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NON-LINEAR INVERSION OF UNDERDETERMINED MIXTURES

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ABSTRACT

Static linear mixtures with fewer sensors than sources are considered. They are encountered for instance in downlink radio communications, if the spectral efficiency is attempted to be improved. The blind source extraction problem is addressed by forming virtual sensor measurements, in order to make it possible to invert linearly the observation model. Virtual measurements are a non linear function of actual measurements, and the choice of this non-linearity depends on the source distribution, assumed to be known and discrete. Two numerical algorithms are proposed, depending on the fact that the mixture is known (or beforehand identified) or not. Computer results are run with both BPSK and MSK sources, and compared to the ultimate separation performances.

1. INTRODUCTION

It is now admitted that the use of diversity based techniques will become necessary in future radio communication systems. In particular, a diversity either based on space or polarization will allow in the near future mobile receivers to take advantage of a diversity equivalent to 2 sensors. This will hopefully permit to improve on the spectral efficiency of downlink communications.

The problem of channel identification when there are fewer sensors than sources has been little addressed up to now. One can mention [17] [10] in the case of dynamic mixtures, and [1] [2] [8] [14] [7] [9] [4] in the case of static ones. The problem of source extraction, once the channel is known or not, has been also very little addressed when the mixture is static, whereas standard algorithms are widely used to achieve source extraction in the case of dynamic mixtures.

On the other hand, blind identification and inversion of static mixtures of discrete sources has been addressed in [11] [18] [16] [13] among others, but under the assumption that the number of sources does not exceed the number of sensors.

This paper focuses on the extraction of sources from static mixtures with fewer sensors than sources. Contrary to most blind identification algorithms, the source extraction itself requires additional assumptions on the source statistics [4].

Two algorithms are proposed: the first one assumes the mixture has been primarily identified, and the second performs source extraction directly. Both algorithms need the knowledge of the source distribution (*e.g.* BPSK or MSK) up to a constant multiple, and make use of polynomial functions of the observations. Performances are compared to the ultimate ones, recently investigated for static mixtures [5].

It is assumed throughout the paper that a $K-$dimensional random vector $\mathbf{y}$ is observed, and that it satisfies the following static linear model:

$$\mathbf{y} = A\,\mathbf{x} + \mathbf{w} \tag{1}$$

where $\mathbf{x}$ is a "source" vector of dimension P with independent components, A is an unknown mixing matrix, and $\mathbf{w}$ represents additive Gaussian noise. In order to simplify the algorithm presentation, we consider the case of $P = 3$ sources and $K = 2$ sensors.

Static models are valid when the channels are non dispersive (flat fading), or when source signals are narrow band (which means, in the case of spatial diversity, that their bandwidth is small compared to the wave celerity divided by the antenna spacing); in the latter case, the model takes the form (1) if expressed in the frequency domain.

The MAP estimate of the source vector is given by

$$(\hat{\mathbf{x}}, \hat{A})_{MAP} = \operatorname{Arg}\operatorname*{Max}_{\mathbf{x},A} p_{x|y,A}(\mathbf{x}, \mathbf{y}, A) \tag{2}$$

where $p_{x|y,A}(\mathbf{x}, \mathbf{y}, A) = p_x(\mathbf{x}) \cdot p_w(\mathbf{y} - A\,\mathbf{x})$. The MAP criterion leads to an exhaustive search for the mixture A, but restricts the search for $\mathbf{x}$ to the allowed constellation. Even for BPSK sources, the computational

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load is heavy. In addition, the technique used in [13] cannot be directly implemented here as there are fewer sensors than sources.

The idea proposed in this paper is to form virtual sensors by exploiting the knowledge of the constellation support of discrete signals. In fact, a discrete constellation in the complex plane can always be fully defined by a polynomial equation $P(z) = 0$.

2. ALGORITHMS FOR BPSK SOURCES

2.1. Case of known mixtures

If the mixture is known, the MAP solution can be obtained within a reasonable computational cost. However, a more economic solution is sought. In absence of noise, each observation takes the form:

$$y_i(n) = a_{i1}\, x_1(n) + a_{i2}\, x_2(n) + a_{i3}\, x_3(n)$$

Yet, since x_p are BPSK, we have that $x_p^2(n) = 1$. This allows to express any odd-degree monomial function of the 2 observations as:

$$y_1(n)^k y_2(n)^\ell = b_1(k,\ell)\, x_1(n) + b_2(k,\ell)\, x_2(n) + b_3(k,\ell)\, x_3(n) + b_4(k,\ell)\, x_1(n)x_2(n)x_3(n) \tag{3}$$

where the coefficients b_i are known functions of k, ℓ, and of the mixing matrix A. Everything happens as if an extraneous source term had been added, namely $x_1(n)x_2(n)x_3(n)$ that appears as a fourth BPSK source. This extraneous source can be seen to be uncorrelated with the 3 previous ones, since $\mathrm{E}\{x_1x_2x_3\, x_i\} = 0$, and $\mathrm{E}\{x_1x_2x_3\, x_i^*\} = 0$, $\forall i \in \{1,2,3\}$. But it is correlated at order 4, as will be seen in a subsequent section.

