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Decomposition of quantics in sums of powers of linear forms

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Abstract

Symmetric tensors of order larger than two arise more and more often in signal and image processing and automatic control, because of the recent complementary use of High-Order Statistics (HOS). However, very few special purpose tools are at disposal for manipulating such objects in engineering problems. In this paper, the decomposition of a symmetric tensor into a sum of simpler ones is focused on, and links with the theory of homogeneous polynomials in several variables (i.e. quantics) are pointed out. This decomposition may be seen as a formal extension of the Eigen Value Decomposition (EVD), known for symmetric matrices. By reviewing the state of the art, quite surprising statements are emphasized, that explain why the problem is much more complicated in the tensor case than in the matrix case. Very few theoretical results can be applied in practice, even for cubics or quartics, because proofs are not constructive. Nevertheless in the binary case, we have more freedom to devise numerical algorithms.

Keywords. Tensors, Polynomials, Diagonalization, EVD, High-Order Statistics, Cumulants.

1 Introduction

In signal processing, mainly second order statistics have been used for a long time. But the potentiality of higher order statistics has clearly emerged during the last decade, and their possible

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role in industrial products is now recognized, with still some distrust however. For the moment, only statistics of moderate order are seriously considered (putting aside rank statistics). Typically, statistics of integer order used in signal processing applications would involve essentially quadrics, cubics, or quartics, but not polynomials of higher degree.

Even if it is clear that HOS such as moments or cumulants should be treated as symmetric tensors, it is much less obvious to know how to do it in practice, preserving fully their properties (e.g. symmetry, multilinearity). This is the reason why cumulants are used in signal processing algorithms after contraction or slicing. In particular whether a tensor can be decomposed into a sum of simpler ones is a relevant question, and surprisingly has received only partial answers to date. Since tensors and homogeneous polynomials are bijectively associated (cf section 2.2), this question is addressed thanks to old results borrowed from algebraic geometry.

In fact, if tensors do not seem to have been widely studied as such in the past, beside specific forms that appear in physics, homogeneous polynomials have. In fact, invariant theory has been one of the major mathematical research topics in the nineteenth century. Over a long period of time, researchers as famous as Gauss, Kronecker, Noether, Cayley, Weyl, Hilbert, or Dieudonné, have contributed to this field. At that time, a homogeneous polynomial of degree d in n variables was called a n-ary d-ic [27] [23]. For $d = 2, 3, 4, 5, 6..$ the adopted terminology was the quadric, the cubic, the quartic, the quintic, the sextic... The same terminology will be subsequently retained.

The goal of this paper is (i) to explain why the optimal use of symmetric tensors is difficult, (ii) to give a overview of the (unappreciated) state of the art, and (iii) to identify what could be new directions of investigations, and in particular towards special purpose numerical algorithms. This work turned out to be very difficult, the literature in the field being very forbidding, perhaps because algebra and its terminology have evolved, as the reader see himself by looking over reference [27].

The paper is organized as follows. Notation, statement of the problem, and link with homogeneous polynomials are established in section 2. The main course is section 3, which reports the results most often applicable, before section 4 succintly addresses the rare cases. The last section concludes with a summary and some perspectives.

2 General

2.1 Tensors

A tensor of dimension n and order d is an object defined in a n-dimensional coordinate system by a table with d indices, $g_{i_1,..i_d}, 1 \leq i_k \leq n$, that follows a particular transformation formula if the coordinate system is changed. More precisely, if a linear transform is applied to the space so that any vector u is changed into a vector $U = Au$, where A is a $n \times n$ invertible matrix, then the tensor is transformed into:

$$g_{i_1,..i_d} \to G_{i_1,..i_d} = \sum_{j_1,..j_d} A_{i_1 j_1}..A_{i_d j_d}\, g_{j_1,..j_d}. \tag{1}$$

This property is often referred to as the *multilinearity property* of tensors. A tensor G is symmetric if $G_{\sigma(ij..k)} = G_{ij..k}$, for any permutation σ. Denote $I\!R$ the set of real numbers. The set $\mathcal{T}(n;d)$ of symmetric tensors of dimension n and order d is a vector space on $I\!R$. It can be checked out that the vector space $\mathcal{T}(n;d)$ is of dimension

$$D(n;d) = \binom{n+d-1}{d}. \tag{2}$$

The set of cumulants of order d of a multichannel real random variable X of dimension n forms a symmetric tensor of order d and dimension n. The same holds true for moments. For instance, if the following table is defined:

$$\mu_{ij..k} = E\{X_i\, X_j\, ..\, X_k\},$$

then this table is symmetric and satisfies the multilinearity property (1). Actually, moments and cumulants are more than tensors, since they satisfy the multilinearity property even under non invertible transformations: matrix A defining the transform may be rank deficient or rectangular. See [15] [17] for more details.

As a consequence, when statistics of order larger than two (HOS) are utilized, the appropriate framework is no longer linear algebra anymore, but multilinear algebra, and the tables representing those statistics are in fact tensors, but not matrices.

Of course for simplicity, most algorithms taking advantage of HOS resort only to slices or contracted forms of those tensors, that can be stored in matrices (see for instance [25] and references therein). But it should be borne in mind that information is discarded when proceeding this way, and symmetry is broken.

2.2 Homogeneous polynomials

As a first obvious statement, it can be pointed out that there exists a bijective relation between the space of tensors $\mathcal{T}(n;d)$ and the space of homogeneous polynomials of degree d in n variables, which will be denoted here $\mathcal{F}(n;d)$. Indeed, let G be a tensor of $\mathcal{T}(n;d)$, then the polynomial

$$p(x_{j_1}, x_{j_2}, .., x_{j_n}) = \sum_{i_1, i_2, ..i_d = 1}^{n} G_{i_1 i_2 ..i_d}\, x_{i_1}\, x_{i_2} .. x_{i_d} \tag{3}$$

can be bijectively associated with G. In the above expression, it is clear that because of the symmetry of G, some terms appear several times. Actually, there is another way of writing polynomials of $\mathcal{F}(n;d)$ by resorting to a standard compact notation [12] [10] [20], widely used in invariant theory.

Let $\mathbb{N}$ be the set of integers $\{0, 1, 2, ..\}$, and $J(n)$ the subset $\{1, 2, ..n\}$. A multi-index of size n is a vector of n indices, $\mathbf{i} \in \mathbb{N}^n$. By convention, if $a \in \mathbb{R}^n$ and $\mathbf{i} \in \mathbb{N}^n$, $a^{\mathbf{i}}$ denotes the product $\prod_k a_k^{\mathbf{i}_k}$, and $(\mathbf{i})! = \prod_k (\mathbf{i}_k!)$. The length of a multi-index $\mathbf{i}$ is defined as $|\mathbf{i}| = \sum_k \mathbf{i}_k$. Lastly, $c(i)$ denotes the multinomial coefficient, namely $c(\mathbf{i}) = |\mathbf{i}|!/(\mathbf{i})!$. With these notations, any homogeneous polynomial of $\mathcal{F}(n;d)$ can be written as

$$p(\mathbf{x}) = \sum_{|\mathbf{i}|=d} \gamma(\mathbf{i};p)\, c(\mathbf{i})\, \mathbf{x}^{\mathbf{i}}. \tag{4}$$

Each coefficient $\gamma(\mathbf{i};p)$ characterizing polynomial $p(\cdot)$ is associated with one entry of the corresponding symmetric tensor, $G_{\mathbf{j}}$. The exact expression of the mapping $f : \mathbf{i} \in \mathbb{N}^n \leftrightarrow \mathbf{j} = f(\mathbf{i})$, $\mathbf{j} \in$

$J(n)^d$ is as follows. The k^{th} component $\mathbf{i}_k$ of multi-index $\mathbf{i}$ represents the number of times index k appears in $\mathbf{j}$. For example, with $d=3$ and $n=4$, we would have $f(2010) = 113$. Even in the community of engineers using HOS, these two notations are simultaneously used [15].

