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# BLOCK METHODS FOR CHANNEL IDENTIFICATION AND SOURCE SEPARATION

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## ABSTRACT

On-line adaptive algorithms are not effective enough in tracking fast varying channels in blind and semi-blind modes. Block methods, long disregarded because of their numerical complexity, now hold appeal. In fact, the performance limitation now comes either from channel non-stationarity or from the absence of long training sequences, or both, but essentially no longer from the computational load. In addition, block methods are well adapted to burst-mode transmissions (TDMA systems).

Some block algorithms are presented in this paper, aiming at identifying communication channels blindly. These algorithms provide analytical solutions (i.e. non iterative) in order to be free of convergence problems. They exploit either the knowledge of the input alphabet, or the constant modulus property. The tools involved in the blind Source Separation (flat fading channels) and Channel Identification (frequency selective channels) problems are reviewed. The under-determined case (number of users exceeding the available diversity) is of particular interest.

This assessment is not a survey of existing techniques, but a partial review of some original recent algorithms, often leading to closed-form solutions, and applicable in the presence of limited diversity.

## 1. INTRODUCTION

In order to increase the performance of telecommunication systems, one tries to take advantage of the *diversity*, with the help of *smart arrays*. In practice, this consists of building several distinct observations, say $K$, either by using actual spatially distant sensors (spatial diversity), or by exploiting other forms of diversity induced by the polarization or by the excess bandwidth.

Thus, after a preprocessing that is beyond the scope of this report, we have at our disposal a $K-$dimensional observation vector, $\mathbf{y}(n)$, for every unit of time $n$, satisfying the model below:

$$\mathbf{y}(n) = \sum_{j=0}^{L} \mathbf{H}(j)\mathbf{x}(n-j) + \mathbf{v}(n), \tag{1}$$

where $\mathbf{x}(n)$ the $P-$dimensional random vector containing the sources (users), assumed to be mutually statistically independent, $\mathbf{H}(j)$ the $K \times P$ matrix taps of the channel impulse response, and $\mathbf{v}(n)$ an additive noise independent from $\mathbf{x}(n)$, standing for background noise and modeling errors.

When the spatio-temporal array response $\mathbf{H}(n)$ is not known, or imprecisely known (loss of calibration), then it needs to be identified before the source vector process $\mathbf{x}(n)$ can be reliably estimated. Note that there are direct equalization algorithms, for instance prediction-based [1] or subspace-based [34], which permit the estimation of the sources without prior identification. The former are very sensitive to a misdetection of the channel order, whereas the latter are less so; see [10] for Cramér-Rao bounds. But the point is that the performance is acceptable only if the diversity, $K$, is sufficiently large compared to the number of sources, $P$. In contrast, it will be assumed throughout that $K \leq P$.

This paper focuses on analytical solutions, that is, algorithms able to provide a closed-form expression of $\mathbf{H}$. Block methods indeed become more and more attractive since computer power no longer appears to be an impediment. On the other hand, adaptive algorithms also benefit from of a block update, by gaining in convergence speed and complexity [29].

Some algorithms are based on second-order (circular or non-circular) statistics only. Others resort to High-Order Statistics (HOS), and perform tensor decompositions. Among the advantages of HOS-based algorithms, is their ability to identify under-determined mixtures (i.e. $K < P$). This will be discussed in sections 2.4 and 3.3.

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Devising algorithms of channel identification depends on the availability of a training sequence. In other words, is part of the source process $\mathbf{x}(n)$ known or not? If so, channel identification can be carried out through semi-blind techniques. In the context of telecommunications, this leads to increased performances in symbol error rate [16] [24], but decreases the throughput. Determining the best compromise remains an open question. This paper, however, is restricted to discussion of blind approaches.