An interesting way to form a virtual sensor measurement consists of eliminating the extraneous source, for instance by computing:

$$y_3(n) = b_4(0,3)\, y_1(n)^3 - b_4(3,0)\, y_2(n)^3$$

This new virtual sensor measurement is indeed a linear combination of the $x_p(n)$'s, and the model becomes (generically) invertible.

Another way consists of using all free monomials of degree 3. This approach has been shown to be more robust [4], because the overdetermined linear system of 6 equations can be solved for the 4 unknowns in the Least Squares (LS) sense:

$$\bar{\mathbf{y}} = B \cdot \bar{\mathbf{x}} \tag{4}$$

where $\bar{\mathbf{y}} = \begin{pmatrix} \mathbf{y} \\ y_1^3 \\ y_1^2\, y_2 \\ y_1\, y_2^2 \\ y_2^3 \end{pmatrix}$ and $\bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ x_1x_2x_3 \end{pmatrix}$

and where B is a given function of A.

2.2. Case of unknown mixtures

If the mixture is unknown, the principle is the same, except that the extraneous sources cannot be eliminated anymore, because coefficients b_i are not known. But as previously, we have $P = 4$ BPSK sources (including one virtual) in the mixture.

Two options are possible. One can either run a deflation procedure based on the AMiSRoF algorithm described in [13], that does not require more than second order decorrelation between sources. But there is also another more attractive option.

Let $x_4 = x_1x_2x_3$; then, x_1, x_2, x_3 being real independent BPSK sources, so is x_4. But x_4 is obviously not independent of the former sources. One can even stress that $\mathrm{Cum}\{x_1, x_2, x_3, x_4\} = 1$. Nevertheless, it can be shown that $\mathrm{Cum}\{x_i, x_i, x_j, x_j\} = 0$, $\mathrm{Cum}\{x_i, x_i, x_i, x_j\} = 0$, $\mathrm{Cum}\{x_i, x_i, x_i, x_i\} = -2$, $\forall i, j \in \{1,2,3,4\}$. This shows that all pairwise source cross cumulants vanish, which is sufficient for applying the ICA algorithm proposed in [3] [6].

In order to discriminate between the actual sources and the extraneous ones, one can compute the correlation between estimated sources and observations $\mathbf{y}(n)$. The 3 actual sources will indeed have the largest correlation (because of the linear link and because of second order decorrelation among sources). See section 4 for computational details.

2.3. Identifiability

As any monomial function of y_1 and y_2 is a linear combination of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_1(n)x_2(n)x_3(n)$, one could think that it is possible to create as many virtual sensors as we want. But it turns out that the number of useful virtual sensors is bounded. In fact, the virtual measurements should be linearly independent of each other, in order for the augmented mixing matrix to be full rank. This identifiability issue will be addressed more deeply in a forthcoming paper. But it is clear that the submatrix of $\bar{G}$

$$\begin{pmatrix} a & b & c & 0 \\ e & f & g & 0 \\ a^3{+}3ab^2{+}3ac^2 & 3a^2b{+}b^3{+}3bc^2 & 3a^2c{+}3b^2c{+}c^3 & 6abc \\ e^3{+}3ef^2{+}3eg^2 & 3e^2f{+}f^3{+}3fg^2 & 3e^2g{+}3f^2g{+}g^3 & 6efg \end{pmatrix}$$

is already generically of rank 4, because the set of matrices A, such that the two last rows of the above submatrix are proportional, is of null measure.

3. MSK AND QPSK SOURCES

In many Radio communication systems, in particular in the current GSM and the future UMTS standards, the modulation utilized is GMSK and QPSK, respectively. It is therefore relevant to extend our procedure to these kinds of modulation.