With these notations, it is also clear that expression (3) can be written in compact form as

$$p(\mathbf{x}) = \sum_{\mathbf{j}\in J(n)^d} G(\mathbf{j})\, \mathbf{x}^{f^{-1}(\mathbf{j})}. \tag{5}$$

Now, as seen in section 2.1, the dimension of $\mathcal{T}(n;d)$ is $D = \binom{n+d-1}{d}$, and so is the dimension of $\mathcal{F}(n;d)$. The set of monomials $\mathcal{B}(n;d) = \{\mathbf{x}^{\mathbf{i}},\ |\mathbf{i}| = d\}$ is chosen as a basis of $\mathcal{F}(n;d)$.

The scalar product between two polynomials of $\mathcal{F}(n;d)$ is defined as:

$$\langle p, q\rangle = \sum_{|\mathbf{i}|=d} c(\mathbf{i})\, \gamma(\mathbf{i};p)\, \gamma(\mathbf{i};q), \tag{6}$$

which means in particular that monomials in the basis $\mathcal{B}$ are orthogonal and have a squared norm $\langle \mathbf{x}^{\mathbf{i}}, \mathbf{x}^{\mathbf{i}}\rangle = (\mathbf{i})!/d! = 1/c(\mathbf{i})$. The projection of any polynomial $p(\mathbf{x})$ onto basis $\mathcal{B}$ yields then its components $\gamma(\mathbf{i})$. Sometimes, the *apolar* scalar product is used instead, and is defined as $d!$ times the previous one.

The choice of this Euclidian scalar product has other advantages. Suppose $\alpha(\cdot)$ is a linear form acting on $\mathbb{R}^n$, and defined by its n-dimensional vector α. Denote $\alpha^{(d)}$ the polynomial of $\mathcal{F}(n;d)$ obtained by raising the form to the d^{th} power. Then its scalar product with any polynomial q of $\mathcal{F}(n;d)$ turns out to be, from (6):

$$\langle q, \alpha^{(d)}\rangle = \sum_{\mathbf{i}} c(\mathbf{i})\, \gamma(\mathbf{i};q)\, \alpha^{\mathbf{i}} = q(\alpha). \tag{7}$$

Moreover, if $\partial_{\mathbf{a}\cdot\mathbf{x}} = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$ and $\mathbf{a}(\mathbf{x}) = a_1\, x_1 + \cdots a_n\, x_n$ then we have $\forall p \in \mathcal{F}(n;d-1)$:

$$d\,\langle q, \mathbf{a}(\mathbf{x})\, p\rangle = \langle \partial_{\mathbf{a}\cdot\mathbf{x}}(q), p\rangle,$$

which is another nice invariant property of this inner-product.

During the last century, one objective of invariant theory was to classify polynomials based on canonical forms, valid up to a change of variables. The methods used at that time were in some way, quite efficient [27] [23] [19]. Then, came the modern theory of algebraic geometry, which gave a very theoretical and general setting for this field [9] [18]. Our discussion borrows results from both frameworks.

2.3 Statement of the problem

It is known that symmetric matrices can be diagonalized by a change of coordinates, and that there are infinitely many ways of doing it; Sylvester's theorem on inertia states an invariance property enjoyed by minimal representations. The question is whether this holds true for tensors of higher order, and in particular of order 3 or 4.

This problem can be rephrased in terms of polynomials. Given a polynomial p of $\mathcal{F}(n;d)$, $d > 2$, under what conditions, if any, can this polynomial be written as a sum of N d^{th} powers of linear forms ? Then several more basic questions can be raised: (i) What is the minimal value of N required in general (the generic case, valid for almost every polynomial) ? (ii) Given p, can one compute the exact minimal value of N (i.e. the width, as defined in definition 2) ? (iii) Once N is determined, how many such decompositions exist ? (iv) Can something be done when N is imposed ? (v) How can a decomposition be computed in practice ? Some answers are attempted to be given in this paper.

It is pointed out in particular that Sylvester's theorem for quadratic forms does not extend easily to higher degrees, and the search for canonical forms is still an open problem for many values of the pair $(n; d)$. The goal of this paper is to show how large is the difficulty, and to give an idea of the state of the art, as accurately as is possible in a few pages. Other approaches have been also proposed recently but are not discussed subsequently. In [14] for instance, an extension of the Singular Value Decomposition to third order tensors is proposed. In [5], a canonical decomposition of tensors of $\mathcal{T}(n; 2k)$ into n^k powers is suggested, based on the assimilation of tensors to linear operators, whereas in [6] an approximate decomposition is described that always yields n powers. See also [7] for a discussion of the two latter approaches.

2.4 Application in array processing

Decompositions of quadrics in sums of powers are already used in antenna processing, and related areas in signal processing. The principle consists of approximating a symmetric matrix (the covariance of the observations) by another of lower rank, allowing to partition the space into signal and noise subspaces [2]. Now in order to apply the same principle to higher order tensors, one would like to approximate a tensor of $\mathcal{F}(n; d)$ by another of lower width, with our terminology.

More precisely, the linear statistical model assumed in array processing is of the form:

$$z(\nu) = \sum_{j=1}^{r} A_j(\nu)\, s_j(\nu) + \rho\, v(\nu),$$

where $z(\nu)$ are observed random vectors of dimension n, $A_j(\nu)$ are unknown deterministic vectors, $s_j(\nu)$ are random scalar variables, also referred to as "sources", and $\rho\, v(\nu)$ accounts for background and measurement noises. Standard identification algorithms exist when the array is known (that is, every $A_j(\nu)$ belongs to a known manifold $A(\theta, \nu)$), and when the number of sensors n, is strictly larger than the number of sources, r. This constraint comes from the fact that only *second order* statistics are utilized.

Here, it is not assumed that $n > r$. On the contrary, n and r are allowed to take any value. Of course, if r is too large, identifiability problems will occur. As will be pointed out in section 3.2, there is in fact an upper bound to r (namely the width of the approximating tensor), depending on n and on the order d of the statistics to be used. Contrary to [16], the array manifold is not required, to be able to detect and estimate the sources s_j, as well as to identify the source vectors A_j. If one desires to perform localization, the array manifold can be used only in a second stage, and an improved robustness (against calibration errors for instance) is expected, compared to standard procedures where the array manifold is utilized right from the beginning.

