldsymbol{x}_1 \circ \boldsymbol{y}_2 \circ \boldsymbol{y}_3] \\ + \varepsilon \boldsymbol{y}_1 \circ \boldsymbol{y}_2 \circ \boldsymbol{y}_3.$$

CanD in the real field. Now for real tensors, if the CanD is sought in $\mathbb{R}$, the rank can be found to be larger than the value found in $\mathbb{C}$, as pointed out in [17]. In other words, we have actually the (rather natural) inequality, for any tensor $\boldsymbol{T}$:

$$r^{\mathbb{C}}(\boldsymbol{T}) \leq r^{\mathbb{R}}(\boldsymbol{T}). \tag{9}$$

Example 7. *In order to demonstrate that the equality does not always hold, define the square symmetric real tensor $\boldsymbol{T}$ of order 3 and dimension 2 as:*

$$\boldsymbol{T}(:,:,1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{T}(:,:,2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

If decomposed in $\mathbb{R}$, it is of rank 3:

$$\boldsymbol{T} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\circ 3} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\circ 3} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\circ 3}$$

whereas it admits a CanD *of rank 2 in $\mathbb{C}$:*

$$\boldsymbol{T} = \frac{\jmath}{2}\begin{bmatrix} -\jmath \\ 1 \end{bmatrix}^{\circ 3} - \frac{\jmath}{2}\begin{bmatrix} \jmath \\ 1 \end{bmatrix}^{\circ 3}, \text{ with } \jmath \overset{def}{=} \sqrt{-1}$$

These decompositions can be obtained with the help of the algorithm described in [9], for instance. Alternatively, this tensor is associated with the homogeneous polynomial in two variables $p(x, y) = 3xy^2 - x^3$, which can be decomposed in $\mathbb{R}$ into

$$p(x, y) = \frac{1}{2}(x + y)^3 + \frac{1}{2}(x - y)^3 - 2x^3.$$

In the case of $2 \times 2 \times 2$ symmetric tensors, or equivalently in the case of binary cubics, the CanD can always be computed [9]. Hence, the rank of any tensor can be calculated, even in the real field. In that case, it can be shown that the generic rank in $\mathbb{C}$ is 2 whereas *there are two typical ranks* in $\mathbb{R}$, which are 2 and 3. Kruskal already noticed that fact, with arrays with random Gaussian inputs.

In fact, in the $2 \times 2 \times 2$ case, there are two 2×2 matrix slices, that we can call $\boldsymbol{A}$ and $\boldsymbol{B}$. Since the generic rank in $\mathbb{C}$ is 2, the CanD is obtained via the EVD of the matrix pencil $(\boldsymbol{A}, \boldsymbol{B})$, which generically exists and whose eigenvalues are those of $\boldsymbol{A}\,\boldsymbol{B}^{-1}$. By generating (four) independent real Gaussian entries, it can be easily checked out with a simple computer simulation that one gets real eigenvalues in 52% of the cases. This means that the real symmetric rank is 3 in 48% of the remaining cases. For asymmetric tensors, the same simulation yields (by generating 8 independent real Gaussian entries) real ranks of 2 and 3, 78% and 22% of the time, respectively. This is the simplest example that can evidence the existence of typical ranks (and hence the lack of generic rank).

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