## 2. BLIND SOURCE SEPARATION

Of particular interest are flat fading channels. In fact, the search for an analytical solution seems less complicated at first glance, since model (1) becomes static and reduces to:

$$\mathbf{y}(n) = \mathbf{H}\,\mathbf{x}(n) + \mathbf{v}(n) \tag{2}$$

where $\mathbf{H}$ is a $K \times P$ matrix. The classic blind source separation problem [11] does not assume spectral diversity, so that in model (2), time $n$ can be considered as a mere realization index. The goal is to identify the channel matrix $\mathbf{H}$, and possibly extract the sources $\mathbf{x}(n)$, from the sole observation of $\mathbf{y}(n)$.

This problem is known to have a unique solution (up to scale and permutation factors) provided that at most one source is circular Gaussian, and $K \geq P$ [11]. However, closed-form solutions *stricto sensu* have been found only for $P = 2$, based on the use of fourth-order cumulants, even if efficient iterative algorithms are now available for $P > 2$ [28] [7] [11] [6]. Nevertheless, there could be analytical solutions, which may require minor additional assumptions, as will be subsequently discussed.

### 2.1. Notation

**Kronecker products.** Given a $d-$dimensional vector $\mathbf{u}$, one defines the $d^2-$dimensional vector $\mathbf{u}^{\otimes} = \mathbf{u} \otimes \mathbf{u}^*$, as the vector containing all the products between components of $\mathbf{u}$ and its complex conjugate. One also defines the vector $\mathbf{u}^{\oslash q}$ of size $\binom{d}{q}$, composed of all distinct cross products of degree $q$, appropriately weighted; the weight of every monomial is assigned the value of the square root of the number of ways to build it. As a consequence, the Hermitian norm is preserved: $||\otimes_{i=1}^{q}\mathbf{u}|| = ||\mathbf{u}^{\oslash q}||$. For instance, $\mathbf{u}^{\oslash 2}$ has $d(d+1)/2$ entries: $d$ squares and $d(d-1)/2$ cross products weighted by $\sqrt{2}$.

Next, given a $p \times r$ matrix $\mathbf{U}$, define the vector $\mathbf{vec}\{\mathbf{U}\}$ of size $pr$, obtained by stacking the columns of $\mathbf{U}$ one below the other. For instance, we have $\mathbf{vec}\{\mathbf{u}\,\mathbf{u}^{\mathrm{H}}\} = \mathbf{u}^{\otimes}$. Conversely, define the inverse operator $\mathbf{Unvec}\{\}$ such that $\mathbf{Unvec}\{\mathbf{vec}\{\mathbf{U}\}\} = \mathbf{U}$.

**Tensors.** A tensor of order $q$ is a table with $q$ indices, enjoying the multilinearity property [33], as will be precised in section 2.4. It is symmetric if the order of indices does not matter. For example, matrices are tensors of order 2.

**Symmetric arrays.** Symmetric matrices of size $p \times p$ have $p(p+1)/2$ degrees of freedom. It is thus necessary to store only $p(p+1)/2$ entries; therefore, define the operators $\mathbf{vecs}\{\}$ and $\mathbf{Unvecs}\{\}$ accordingly. We have, for instance, $\mathbf{vecs}\{\mathbf{u}\,\mathbf{u}^{\mathrm{T}}\} = \mathbf{u}^{\oslash}$. More generally, a symmetric tensor $\mathbf{T}$ of order $q$ and dimension $p$ can be stored in a vector of size $\binom{p}{q}$; let $\mathbf{vecs}\{\mathbf{T}\}$ denote this vector, and $\mathbf{Unvecs}_q\{\cdot\}$ the corresponding inverse operator. With these notations, $\mathbf{Unvecs}_q\{\mathbf{u}^{\oslash q}\}$ is a tensor of rank 1.

**Definitions of rank.** A $q-$th order tensor has a rank equal to 1 simply if it is the outer product of $q$ vectors. We define the *array rank* [30] of a symmetric tensor $\mathbf{T}$ (or just *rank* in short) as the minimal number of rank-one tensors necessary to add in order to yield $\mathbf{T}$. Note that other terminologies such as *tensor rank* [17] or *polynomial width* [36] are also encountered.

Other definitions of rank have been proposed [30, 2, 20], and are related to matrix ranks, but are not relevant in our framework.

