NORMALITY TESTS FOR COLOURED SAMPLES

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ABSTRACT

The complementary use of High-Order Statistics (HOS) is useful if the data are not Gaussian. Yet, most normality tests available in the literature are applicable only for independent and identically distributed (i.i.d.) samples, so that they are rarely of practical value. In this paper, bias and variance of Mardia's multivariate kurtosis are calculated in the case of correlated (coloured) samples, based on expansions up to the second order, when the sample size is large. This allows to obtain a more accurate test statistic for large samples in realistic situations.

1. INTRODUCTION

The advantage in resorting to high-order statistics in various applications is now established in various domains (*e.g.* telecommunications), because data are clearly far from Gaussian. However the conclusions are less obvious in some applications, such as sonar. The problem is to know whether it is worthwhile to invest efforts in developing special purpose HOS-based algorithms, when many more standard ones give disappointing results. It is thus important to detect what are the situations where the data are sufficiently far from Gaussian. This question is not obvious, since we are dealing with wide-band random processes.

The (univariate) kurtosis is a standard measure of gap to normality, and takes the form for a **zero-mean** random variable, x:

$$\mathcal{B}_1 = K/S^2,\ K \stackrel{\text{def}}{=} E\{x^4\},\ S \stackrel{\text{def}}{=} E^2\{x^2\}. \qquad (1)$$

In practice, if a finite sample $\{x_1, .., x_N\}$ is given, the moments can be replaced by their sample counterparts:

$$\hat{K} = \frac{1}{N}\sum_{n=1}^{N} x(n)^4, \hat{S} = \frac{1}{N}\sum_{n=1}^{N} x(n)^2, \qquad (2)$$

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defining a sample kurtosis, $\hat{\mathcal{B}}_1(N)$. Under the null hypothesis (normal x(n)'s), this estimated kurtosis is asymptotically normal, with mean 3 and variance $24/N$, if samples are i.i.d. It is then easy to devise a test with an arbitrary power [17]. But using such a test has two limitations:

L1 If samples $x(n)$ are not i.i.d, the variance of the test variable can be underestimated. Consequently the test is biased and yields a too frequent decision in favour of "non Gaussian". This bias is also inherited by multivariate tests.

L2 Multivariate Gaussianity implies marginal Gaussianity, but the reverse is untrue. In other words, univariate tests have the tendency to conclude "Gaussian" more often than the multivariate ones. Actually, the only correct test would be the one having infinite dimension.

These two limitations are biasing the decision, and unfortunately the effects are opposite. Additionally, according to Moore [15], the colourness of the data may yield a loss in apparent normality of scalar tests.

The goal is to cope with these two origins of bias, by using multivariate tests (at least of dimension $p = 2$), whose mean and variance are calculated under the assumption that the process $x(n)$ is not i.i.d.

2. BRIEF REVIEW OF EXISTING TESTS

One should distinguish between two kinds of tests: scalar (univariate), and vector (multivariate) tests. Most normality tests fall in the first category. The oldest one is the Chi-square test, based on a proof due to Fisher (1922). Improvements of the Chi-square test include the extension to variable cells due to Moore (1971).

In historical order, Geary (1935) next proposed a test variable based on the ratio of an estimation of E$|x|$ and the sample standard deviation, that is equal to $\sqrt{2/\pi}$ in the normal case. Kolmogorov proposed in 1933 a test based on the rank statistic of order

1, and gave the distribution of the test in the form of a series; this test became usable with the tables given by Smirnov in 1948, and improved later by Lilliefors (1967) and Stephens (1974). Pearson and Hartley published a useful summary of tables in 1962, including quantiles of the estimated kurtosis [17]. Then appear various tests including Shapiro-Wilk (1965) and d'Agostino (1971), dedicated essentially to small samples. A survey of other scalar tests can be found in [11].

One more recent test has been proposed in [8], that is based on a theorem proved by Cramér, stating that sample mean and sample variance are independent if and only if the sample is Gaussian. Other characterizations of Gaussianity can be found in [5].

The first extension proposed to higher dimension was to perform a projection onto an arbitrary one-dimensional subspace [9] [1]. But in our opinion, the real extension was first proposed in 1982 by Hinich with his famous test based on the bispectrum [4]. Then came the tests based on multivariate kurtosis and skewness [12], those based on the characteristic function [2] [3], and those based on entropy [18]. Inspired from Epps [3], Moulines et al developed a test based on a vector of sample moments [16], that can also pretend to be omnibus.