Thus, a cumulants-based approach would not only permit to get rid of Gaussian noise, or to improve on robustness, but also to identify $r > n$ signal components. However, several problems need to be fixed before the framework proposed in this paper can be efficiently utilized in array processing. Potential applications, currently under study in the case where $r = n$, include channel identification and equalization, Air traffic control in Radar, Super-resolution in Sonar, Speech deconvolution, Texture analysis, Object recognition, or Reactor monitoring. Some of them are resorting presently to Independent Component Analysis (ICA), a suboptimal decomposition; see [6] and references therein.

3 Decompositions in the generic case

3.1 Introduction

Denote by $I\!K$ the field on which we are working. From a practical point of view it will be $I\!R$, but if we need to use geometrical properties (and find all the roots of a polynomial for instance), it will be $\mathbb{C}$. Let p be a polynomial of $\mathcal{F}(n;d)$, and L_k, $1 \leq k \leq N$, N linear forms such that:

$$p(\mathbf{x}) = \sum_{j=1}^{N} L_j^d(\mathbf{x}), \quad L_k(\mathbf{x}) = \sum_{j=1}^{n} a_{k,j} x_j. \tag{8}$$

The expansion of this sum of powers in the basis of monomials $\mathcal{B}(n;d)$ defines a map Φ from the set $\mathcal{X} = I\!K^{nN}$ of coefficients $a_{k,j}$ onto $\mathcal{Y} = I\!K^D$, with D defined as in (2):

$$\begin{array}{rrcl} \Phi : & \mathcal{X} = I\!K^{nN} & \rightarrow & \mathcal{Y} = I\!K^D \\ & \mathbf{a} = ((a_{1,i}), \dots (a_{N,i})) & \mapsto & (c_I(a_{i,j})) \end{array}$$

where coefficients $c_I(a_{i,j})$ are given by $c(\mathbf{i})$, as defined in (4). The image of this polynomial map contains a dense open subset U of the algebraic manifold (or variety) $\overline{\Phi(\mathcal{X})}$ (the closure of $\Phi(\mathcal{X})$) in $\mathcal{Y}$ (see [24, th6, p. 60]). The complementary of U in $\Phi(\mathcal{X})$ is then a closed subset defined by algebraic equations. If the closure $\Phi(\mathcal{X})$ is whole space $\mathcal{Y}$, then the image is dense in $\mathcal{Y}$. In applications, coefficients are always given with some uncertainty, so that we are interested in properties that are true only on a (open) dense subset of $\mathcal{Y}$. To precise this notion, we introduce the following definition:

Definition 1 — *A property will be true in the generic case, or for generic polynomials, if it is true in a dense algebraic open subset of $\mathcal{Y}$.*

Example. Generic quadratic forms of a vector space of dimension n are sums of n squares. The case where this is not true corresponds to quadratic forms whose determinant vanishes. This determinant defines a closed set and its complementary is an open dense subset of the set of quadratic forms.

Definition 2 — *Given a polynomial p of $\mathcal{F}(n;d)$, the "width" of p refers to the minimal number of forms, $\omega(p)$, necessary to write p as a sum d^{th} powers of linear forms. The width of a generic polynomial of $\mathcal{F}(n;d)$ is denoted $g(n;d)$.*

Thus $g(n,d)$ denotes the minimal value to be given to N so that (8) holds true in the generic case [20]. Then $g(n,d)$ is obviously smaller than D, by definition. On the other hand, it is also larger than D/n. In fact the dimension of the image cannot be greater than the number of parameters in function Φ (which is nN). If nN were smaller than D then the image would lie in an hypersurface and would not be dense. But these bounds are clearly too loose to be really useful.

It has been shown recently by Reznick [21] that

$$\forall p \in \mathcal{F}(n;d),\ \omega(p) \leq \binom{n+d-2}{d-1}, \tag{9}$$

which is a much tighter bound. Moreover, this bound holds true when p ranges in the whole space $\mathcal{F}(n;d)$, and not only in a dense subset (i.e. the inequality is valid not only in the generic case).

There is no general expression that gives the exact value of $g(n,d)$. To be accurate, it is necessary to study each case separately, and this has been done mostly during the nineteenth century in the frame of invariant theory. Table 1 summarizes known values of $g(n,d)$. Again, these values correspond to the generic case, and there are smaller and larger reachable values (see example in section 4).

Example. To show how careful we have to be, consider for instance a generic ternary quartic. Counting the number of parameters on each side, we would expect that it could be decomposed into 5 linear forms since $5 \times 3 \geq \binom{6}{4}$, but the correct number of linear forms is 6 (see Clebsh's Theorem [10, p 26] and table 1).

3.2 Number of forms required: the generic width

The generic width, $g(n,d)$, is known for some values of degree d and dimension n. The easy case is when $d = 2$, since it is dealt with quadratic polynomials (quadrics), and the decomposition into a sum of n squares is possible, though not unique. This is equivalent to saying that the rank of a quadratic form is generically n. Another case has already been well studied, namely the case of binary forms. It is handled by the following theorems.

Theorem 3 (Sylvester) — *A generic binary form of odd degree $2m-1$ can be decomposed into a sum of m powers of linear forms.*

For binary forms of even degree $d = 2m$, there are infinitely many such decompositions in $m+1$ powers [20, section 5], unless some determinant is null (as explained in the theorem below), and a decomposition in m powers is in general impossible. Unicity can be insured by various constraints. The other cases (non-generic) are treated by the following theorem:

Theorem 4 (Sylvester) — *A polynomial $p(x,y) = \sum_i \gamma_i c(i) x^i y^{d-i}$ can be decomposed into a sum of r powers as $p(x,y) = \sum_{j=1}^{r} \lambda_j (\alpha_j x + \beta_j y)^d$ if and only if the form*

$$q_c(x,y) = \prod_{j=1}^{r} (\beta_j x - \alpha_j y) = \sum_{l=0}^{r} g_l x^l y^{r-l}$$

$$\text{satisfies} \quad \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{r+1} \\ \vdots & & & \vdots \\ \gamma_{d-r} & & \cdots & \gamma_d \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_r \end{bmatrix} = 0.$$

and has r distinct roots (real if the problem is real).

See [23, p 153], [12, p 63], or [21, section 6, p 28]. In fact, the last theorem implies an algorithm to decompose any binary polynomial into a minimal sum of powers, as will be seen in section 3.4.3.

Partial proof. If $p(x,y)$ is decomposable as $p(x,y) = \sum_{j=1}^{r} \lambda_j\,(\alpha_j\,x + \beta_j\,y)^d$ with $r \leq d$ then it is not hard to see that for any monomial m of degree $d-r$ in (x,y), we have $\langle m\,q_c(x,y), p\rangle = 0$. These equations correspond to the rows of the previous matrix. The determinant of this matrix is called the *catalecticant.*

Example. A normal form of any binary quartic is $u^4 + v^4 + \lambda\,u^2 v^2$. If the catalecticant of order 3 vanishes then $\lambda = 0$ and the polynomial is a sum of two squares. [23, p 157].