### 2.2. Constant modulus sources

Van der Veen was the first to propose an analytical algorithm to separate CM sources [39]. The principle of his algorithm is best understood through the Noiseless Case. Assume the observations have been spatially prewhitened, so that the data matrix is of size $P \times N$ and full rank.

One attempts to extract each source with a linear filter $\mathbf{f}$ as: $s(n) = \mathbf{f}^{\mathrm{T}}\mathbf{y}(n)$. Thus the constraint of unit modulus can be written as:

$$\mathbf{f}^{\mathrm{T}}\mathbf{y}(n)\mathbf{y}(n)^{\mathrm{H}}\mathbf{f} = 1, \forall n. \tag{3}$$

With the notation introduced in section 2.1, this yields $\mathbf{f}^{\otimes\mathrm{T}}\mathbf{y}(n)^{\otimes} = 1$, for all time samples $n$, $1 \leq n \leq N$. Stacking all these equations one above the other, we get a $N \times P$ formally linear system: $\mathbf{Y}^{\otimes}\mathbf{f}^{\otimes} = \mathbf{1}$. It can be shown that, in the absence of noise, and if $N \geq P^2$, the null space of $\mathbf{Y}^{\otimes}$ is of dimension $P - 1$. Among the infinite number of solutions to this system, only $P$ have the structure requested by vectors of the form $\mathbf{f}^{\otimes}$. In order to calculate them, simply construct the null space equation $\mathbf{N}\mathbf{f}^{\otimes} = \mathbf{0}$, where $\mathbf{N}$ is a given $N-1 \times P^2$ matrix. Let $\{\mathbf{u}^{(1)}, \ldots \mathbf{u}^{(P)}\}$ be a basis of $\ker\{\mathbf{N}\}$. Then any of the $P$ extractors $\mathbf{f}_j^{\otimes}$ is a linear combination of the $\mathbf{u}^{(p)}$'s, and vice-versa. Rearranging the system of equations leads eventually to $\mathbf{u}^{(p)} = \sum_{j=1}^{P} \lambda_j^{(p)} \mathbf{f}_j^{\otimes}$, or, in closed form, by applying the operator $\mathbf{Unvec}\{\cdot\}$ :

$$\mathbf{U}^{(p)} = \mathbf{F}\Lambda^{(p)}\mathbf{F}^{\mathrm{H}} \tag{4}$$

In conclusion, the problem amounts to jointly diagonalize $P$ matrices, which recalls the Jade algorithm proposed in [7]; the difference here is that $\mathbf{F}$ is not constrained to be unitary. The numerical algorithm proposed in [39] is iterative, and is inspired from the QZ iteration usually dedicated to the diagonalization of matrix pencils. Another analytical solution to this type of problem, applicable to CM sources, has been introduced in [26] for tensors, and is described in the next section.

### 2.3. Discrete sources

In [38], the above algorithm has been specialized to real data, that is, to BPSK sources. This algorithm cannot handle complex constellations. Here, we shall examine a related procedure for $q-$PSK modulations, where the support of the distribution reduces to $q$ masses, $c_j$.

In the present case, we impose the constraint $[\mathbf{f}^{\mathrm{T}}\mathbf{y}(n)]^q = 1$, so that:

$$\mathbf{y}(n)^{\oslash q \mathrm{T}}\mathbf{f}^{\oslash q} = 1. \tag{5}$$

For $q = 2$, this is still different from equation (3) because of the absence of complex conjugates, and because of the smaller vector dimension.

By stacking equations (5) for $1 \leq n \leq N$, one obtains a formally linear system $\mathbf{Y}^{\oslash q}\mathbf{f}^{\oslash q} = \mathbf{1}$. Matrix $\mathbf{Y}^{\oslash q}$ is generically of rank $P-1$ if $P$ sources are present [26]; there are thus again infinitely many solutions. Denote $\mathbf{f}_{min}^{\oslash q}$ the minimum norm solution, and $\{\mathbf{u}^{(p)}\}$ a basis of the null space. Then extractors take the form: $\mathbf{f}^{\oslash q} = \mathbf{f}_{min}^{\oslash q} + \sum_{p=1}^{P-1} \lambda_p \mathbf{u}^{(p)}$. The coefficients $\lambda_p$ are determined by imposing the structure on $\mathbf{f}^{\oslash q}$. Unlike the algorithm described in section 2.2, here one uses the property that $\mathbf{Unvecs}_q\{\mathbf{f}^{\oslash q}\}$ must have a rank 1. An analytical solution is proposed in [26].