For most tests mentioned above, the power has been evaluated only in the i.i.d. case, prohibiting then their application to a large part of records in the real world. On the other hand, the multivariate tests applicable in the coloured case, as [4], [3], or [16], are not only very computationally heavy, but also often require very long data records (we are talking about several hundreds of thousands). Our goal is to devise a test that could be applied to coloured samples of moderate length, say ten thousands, and that would require limited computer resources (*i.e.* run on a PC).

3. ASYMPTOTICAL STATISTICS OF THE KURTOSIS

Let $X(n), 1 \leq n \leq N$ be N realizations of a zero-mean random vector of dimension p, with finite moments up to order four. Mardia proposed the following definition of the multivariate kurtosis around 1970 [9]:

$$\mathcal{B}_p(N) = \frac{1}{N} \sum_{n=1}^{N} \left(X(n)^T S^{-1} X(n) \right)^2, \qquad (3)$$

Nevertheless, he assumed the process was not necessarily zero-mean, and subtracted the sample mean. For convenience, we simplified the above definition, since $X(n)$ is zero-mean by hypothesis (but there is apparently no difference in the results obtained after approximation up to second order). In the scalar case ($p = 1$), this definition reduces to (1). Note that many authors use the term kurtosis to denote a cumulant; we assumed here Mardia's choice (also shared by a few other statisticians): in the Gaussian case, the multivariate kurtosis is equal to $\mathcal{B}_p(N) = p(p+2)$, instead of zero. Below is another way of writing this definition, that will be subsequently useful:

$$\mathcal{B}_p(N) = \sum_{i,j,k,l=1}^{p} \hat{K}_{ijkl}\, G_{ij}\, G_{kl}, \qquad (4)$$

where $\hat{K}$ denotes the sample fourth order moment tensor, and G the inverse of the covariance matrix, S. Note that if we set K to be the fourth-order cumulant tensor instead, then we get the cumulant definition of the multivariate kurtosis. This definition seems to be the only consistent one, *i.e.* , invariant by linear transform [14].

In practice, the covariance matrix S is also replaced by the sample covariance, $\hat{S}$, and the test will be denoted $\hat{\mathcal{B}}_p(N)$. In the **i.i.d. case**, the 3 first moments of $\hat{\mathcal{B}}_p(N)$ have been calculated under the Gaussian hypothesis [13]:

$$\begin{aligned} \mu_{1,\mathcal{B}} &= \mathrm{E}\{\hat{\mathcal{B}}_p(N)\} = p(p+2)\,\frac{N-1}{N+1}, \\ \mu_{2,\mathcal{B}} &= \mathrm{Var}\{\hat{\mathcal{B}}_p(N)\} = \frac{8p(p+2)}{N} + o(N^{-2}), \qquad (5) \\ \mu_{3,\mathcal{B}} &= 64\,\frac{p(p+2)(p+8)}{N^2} + o(N^{-3}). \end{aligned}$$

And for large enough N, that is $N = 1000$ for instance, the variable $\hat{\mathcal{B}}_p(N)$ can be considered Gaussian without biasing the power of the test (the asymptotic normality is well known, but it is useful to know from what value of N the approximation is acceptable). See also the excellent paper [11], and [6, ch.12] for more details.

Let $x(t)$ be a zero-mean random process with finite moments up to order 4. Now given a finite sample $\{x(1), ..x(N)\}$, it is wished to test whether $x(t)$ is Gaussian or not. In practice, one can only test for the joint Gaussianity of a random vector $X(n)$ of reduced dimension p:

$$X(n) = \begin{bmatrix} x(n\,\Delta + 1) \\ \vdots \\ x(n\,\Delta + p) \end{bmatrix}, \quad 1 \leq \Delta << N. \qquad (6)$$

Parameter Δ is introduced to control overlapping. The choice $\Delta = p$ seems natural, but by playing with the value of Δ, it is also possible to interlace the values contained in successive $X(n)$'s. This will however not be investigated here. Realizations $X(n)$ are now **not i.i.d. anymore**.

