Analytical Blind Channel Identification

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Abstract— **In this paper, a novel analytical blind SISO identification algorithm is presented, based on the non-circular second-order statistics of the output. It is shown that statistics of order higher than two are not mandatory to restore identifiability. Our approach is valid, for instance, when the channel is excited by Phase Shift Keying (PSK) inputs. It is shown that the channel taps need to satisfy a polynomial system of degree 2, and that identification amounts to solving the system. We describe the algorithm able to solve this particular system entirely analytically, thus avoiding local minima. Computer results eventually show the robustness with respect to noise and to channel length over-determination. Identifiability issues are also addressed.**

Keywords— **Blind channel estimation, Second order statistics, Non Circularity, Minimum Shift Keying (MSK), Multipath channels, Random Time-varying channels.**

I. Introduction

Blind identification methods depend on the characteristics of the input sources. For example, it is known that a system can only be identified up to an all-pass filter when its input is Gaussian circular. Consequently, particular attention has been paid to the non-Gaussian inputs during the last two decades. In those situations, the phase information can be accessed using high-order statistics of the observations, and in the Single Input Single Output (SISO) case, the system is identified up to a scalar factor only. This has been studied in numerous papers, including the works of Shalvi-Weinstein [24] or Tugnait [28]. Here, we rather focus our attention on the non-circular character of inputs.

An interesting class of non-circular signals is the discrete, which appears in wireless communications. In the SISO case, the discrete character has been used by few authors; Li [15], and Yellin and Porat [31] proposed deterministic approaches. The former is valid for binary inputs and is iterative. To our knowledge, the latter, quite complicated, is the only work available in open literature addressing analytical blind identification of SISO channels with discrete inputs; it includes a clustering stage rather sensitive to noise. On the other hand, the discrete character has been broadly used for equalization [15], but often in an iterative manner [4]; key references are not cited here since equalization is out of the scope of the present paper. The Constant Modulus property, widely used in blind equalization, can hardly be used in blind identification.

The studied signals also have non-zero cyclo-stationary statistics, which allows identification using second-order statistics only [16] [11]. But for those signals it is more interesting to use the cyclo-stationarity as a time diversity, which leads to the study of SIMO systems.

Tong et al. in [26], and Slock in [25] have first taken advantage of oversampling of cyclo-stationary sources. With SIMO systems, second-order statistics only can be used, provided that the channels do not share a common root [23] [1] [30]. In this sense, the SIMO problem can be considered as easier than the SISO, in which one conventionally resorts either to cyclo-stationarity (which induces diversity), or to high-order moments, *e.g.* Constant Modulus Algorithms (CMA). SIMO second order methods can be divided into three families : the Subchannel Response Matching approach (SRM) introduced by Xu [30], subspace methods [18], and linear prediction techniques [25] [2].

In this paper, the oversampling method (inducing a diversity) is not used, *i.e.* only SISO systems are studied. The novelty of our contribution is two-fold. First, only second-order moments are used; they are shown to be sufficient to restore identifiability, without resorting to higher order statistics. Second, an algebraic solution to a class of polynomial systems, constructed from a block of data, is introduced. Our approach is described mainly in the case of Minimum Shift Keying (MSK) modulations, effectively approximating the digital modulation used in the GSM standard, but it holds valid for Differential Binary PSK (DBPSK) or Quadrature PSK (DQPSK) modulations. In addition, block methods are well matched to burst-mode communication systems (TDMA).

For instance, at 900MHz and 190km/h, the coherence time is of order 2ms; in the GSM system, this corresponds to only two bursts, or about 300 symbol periods. This example shows that block algorithms become necessary in a blind context and for reduced coherence times.

The paper is organized as follows. Section II introduces the assumptions made on the input, and the related second-order properties. Section III describes the principles of the novel procedure used to solve polynomial systems; this procedure is detailed in appendix VIII-C, whereas the standard technique of resultants is recalled in appendix VIII-A, but is not used in the paper. The selection of the best solution is described in section IV. Some identifiability results are proved in section V, and computer experiments are eventually presented in section VI.

II. Model and basic properties

Assume a finite sequence of input samples $x(m)$ is fed into a Finite Impulse Response (FIR) linear system of length M, with (a priori complex) taps $h(m), 0 \leq m \leq M-1$. Denote as $y(n)$ the corresponding output sequence

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of length N, satisfying:

$$y(n) = \sum_{m=0}^{M-1} h(m)\, x(n-m) + w(n) \stackrel{\text{def}}{=} \boldsymbol{x}(n;M)^{\mathrm{T}}\boldsymbol{h} + w(n)$$

where $w(n)$ stands for a noise with unknown distribution, and $(^{\mathrm{T}})$ denotes transposition. In a standard manner, multidimensional variables are stored in column vectors and denoted by boldface letters; for instance, $\boldsymbol{x}(n;M) = [x(n), \ldots x(n-M+1)]^{\mathrm{T}}$, by construction.

The input sequence $x(m)$ is i.i.d. and assumed to follow a discrete distribution, stemming from BPSK, MSK, or QPSK digital modulations [22] [5], and the channel $\boldsymbol{h}$ is supposed time-invariant during the observation record, which can be very short. The noise is introduced to take into account modeling errors, and computer experiments are run in section VI for various noise levels. However, noise is ignored in the theoretical developments, so that it is considered only as a nuisance, for its distribution is assumed to be unknown.

Complex Gaussian random variables are nothing else but a pair of real random variables. What allows simpler expressions of its distribution, moments, and related statistical objects, is its circularity, which induces a correlation between real and imaginary parts [29] [12]. For a scalar random variable Z, the circularity property at order 2 is characterized by the equation $\mathrm{E}\{Z^2\} = 0$. A random variable is referred to as *non circular* at order 2 if the latter moment is non-zero.

For non-Gaussian random variables, strict-sense circularity means invariance of the distribution by multiplication of a unit modulus complex number (that is, a rotation in the complex plane), hence the terminology. The concept has been introduced independently in [6] and [21]. Various properties are investigated in depth in [21]. Some statistical aspects have been addressed in [3]. Random variables whose distribution is not circularly invariant are referred to as *non-circular*.

The key statistical property used in this paper is that discrete signals are non-circular at given orders (at order k for a PSK$-k$ random variable [14] [3]). But only second-order statistics are used, so that only **non-circularity at order 2** will be exploited. More precisely, for DBPSK modulated signals, non-circular and circular second-order correlations are given by :

$$\begin{aligned} \mathrm{E}\{x(n)x(n-\ell)|x(0)\} &= x(0)^2\delta(\ell) \\ \mathrm{E}\{x(n)x(n-\ell)^*\} &= \delta(\ell), \end{aligned} \tag{1}$$

respectively. Next, we have for MSK signals :

$$\mathrm{E}\{x(n)x(n-\ell)|x(0)\} = (-1)^n x(0)^2\delta(\ell) \tag{2}$$
$$\mathrm{E}\{x(n)x(n-\ell)^*|x(0)\} = \delta(\ell), \tag{3}$$

and lastly for DQPSK modulated signals:

$$\begin{aligned} \mathrm{E}\{Re\,[x(n)]\,Re\,[x(n-\ell)]\,|x(0)\} &= Re\,[x(0)]^2\,\delta(\ell) \\ \mathrm{E}\{Im\,[x(n)]\,Im\,[x(n-\ell)]\,|x(0)\} &= Im\,[x(0)]^2\,\delta(\ell) \\ \mathrm{E}\{x(n)x(n-\ell)^*\} &= \delta(\ell), \end{aligned}$$

and where $\delta(\ell) \stackrel{\text{def}}{=} 1$ if $\ell = 0$ and $\delta(\ell) = 0$ elsewhere. Note the conditional expectation, necessary under the assumption that the initial value $x(0)$ is uniformly distributed, exhibiting cyclostationarity in the non-circular moment of MSK inputs.