It is well known that the Gaussian Minimum Shift Keying (GMSK) modulation, utilized in the GSM standard, can be approximated by a Minimum Shift Keying (MSK) modulation [15], which in turn can be viewed as a QPSK modulation with transition constraints (see [12] and references therein). This motivates the study of the MSK source separation problem.

3.1. Known mixture of MSK sources

Similarly to the case of BPSK sources, one can show that any odd-degree monomial function of the 2 observations satisfies:

$$(-1)^n\, y_1(n)^k y_2(n)^\ell = b_1(k,\ell)\, x_1(n) + b_2(k,\ell)\, x_2(n) + b_3(k,\ell)\, x_3(n) + b_4(k,\ell)\, (-1)^n\, x_1(n)x_2(n)x_3(n) \quad (5)$$

where the coefficients b_i are known functions of k, ℓ, and A.

As in the BPSK case, it can be shown that the extraneous term $(-1)^n\, x_1(n)x_2(n)x_3(n)$ is an MSK source, uncorrelated with the actual ones. The sources can then be recovered along the same lines as for BPSK sources.

3.2. Unknown mixture of MSK sources

MSK sources can be split into two *independent* BPSK sources: MSK signals are alternatively real and imaginary (up to some fixed complex phase). To simplify the notation, one can thus rearrange the order of samples and write the observation model as:

$$[\mathbf{y}_{odd} \quad -\jmath\, \mathbf{y}_{even}] = A \cdot [\mathbf{x}_{odd} \ \ \mathbf{x}_{even}] \qquad (6)$$

where $\mathbf{x}_{odd}$ and $\mathbf{x}_{even}$ are real BPSK sources. Now with this writing, 3 BPSK sources remain to be found, the first (resp. second) half corresponding to the odd (resp. even) samples of MSK sources. The ICA algorithm working with pairwise cumulants [3] [6] will consequently successfully separate those sources.

3.3. Extraction of QAM4 sources

The case of QPSK sources is more complicated, but we sketch the procedure in this section. In fact, a QAM4 source is a sum of two independent BPSK sources in quadrature:

$$x_p = \varepsilon_p + \jmath\, \varepsilon'_p$$

Hence in the case of QAM4 sources, equation (1) can be rewritten as:

$$\begin{pmatrix} \mathcal{R}e[\mathbf{y}] \\ \mathcal{I}m[\mathbf{y}] \end{pmatrix} = \begin{pmatrix} \mathcal{R}e[A] & -\mathcal{I}m[A] \\ \mathcal{I}m[A] & \mathcal{I}m[A] \end{pmatrix} \begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix}$$

As a consequence, 6 independent BPSK inputs need now to be recovered from 4 sensor measurements.

Let us state a general result now. If we have K sensor measurements, and P sources, satisfying $x_i^d = x_i$. Then raising the measurements to the dth power yields $\binom{K+d-1}{d}$ additional equations, but introduces $\binom{P+d-1}{d}$ distinct source monomials, among which P are of the form $x_i^d = x_i$ and $P(P-1)$ of the form $x_i^{d-1}x_j = x_j$. Thus, the augmented linear system can be solved only if the necessary condition below is satisfied:

$$K + \begin{pmatrix} K+d-1 \\ d \end{pmatrix} \geq \begin{pmatrix} P+d-1 \\ d \end{pmatrix} - P(P-1)$$

It can be shown that if $P > K$, this inequality is satisfied only for $d \leq 3$. One can also prove that for $d = 3$, the inequality holds true only for $P \leq K+1$.

This result shows that with 6 BPSK sources, the approach followed in the previous sections cannot apply with 4 sensor measurements: by raising to the third power, one introduces 20 equations, but also 20 new unknowns. On the other hand, it works if we raise the sensor measurements to the fifth power. In fact, in that case, one gets $\binom{8}{5} = 56$ equations and introduces only 26 new unknowns (20 of the form $\varepsilon_1\varepsilon_2\varepsilon_3$ and 6 of the form $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5$), among the $\binom{10}{5} = 252$ source monomials, because $\varepsilon_p^4 = \varepsilon_p^2 = 1$.

The problem is generically solvable in the LS sense (under the rank conditions guaranteeing identifiability). The process of choosing the interesting 6 BPSK sources among the 32 extracted ones remains the same as in the previous cases. However, the association of two BPSK sources to form an actual QPSK source is somewhat more complicated, but can be performed by taking advantage of the mixture structure, as will be explained in a companion paper.