3.1. Mathematical preliminaries

Denote $\delta = \hat{S} - S$ and $\varepsilon = \hat{K} - K$. The variance of these quantities is of order N^{-1}. More precisely in the Gaussian case we have, assuming the bracket notation intoduced by McCullagh [14]:

$$\mathrm{Cov}\{\hat{S}_{ab}, \hat{S}_{cd}\} = \frac{1}{N^2}\sum_{m,n=1}^{N}([3]R_{a_n b_n}R_{c_m d_m} - R_{a_n b_n}R_{c_m d_m}),$$

$$\mathrm{Cov}\{\hat{K}_{abcd}, \hat{K}_{efgh}\} = \frac{1}{N}\sum_{m,n=1}^{N} ([105]R_{a_n b_n}R_{c_n d_n}R_{e_m f_m}R_{g_m h_m} - [3]R_{a_n b_n}R_{c_n d_n}\cdot[3]R_{e_m f_m}R_{g_m h_m}).$$

where $R_{i_n j_m} \stackrel{\mathrm{def}}{=} \mathrm{E}\{X_i(n)X_j(m)\} = C_{(n-m)\Delta+i-j}$. And the cross-covariance is also of order N^{-1}. In particular in the scalar case, it can be checked out that

$$\frac{\mathrm{Var}\{\hat{S}\}}{S^2} = \frac{2}{N^2}\sum_{m,n=1}^{N}\frac{k_0^2(n-m)}{C_0^2},$$

$$\frac{\mathrm{Var}\{\hat{K}\}}{K^2} = \frac{8}{N^2}\sum_{m,n=1}^{N}\frac{k_0^2(n-m)}{C_0^2}\left[1+\frac{1}{3}\frac{k_0^2(n-m)}{C_0^2}\right],$$

if C_s denotes the correlation of the process $x(t)$, and $k_0(s) = C_{s\Delta}$. In the scalar i.i.d. case, we find in fact $2/N$ and $32/(3N)$ as expected. For large values of N, it is thus legitimate to assume that both δ and ε are small, since we are dealing with wide-band signals (the correlation has a reduced support).

The expansion of $\hat{G}$ up to second order in δ takes the form:

$$\hat{G} = G - G\delta G + G\delta G\delta G + O(\delta^3). \tag{7}$$

Next, using the expression (4), the estimated kurtosis can be approximated, after some manipulations, by:

$$\begin{aligned}\hat{\mathcal{B}}_p = \mathcal{B}_p &+ \sum_{abcd}\sum_{ijkl}\frac{1}{p^4}\varepsilon_{abcd}G_{ab}G_{cd} \\ &-\frac{2}{p^2}K_{abcd}G_{ab}G_{ci}\delta_{ij}G_{jd} \\ &-\frac{2}{p^2}\varepsilon_{abcd}G_{ab}G_{ci}\delta_{ij}G_{jd} \\ &+K_{abcd}G_{ai}\delta_{ij}G_{jb}G_{ck}\delta_{kl}G_{ld} \\ &+2K_{abcd}G_{cb}G_{ai}\delta_{ij}G_{jk}\delta_{kl}G_{ld} + o(N^{-1}).\end{aligned} \tag{8}$$

This generic expression allows to compute the mean and the variance of $\hat{\mathcal{B}}_p$ up to order two. After some lengthy calculations, the results obtained are:

$$\begin{aligned}\mathrm{E}\{\hat{\mathcal{B}}_p\} = \mathcal{B}_p + o(N^{-1}) &+ \\ \sum_{abcd}\sum_{ijkl} &-\frac{2}{p^2}G_{ab}G_{ci}G_{jd}\mathrm{E}\{\varepsilon_{abcd}\delta_{ij}\} \\ &+K_{abcd}G_{ai}G_{ld}\mathrm{E}\{\delta_{ij}\delta_{kl}\}(G_{jb}G_{ck}+2G_{cb}G_{jk}).\end{aligned} \tag{9}$$

and

$$\begin{aligned}\mathrm{Var}\{\hat{\mathcal{B}}_p\} = o(N^{-1}) &+ \\ \sum_{\substack{abcd\\efgh}}\sum_{ijkl} &\frac{1}{p^4}G_{ab}G_{cd}G_{ef}G_{gh}\mathrm{E}\{\varepsilon_{abcd}\varepsilon_{efgh}\} \\ &-\frac{4}{p^2}K_{efgh}G_{ab}G_{cd}G_{ef}G_{gi}G_{jh}\mathrm{E}\{\varepsilon_{abcd}\delta_{ij}\} \\ &+4K_{abcd}K_{efgh}G_{ab}G_{ci}G_{jd}G_{ef}G_{gk}G_{lh}\mathrm{E}\{\delta_{ij}\delta_{kl}\}.\end{aligned} \tag{10}$$

In the scalar case ($p = 1$), the latter two expressions can be simplified to:

$$\mathrm{E}\{\hat{\mathcal{B}}_1\} \approx \mathcal{B}_1 - \frac{2}{S^3}\mathrm{E}\{\varepsilon\delta\} + \frac{3K}{S^4}\mathrm{E}\{\delta^2\}, \tag{11}$$

$$\mathrm{Var}\{\hat{\mathcal{B}}_1\} \approx \frac{\mathrm{E}\{\varepsilon^2\}}{S^4} + \frac{4K^2}{S^6}\mathrm{E}\{\delta^2\} - \frac{4K}{S^5}\mathrm{E}\{\varepsilon\delta\}. \tag{12}$$

We shall now see how to use these generic results when vector $X(n)$ has the particular form (6).