Based on these properties, it is possible to derive a set of polynomial equations that the channel must satisfy. In the MSK case, we obtain:

$$\mathrm{E}\,[y(n)y(n-\ell)|x(0)] = x(0)^2 \sum_{m=0}^{M-1} (-1)^m h(m)h(m+\ell) \tag{4}$$

and in the BPSK case:

$$\mathrm{E}\,[y(n)y(n-\ell)|x(0)] = x(0)^2 \sum_{m=0}^{M-1} h(m)h(m+\ell). \tag{5}$$

In the QPSK case, we consider the equivalent problem (up to a rotation of $\pi/4$) of a QAM4 distributed source, where $x(n)$ is the sum of purely real and purely imaginary binary white processes. It is necessary to consider real and imaginary parts separately at the receiver, because $x(0)^2$ is not deterministic, whereas the real and imaginary parts of $x(n)$ have a deterministic square. This yields four families of equations. For simplicity, setting $Re[x(0)]^2 = Im[x(0)]^2 = 1$ without restricting the generality, one gets :

$$\begin{aligned} E\left[y^r(n)\, y^r(n-\ell)\,|x(0)\right] &= \textstyle\sum_q a(q)a(q+\ell) + b(q)b(q+\ell) \\ E\left[y^r(n)\, y^i(n-\ell)\,|x(0)\right] &= \textstyle\sum_q a(q+\ell)b(q) - a(q)b(q+\ell) \\ E\left[y^i(n)\, y^r(n-\ell)\,|x(0)\right] &= \textstyle\sum_q a(q)b(q+\ell) - a(q+\ell)b(q) \\ E\left[y^i(n)\, y^i(n-\ell)\,|x(0)\right] &= \textstyle\sum_q b(q)b(q+\ell) + a(q)a(q+\ell) \end{aligned} \tag{6}$$

where $a(m)$ and $b(m)$ denote the real and imaginary parts of $h(m)$, and $y^r(n)$ and $y^i(n)$ those of $y(n)$, respectively.

Another obvious possibly would be to use fourth-order moments, which would yield the family of equations :

$$\begin{aligned} &E\left[y(n)y(n-\ell_1)y(n-\ell_2)y(n-\ell_3)|x(0)\right] = \\ &\quad x(0)^4 \textstyle\sum_{m=0}^{M-1} h(m)h(m+\ell_1)h(m+\ell_2)h(m+\ell_3). \end{aligned}$$

In this paper, only polynomial systems of degree 2 will be considered, so that the latter property will not be utilized.

III. Solving the polynomial system

So as to concentrate on principles, we shall explain in details the algorithm in the case of an MSK input, which seems to be a good compromise between simplicity of developments and generality. The algorithm described in section III-D is nevertheless valid for other cases. Without restricting the generality, assume a channel of length $M = 3$. Then from (4) the polynomial system given above based on non-circular statistics can be explicitly written as:

$$\left\{ \begin{aligned} f_1(\boldsymbol{h}) = h(0)^2 - h(1)^2 + h(2)^2 - \alpha_0 &= 0 \\ f_2(\boldsymbol{h}) = h(0)h(1) - h(1)h(2) - \alpha_1 &= 0 \\ f_3(\boldsymbol{h}) = h(0)h(2) - \alpha_2 &= 0 \end{aligned} \right. \tag{7}$$

where $\boldsymbol{h} = \{h(0), h(1), h(2)\}$ denotes the taps vector. The goal of this paper is to solve this polynomial system for taps $h(m)$. A polynomial system having generally several solutions in a finite number, equations provided by standard circular statistics (covariance matching) are used to pick up the best solution in a final stage.

A. Example in the case of a real channel

As a simple particular case, consider a real channel; but outside this section, the channel is **always assumed to be complex with no real roots**, in accordance with identifiability results proved in section V. Then circular statistics yield:

$$\begin{cases} g_1(\boldsymbol{h}) = h(0)^2 + h(1)^2 + h(2)^2 - \beta_0 & = & 0 \\ g_2(\boldsymbol{h}) = h(0)h(1) + h(1)h(2) - \beta_1 & = & 0 \\ g_3(\boldsymbol{h}) = h(0)h(2) - \alpha_2 & = & 0 \end{cases}$$

where $\alpha_i = \mathrm{E}\{y(n)y(n-i)|x(0)\}$ and $\beta_i = \mathrm{E}\{y(n)y(n-i)^*|x(0)\}$ are given (they depend on statistics of observations y). Grouping of those equations results in:

$$\begin{cases} h(0)^2 + h(2)^2 & = & (\alpha_0 + \beta_0)/2 \\ h(0)h(1) & = & (\alpha_1 + \beta_1)/2 \\ h(0)h(2) & = & \alpha_2 \end{cases}$$

Using the first and third equations, one obtains:

$$\begin{array}{rcl} (h(0) - \jmath h(2))^2 & = & h(0)^2 + h(2)^2 - 2\jmath h(0)h(2) \\ & = & (\alpha_0 + \beta_0)/2 - 2\jmath\alpha_2 \end{array}$$

This equation eventually allows to calculate $h(0)$ and $h(2)$ up to a sign, and then $h(1)$.

This particular example shows that it is possible to identify a real channel by using the **non-circular second-order** statistics together with **circular second-order** ones, using a simple elimination procedure. Of course, this was valid only for real channels. For general complex FIR channels, which is the case we are actually interested in, the elimination procedure is more complicated, and is described in section III-D.

B. Preliminaries

Consider the ring $\mathcal{R} = \mathbb{C}[\boldsymbol{\xi}]$ of polynomials in variables $\boldsymbol{\xi} \overset{\text{def}}{=} [\xi(0), \xi(1), \dots \xi(M-1)]$ with coefficients in the complex field $\mathbb{C}$; the dual space of $\mathcal{R}$ is the set of linear forms from $\mathcal{R}$ to $\mathbb{C}$, denoted as $\widehat{\mathcal{R}}$. The evaluation of a polynomial p at a point $\zeta \in \mathbb{C}^M$, denoted as $\mathbf{1}_\zeta : \mathbf{p} \mapsto \mathbf{p}(\zeta)$, is the linear form which most interests us.

Let $\{f_1, \dots, f_M\}$ be polynomials of degree D belonging to $\mathcal{R}$.

Definition III.1: The ideal $\mathcal{I}$ spanned by polynomials $\{f_1, \dots, f_M\}$ is the set of polynomials $p \in \mathcal{R}$ of the form :

$$p = \sum_{i=1}^{M} f_i q_i, \quad \text{with} \quad q_i \in \mathcal{R}.$$

The quotient ring $\mathcal{A} = \mathcal{R}/\mathcal{I}$ is then defined as follows :

Definition III.2: For any ideal $\mathcal{I}$ included in $\mathcal{R}$, the quotient algebra $\mathcal{A} = \mathcal{R}/\mathcal{I}$ is the set of polynomial classes $p \in \mathcal{R}$ modulo ideal $\mathcal{I}$, viz :

$$p \equiv q \quad \text{iff} \quad p - q \in \mathcal{I}$$

The dual space $\widehat{\mathcal{A}}$ of $\mathcal{A}$ is the subspace of $\widehat{\mathcal{R}}$ of linear forms vanishing on the ideal $\mathcal{I}$. In particular, the evaluation $\mathbf{1}_\zeta$ is in $\widehat{\mathcal{A}}$ if and only if ζ is a root of all polynomials belonging to $\mathcal{I}$. This is the fundamental property on which our approach is based.

Given a polynomial $a \in \mathcal{A}$, define the multiplication operator by a as the mapping $\mathcal{M}_a$ that associates q with qa:

$$\begin{array}{rccl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ q & \mapsto & q\,a \end{array} \tag{8}$$

The transposed operator, $\mathcal{M}_a^{\mathsf{T}}$, is by definition the mapping from $\widehat{\mathcal{A}}$ onto itself such that $(\mathcal{M}_a^{\mathsf{T}}\Lambda)(q) = \Lambda(\mathcal{M}_a q)$, $\forall \Lambda \in \widehat{\mathcal{A}}$, $\forall q \in \mathcal{A}$, or equivalently $(\mathcal{M}_a^{\mathsf{T}}\Lambda)(q) = \Lambda(q\,a)$.

C. Lemmas

Let $\mathcal{P}$ be the subset of polynomials $\{f_1, \dots, f_M\}$ of degree D and belonging to $\mathcal{R}$. Bézout's theorem [13, p.227] states that such a system

$$\mathcal{P} \ : \ \{f_m(\boldsymbol{\xi}) = 0, \ 1 \leq m \leq M\}. \tag{9}$$

where $\boldsymbol{\xi} \overset{\text{def}}{=} [\xi(0), \xi(1), \dots \xi(M-1)]$, has either an infinity of solutions, or a number of solutions smaller than or equal to D^M. This extends more well known results for a single polynomial ($M = 1$) or for linear systems ($D = 1$). In the remaining, we consider *generic systems*, having exactly D^M solutions.

When the system has a finite number of solutions, the quotient $\mathcal{A}$ is of finite dimension (in fact, the variety of solutions is zero-dimensional, which implies that $\mathcal{A}$ is of finite dimension because Hilbert's polynomial [8] is of degree zero). Therefore, one conventional way to compute the solutions is to reduce the problem to an eigenvector computation, as shown by the following lemma.

Lemma III.3: Let a be any given polynomial in $\mathcal{R}$. Then the eigenvalues of the multiplication map $\mathcal{M}_a$ in $\mathcal{A}$ are the values of a at the roots of the polynomial system $\mathcal{P}$.