Example. The case of ternary cubics is also well known. Their general normal form is $u^3 + v^3 + w^3 + 6\,\lambda\,u\,v\,w$ [23, p 157] [3, p 278], but there are seven other possible forms [22] [21, section 7, p29]. Knowing the minimal number of powers in the decomposition of a polynomial p, one can then determine a canonical form of p, that makes it possible to classify polynomials. The case of ternary quartics is discussed in [20] for instance.

For larger values of n or d, in order to know whether the dimension of $\mathcal{Y}$ reaches $D(n;d)$ or not, we compute the rank of the Jacobian of Φ (defined in section 3.1), which gives the dimension of a generic tangent space to this variety, or equivalently the dimension of the variety. If this rank is maximal (equal to D) then the image is dense. Else the image is an open-subset of an algebraic variety of dimension strictly less than $D(n;d)$. The Jacobian of $\Phi\ : \mathbf{a} \mapsto \sum_i L_i^d$ can be computed in the following way : differentiating Φ with respect to $a_{1,i}$ yields to $d\,x_i\,L_1^{d-1}$, so that the Jacobian of Φ is the matrix of $(x_1 L_1^{d-1}, \ldots, x_n\,L_1^{d-1},\ x_1 L_2^{d-1}, \ldots, x_n\,L_2^{d-1}, \ldots,\ x_1 L_N^{d-1}, \ldots, x_n\,L_N^{d-1})$ in $\mathcal{B}(n;d)$. This yields the following theorem:

Theorem 5 (Lasker-Wakeford) — *A generic polynomial of degree d in n variables can be decomposed minimally in a sum of N powers of linear forms if and only if there exist linear forms $L_1, \ldots, L_N$ such that the rank of $(x_1 L_1^{d-1}, \ldots, x_n\,L_1^{d-1},\ x_1 L_2^{d-1}, \ldots, x_n\,L_2^{d-1}, \ldots, x_1 L_N^{d-1}, \ldots, x_n\,L_N^{d-1})$ is equal to $D(n;d)$.*

See [26], [13], [21], [10] for more details. Another way to formulate this theorem is to say that there exist linear forms $L_1, \ldots, L_N$ such that there is no polynomial of degree d orthogonal (for the scalar product defined in (6)) to the forms $L_1^{d-1}, \ldots, L_N^{d-1}$.

Remark that it is enough to find a point $\mathbf{a}$ such that the corresponding Jacobian is of maximal rank, for the rank will be the same in a neighborhood (for the Zariski topology) of this point. In other words, the rank will be generically $D(n;d)$ if we can find one point for which it is true.

An incremental algorithm for computing $g(n,d)$

Here a probabilistic algorithm is described, that computes the generic width of polynomials of $\mathcal{F}(n;d)$. According to theorem 5, we have to check the rank of the matrix

$$M(\mathbf{x};N) = (x_1 L_1^{d-1}, \ldots, x_n\,L_1^{d-1},\ x_1 L_2^{d-1}, \ldots, x_n\,L_2^{d-1}, \ldots,\ x_1 L_N^{d-1}, \ldots,\ x_n\,L_N^{d-1}) \qquad (10)$$

in the basis of all monomials of degree d.

This is done incrementally, adding at each step a block $(x_1 L_k^{d-1}, \ldots, x_n\,L_k^{d-1})$ and find values of the coefficients such that the rank and the generic rank are equal at this step. When the iteration stops, we are left with $g(n,d)$ linear forms such that the corresponding matrix is of maximal rank.

1. Take for the first n linear forms : $L_1 = x_1, \dots, L_n = x_n$. Set initially $k = n + 1$.
2. Take a new form $L_k := \sum_{i=1}^{n} a_{i,k} x_i$ and compute the rank of the matrix $M(\mathbf{x}; k)$ defined in (10), which depends only on the variables $a_{k,i}$ (the other coefficients are numerical).
3. Find randomly numerical values for $a_{k,i}$ such the corresponding matrix has the same rank as the matrix in variables $a_{k,i}$.
4. If the rank of matrix $M(\mathbf{x}; k)$ is full, then stop and set $g(n; d) = k$. Else go to step 2 with $k \leftarrow k + 1$.

This algorithm has been implemented in MAPLE, and the results are reported in table 1. The values of $g(n, d)$ obtained coincide with those obtained already algebraically, when they were indeed known. But the program also allowed to fill the values that were yet unknown[1] (indicated in bold face).

3.3 Number of solutions

Now, given a generic polynomial, we want to know how many decompositions there are. In fact, as we are dealing with algebraic varieties, this means that we want to know what is the dimension and the degree of these varieties.

Proposition 6 — *Given a generic polynomial p, the solutions* $\mathbf{a}$ *such that* $\Phi(\mathbf{a}) = p$ *form an algebraic variety of dimension* $nN - D$.

Proof. The set of coefficients $\mathbf{a}$ such that $\Phi(\mathbf{a}) = p$ is the fiber $\Phi^{-1}(p)$ of Φ over p. This map between two affine spaces of dimension nN and D (with $nN \geq D$) is regular. According to [24, th 7, p 60], the dimension of the fiber is at least $nN - D$. The latter bound is reached for generic polynomials (on a non-empty open subset of $\mathcal{Y}$). ■

Example. Consider the case of polynomials of degree 2 in n variables. A well-known theorem of Sylvester tells that a generic quadric is a sum of n signed squares. Consequently, as the dimension of $\mathcal{X}$ is n^2 and the one of $\mathcal{Y}$ is $\frac{1}{2}n\,(n+1)$, the dimension of a generic fiber is $n^2 - \frac{1}{2}\,n\,(n+1) = \frac{1}{2}\,n\,(n-1)$. It is the dimension of the orthogonal group of the corresponding quadric (set of matrices that leaves the quadratic form unchanged).

In the case where the dimension of a generic fiber is null, it contains a finite number of points. This number of points (by definition) is the degree of the map Φ. As it is defined by D polynomials of degree d, a rough bound on the degree is d^D (according to Bézout's theorem [24]).

3.4 Calculation of a decomposition

To date, constructive algorithms for calculating a decomposition into powers of linear forms exist only for cubics and binary forms.

[1] We discovered recently a work borrowing tools from another area of algebraic geometry [1]. Although it was addressing a different question (interpolation), the results presented allow to compute the generic width in a different way, especially for $d > 4$.

3.4.1 Completion of the cube

The first (well known) constructive method is based on a simple idea that is usually applied to quadrics.

Theorem 7 — *Any polynomial of $\mathcal{F}(n;3)$ can be decomposed as*

$$p(\mathbf{x}) = \sum_{j=1}^{n} y_j^3 + q(\mathbf{x}), \tag{11}$$

where y_j are linear forms in the x_i's, and q a polynomial of partial degree in x_1 equal to $d-3=0$.

Proof. As now briefly explained, the proof may be described in 4 stages in the case of cubics ($d=3$). Even if the principle of the proof could be generalized to $d>3$, the theorem would not yield a means of recursion, and thus would not give any possibility of computing the desired decomposition.

1. Any polynomial $\varphi(\mathbf{x})$ of $\mathcal{F}(n;3)$ can be written $p(\mathbf{x}) = u_1^3 + 3\,u_1^2\,h_1 + 3\,u_1\,h_2 + h_3$, where h_k are polynomials of $\mathcal{F}(n-1;k)$ in variables $u_2, ..u_n$.