3.2. Statistics with estimated covariance

In order to apply the results obtained in the previous section, it is necessary to develop the expressions of both K_{abcd} and G_{ef} as functions of the correlation of the process, C_τ. Since these calculations are very complicated and tedious, we have used MAPLE to assist their development. For conciseness, the notation $k_i(s) = C_{s\Delta+i}$ has been assumed.

3.2.1. Case $p = 1$

The simplest case is when $p = 1$, because matrices and tensors reduce to scalars. One obtains after simplifications:

$$\mathrm{E}\{\hat{\mathcal{B}}_1\} = 3\left[1 - \frac{2}{N} - \frac{4}{N^2}\sum_{s=1}^{N-1}(N-s)\frac{k_0^2(s)}{C_0^2}\right], \tag{13}$$

$$\mathrm{Var}\{\hat{\mathcal{B}}_1\} = \frac{24}{N}\left[1 + \frac{2}{N}\sum_{s=1}^{N-1}(N-s)\frac{k_0^4(s)}{C_0^4}\right]. \tag{14}$$

For verification, in the i.i.d. case, one can check out that the bias is $-6/N$ and the variance $24/N$, as known from the literature [7, vol.5, page 219], [10].

3.2.2. Case $p = 2$

The dimension 2 is also an intersting case because the inverse matrix G has still a rather simple form, and the result can be presented explicitely:

$$\mathrm{E}\{\hat{\mathcal{B}}_2\} = 4\left[2 - \frac{4}{N} - \frac{1}{N^2}\frac{\sum_{s=1}^{N-1}(N-s)q_1(s)}{(C_0^2 - C_1^2)^2}\right], \quad (15)$$

$$\mathrm{Var}\{\hat{\mathcal{B}}_2\} = \frac{16}{N}\left[4 + \frac{1}{N}\frac{\sum_{s=1}^{N-1}(N-s)q_2(s)}{(C_0^2 - C_1^2)^4}\right], \quad (16)$$

where $q_1(s)$ and $q_2(s)$ are given in figure 1.

3.2.3. Larger dimensions

The complexity of the result increases rapidly as p gets larger. Even if our approach is still valid, and if our MAPLE codes are theoretically usable, the expressions obtained cannot be presented for $p > 2$ because of their length, and can hardly be obtained on usual workstations for $p > 3$ because of both memory and computational limitations (we used a SUN4 sparc5).

3.3. Statistics with known covariance

When the covariance S is known, the preliminaries presented in section 3.1 are not necessary. The bias is null, and it suffices to expand $\hat{K} - K$ as a function of C_τ to obtain the variance of $\mathcal{B}_p$. Perhaps surprisingly, the results may seem more complicated than in the case where the sample covariance $\hat{S}$ was utilized instead, as in section 3.2. Means are $p(p+2)$, and variances:

$$\mathrm{Var}\{\mathcal{B}_1(N)\} = \frac{48}{N}\left[2 + \frac{1}{N}\sum_{s=1}^{N-1}(N-s)\,\frac{k_0^2(s)}{C_0^2}(3 + \frac{k_0^2(s)}{C_0^2})\right], \quad (17)$$

$$\mathrm{Var}\{\mathcal{B}_2(N\}\} = \frac{16}{N}\left[20 + \frac{1}{N}\frac{\sum_{s=1}^{N-1}(N-s)q_3(s)}{(C_0^2 - C_1^2)^4}\right] \quad (18)$$

where $q_3(s)$ is given in figure 1.

Studentized tests described in section 3.2 have consequently a smaller variance, and should then be preferred. This is a known property that comes from the fact that $\hat{S}$ is correlated with the data (here with $\hat{K}$).

4. COMPUTER RESULTS

For each data record, all tests are using the same estimate of the correlation function C_τ for a fixed p, so that only estimates of mean and variance of the test variable are different. It has been chosen $\Delta = p$ in all tests.