Proof: Consider the polynomial $b(\boldsymbol{\xi}) = \prod_{\boldsymbol{z} \in \mathcal{P}} (a(\boldsymbol{\xi}) - a(\boldsymbol{z}))$. It vanishes at all the roots $\boldsymbol{z} \in \mathcal{P}$. Thus by Hilbert's zero theorem (Nullstellensatz) [8], there exists a positive integer N, such that $b^N \in \mathcal{I}$ or equivalently such that $\prod_{\boldsymbol{z} \in \mathcal{P}} (a(\boldsymbol{\xi}) - a(\boldsymbol{z}))^N \equiv 0$ in $\mathcal{A}$. In terms of operators, this means that

$$\prod_{\boldsymbol{z} \in \mathcal{P}} (\mathcal{M}_a - a(\boldsymbol{z})I)^N \equiv 0$$

where I denotes the identity operator of $\mathcal{A}$. Thus for any eigenpair $(\lambda, \boldsymbol{v})$ of $\mathcal{M}_a$, we have $\prod_{\boldsymbol{z} \in \mathcal{P}} (\mathcal{M}_a - a(\boldsymbol{z})I)^N \cdot \boldsymbol{v} = 0$ and thus

$$\prod_{\boldsymbol{z} \in \mathcal{P}} (\lambda - a(\boldsymbol{z}))^N \cdot \boldsymbol{v} = 0.$$

As $\boldsymbol{v} \neq 0$, λ must be one the values $a(\boldsymbol{z})$, $\boldsymbol{z} \in \mathcal{P}$, which completes the proof. ∎

The reverse inclusion will be proved by lemma III.4, among others.

Besides, if $a = \xi(0)$, the eigenvalues of matrix $\boldsymbol{M}_a$ of operator $\mathcal{M}_a$ give the values of coordinate $\xi(0)$ of the D^M solutions. If we repeat this operation for each tap $\xi(m)$, we have the D^M solutions. However, a somewhat simpler solution is introduced by the following lemma, and avoids the computation of all eigenvalues of every multiplication operator $\mathcal{M}_{\xi(m)}$. In fact, *all* eigenvectors of a *single* operator $\mathcal{M}_a$ are actually required.

Lemma III.4: Linear forms $\mathbf{1}_{\boldsymbol{z}} : \mathbf{p} \mapsto \mathbf{p}(\boldsymbol{z})$, where $\boldsymbol{z}$ is any solution of $\mathcal{P}$, are the eigenvectors of all matrices $(\boldsymbol{M}^{\mathrm{T}}_a)_{a\in\mathcal{A}}$ associated with the eigenvalues $a(\boldsymbol{z})$, $a \in \mathcal{A}$.

Proof: Using the definition of matrix $\boldsymbol{M}^{\mathrm{T}}_a$ and applying it to the linear form $\mathbf{1}_{\boldsymbol{z}}$, we get :

$$\boldsymbol{M}^{\mathrm{T}}_a(\mathbf{1}_{\boldsymbol{z}})(q) = \mathbf{1}_{\boldsymbol{z}}(\mathbf{aq}) = \mathbf{a}(\boldsymbol{z})\mathbf{1}_{\boldsymbol{z}}(\mathbf{q}) \quad \forall \mathbf{q} \in \mathcal{A}$$

in other words, we have : $\boldsymbol{M}^{\mathrm{T}}_a(\mathbf{1}_{\boldsymbol{z}}) = \mathbf{a}(\boldsymbol{z})\mathbf{1}_{\boldsymbol{z}}$. Therefore, $\mathbf{1}_{\boldsymbol{z}}$ and $a(\boldsymbol{z})$ are respectively the eigenvectors and eigenvalues of matrix $\boldsymbol{M}^{\mathrm{T}}_a$. This proves the lemma. ∎

If eigenvalues are not distinct, some eigenvectors are not uniquely determined, which makes the previous lemma less useful. Yet, it can be proved that the common eigenvectors of all operators $\mathcal{M}^{\mathrm{T}}_a$ are exactly the evaluation forms $\mathbf{1}_{\boldsymbol{z}}$ [19]. So a solution consists of taking several forms, say $a(\xi)$ and $b(\xi)$, instead of a single one, $a(\xi)$. $\boldsymbol{M}_a$ and $\boldsymbol{M}_b$ commute, and have the same eigenspaces. The indeterminacy can be handled in this way, but it is not reported here in detail.

Hence, the computation of the multiplication matrix $\boldsymbol{M}_a$ appears as a key step in the proposed algorithm, since the eigenvectors of $\boldsymbol{M}^{\mathrm{T}}_a$ allow to find all the solutions of $\mathcal{P}$. Indeed, if we take for a basis of $\mathcal{A}$ the set $\mathcal{B} = \{1, \xi(0), \xi(1), \cdots, \xi(0)\xi(1), \cdots\}$ of monomials of global degree at most D, the entries of the eigenvector $\mathbf{1}_{\mathbf{z}}$ are equal to $\{1, z(0), z(1), \cdots, z(0)z(1), \cdots\}$ in the dual basis of $\mathcal{B}$, where $\boldsymbol{z}$ stands for any possible solution of $\mathcal{P}$. More precisely, entries 2 to $M+1$ of any eigenvector of $\boldsymbol{M}^{\mathrm{T}}_a$, whose first entry is normalized to 1, yield a solution $\boldsymbol{h}$ to $\mathcal{P}$. This property is used in the numerical algorithm of section III-D by merely choosing $a = \xi(0)$. Any other polynomial could have made it.

D. Computing $\boldsymbol{M}_a$ directly from the polynomial system

As already mentioned, we prefer a more direct approach than that (more standard) described in section VIII-A, for computational reasons. In order to simplify the discussion, we will use the following example: suppose that a channel of length $M = 3$ is excited by a MSK input. System $\mathcal{P}$ is then equal to that in (7). In that case, the computation of the multiplication matrix can be split into 5 steps.

First step : Change in variables

Suppose we use the following change in variables : $\boldsymbol{z} = \boldsymbol{T}^{-1}\boldsymbol{h}$. The system in $\boldsymbol{z}$ is implicitly defined by the system $\mathcal{P}$ in $\boldsymbol{h}$:

$$\boldsymbol{P}\mathcal{H} = \mathbf{0} \tag{10}$$

where $\mathcal{H} = [1, h(0), h(1), h(2), h(0)h(1), h(0)h(2), h(1)h(2), h(0)^2, h(1)^2, h(2)^2]$. The matrix $\boldsymbol{T}$ is chosen so that a simple basis can always be found for $\mathcal{A}$ (that basis will contain monomials of degree at most 1 in every variable). Any choice of $\boldsymbol{T}$ (by drawing it randomly) would lead to a system that can be solved by $\mathcal{B}_3$ with probability one. Moreover, if $\boldsymbol{T}$ is not well chosen, the algorithm detects it because one of the matrices involved in the subsequent steps is rank deficient.

The next steps consist of expressing second, third, and fourth degree monomials in the basis. We give below the general procedure, but for more clarity, these steps are explained in details in the particular case of system (7), of 3 equations of degree 2 in 3 variables, in appendix VIII-C.

Second step : choosing a basis

A basis that can generically solve our kind of polynomial systems is composed of the neutral element 1, the M unknowns $z(m)$, and all the cross monomials of degree less than or equal to M, where each unknown appears with a power equal to 1 or 0. In fact, it has been proved by Macaulay that such a basis is sufficient when there are no zeros at infinity (generic case) [27]. Here are two examples : $M = 2$ and $M = 3$.

- Channel length $M = 2$:

$$\mathcal{B}_2 = \{1, z(0), z(1), z(0)z(1)\}$$

- Channel length $M = 3$:

$$\mathcal{B}_3 = \{1, z(0), z(1), z(2), z(0)z(1), z(0)z(2), z(1)z(2), z(0)z(1)z(2)\}$$

In the following, the column vector containing the elements of the basis $\mathcal{B}_i$ will be denoted as $\boldsymbol{b}_i(\boldsymbol{z})$. In this example, $\boldsymbol{b}_3(\boldsymbol{z})$ is of size 8. System (10) can then be rewritten as $\boldsymbol{A}\,\boldsymbol{b}_3(\boldsymbol{z}) = \mathbf{0}$, for some matrix $\boldsymbol{A}$ depending on $\boldsymbol{P}$ and $\boldsymbol{T}$ only.

This basis can always be used in our problem [27] because the system is a complete intersection (there are a finite number of solutions), and we first applied a generic change in the variables (there are no zeros at infinity anymore). The reason is that in (7), some equations linked monomials $h(i)h(j)$ and 1, which would have not been linearly independent in $\mathcal{A}$.