In the presence of noise, one can attempt to satisfy equations (5) in the LS sense. Actually, since source distributions are known, a Bayesian approach is possible. With this goal, it has been proved in [25] that the Maximum A Posteriori (MAP) criterion is asymptotically equivalent to the Mean Square Error (MSE) $\sum_n \prod_j |\mathbf{f}^{\mathrm{T}}\mathbf{y}(n) - c_j|^2$, which is itself equal to $\sum_n |\mathbf{f}^{\oslash q \mathrm{T}}\mathbf{y}(n)^{\oslash q} - 1|^2$.

### 2.4. Non circular sources, under-determined mixtures

Mixtures in which the number of sources, $P$ exceeds the available diversity, $K$, are referred to as under-determined; some authors also refer to over-complete representations [32].

**Why resorting to tensors?** Reasons for using higher order arrays are usually three-fold. First, the sole use of second order statistics is often seen as insufficient; the orthogonality constraint between the columns of $\mathbf{H}$ may indeed not be in accordance with the actual structure of the data. Second, data are often arranged in many-way arrays, for instance three-way [17] [3]; the reduction to a 2-way array represents a loss of information. Third, the number of factors that can be identified is limited to the rank of the data matrix, itself bound by the smallest dimension. Yet, there may very well be more factors than the smallest dimension. Clearly, as we have seen in the previous sections, the first two arguments do not always hold. On the contrary, the latter alone suffices to justify that we resort to high order tensors [17] [12].

**Cumulants.** Define the following fourth order cumulant tensors of observations:

$$\mathbf{G}_{ijk\ell} = \mathrm{Cum}\{y_i, y_j, y_k, y_\ell\}$$
$$\tilde{\mathbf{G}}_{ijk\ell} = \mathrm{Cum}\{y_i, y_j, {y_k}^*, {y_\ell}^*\}.$$

Then, taking the fourth order cumulants of both sides of (2) leads to the multilinear relations, up to a noise term:

$$\mathbf{G}_{ijk\ell} = \sum_{p=1}^{P} \gamma(p)\, H_{ip}\, H_{jp}\, H_{kp}\, H_{\ell p} \tag{6}$$

$$\tilde{\mathbf{G}}_{ijk\ell} = \sum_{p=1}^{P} \tilde{\gamma}(p)\, H_{ip}\, H_{jp}\, {H_{kp}}^*\, {H_{\ell p}}^*, \tag{7}$$

where $\gamma(p) = \mathrm{Cum}\{x_p, x_p, x_p, x_p\}$ and $\tilde{\gamma}(p) = \mathrm{Cum}\{x_p, x_p, {x_p}^*, {x_p}^*\}$. Clearly, equations (6) and (7) show that the observation cumulant tensors are a sum of $P$ rank-one tensors, whose latent vector corresponds to a column of the mixing matrix, $\mathbf{H}$.

Formally, this results recalls the eigen-decomposition of symmetric and Hermitian matrices. But the key difference lies in the fact that the rank of a tensor can exceed its dimension [12], as will be soon discussed.

**State of the art.** Bergman [2] has been probably the first to notice that the concept of *rank* is difficult to extend from matrices to higher order arrays. Caroll [8] provided the first *canonical decomposition* algorithm of a three-way array, later referred to as CANDECOMP model. Kruskal [30] conducted several years later a detailed analysis of uniqueness, and relates several definitions of rank. The algorithm CANDELINC [9] was devised by Caroll and others in the eighties; it allows to compute a canonical decomposition subject to a priori linear constraints. Leurgans [31] derived sufficient identifiability conditions for the 3-way array decomposition; as opposed to Kruskal, his proof is constructive and yields a numerical algorithm running several matrix SVD's. A solid account on decompositions of 3-way arrays can also be found in DeLathauwer's PhD thesis [17].