Test	$\hat{\mathcal{B}}_{1,iid}$	$\mathcal{B}_p$		$\hat{\mathcal{B}}_p$	
Dim. p	1	1	2	1	2
Formula	(5)	(17)	(18)	(14)	(16)
Gauss.					
iid	-0.165	-0.089	0.275	-0.165	0.643
MA(1)	0.248	0.115	0.251	0.249	0.598
MA(9)	-0.255	-0.062	0.043	-0.117	0.121
AR(2)	-0.106	-0.027	-0.139	-0.029	-0.279
Uniform					
iid	-24.79	-12.39	-9.655	-24.79	-21.59
MA(1)	-22.86	-11.18	-9.191	-22.84	-20.79
MA(9)	-2.722	-0.666	-5.182	-1.620	-11.24
AR(2)	-2.252	-0.494	-1.281	-1.283	-2.859
Sea					
mer201	-1.067	-0.193	1.433	-0.630	3.197
mer202	1.661	0.355	2.009	1.339	4.613
mer204	1.482	0.281	1.591	1.154	3.666
mer205	1.866	0.340	7.074	1.316	15.59

Table 1: Values of the ratio t.

Table 1 presents the values of the ratio

$$t = \frac{\mathcal{B} - \mathrm{E}\{\mathcal{B}\}}{Var\{\mathcal{B}\}^{1/2}}, \quad (19)$$

for $\mathcal{B}$ either equal to $\hat{\mathcal{B}}_p$ or $\mathcal{B}_p$, and with mean and variance estimated by the various approaches previously described (see numbers of formulas in table on top of columns for the sake of clarity). Recall that asymptotically, $t = \pm 1.960$ corresponds to a power of 95%, and $t = \pm 1.645$ to a power of 90%. The sample size was $N = 10000$. The correlation function was estimated over the full data length, but correction coefficients were calculated up to 200 correlation lags (that is, the correlation has been neglected beyond lag 200).

Coloured processes driven by a uniform i.i.d. sequence have often been assimilated to Gaussian when the correlation tails were long (e.g. MA(9) or AR(2)), except for the test $\hat{\mathcal{B}}_2$. For sea noises, test $\hat{\mathcal{B}}_2$ always concludes to non gaussianity, whereas the result is variable with the *i.i.d.* and $\mathcal{B}_2$ tests. In a full paper, deeper analyses will report Monte Carlo experiments.

5. CONCLUDING REMARKS

When a sample is Gaussian, all tests give the correct decision, regardless of the color. On the other hand, non Gaussian processes can be recognized as Gaussian when the correlation tails are long; in that case, bivariate tests perform better than univariate ones.

Other approaches to testing for Gaussianity of colored samples can be thought of. One is to whiten the data in a first step, by a non minimum phase linear finite response filter (the criterion to determine it is then

$$
q_1(s) = \left[(k_1+k_{-1})^2 + 8\,k_0^2\right] C_0^2 - 12\,k_0(k_1+k_{-1})C_1C_0 + 2\,\left[(k_1+k_{-1})^2 + 2k_0^2\right] C_1^2,
$$

$$
\begin{aligned}
q_2(s) = {} & \left[8(k_0^2 - k_1k_{-1})^2 + 3(k_1^2+k_{-1}^2)^2 + 12k_0^2(k_1-k_{-1})^2\right] C_0^4 \\
& + 4\,\left[8k_0^4 + 3(5k_0^2 + k_1k_{-1})(k_1+k_{-1})^2 - 4k_{-1}k_1(k_0^2+k_1k_{-1})\right] C_0^2C_1^2 \\
& + 8\,\left(k_{-1}^2k_1^2 + k_0^4 + 4\,k_0^2k_1\,k_{-1}\right) C_1^4 \\
& - 24\,k_0\,(k_{-1}+k_1)\left[(k_1^2+k_{-1}^2+2\,k_0^2)C_0^2 + 2\,(k_{-1}\,k_1 + k_0^2)C_1^2\right] C_0C_1,
\end{aligned}
$$

$$
q_3(s) = q_2(s) + 16(C_0^2 - C_1^2)^2\left[(2k_0^2 + k_1^2 + k_{-1}^2)C_0^2 - 4k_0(k_1+k_{-1})C_0C_1 + 2(k_0^2+k_1k_{-1})C_1^2\right].
$$

Compact notation: k_i has to be translated into $k_i(s)$.

Figure 1: Exact expressions of terms involved in formulas (15), (16) and (18).

the marginal kurtosis maximization). In a second step, a multivariate test should be applied.

To conclude, the use of multivariate tests is recommended, even when correlation must be neglected beyond lag p (for reasons of computational load for instance).

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