Third step : Expression of the second degree monomials

Suppose we want to find in $\mathcal{B}_3$ the matrix associated with multiplication by $z(0)$. The monomials to be expressed are, if $M = 3$:

Monomials of the basis		Monomials to be expressed
1		$z(0)$
$z(0)$		$z(0)^2$
$z(1)$		$z(0)z(1)$
$z(2)$		$z(0)z(2)$
$z(0)z(1)$	$\xmapsto{\times z(0)}$	$z(0)^2z(1)$
$z(0)z(2)$		$z(0)^2z(2)$
$z(1)z(2)$		$z(0)z(1)z(2)$
$z(0)z(1)z(2)$		$z(0)^2z(1)z(2)$

Some of these monomials are already in the basis, such as $z(0)$, $z(0)z(1)$, $z(0)z(2)$ and $z(0)z(1)z(2)$. The other monomials, $z(0)^2$, $z(0)^2z(1)$, $z(0)^2z(2)$ and $z(0)^2z(1)z(2)$, have to be expressed using the polynomial system.

According to equation (10), monomials $z(0)^2$, $z(1)^2$ and $z(2)^2$ can be expressed directly as a function of 1, $z(0)$, $z(1)$, $z(2)$, $z(0)z(1)$, $z(0)z(2)$ and $z(1)z(2)$, provided that $\boldsymbol{T}$ is chosen correctly. In other words, monomials $z(0)^2$, $z(1)^2$ and $z(2)^2$ can be expressed directly as a function of the basis using equation (10) :

$$\begin{bmatrix} z(0)^2 \\ z(1)^2 \\ z(2)^2 \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} 1 \\ z(0) \\ z(1) \\ z(2) \\ z(0)z(1) \\ z(0)z(2) \\ z(1)z(2) \end{bmatrix}, \tag{11}$$

for some matrix $\boldsymbol{B}$ depending on $\boldsymbol{A}$ only. Therefore, the monomial $z(0)^2$ is now expressed in the basis $\mathcal{B}_3$; in fact $z(0)z(1)z(2)$ has here a null coefficient.

Fourth step : expression of the third degree monomials

Monomials $z(0)^2z(1)$ and $z(0)^2z(2)$ are now of interest. These monomials can be written using the expression of the monomial $z(0)^2$ in the first row of equation (11). If we multiply this equation by $z(1)$, monomial $z(0)^2z(1)$ appears in the left-hand side and monomials $z(1)$, $z(0)z(1)$, $z(1)^2$, $z(2)z(1)$, $z(0)z(1)^2$, $z(0)z(1)z(2)$ and $z(1)^2z(2)$ appear in the right-hand side. Among these monomials, one can distinguish those that are in the basis like $z(1)$, $z(0)z(1)$, $z(2)z(1)$ and $z(0)z(1)z(2)$, those which have already been expressed in the basis, like $z(1)^2$, and those which are unknown, like $z(1)^2z(2)$ and $z(0)z(1)^2$. However, these unknown monomials are of the same type as monomials $z(0)^2z(1)$ and $z(0)^2z(2)$, and one can show that expressing monomials $z(0)^2z(1)$, $z(0)^2z(2)$, $z(1)^2z(0)$, $z(1)^2z(2)$, $z(2)^2z(0)$ and $z(2)^2z(1)$ all together using (11) leads to :

$$\begin{bmatrix} z(0)^2z(1) \\ z(0)^2z(2) \\ z(1)^2z(0) \\ z(1)^2z(2) \\ z(2)^2z(0) \\ z(2)^2z(1) \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} 1 \\ z(0) \\ z(1) \\ z(2) \\ z(0)z(1) \\ z(0)z(2) \\ z(1)z(2) \\ z(0)z(1)z(2) \end{bmatrix} \tag{12}$$

for some matrix $\boldsymbol{C}$ depending on $\boldsymbol{B}$ only.

Fifth step : expression of the fourth degree monomials

Using the same method as before, one can express monomials such as $z(0)^2z(1)z(2)$ using equation (12). See appendix VIII-C for more details.

Having expressed all the monomials in the chosen basis, the multiplication matrix can be constructed. Once the multiplication matrix $\boldsymbol{M}_a$ is found for the arbitrarily chosen $a = z(0)$, one computes the 2^M eigenvectors of $\boldsymbol{M}_a^{\mathrm{T}}$, $\boldsymbol{v}^{(m)}$. Next, all the possible solutions $\boldsymbol{h}^{(m)}$ to the polynomial system are obtained as $\boldsymbol{z}^{(m)} = \boldsymbol{v}^{(m)}(2 : M + 1)/\boldsymbol{v}^{(m)}(1)$. Then, the solutions in the original coordinate system are given by $\boldsymbol{h}^{(m)} = \boldsymbol{T}\boldsymbol{z}^{(m)}$. See section VIII-C for details.

For the sake of clarity, we have described above the elimination algorithm for a channel length of $M = 3$, but the principles hold exactly the same for larger values of M (with a larger number of unknowns), such as $M = 5$ as in some subsequent computer simulations. As explicited in [27], the general algorithm goes along the lines described in table I.

IV. Estimation of the channel

Selection of a solution. In a final step, one chooses the solution $\boldsymbol{h}$ among $\{\boldsymbol{h}^{(m)}, 1 \le m \le 2^M\}$ that best matches the actual channel, by a moment matching method, as we explain now.

A polynomial system rarely admits a unique solution, regardless of the number of unknowns. So it is very likely that we obtain in practice 2^M distinct solutions, that we can denote as $\boldsymbol{h}^{(m)} = [h^{(m)}(0), ...h^{(m)}(M-1)]$, $1 \le m \le 2^M$. However, circular statistics (3) have yet to be utilized. With this goal, denote:

$$R_y(\ell) \stackrel{\text{def}}{=} \mathrm{E}\{y(n)y(n-\ell)^*|x(0)\}. \tag{13}$$

Then, because of (3), we have the well known phase-blind relation

$$R_y(\ell) = \sum_{i=\ell}^{M-1} h(i)h(i-\ell)^* + R_w(\ell), 0 \le \ell \le M-1 \tag{14}$$

The procedure proposed in this paper consists of choosing the solution $\boldsymbol{h}$ minimizing the distance:

$$\boldsymbol{h} = Arg \min_{\boldsymbol{h}^{(m)}} \sum_{\ell=0}^{M-1} |R_y(\ell) - \sum_{i=\ell}^{M-1} h^{(m)}(i)h^{(m)}(i-\ell)^*|^2, \tag{15}$$

hence the name of *moment matching* method. The whole algorithm is summarized in Table II.

Computational complexity. It is worth noting that the largest part of the computational load consists of building the multiplication matrix, $\boldsymbol{M}_a$, which depends on the modulation (discrete alphabet and trellis). Yet, it can be shown [9] that this matrix $\boldsymbol{M}_a$ is itself a polynomial function of the data moments, when the polynomial system $\mathcal{P}$ has no infinite solution. Thus in an operational context, for a given modulation, one can only store the polynomial coefficients of $\boldsymbol{M}_a$ in a ROM; as a consequence, the computation of $\boldsymbol{M}_a$ becomes negligible, and the overall data-dependent computations are dominated by the calculation of the eigenvectors of $\boldsymbol{M}_a^{\mathrm{T}}$.

V. Identifiability

It is well known that blind identifiability can be carried out only up to a complex multiplicative factor. Therefore, there are infinitely many solutions. However, if we arbitrarily fix the source variance to 1, and the scalar phase inherent indeterminacy, which contains a fixed phase and a delay z^γ, then identifiability results can be stated. Thus, we may assume from now on that the channel is causal of degree $M-1$, and we want to prove that the joint use of circular and non-circular second-order output correlation functions yields an *essentially* unique solution (*i.e.* up to a unit modulus multiplicative factor and up to a time delay).

Lemma V.1: Suppose we look for a causal FIR channel H of length M, from given second-order circular statistics; then the number of solutions is *essentially* finite and bounded by 2^{M-1}.