4. PERFORMANCES OF SOURCE SEPARATION

4.1. Extraction algorithm

For the three source modulations that we have studied, it was possible to solve an equivalent problem (of larger dimension) where BPSK sources needed to be extracted. Among these sources, the actual ones are mutually independent, whereas the extraneous ones introduced by monomial transforms are not. But the latter are uncorrelated at order 2, and still have null pairwise

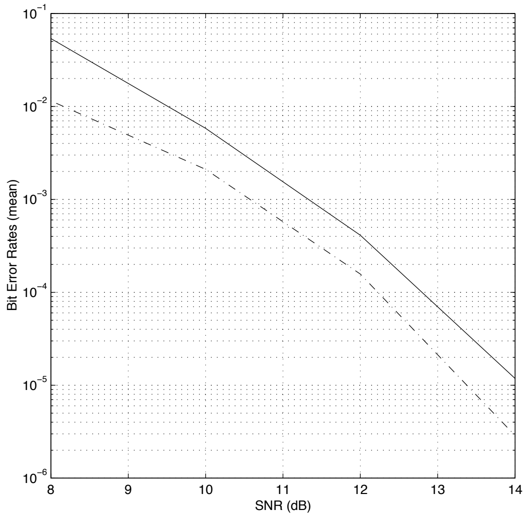

Figure 1: Bit Error rates (BER) for the separation of 3 BPSK sources from 2 sensors.

cross-cumulants of order four, allowing the computation of an Independent Component Analysis (ICA) via the algorithm proposed in [6].[2]

4.2. BPSK

The most favorable mixtures have been chosen to run the computer experiments; in fact, this is the only means to obtain mixture independent performances, comparable to the ultimate ones [5]. The mixture has the following structure :

$$A = \begin{pmatrix} a_1 & ia_2 & 0 \\ 0 & ia_3 & a_4 \end{pmatrix}$$

where the values of a_i are given by table 1. Computer experiments have been run for data length $N = 1000$, and error rates have been estimated over 5000 snapshots. The results reported in figure 1 show good performances, compared to the ultimate BER performances plotted in dashed line.

4.3. MSK

Since MSK sources can be assumed to be alternatively imaginary and real, and can be split into two independent BPSK sources, it is quite clear that the most favorable mixtures are the same as in the case of BPSK sources :

$$A = \begin{pmatrix} a_1 & ia_2 & 0 \\ 0 & ia_3 & a_4 \end{pmatrix}$$

[2]The matlab code can be found on the URL `www.i3s.unice.fr/~comon`.

SNR	a_1	a_2	a_3	a_4
10	0.900	0.4254	0.4254	0.905
12	0.900	0.4254	0.4146	0.910
14	0.900	0.4254	0.4146	0.910
16	0.910	0.4146	0.4146	0.910
18	0.910	0.4146	0.4146	0.910
20	0.830	0.5578	0.5578	0.830

Table 1: Best mixture for 2 sensors and 3 BPSK inputs for various SNRs

Again, the data length is $N = 1000$, and BER's are averaged over 5000 trials. The results presented in figure 2 demonstrate a good behavior of our algorithm, only 2dB above the ultimate bound given in the previous section.

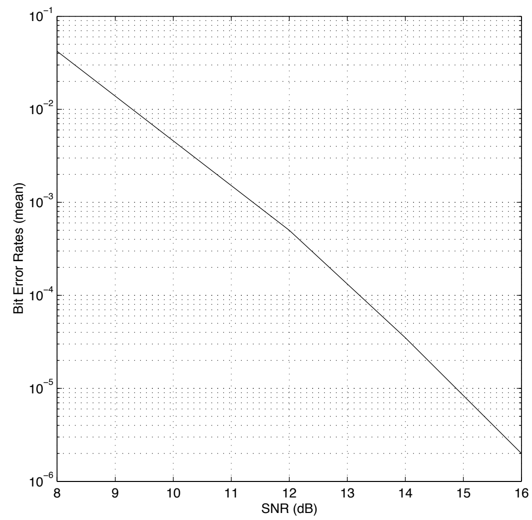

Figure 2: Bit Error Rates (BER) for the separation of 3 MSK sources from 2 sensors.

5. CONCLUSION

The source separation in the case of static mixtures with fewer sensors than sources has been addressed. In order to restore linear invertibility, the algorithms proposed are based on the construction of virtual sensor measurements, that are polynomial functions of the actual sensor records. These algorithms are *closed form*, and exhibit promising performances compared to ultimate bounds for both BPSK and MSK sources. The procedure is also applicable for QAM4 sources, though not as simple. Computer results need to be deepened in the QAM4 case, and have not been reported in this paper.

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