As shown in (8), symmetric arrays of order $q$ and dimension $K$ can be associated bijectively to homogeneous polynomials of degree $q$ in $K$ variables. Based on this remark, decomposing a symmetric array is equivalent to decomposing a homogeneous polynomial into a sum of linear forms raised to the $q-$th power [15]. The advantage, compared to CANDECOMP, is that the symmetry is not broken. As a consequence, the problem can then be connected to early works in invariant theory [35] and multilinear algebra [22]. The first results go back to the beginning of the century with the works of Sylvester and Wakeford [41]. Also related are the works of Rota [23] on binary quantics, and those of Reznick on quantics of even degree [36].

**Quantics and identifiability.** The first results on identifiability stated that, in a neighborhood of a solution, imposing both Hermitian symmetry and rank-one when decomposing $\tilde{\mathbf{G}}$ ensure local identifiability [5]. This led to an iterative algorithm, consisting of alternate projections on both sets [5]. Other iterative algorithms have been proposed to identify the dominant rank-one tensor, for instance based on the power method principle [19] [17], or rank-one slices [26].

Recent results in identifiability are more general and accurate [12]. The set of symmetric arrays of order $q$ and dimension $K$ is a vector space of dimension $\binom{K+q-1}{q}$. Now, every symmetric array $G$ of order $q$ and dimension $K$ can be associated with a homogeneous polynomial $p$ of degree $q$ in $K$ variables as follows:

$$p(x_1, \cdots x_K) = \sum_{k_1..k_q=1}^{K} G_{k_1..k_q}\, x_{k_1} \cdots x_{k_q} \qquad (8)$$

A number of results are already known, especially concerning the *rank* of tensors (or the *width* of quantics [36]). More precisely, for $2 \leq K \leq 8$, it has been shown that the generic value of the rank $r$ is given by the following table [15]:

| $r$ | $K$ | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| $q$ | **3** | 2 | 4 | 5 | 8 | 10 | 12 | 15 |
| | **4** | 3 | 6 | 10 | 15 | 22 | 30 | 42 |

The number of free parameters that remain to be fixed in order to ensure uniqueness (in the sense that a finite set of equivalent solutions can be obtained) is given by the dimension of the manifold of solutions [15]:

| | $K$ | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| $q$ | **3** | 0 | 2 | 0 | 5 | 4 | 0 | 0 |
| | **4** | 1 | 3 | 5 | 5 | 6 | 0 | 6 |

One can check out, for instance, that for $K = 2$, we have indeed 1 free parameter to fix when decomposing 4-way arrays, whereas there are a *finite number* of solutions in the case of 3-way arrays. This would encourage us to use third order tensors. Unfortunately, cumulants of order 3 are often null or very ill-conditioned. The difficulty in using fourth order cumulant tensors can be already seen to be the lack of uniqueness of the decomposition. The idea developed by Comon in the next section is to use jointly $\mathbf{G}$ and $\tilde{\mathbf{G}}$, which both have non unique decompositions, but whose manifolds of solutions intersect in a finite set of points.

**An identification algorithm.** When $K = 2$ and $P = 3$, Comon proposed an identification algorithm based on Sylvester's theorem [12], to apply to the polynomial associated with tensor $\mathbf{G}$ of order $q = 4$:

A binary quantic $p(x, y) = \sum_{i=0}^{q} \gamma_i\, c(i)\, x^i\, y^{q-i}$ can be written as a sum of $q$th powers of $r$ distinct linear forms:

$$p(x, y) = \sum_{j=1}^{r} \lambda_j\, (\alpha_j\, x + \beta_j\, y)^q,$$

if and only if (i) there exists a vector $\mathbf{g}$ of dimension $r + 1$, with components $g_\ell$, such that

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r \\ \vdots & & & \vdots \\ \gamma_{d-r} & \cdots & \gamma_{d-1} & \gamma_q \end{bmatrix} \mathbf{g} = \mathbf{0}. \qquad (9)$$

and (ii) the polynomial $q(x, y) \stackrel{\text{def}}{=} \sum_{\ell=0}^{r} g_\ell\, x^\ell\, y^{r-\ell}$ admits $r$ distinct roots.