Proof: The z-transform of the circular covariance $c(n)$ of the output $y(n)$ is equal to $C(z) = H(z)H^*(1/z^*)$, since the input is white and of unit variance. This shows that if $H(z)$ is causal, it can be determined from $C(z)$ up to two kinds of indeterminacies: first, $H(z)$ can only be determined up to a multiplicative constant phase factor. Second, if $H(z)$ is transformed into $H(z)\Phi(z)$ where $\Phi(z)$ verifies $\Phi(z)\Phi^*(1/z^*) = 1$, $C(z)$ remains the same. It is well known that $\Phi(z)$ is then an all-pass filter, *i.e.* of the following form, up to a delay :

$$\Phi(z) = \prod_{i=1}^{Q} \frac{1-a_i^* z^{-1}}{a_i - z^{-1}}$$

Since $H(z)$ must be FIR and since $\Phi(z)$ is not FIR, $H(z)\Phi(z)$ is FIR only if each pole of $\Phi(z)$ is associated with one of the $M-1$ roots of $H(z)$. As a consequence, there is a finite number of all-pass filters such that $H(z)\Phi(z)$ is FIR. Therefore, if the phase indeterminacy is fixed, there are at most 2^{M-1} possible FIR filters that correspond to $C(z)$. If the roots of $H(z)$ are not all distinct, or of unit modulus, there are indeed fewer possibilities than 2^{M-1}. This proves lemma V.1, which has been known for many years. ∎

Theorem V.2: Suppose we look for a FIR channel H of length M from given second-order circular and non-circular statistics of the output when the input is stationary white. Then, up to complex phase and time delay indeterminacies, we have :

- a unique solution if H has no real root,
- 2^Q solutions if H has Q distinct real roots.

Proof: Suppose now that we also use the non-circular covariance $\overline{c}(\ell) \stackrel{\text{def}}{=} \mathrm{E}\{y(n)y(n-\ell)|x(0)\}$. Its z-transform is equal to $\overline{C}(z) = x(0)^2\, H(z)H(1/z)$. Yet, one can easily show that rational filters satisfying $\Psi(z)\Psi(1/z) = 1$ are of the following form, up to a delay :

$$\Psi(z) = \prod_{j} \frac{1-b_j z^{-1}}{b_j - z^{-1}}$$

Now using statements made in the proof of lemma V.1, all-pass rational filters $\Phi(z)$ that also satisfy $\Phi(z)\Phi(1/z) = 1$ must have real poles (and zeros). As a consequence, if $H(z)$ has no real roots, $\Phi(z)$ cannot have real poles since $H(z)\Phi(z)$ is FIR, and the all-pass filter $\Phi(z)$ must be equal to ± 1. In this case, there is an *essentially* unique solution for $H(z)$, up to a sign. The fact that $x(0)$ is known or not is of no importance, since it is of unit modulus and can be pulled into the inherent indeterminacies. If $H(z)$ has Q real roots, one must use the result of lemma V.1. The number of solutions is then *essentially* equal to 2^Q. ∎

When the input source is MSK, it is white but not stationary. However, identifiability still holds.

Corollary V.3: When the input source is white MSK, the joint use of second-order circular and non-circular statistics of the output yield a unique solution for $H(z)$ up to a sign.

The proof is given in appendix VIII-B. These identifiability results justify the algorithm we have proposed, summarized in section IV.

VI. Computer results

Tests are run on a random FIR channel ($M = 5$). At each run the channel is a realization of a Clarke filter in the Typical Urban mode. Every channel generated is specular, and contains 6 paths, whose delays and attenuations are given in the table below, according to the ETSI norm (GSM 05.05, December 1995, 6 taps TU setting):

τ_i (μs)	0	0.2	0.5	1.6	2.3	5.0
$\mathrm{E}\{a_i^2\}$ (dB)	-3	0	-2	-6	-8	-10

In other words, each coefficient is a random complex circular Gaussian variable whose standard deviation is given in the table. Note that the first path is not the strongest. The symbol rate is 271 ksymbol/s, which corresponds to a symbol period of 3.68μs, whereas the delays above are also given in μs (these cases are those of GSM). The transmit filter is a raised cosine with roll-off $\beta = 0.1$, and four samples have been generated per symbol period. The channel is excited by a MSK input. The performance is presented as a function of the SNR and of the length N of the observation block, and averaged over 500 independent channel trials.

Figure 1 shows the average Bit Error Rate obtained at the output of a Viterbi equalizer that uses our channel estimate. The solid, dashed and dash-dotted lines correspond to block lengths $N = 200$, $N = 500$ and $N = 1000$ respectively. These performances are compared to the average BER obtained with the actual channel (dotted line).

For high SNRs and $N = 200$ or $N = 500$ the results show the effects of moment estimation errors. These effects disappear for $N = 1000$, where the performances exhibit a loss of $2dB$ compared to the actual channel results.

Figure 2 shows the average mean squares distance between the actual and the estimated channel. The solid, dashed and dash-dotted lines correspond again to block lengths $N = 200$, $N = 500$ and $N = 1000$ respectively.

It appeared relevant to test the robustness of the algorithm with respect to channel over-estimation.

Keeping $M = 5$ in the algorithm, tests have been run on a random FIR channel of actual length $M = 3$. Figure 3 shows the average Bit Error Rate obtained at the output of a Viterbi equalizer that uses our channel estimate when $\hat{M} = 3$ (solid line) and $\hat{M} = 5$ (dashed line). These two performances are compared to the one obtained when the actual channel is used (dash-dotted line). Hence, this test illustrates the loss in performance encountered in the presence of channel order over-determination.

Experiments for QPSK modulated inputs have not been run; but one can either treat real and imaginary parts separately, as explained in section II, or use fourth order moments. Of course, the former is preferred (*e.g.* for IS95 standard), but the latter seems unavoidable in the case of $\pi/4-$QPSK modulations, found for example in the IS54 standard.

VII. Concluding remarks

The blind identification method described in the paper is based only on second-order statistics of the observation. We proved that it is not necessary to resort to higher order statistics, explicitly or implicitly (like in CMA), to identify a SISO channel; next, we gave an analytical solution, thus free of local extrema problems. In addition, identifiability results were derived when a FIR channel is searched for, from both circular and non-circular moments. Finally, computer results showed that the behavior of our algorithm depends on the estimation quality of the moments, and on the accuracy of the determination of the channel length.

The extension to SIMO channels is rather straightforward, and consists essentially of code writing and computer experiments. In fact, there is no fundamental change in the algorithm. However, there exist specific algorithms that work in the SIMO case and not in the SISO case. For this reason, we believe our algorithm is more attractive in the SISO case.

VIII. Appendix

A. Computation of matrix $\boldsymbol{M}_a$ by resultants

Many methods can be used to compute the multiplication matrix. Among them, the Gröbner bases are the most well known. They can be used to build a base of the quotient algebra $\mathcal{A}$ and then to compute the multiplication matrix with normal forms computation. However, this method must be run in exact arithmetic and thus implies the use of big numbers. Another use of Gröbner basis consists of the elimination of $M-1$ variables and leads to the computation of the roots of a mono-variate polynomial of degree D^M. Since these methods are too expensive in terms of computation load, we prefer to use a modification of the old method by Macaulay [17], used to build resultants.

This approach, discussed in this section for the sake of completeness, can be considered as an extension of the Sylvester theorem to multivariate polynomials, and thus could seem natural. However, another simpler (but less standard) approach is discussed in sections III-D and VIII-C, and has been used throughout the paper. We now present Sylvester and Macaulay matrices and their properties.

A.1 Sylvester matrices

Sylvester matrices are matrices S that represent maps of the form :

$$\begin{array}{rcl} \mathcal{S}:\ V_0\times V_1 & \to & V \\ (q_0,q_1) & \mapsto & q_0 f_0 + q_1 f_1 \end{array}$$

where V_0, V_1 and V are the spaces generated by monomials $\{1,\cdots,x^{d_1-1}\}$, $\{1,\cdots,x^{d_0-1}\}$ and $\{1,\cdots,x^{d_0+d_1-1}\}$, respectively. Matrix S, of size $d_0+d_1\times d_0+d_1$, has thus the following structure :

$$S = \begin{bmatrix} \boldsymbol{f}_0 & & & \boldsymbol{f}_1 & & \\ & \ddots & & & \ddots & \\ & & \boldsymbol{f}_0 & & & \boldsymbol{f}_1 \end{bmatrix}$$

where the entries of $\boldsymbol{f}_i$ are the coefficients of the polynomials f_i, so that $f_0 = [1,\cdots,x^{d_0}]\,\boldsymbol{f}_0$ and $f_1 = [1,\cdots,x^{d_1}]\,\boldsymbol{f}_1$. The determinant of S is also called the resultant of polynomials f_0 and f_1. Of course, the construction procedure also applies for more than 2 polynomials.