2. Letting $v_1 = u_1 + h_1$ and $v_i = u_i$ for $i>1$, yields an expression where the term in v_1^2 disappears: $p(\mathbf{x}) = v_1^3 + 3\,v_1\,H_2 + H_3$, where $H_2 = h_2 - h_1^2$ and $H_3 = 2\,h_1^3 - 3\,h_1 h_2 + h_3$.

3. The quadratic form H_2 can be diagonalized after an appropriate linear transformation in the variables $v_2, ..v_n$, keeping v_1 unchanged. Denoting y_i the new variables, and k the rank of H_2, $k<n$: $p(\mathbf{x}) = y_1^3 + 3\,y_1 \sum_{j=2}^{k} y_j^2 + G_3$, where G_3 is a polynomial of $\mathcal{F}(n-1;3)$ in the variables $y_2, ..y_n$.

4. Lastly, the two first terms of the last expression of p above can be transformed into the sum $\sum_{i=1}^{k} f_i^3$, by defining the linear forms $f_i = k^{-1/3} y_1 + k^{-1/6}\,(y_i + \lambda \sum_{j=2}^{k} y_j)$, where λ is a root of $(k^2+k)\,\lambda^2 + 2(1+k)\,\lambda + 1 = 0$ [21]. ■

The conclusion is that any polynomial p of $\mathcal{F}(n;3)$ can always be decomposed into a sum of at most $n(n+1)/2-1$ powers of linear forms.

3.4.2 Simultaneous diagonalization

Now the inconvenience of the previous approach is that the number of forms obtained in the decomposition is in general much larger than the achievable bound, D. The approach described below leads to a smaller number of forms, and was proposed by Reznick [21]. Our attempts to extend this result to the real case have not succeeded.

Theorem 8 (Reichstein's canonical form) — *Any polynomial of $\mathcal{F}(n;3)$ can be decomposed as*

$$p(\mathbf{x}) = \sum_{j=1}^{n} y_j^3 + q(x_2, ..x_n), \tag{12}$$

where y_j are linear forms in the x_i's, and q a polynomial of $\mathcal{F}(n-2;3)$.

Proof. This proof was given by Reznick in [21]. Consider the partial derivatives $f_1 = \partial p/\partial x_1$ and $f_2 = \partial p/\partial x_2$. Since these are quadratic forms, they can be written as $f_1 = \mathbf{x}^T S_1 \mathbf{x}$ and $f_2 = \mathbf{x}^T S_2 \mathbf{x}$ where S_1 and S_2 are symmetric. Yet, the pencil (S_1, S_2) admits generically distinct eigenvalues. Thus, there exist a basis of (possibly complex) vectors $\mathbf{a}_i$ such that $f_2 = \sum \lambda_i\, y_i^2$, $f_1 = \sum_i y_i^2$, and $y_i = \mathbf{a}_i^T \mathbf{x}$. Integrating f_1 and f_2 with respect to x_1 and x_2, respectively, leads to two expressions of p:

$$p(\mathbf{x}) = \sum_i \frac{1}{3a_{i1}}\, y_i^3 + q_1(x_2, x_3, .., x_n) = \sum_i \frac{\lambda_i}{3a_{i2}}\, y_i^3 + q_2(x_1, x_3, .., x_n).$$

On the other hand, it can be seen that $\lambda_i\, a_{i1} = a_{i2}$, by checking out the second derivative. As a consequence the first terms coincide in both sides, and so do the remainders q_1 and q_2, which must then be independent of both x_1 and x_2. ■

Thus, this theorem allows to reduce by half the number of variables recursively. As a consequence, the polynomial p of $\mathcal{F}(n;3)$ can be decomposed into a sum of at most $n(n+2)/4$ cubes of linear forms if n is even, and at most $(n+1)^2/4$ if n is odd. This is about twice as less as the number of forms obtained in section 3.4.1.

3.4.3 Binary forms

Theorem 4 actually gives an algorithm to compute the decomposition of any binary form in sums of powers. We have implemented the MATLAB program listed in figure 5.

We show now the results obtained with some examples. Take $p(x, y) = (2\,x + y)^4 + (x + 4\,y)^4$:

```
> p=convd([2,1],4)+convd([1,4],4)
  p = 17    48   120   264   257
> [lambda,q]=binarydec2(p)
  err = 6.70011e-14
  lambda =
    1.0000
  256.0000
  q =
    2.0000    1.0000
    0.2500    1.0000
```

So we find the original canonical expression as expected, as shows the small reconstruction error, **err**. Let's take now another example with $p(x, y) = (2x + y)^5 + (-2x + y)^5 + (3x + 3y)^5$.

```
> p=convd([2,1],5)+convd([-2,1],5)+convd([3,3],5)
  p = 243  1375  2430  2510  1215  245
> [lambda,q]=binarydec2(p);
  err =  7.52827e-13
> [lambda,q] =
    1.0000    2.0000    1.0000
    1.0000   -2.0000    1.0000
  243.0000    1.0000    1.0000
```

The polynomial is reconstructed correctly with 3 linear forms ($3^5 = 243$). Now let's see a last example. Take $p(x, y) = (2x + y)^6 + (3x - 5y)^6 + (-2x + y)^6 + (x + y)^6$, and get:

```
> p=convd([2,1],6)+convd([3,-5],6)+convd([-2,1],6)+convd([1,1],6)
  858 -7284  30870 -67480  84510 -56244 15628
> [lambda,q]=binarydec2(p);
   err = 16.9017
```

In the present case, the reconstructed polynomial is different from the original, `p`. In fact, we are here in the generic case and the fiber is of dimension $2 \times 4 - (6+1) = 1$ (see table 2). So we have one degree of freedom to choose a solution.

Another interesting analysis is to look at the effect of measurement noise on the robustness of this algorithm. For this purpose, we took the polynomial $p_0(x, y) = (2\,x + y)^4 + (x + 4\,y)^4$, and tried to identify the linear forms generating the polynomial $p(x, y) = p_0(x, y) + \rho\, g(x, y)$, where the coefficients of $g(x, y)$ were generated randomly. Two noise distributions were envisaged: uniform in the interval $[0, 1]$, and Gaussian.

The algorithm outputs r linear forms $q_i(x, y)$ and r coefficients λ_i. This allows to reconstruct the polynomial $\hat{p}(x, y) = \sum_{i=1}^{r} \lambda_i\, q_i(x, y)^d$. As ρ increases, we can measure two kinds of errors. The first one is $\hat{p}(x, y) - p(x, y)$, and measures the ability of the algorithm to reconstruct an arbitrary polynomial. The second error is $\hat{p}(x, y) - p_0(x, y)$, and represents the deviation to the original noise free polynomial. This error also accounts for the ability to reject additive noise. In table 3, the norm of the errors are reported, in the canonical metric previously introduced in (6), and detailed in the function `binarydec2`. Since the null space of the Hankel matrix is estimated with a given tolerance thanks to the use of SVD, the algorithm proves some robustness. It is more robust against uniform noise, but on the other hand Gaussian noise is more realistic if the tensor is formed of sample cumulants.