Sylvester's theorem is not only proving the existence of the $r$ forms, but also gives a means to calculate them. For odd values of $q$, we have thus a generic rank of $r = \frac{q+1}{2}$, whereas for even values of $q$, $r = \frac{q}{2}+1$, generically. It is then clear that when $q$ is even, there are at least two vectors satisfying (9), because the Hankel matrix is of size $\frac{q}{2} \times (\frac{q}{2} + 2)$. To be more concrete, take as example $q = 4$. The first Hankel matrix having more columns than rows is of size $2 \times 4$, and obviously has generically 2 vectors in its null space.

As a conclusion, when $q = 4$, there are generically two vectors $\mathbf{g}$ satisfying (9). This means that there are infinitely many decompositions of $\mathbf{G}$, all lying on a manifold of dimension 1 in the complex space. As said above, the idea proposed in [12] is to use another array, namely $\tilde{\mathbf{G}}$, to fix this indeterminacy.

DeLathauwer and Comon proposed in [18] a significant improvement of the above algorithm, decreasing its numerical complexity. Identification algorithms with $2 < K < P$, or $3 < P$ and $K < P$, still remain to be devised.

**Source extraction.** As far as source extraction is concerned, Cao and Liu [4] derived in detail the separability conditions, and the related underlying statistical results. Comon and Grellier proposed an analytical extraction algorithm when sources are discrete [13].

## 3. CHANNEL IDENTIFICATION

### 3.1. Discrete sources

A large number of algorithms can be found in the literature to solve the blind identification problem. However, most of them are iterative, *e.g.* [40]. For instance, Yellin and Porat proposed a non iterative algorithm, which is fully deterministic [42]. In their approach, it is assumed that during the observation time, the source process $x(n)$ takes several times the same value on whole blocks of length $L$. This allows the establishment of simple elimination procedures. One of the main difficulties is to test (and find) such blocks in the measured series, $y(n)$.

### 3.2. Non circular sources

Another alternative is to exploit the non-circular source statistics, which are non-zero for discrete distributions. More precisely, if $x$ is $q-$PSK, then it is circular at order $q-1$, which means that its moments up to order $q-1$ are zero, but $\mathrm{E}\{x^q\} \neq 0$.

This property has been used in [14] to blindly identify BPSK and MSK channels, with the help of second-order non-circular moments. This results in a very particular polynomial system of degree 2 in several variables, which is solved entirely analytically. The semi-blind context can also be approached in this way [24].

### 3.3. Under-determined mixtures

In [37], Tong showed that if the columns of $\mathbf{H}(0)$ are not pairwise collinear, and if the matrix $[\mathbf{H}(0)^{\mathrm{T}}, \cdots, \mathbf{H}(L)^{\mathrm{T}}]^{\mathrm{T}}$ is full rank, then model (1) can be identified with the help of output fourth-order cumulants. This result is very significant, because it does not impose that $K \geq P$, which opens the door to the discovery of new algorithms. Efficient numerical algorithms have apparently not yet been published, although the proof of [37] was constructive.

## 4. CONCLUDING REMARKS

In the case of static mixtures, there are basically two techniques. One leads to a joint approximate diagonalization of matrices, and the other to the decomposition of a tensor into a sum of rank-one tensors. The two techniques are usable and equivalent when the number of sources, $P$ does not exceed the diversity, $K$. But when $K < P$, only the latter technique is valid, but also becomes much more difficult to implement.

Deterministic approaches seem very attractive for discrete sources, especially for $q-$PSK sources. A combination of tensor techniques and deterministic approaches can be envisaged.

For static as well as dynamic under-determined mixtures, the identification problem appears now solvable, but the source extraction (equalization) remains much more difficult, even if solved in some particular cases.

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