A.2 Macaulay matrices

The approach described above can be generalized to multi-variate polynomials. Let $f_0,\ f_1,\cdots,\ f_M\ \in \mathcal{R}$, be $M+1$ polynomials of degree $d_0,\cdots,d_M$ in variables $\boldsymbol{\xi} = \{\xi(1),\cdots,\xi(M)\}$ (Note that in the algorithm of section III-D, all degrees d_i are equal to D, except $d_0 = 1$). Macaulay matrices [17] used to build the resultant of these polynomials are matrices associated with maps of the form :

$$\begin{array}{rcl} \mathcal{S}:\ V_0\times\cdots\times V_M & \to & V \\ (q_0,\ldots,q_M) & \mapsto & \displaystyle\sum_{i=0}^{M} q_i f_i \end{array}$$

where the V_m are subspaces spanned by a finite number of monomials in variables $\{\xi(1),\cdots,\xi(M)\}$, which are denoted as $V_m =<\boldsymbol{\xi}^{E_m}>$, where $\boldsymbol{\xi}^{E_m} = \{\xi^{\boldsymbol{\beta}},\boldsymbol{\beta}\in E_m\}$ and $\xi^{\boldsymbol{\beta}} = \xi_1^{\beta_1}\ldots\xi_M^{\beta_M}$. In this notation, E_m is the set of respective powers of the variables of the monomials that span the subspace V_m, $E_m = \{\boldsymbol{\beta}_{m,1},\cdots,\boldsymbol{\beta}_{m,N_m}\}$, each $\boldsymbol{\beta}_{m,n}$ containing M integers. In a similar manner, define F such that $V =<\boldsymbol{\xi}^F>$.

The matrix S associated with this multi-variate map, $\mathcal{S}$, is such that its columns are the image, in the canonical basis, of the monomials $\xi^{\beta_{m,i}}$ in the polynomials f_m. If d_m denote the degree of polynomial f_m, it can be shown that $\mathcal{S}$ is of dimension $\binom{\nu+M}{M}$, where $\nu = \sum_i d_i - M$. In other words, its size is that of the basis $\mathcal{B}$.

Matrix S can be split in $M+1$ blocks : $S = [S_0,S_1,\cdots,S_M]$, where block S_m is the canonical representation of the image subspace of the monomials $\boldsymbol{\xi}^{E_m}$

by the polynomial f_m. Let us now explain the construction procedure on an example.

Example. Suppose that $M = 2$, $D = 2$, $f_0(x, y) = a_0 x$, $f_1(x, y) = a_1 x^2 + b_1 y$, and $f_2(x, y) = a_2 y + b_2$.

The critical degree is $\nu = 1 + 2 + 1 - 2 = 2$ in this example, and we arbitrarily choose the basis $\xi^F = \{1, x, y, x^2, xy, y^2\}$.

At each step $M - m + 1$ of the procedure, $M \geq m \geq 1$, one looks for monomials left in ξ^F that can be divided by $\xi_m^{d_m}$; in order to construct a basis of space V_m, one divides these monomials by $\xi_m^{d_m}$. *Step 1*: monomials of ξ^F divisible by $\xi_2^{d_2} = y$ are $\{y, xy, y^2\}$. Ones sets $V_2 = \{1, x, y\}$, removes $\{y, xy, y^2\}$ from ξ^F, and go to the next step. *Step 2*: monomials of $\{1, x, x^2\}$ divisible by $\xi_1^{d_1} = x^2$ reduce to $\{x^2\}$. Thus, one sets $V_1 = \{1\}$ and removes $\{x^2\}$ from what remained in ξ^F. It remains eventually $\{1, x\}$ in ξ^F, which consists of the basis of V_0.

Every row of S is associated with a monomial of ξ^F, in a preassigned order; here we choose arbitrarily $\xi^F = \{1, x, y, x^2, xy, y^2\}$. The first 2 columns span the space $f_0(V_0)$, the third one spans $f_1(V_1)$, and the last 3 columns span $f_2(V_2)$:

$$S = \left|\begin{array}{cc|c|ccc} 0 & 0 & 0 & b_2 & 0 & 0 \\ a_0 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & b_1 & a_2 & 0 & b_2 \\ 0 & a_0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 \end{array}\right|$$

The sub-matrices comprised within vertical bars can be denoted as S_0, S_1 and S_2, respectively.

The sets of exponents E_m associated with V_m are the following: $E_0 = \{[0, 0], [1, 0]\}$, $E_1 = \{[0, 0]\}$, $E_2 = \{[0, 0], [1, 0], [0, 1]\}$, and the set F associated with the whole space V is $F = \{[0, 0], [1, 0], [0, 1], [2, 0], [1, 1], [0, 2]\}$, and actually describes $\mathcal{B}$.

A.3 Properties of Macaulay's matrices

The proof of the following theorems are not given but can be found in [10] [17] [20].

Theorem VIII.1: Consider the generic polynomial system $f_1, \cdots, f_M$, choose a polynomial f_0 and build the Macaulay matrix S associated with these polynomials. The set of monomials $\boldsymbol{\xi}^{E_0}$ is a basis of the quotient algebra $\mathcal{A} = \mathcal{R}/\left(f_1, \cdots, f_M\right)$, and where E_0 is defined in section VIII-A.2.

Hence, the construction of the Macaulay matrix yields a basis of the quotient algebra. The following theorem gives a means to compute the matrix associated with the multiplication by f_0 thanks to the Macaulay matrices.

Theorem VIII.2: Suppose E_0 is a subset of F, the set of exponents indexing the rows of S; this is the case if f_0 includes a constant term. Then matrix S rewrites :

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the rows and the columns of A are indexed by the monomials $\boldsymbol{\xi}^{E_0}$, and the columns of B and D are indexed by the monomials $\{\boldsymbol{\xi}^{E_m}\}$, where m is different from zero. Then, for all polynomial systems $f_1, \cdots, f_M$ the matrix of the multiplication by f_0 in the quotient algebra $\mathcal{A} = \mathcal{R}/\left(f_1, \cdots, f_M\right)$ is the Schur complement of the block D in the matrix S :

$$\boldsymbol{M}_{f_0} = A - BD^{-1}C$$

Thus, matrix $\boldsymbol{M}_a$ can be directly computed from the Macaulay matrix associated with polynomials $\{f_1, \ldots, f_M\}$ [7]. However, if we take into account the relationships between the monomials introduced in the polynomial system $\mathcal{P}$, there exists a much simpler procedure to compute $\boldsymbol{M}_a$, explained in sections III-D and VIII-C.

B. Proof of Corollary V.3

Proof: When the input source is MSK, let us first show that the non-circular covariance $\overline{c}(\ell; n) \stackrel{\text{def}}{=} \mathrm{E}\{y(n)y(n-\ell)|x(0)\}$ has as $z-$transform given by:

$$\overline{C}(z; n) = (-1)^n x(0)^2 H(-z) H(1/z) \tag{16}$$

In fact, $\overline{c}(\ell; n) = \sum_p \sum_q h(p)h(q)\mathrm{E}\{x(n-p)x(n-q-\ell)|x(0)\}$, which, from (2) yields:

$$\overline{c}(\ell; n) = (-1)^n x(0)^2 \sum_p \sum_q (-1)^p h(p)h(q)\delta(p - \ell - q)$$

or $\overline{c}(\ell; n) = (-1)^n x(0)^2 \sum_p (-1)^p h(p)h(p-\ell))$. Now take the $z-$transform over time index ℓ, and get $\overline{C}(z; n) = (-1)^n x(0)^2 \sum_\ell \sum_p (-1)^p h(p)h(p-\ell)z^(- \ell)$. Making the change of variables $k = p - \ell$ eventually gives (16).

Now, suppose we have one solution $H_o(z)$ satisfying (16). From the result above, we see that the whole set of rational filters satisfying (16) is then generated by $H_o(z)\Upsilon(z)$, where $\Upsilon(-z)\Upsilon(1/z) = 1$ and where poles of $\Upsilon(z)$ coincide with zeros of $H_o(z)$.

Yet, it can be easily shown that such filters take the following form, up to an even delay :

$$\Upsilon(z) = \prod_k \jmath \frac{1 + b_k z^{-1}}{b_k - z^{-1}}$$

Now it can be seen that the only all-pass filter, thus satisfying $\Phi(z)\Phi^*(1/z^*)$, also satisfying $\Phi(-z)\Phi(1/z) = 1$, is $\Phi(z) = \pm 1$. Therefore, the whole set of solution reduces to $\pm H_o(z)$, up to a delay. ■

C. Computational details

In this section, we detail the steps described in section III-D in the particular case of system (7), which contains 3 equations of degree 2 in 3 variables. Computer codes can be downloaded from the web page of the second author. From the Bézout lemma quoted in section III-C, we already know that this system will have generally $2^3 = 8$ solutions.

C.1 First step.