3.4.4 General case

In the general case, to date, there is no really efficient way to find a decomposition of a generic polynomial. Given a polynomial p, the problem is equivalent to finding a solution of a polynomial system in the coefficients $a_{i,j}$ of the linear forms. In practice, the size of the polynomial system is so huge that usual techniques based on resultants and elimination [4] cannot work. For instance for a generic polynomial of degree 4 in 3 variables, we need to consider 6 linear forms or 18 variables. The system corresponds to 15 equations of degree 4 in the variables $a_{i,j}$, so that a classical multivariate resultant would yield a polynomial in one variable of degree $4^{15} = 1073741824$.

A more feasible approach to this problem is now given. It is based on a classical Least-Squares method, that starts from a given point and minimizes the square of the Euclidian distance (defined in section 2.2) between the polynomial $\sum_i L_i^d$ and the polynomial p that we want to decompose.

This program as been implemented in MAPLE and the C-language. The first system computes the Jacobian of the norm and its Hessian with respect to variables $a_{i,j}$, and generates a C-code that evaluates these matrices. This C-code is then linked with a general-purpose minimization algorithm (developed by J. Grimm, SAFIR Project). One could also use a more sofisticated method [11].

We illustrate this method on the following polynomial:

$$\begin{aligned} & {x_1}^4 + {x_1}^3\, x_2 + {x_1}^3\, x_3 + {x_1}^2\, {x_2}^2 \\ & \quad + {x_1}^2\, x_2\, x_3 + {x_1}^2\, {x_3}^2 + x_1\, {x_2}^3 + x_1\, {x_2}^2\, x_3 + x_1\, x_2\, {x_3}^2 + x_1\, {x_3}^3 + {x_2}^4 + {x_2}^3\, x_3 \\ & \quad + {x_2}^2\, {x_3}^2 + x_2\, {x_3}^3 + {x_3}^4 \end{aligned}$$

The decomposition found is

$$(0.6307218544\, x_1 + 0.541354088\, x_2 + 0.3534113236\, x_3)^4$$

$$
\begin{aligned}
&+ (0.02997684794\, x_1 + 0.6869227237\, x_2 + 0.4622302052\, x_3)^4 \\
&+ (0.1770527171\, x_1 + 0.9112083062\, x_2 + 0.05298312847\, x_3)^4 \\
&+ (0.8287429345\, x_1 + 0.03202241062\, x_2 + 0.329541934\, x_3)^4 \\
&+ (0.2009353301\, x_1 + 0.166255501\, x_2 + 0.9812070694\, x_3)^4 \\
&+ (0.7785569235\, x_1 + 0.1896890002\, x_2 - 0.07313337321\, x_3)^4
\end{aligned}
$$

and the error is

$$0.60038718147983510^{-21}.$$

The figure shows the projection of the approximation on the plane of the first two coordinates. It ends on the point $(0.6307\ldots, 0.5413\ldots)$ which corresponds to the first coordinates of the first linear form in the decomposition. This method converges more easily when the initial polynomial is "generic" (the number of linear form is the generic number) and when the number of solution is big. When the initial polynomial is not a sum of the generic number of linear forms, the convergence often fails.

The advantage of the proposed method is that the larger the dimension of the fiber, the more chances to converge to a solution. The drawback is that we are not sure to find an acceptable solution and that the formula of the norm, the Jacobian, and the Hessian, become huge when the dimension and the degree increase. It is expected that combining algebraic methods (using the symmetry of the problem) and numerical techniques such as homotopies, would help to achieve this decomposition in generic cases.

4 Minimal decompositions in non-generic cases

We consider now the problem of determining the minimal number of powers, $\omega(p)$, that needs to be used to decompose a given polynomial p. This happens to be a problem closely related to the classification of orbits of polynomials under linear change of variables, which is in itself a hard problem of algebraic geometry (see for instance [9, p 53]).

As we have seen in section 3.4.3, we can describe explicitly when a binary form can be decomposed in a sum of r powers and this decomposition can be achieved by a simple algorithm. One surprising property of this decomposition is that in some non-generic case the number of powers can be larger than in the generic case. For instance, the binary polynomial $x^2\, y$ cannot be decomposed in less than a sum of 3 powers : $x^2\, y = \frac{1}{6}((x+y)^3 + (-x+y)^3 - 2\, y^3)$. In the other cases, the characterization of width r polynomials in $\mathcal{F}(n;d)$ is not so simple.

Let $\mathcal{L}_r$ be the set of polynomials of width $\omega(p) = r$. We propose here a general method to characterize the closure of these sets. Denote $\mathcal{R} = I\!\!K[\gamma_{\mathbf{i}}, a_{k,l}]$, $1 \le k \le n, 1 \le l \le r$, the set of polynomials in the variables $\gamma_{\mathbf{i}}, a_{k,l}$. We note $\mathbf{a}_j = (a_{j,1}, \ldots, a_{j,n})$ the coefficients of the j^{th} linear form and $\mathbf{a}_j^{\mathbf{i}} = a_{j,1}^{i_1} \cdots a_{j,n}^{i_n}$. Let $\mathcal{I}_r$ be the ideal of $\mathcal{R}$ generated by the polynomial $\gamma_{\mathbf{i}} - \sum_{l=1}^{r} \mathbf{a}_l^{\mathbf{i}}$.

In the case of one power ($\omega = 1$), the map Φ defines another map between projective spaces:

$$
\begin{array}{cccc}
\overline{\Phi}: & I\!\!P^{n-1} & \rightarrow & I\!\!P^{D-1} \\
 & \mathbf{a} = (a_1, \ldots a_n) & \mapsto & (a_1^{i_1} \cdots a_n^{i_n})_{i_1 + \cdots + i_n = d}
\end{array},
$$

where $I\!\!P^{m-1}$ is the projective space associated with $I\!\!K^m$. This map is known as the *Veronese* map and its image is called the *Veronese Variety*. It is a closed variety whose ideal is generated by all the polynomials satisfying:

$$\gamma(i_1,\ldots,i_n)\gamma(j_1,\ldots,j_n)-\gamma(k_1,\ldots,k_n)\gamma(l_1,\ldots,l_n)=0 \tag{13}$$

with $i_s+j_s=k_s+l_s$ for all $1\le s\le n$ (see [24, p 40]). So a polynomial p is a power of a linear form if and only if its coefficients satisfy the previous relations (13).