The identity matrix is not suitable for $\boldsymbol{T}$ because some steps in the procedure will involve singular matrices. This comes from the fact that pure squares all appear in the first equation only. Thus, the following choice is made:

$$\boldsymbol{h} = \boldsymbol{T}\boldsymbol{z}, \quad \boldsymbol{T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \tag{17}$$

C.2 Second step.

One chooses the basis $\mathcal{B}_3$, which contains the eight monomials below, that we can arrange in a vector:

$$\begin{aligned} \boldsymbol{b}_3 \;=\; & [1, z(0), z(1), z(2), z(0)z(1), \\ & z(0)z(2), z(1)z(2), z(0)z(1)z(2)]^{\mathrm{T}} \end{aligned}$$

The system (7) can now be expressed in terms of the new variables, $z(i)$, as:

$$\boldsymbol{A} \begin{bmatrix} 1 \\ z(0)z(1) \\ z(0)z(2) \\ z(1)z(2) \\ z(0)^2 \\ z(1)^2 \\ z(2)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{18}$$

with:

$$\boldsymbol{A} = \begin{bmatrix} \alpha_0 & -4 & 2 & 2 & -3 & -2 & 0 \\ \alpha_1 & -2 & -1 & -2 & -2 & 0 & 1 \\ \alpha_2 & 2 & -2 & -1 & 0 & 1 & 0 \end{bmatrix}$$

The goal is now to express the matrix $\boldsymbol{M}_{z(0)}$ of operator $\mathcal{M}_{z(0)}$ in the basis $\mathcal{B}_3$. For doing this, we need to have the expression of all monomials of $z(0)\boldsymbol{b}_3$, in particular monomials such as $z(0)^2$, $z(0)^2z(1)$, or $z(0)^2z(1)z(2)$, as a function of monomials of $\boldsymbol{b}_3$ itself.

C.3 Third step.

The pure squares $z(i)^2$ are not in $\mathcal{B}_3$, but we can obtain from (18), by isolating the squares in the left-hand side:

$$\begin{bmatrix} z(0)^2 \\ z(1)^2 \\ z(2)^2 \end{bmatrix} = \boldsymbol{A}_1^{-1}\,\boldsymbol{B}_1\,\boldsymbol{b}_3(\boldsymbol{z}) \stackrel{\text{def}}{=} \boldsymbol{C}_1\,\boldsymbol{b}_3(\boldsymbol{z}) \tag{19}$$

where matrices $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ are functions of $\boldsymbol{A}$:

$$\begin{aligned} \boldsymbol{A}_1 &= [A(:,5)\,A(:,6)\,A(:,7)] \\ \boldsymbol{B}_1 &= -[A(:,1)\,\mathbf{0}\,\mathbf{0}\,\mathbf{0}\,A(:,2)\,A(:,3)\,A(:,4)\,\mathbf{0}] \end{aligned}$$

and where $A(:,i)$ denotes the ith column of $\boldsymbol{A}$ (as in the Matlab notation).

C.4 Fourth step.

Let us turn now to monomials of the form $z(i)^2z(j)$, $i \neq j$. Take the first row of (19), and multiply it by $z(1)$. A careful inspection then shows that $z(0)^2z(1)$ depends only on $\{z(1), z(0)z(1)^2, z(0)z(1)z(2), z(1)^2z(2)\}$, because of the presence of zeros in matrix $\boldsymbol{C}_1$. Similarly, $z(0)^2z(2)$ depends only on $\{z(2), z(0)z(1)z(2), z(0)z(2)^2, z(1)z(2)^2\}$. In order to solve this problem, it suffices to write jointly the other equations, obtained on one hand by multiplying the second row of (19) by $z(0)$ and $z(2)$, and on the other hand by multiplying the third row of (19) by $z(0)$ and $z(1)$. This eventually yields the following system, after some manipulations:

$$\begin{bmatrix} z(0)^2z(1) \\ z(0)^2z(2) \\ z(1)^2z(0) \\ z(1)^2z(2) \\ z(2)^2z(0) \\ z(2)^2z(1) \end{bmatrix} = -\boldsymbol{A}_2^{-1}\,\boldsymbol{B}_2\,\boldsymbol{b}_3(\boldsymbol{z}) \stackrel{\text{def}}{=} \boldsymbol{C}_2\,\boldsymbol{b}_3(\boldsymbol{z}) \tag{20}$$

where

$$\boldsymbol{A}_2 = \begin{bmatrix} -1 & 0 & C_1(1,5) & C_1(1,7) & 0 & 0 \\ 0 & -1 & 0 & 0 & C_1(1,6) & C_1(1,7) \\ C_1(2,5) & C_1(2,6) & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & C_1(2,6) & C_1(2,7) \\ C_1(3,5) & C_1(3,6) & 0 & 0 & -1 & 0 \\ 0 & 0 & C_1(3,5) & C_1(3,7) & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{B}_2 = \begin{bmatrix} 0 & 0 & C_1(1,1) & 0 & 0 & 0 & 0 & C_1(1,6) \\ 0 & 0 & 0 & C_1(1,1) & 0 & 0 & 0 & C_1(1,5) \\ 0 & C_1(2,1) & 0 & 0 & 0 & 0 & 0 & C_1(2,7) \\ 0 & 0 & 0 & C_1(2,1) & 0 & 0 & 0 & C_1(2,5) \\ 0 & C_1(3,1) & 0 & 0 & 0 & 0 & 0 & C_1(3,7) \\ 0 & 0 & C_1(3,1) & 0 & 0 & 0 & 0 & C_1(3,6) \end{bmatrix}.$$

Consequently, we have indeed expressed monomials $z(0)^2z(1)$ and $z(0)^2z(2)$ in the basis $\mathcal{B}_3$.

C.5 Fifth step.

Lastly, let us turn to monomial $z(0)^2z(1)z(2)$. From (20), it can be seen that

$$z(0)^2z(1) = [C_2(1,2:4),\ C_2(1,8)] \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ z(0)z(1)z(2) \end{bmatrix}$$

Multiplying this relation by $z(2)$ shows that $z(0)^2z(1)z(2)$ can be expressed as a function of monomials $z(2)^2$, $z(0)z(1)$, $z(0)z(2)$, and $z(0)z(1)z(2)^2$; the first monomial is expressed in $\mathcal{B}_3$ thanks to (19), the second and the third thanks to (20), and the fourth is of same nature as $z(0)^2z(1)z(2)$. In order to obtain its expression, we shall proceed along the same lines as in the previous paragraph: we shall express jointly all the 3 monomials of the same type. This leads eventually to the linear system:

$$\begin{bmatrix} z(0)^2z(1)z(2) \\ z(1)^2z(0)z(2) \\ z(2)^2z(0)z(1) \end{bmatrix} = \boldsymbol{A}_3^{-1}\,\boldsymbol{B}_{31}\,\boldsymbol{B}_{32}\,\boldsymbol{b}_3(\boldsymbol{z}) \stackrel{\text{def}}{=} \boldsymbol{C}_3\,\boldsymbol{b}_3(\boldsymbol{z}) \tag{21}$$

where

$$\boldsymbol{A}_3 = \begin{bmatrix} 1 & 0 & -C_2(1,8) \\ 0 & 1 & -C_2(3,8) \\ -C_2(4,8) & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B}_{32} = \begin{bmatrix} \boldsymbol{I}_7 & & 0 \\ & \boldsymbol{C}_1 & \\ 0 & & 1 \end{bmatrix}$$

$$\boldsymbol{B}_{31} = \left[\begin{matrix} 0 & 0 & 0 & C_2(1,1) & 0 & C_2(1,2) \\ 0 & 0 & 0 & C_2(3,1) & 0 & C_2(3,2) \\ 0 & C_2(4,1) & 0 & 0 & C_2(4,3) & C_2(4,4) \end{matrix} \right.$$
$$\left. \begin{matrix} C_2(1,3) & 0 & 0 & C_2(1,4) & C_2(1,5) \\ C_2(3,3) & 0 & 0 & C_2(3,4) & C_2(3,5) \\ 0 & C_2(4,2) & 0 & 0 & C_2(4,7) \end{matrix} \right].$$

C.6 Solutions to the polynomial system.