For a sum of two powers ($\omega=2$), the map cannot be extended as a map between polynomial spaces in this way, because we can find non-zero elements $(a_{i,j})$ such that the image by Φ is the zero polynomial in x_i. If d is odd, take for instance $L_2=-L_1$. Thus the image is not necessarily closed, and cannot be defined by equations but also needs inequalities. A polynomial p is a sum of two powers if by a change of variables, it is of the form $x_1^d+x_2^d$. Conversely, the orbit of the previous polynomial under the action of Gl_n (the group of invertible $n\times n$ matrices) is the set $\mathcal{L}_2$. The closure of $\mathcal{L}_2$ is defined by the polynomials in $I\!K[\gamma_{\mathbf{i}}]\cap\mathcal{I}_2$ where $\mathcal{I}_2$ is the ideal of $\mathcal{R}$ generated by the equations $\gamma_{\mathbf{i}}-(\mathbf{a}_1^{\mathbf{i}}+\mathbf{a}_2^{\mathbf{i}})=0$. Indeed these are the polynomials which vanish by substitution $(\mathbf{a}_1^{\mathbf{i}}+\mathbf{a}_2^{\mathbf{i}})$ for $\gamma_{\mathbf{i}}$. Moreover, a polynomial vanishes on $\overline{\mathcal{L}_2}$ (resp. $\mathcal{L}_2$) if and only if it vanishes by substitutions $(\mathbf{a}_1^{\mathbf{i}}+\mathbf{a}_2^{\mathbf{i}})$ for $\gamma_{\mathbf{i}}$. These relations can be computed by elimination techniques (for instance using Gröbner Bases [8]).

This technique extends naturally to sums of any powers in the following way. The relations satisfied by the points in $\overline{\mathcal{L}_r}$ are the polynomials of $I\!K[\gamma_{\mathbf{i}}]\cap\mathcal{I}_r$. They give information only on the *closure*, and as we be seen with an example below, more information is needed to compute the width of a polynomial.

As many forms as the dimension

The special case where a polynomial p into n variables is decomposable in a sum of n powers of independent linear forms is worth considering. By a change of variables, it can be written in the form $x_1^d+\cdots+x_n^d$. The Hessian of this polynomial in these variables is $det(\partial_{x_i,x_j}(p))=(d\,(d-1))^n\prod_{i=1}^n x_i^{d-2}$. If p is decomposable in a sum of n powers then its Hessian, being a covariant, will be the product of n linear forms with multiplicity $d-2$ after any change of variables. The linear forms that appear as factors of it are precisely (up to a scalar) the forms that appear in the decomposition of p. Geometrically, the hypersurface defined by the Hessian is the union of hyperplanes with multiplicity $d-2$. This can be checked easily by taking the intersection of a varying line with this hypersurface. The intersection points should vary "linearly" with the line. Once these hyperplanes are known $L_i(x)=0$, one has to compute the scalars λ_i such that $p=\sum_{i=1}^n\lambda_i L_i^d$.

An algorithm for cubics

We illustrate this problem with polynomials of degree 3 in 3 variables. In this case, we have three varieties, $\overline{\mathcal{L}_1}$, $\overline{\mathcal{L}_2}$, $\overline{\mathcal{L}_3}$, and $\overline{\mathcal{L}_4}=F(3,3)$, since the generic width is 4. Of course, we have $\overline{\mathcal{L}_1}\subset\overline{\mathcal{L}_2}\subset\overline{\mathcal{L}_3}\subset F(3,3)$.

Equations of these varieties have been computed by the preceding technique, but are not reported here for reasons of space. This allows to classify all the possible orbits of a polynomial of degree 3 in 3 variables by considering all possible forms of decomposition up to a linear change of coordinates:

- A polynomial of the orbit of x_1^3 is in $\overline{\mathcal{L}_1}$, and any polynomial of $\overline{\mathcal{L}_1}$ is in this (closed) orbit.

- A polynomial of the orbit of $x_1^3 + x_2^3$ is in $\overline{\mathcal{L}_2}$.
- A polynomial of the orbit of $p = x_1^3 + x_2^3 + (a\,x_1 + b\,x_2)^3$ also satisfy the equations of $\overline{\mathcal{L}_2}$. In this case the variety of $I\!P^{n-1}$ defined by $p = \prod_{i=1}^{3}(\alpha_i x_1 + \beta_i x_2)$ is the the union of 3 "parallel" hyperplanes. In other words, these polynomials lie in $\overline{\mathcal{L}_2} \cap \mathcal{L}_3$.
- A polynomial of the orbit of $x_1^3 + x_2^3 + x_3^3$ is in $\overline{\mathcal{L}_3}$. Its Hessian is a product of 3 linear forms.
- A polynomial of the orbit of $x_1^3 + x_2^3 + x_3^3 + (a\,x_1 + b\,x_2 + c\,x_3)^3$ (with $(a, b, c) \neq (0, 0, 0)$) is a generic polynomial, thus of width 4.
- The other polynomials are of width 5, and are in the orbit of $x_1\,(x_1\,x_2 + x_3^2)$, according to [21]. These polynomials are in $\overline{\mathcal{L}_3}$, which means that $\mathcal{L}_5 \subset \overline{\mathcal{L}_3}$.

Other cases such as $x_1^3 + x_2^3 + (a\,x_1 + b\,x_2)^3 + (a'\,x_1 + b'\,x_2)^3$ can be reduced to sums of less powers and do not appear in this list. In the previous case for example, a sum of 4 powers in two variables can be rewritten as a sum of at most 3 powers.

Given a homogeneous polynomial p of degree 3 in 3 variables, we proceed as follows to determine its width.

1. If its coefficients satisfy the equations of $\overline{\mathcal{L}_1}$, then p is the cube of a linear form.
2. Else if they satisfy the equations of $\overline{\mathcal{L}_2}$, then p can be factorized in a product of 3 *dependent* linear forms: $p = L_1 \times L_2 \times L_3$.
   (a) either the linear forms L_1, L_2, L_3 are distinct and p is in the orbit of $x_1\,x_2\,(x_1 + x_2)$ which admits a decomposition in a sum of 2 cubes. The width of p is 2.
   (b) or two linear forms coincide and p is in the orbit of $x_1^2 x_2$ (with $a, b \neq 0$). The width of p is 3.
3. Else if the coefficients of p satisfy the equation of $\overline{\mathcal{L}_3}$, then
   (a) either the Hessian of p is a product of 3 independent linear forms and p is in the orbit of $x_1^3 + x_2^3 + x_3^3$,
   (b) or its Hessian is a cube (in $\overline{\mathcal{L}_1}$), the Hessian of $x_1\,(x_1\,x_2 + x_3^2)$ being $-8\,x_1^3$, and the polynomial p is of maximal width 5.
4. The remaining cases corresponds to generic polynomials of width 4.

More generally, for any dimension and degree the number of orbits $\mathcal{L}_r$ will be related to the possible relative configurations of r linear forms in a space of dimension n and this classification of orbits remains a hard algebraic open problem.

Example. Consider for instance,

$$p = 4\,{x_3}^3 + 18\,{x_3}^2 x_2 + 12\,{x_3}^2 x_1 + 28\,x_3 {x_2}^2 + 36\,x_3 x_2 x_1 + 12\,x_3 {x_1}^2 + 15\,{x_2}^3 + 28\,{x_2}^2 x_1 + 18\,x_2 {x_1}^2 + 4\,{x_1}^3$$

which factors through 3 distinct linear forms:

$$((1-\mathbf{i})\,x_1 + (2-\mathbf{i})\,x_2 + (1-\mathbf{i})\,x_3)\,((1+\mathbf{i})\,x_1 + (2+\mathbf{i})\,x_2 + (1+\mathbf{i})\,x_3)\,(2\,x_1 + 3\,x_2 + 2\,x_3)$$

So, we are in the case (2.a) and p has the following approximated decomposition

$$\begin{aligned}&0.9811252246\,(1.267949192x_1 + 1.535898385x_2 + 1.267949192x_3)^3\\ +\quad &0.0188747754\,(4.732050808x_1 + 8.464101616x_2 + 4.732050808x_3)^3\end{aligned}$$

which has been computed using the algorithm on binary forms.