The matrix of the operator $\mathcal{M}_{z(0)}^{\mathrm{T}}$ can now be obtained as $M_{z(0)}^{\mathrm{T}} =$

$$\begin{bmatrix} 0 & & 0 & 0 & & & 0 & \\ 1 & & 0 & 0 & & & 0 & \\ 0 & & 0 & 0 & & & 0 & \\ 0 & C_1(1,:)^{\mathrm{T}} & 0 & 0 & C_2(1,:)^{\mathrm{T}} & C_2(2,:)^{\mathrm{T}} & 0 & C_3(1,:)^{\mathrm{T}} \\ 0 & & 1 & 0 & & & 0 & \\ 0 & & 0 & 1 & & & 0 & \\ 0 & & 0 & 0 & & & 0 & \\ 0 & & 0 & 0 & & & 1 & \end{bmatrix}^{\mathrm{T}}$$

It admits generally 8 eigenvectors, $\boldsymbol{v}^{(m)}$, $1 \leq m \leq 8$. Each of these eigenvectors gives a solution $\boldsymbol{z}^{(m)} = \boldsymbol{v}^{(m)}(2:4)/v^{(m)}(1)$, because the 4 first entries of $\boldsymbol{b}_3$ are $[1, z(0), z(1), z(2)]$. The corresponding solutions $\boldsymbol{h}^{(m)}$ are obtained by transforming back to the original coordinate system: $\boldsymbol{h}^{(m)} = \boldsymbol{T}\boldsymbol{z}^{(m)}$.

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1. Make a change of variables $\boldsymbol{z} = \boldsymbol{T}^{-1}\,\boldsymbol{h}$
2. Choose the basis $\mathcal{B}_M$ containing all 2^M monomials of global degree at most M, and partial degree at most 1, in variables $z_0, z_1, \ldots, z_{M-1}$
3. For $d = 2$ to M, express monomials of degree d, that are at Hamming distance 1 of basis $\mathcal{B}_M$, as a function of monomials of $\mathcal{B}_M$

TABLE I

Steps of the elimination procedure

- Choose a presumed channel length, $\widehat{M} = L$
- Compute the sample correlation

$$R_y(\ell) = \frac{1}{N-\ell}\sum_{n=\ell+1}^{N} y(n)y(n-\ell)^*,\ 0 \le \ell \le L-1$$

- Compute the non-circular sample correlation

$$\alpha(\ell) = \frac{(-1)^\ell}{N-\ell}\sum_{n=\ell+1}^{N} (-1)^n y(n)y(n-\ell),\ 0 \le \ell \le L-1$$

- Choose a transform matrix, $\boldsymbol{T}$.

From $\alpha(\ell)$, calculate the multiplication matrix, $\boldsymbol{M}^{\mathrm{T}}_{z(0)}$ according to the procedure described in sections III-D and VIII-C, as well as table I.
- Compute all its 2^L eigenvectors, $\boldsymbol{v}^{(m)}$
- Compute the 2^L possible solutions, $\boldsymbol{z}^{(m)}$, by normalizing the first entry of $\boldsymbol{v}^{(m)}$ to 1, and retaining entries 2 to $L+1$
- Transform back every candidate into the original coordinate system: $\boldsymbol{h}^{(m)} = \boldsymbol{T}\boldsymbol{z}^{(m)}$
- For each of these candidates, compute the corresponding theoretical correlation (14)
- Select the solution $\boldsymbol{h}$ minimizing (15)
- Equalize the channel and compute the BER of the estimated input.

TABLE II

Summary of the algorithm

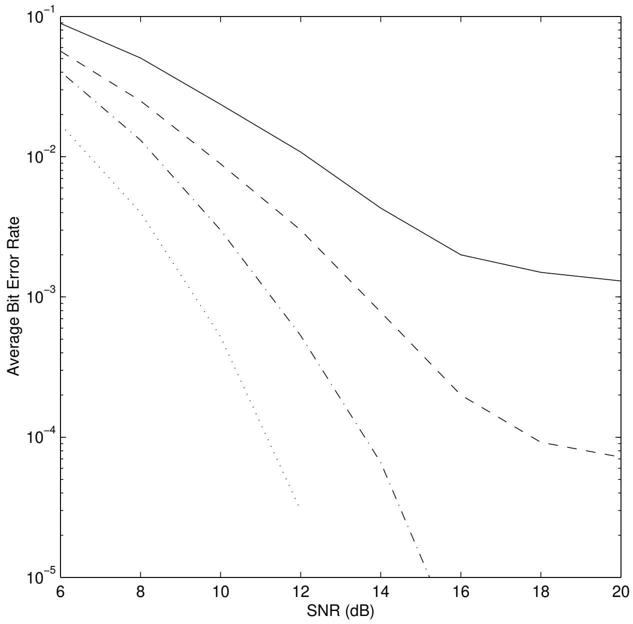

Fig. 1. Average Bit Error Rate at the output of a Viterbi algorithm using our channel estimate as a function of the SNR for various block lengths.

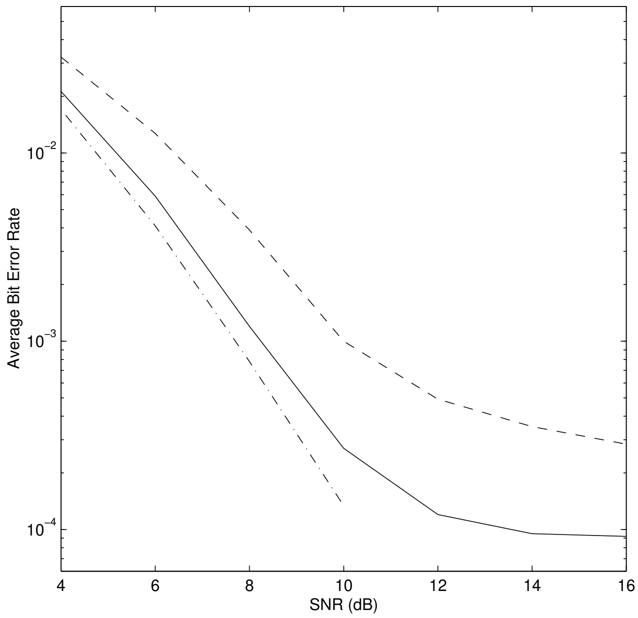

Fig. 3. Average Bit Error Rate at the output of a Viterbi algorithm using our channel estimate as a function of the the SNR and in presence of over-determination.

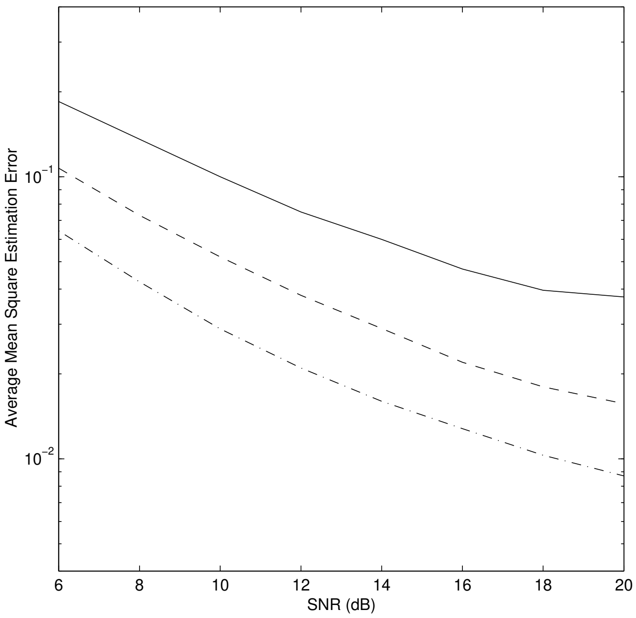

Fig. 2. Average Mean Square Estimation Error of the channel as a function of the SNR for Bit Error Rate at the output of the SNR for various block lengths.

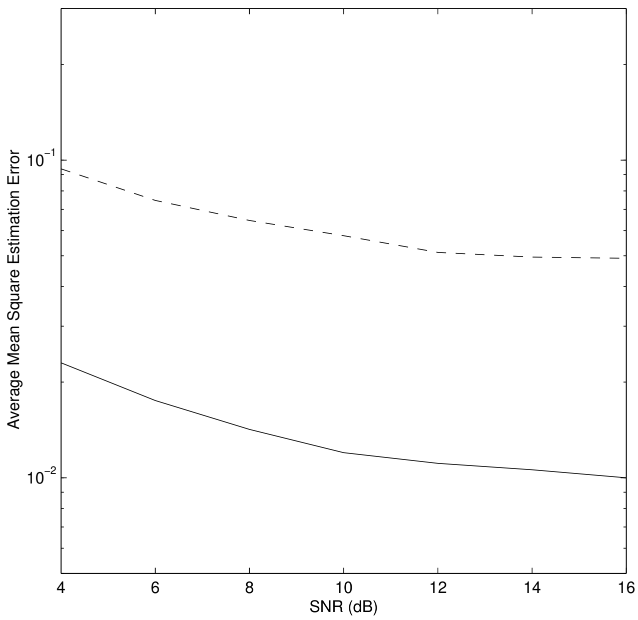

Fig. 4. Average Mean Square Estimation Error of the channel as a function of the SNR and in presence of over-determination.