Example. Consider now

$$p = 2\,{x_3}^2x_2 + {x_3}^2x_1 + 8\,x_3{x_2}^2 + 2\,x_3x_2x_1 + 3\,x_3{x_1}^2 + 6\,{x_2}^3 + 5\,{x_2}^2x_1 + x_2{x_1}^2 + 2\,{x_1}^3$$

which does not satisfy the equations of $\overline{\mathcal{L}_1}$ nor $\overline{\mathcal{L}_2}$ but is in $\overline{\mathcal{L}_3}$. Its Hessian is

$$\begin{aligned}&-128\,{x_3}^3 - 384\,{x_3}^2x_2 - 384\,{x_3}^2x_1 - 384\,x_3{x_2}^2\\ &\quad -768\,x_3x_2x_1 - 384\,x_3{x_1}^2 - 128\,{x_2}^3 - 384\,{x_2}^2x_1 - 384\,x_2{x_1}^2 - 128\,{x_1}^3\end{aligned}$$

which factors through

$$-128\,\left(x_1 + x_2 + x_3\right)^3.$$

Consequently, we are in the case (3.b) and p is a sum of 5 linear forms.

5 Concluding remarks

In this paper, some results of invariant theory have been surveyed. In particular, we emphasized the (perhaps surprising) fact that a symmetric tensor has generally a width larger than its dimension. This is to be compared to matrices, that cannot have a rank larger than their dimension. Another even more striking fact is that the generic width is difficult to compute in some cases, and that the maximal achievable width is known only through upper bounds in most cases.

Four new algorithms have been proposed, that solve (yet only very partially) the problem. In section 3.2, an incremental algorithm is suggested for computing the generic width; in section 3.4.3, an algorithm is described that is able to compute explicitly the decomposition of a binary polynomial of any degree; in section 3.4.4, the technique described allows to find the width of a non generic polynomial; lastly in section 4, an iterative algorithm allows to compute the corresponding decomposition explicitly.

Applications have been pointed out in section 2.4. The fact that tensors can have a width larger than their dimension is a richness that can be exploited in array processing to detect and identify more sources than sensors. However, difficulties remain to be overcome before higher order decompositions become feasible in really useful situations, and several directions of research are worth mentioning. The first is to handle optimally more than n forms, since they would not define a partition anymore. Second, it is wished to define decompositions where forms can be sorted by decreasing importance, in order to cope with noise, as we suggested in the binary case. Third, efficient numerical algorithms are still lacking, even for moderate degrees, in particular for cubics or quartics. Fourth, it is also possible to decompose quantics into sums of powers of quadrics (instead of linear forms). All these directions are being currently investigated.

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n\d	2	3	4	5	6	7	8
2	2	2	3	3	4	4	5
3	3	4	6	7	10	12	**15**
4	4	5	10	14	**22**	**30**	**42**
5	5	8	15	**26**	**42**	**66**	**99**
6	6	10	**22**	**42**	**77**	**132**	**215**
7	7	12	**30**	**66**	**132**	**246**	**429**
8	8	**15**	**42**	**99**	**215**	**429**	**805**

Table 1: *Generic width $g(n;d)$ of polynomials of degree d in n variables.*

n\d	2	3	4	5	6	7	8
2	1	0	1	0	1	0	1
3	3	2	3	0	2	0	0
4	6	0	5	0	4	0	3
5	10	5	5	4	0	0	0
6	15	4	6	0	0	0	3
7	21	0	0	0	0	6	0
8	28	0	6	0	4	0	5

Table 2: *Generic dimension of the fiber of solutions.*

$log_{10}\rho$	-16	-4	-3	-2	-1
μ	0.0000	0.0000	0.0001	0.0045	0.0377
μ_0	0.0000	0.0001	0.0008	0.0087	0.0904
σ	0	0.0001	0.0024	0.0260	0.2254
σ_0	0	0.0002	0.0019	0.0180	0.2042
μ	0.0000	0.0001	0.0012	0.0128	0.1090
μ_0	0.0000	0.0002	0.0016	0.0178	0.1550
σ	0	0.0003	0.0080	0.0729	0.8553
σ_0	0	0.0005	0.0065	0.0612	0.5909

Table 3: median μ (resp. μ_0) of reconstruction error $\hat{p}-p$ (resp. $\hat{p}-p_0$), and standard deviation σ (resp. σ_0) with respect to μ (resp. μ_0), over 61 trials of additive noise. Top: uniform noise. Bottom: Gaussian noise.

```
function [mu,Q]=binarydec2(p)
% Decomposition of a generic binary polynomial p
% into the sum of N dth powers of linear forms
% mu: vector of N coefficients
% Q: N by 2 matrix whose rows are the sought forms
s=1;r=0;d=length(p)-1;eta=1.e-4;
fd=facto(d);c=ones(1,d+1);
for i=1:d-1,c(i+1)=fd/facto(i)/facto(d-i);end;
p0=p;p=p./c;v=[];
while s>eta&r<d-r+2, r=r+1;
  M=hankel(p(1:d-r+1),p(d-r+1:d+1));
  [U,S,V]=svd(M);
  J=find(diag(S)<eta);
    if length(J)>0,s=S(J,J);;J=J(1);
    elseif r+1>d-r+1,s=0;J=r+1;
    end;
end;
v=V(:,J);q=roots(v);
Q=[q,ones(length(q),1)];
mu=convd(Q,d)'\p0';
sol=(mu'*convd(Q,d));W=diag(ones(1,d+1)./c);
% Output of the reconstruction error
err=sqrt((sol-p0)*W*(sol-p0)')

function P=convd(q,d)
% Raising of a polynomial q to the dth power
[a,b]=size(q);P=[];
for i=1:a,
  pd=q(i,:);for t=1:d-1,pd=conv(pd,q(i,:));end;
  P=[P;pd];
end;
```

Figure 1: MATLAB code of the algorithm proposed in the binary case.

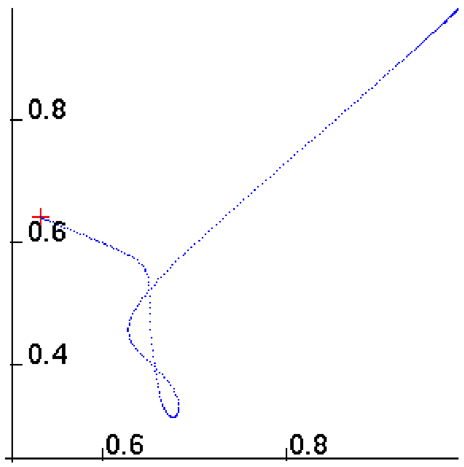

Figure 2: Projection of the approxmation on the plane of the 2 first coordinates. This illustrates the trajectory of the algorithm in a particular case.