Obstacle Avoidance & Simultaneous Target Set Stabilization

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CIICADA

Australian National University

Motivation: Obstacle avoidance & target set stabilization

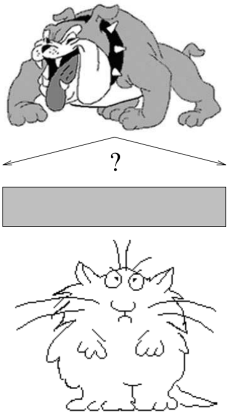

Figure borrowed from: E. D. Sontag, *Nonlinear Feedback Stabilization Revisited*, volume 25 of Progress in Systems and Control Theory, pages 223-262. Birkhäuser, 1999

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- Dynamical system

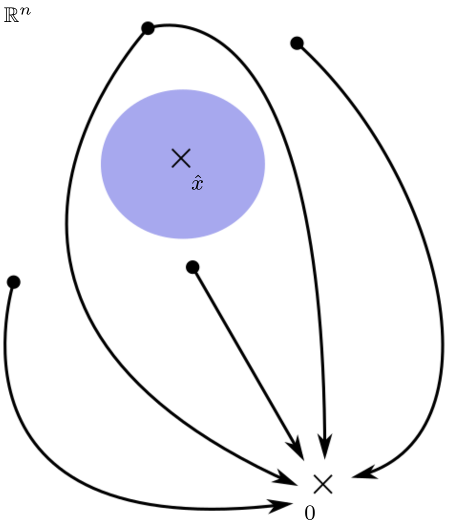
$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) \in \mathbb{R}^n, \ \ \mathbb{R}^m$$

- Obstacle: $\mathcal{B}_\delta(\hat{x}) \subset \mathbb{R}^n \backslash \{0\}$
- Target set: $0 \in \mathbb{R}^n$

Problem formulation:

Define $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ such that

1. $\lim_{t\to\infty} x(t; u(t)) = 0$
2. $x(t; u(t)) \notin \mathcal{B}_\delta(\hat{x}) \; \forall \; t \in \mathbb{R}_{\geq 0}$ (and $\delta > 0$)



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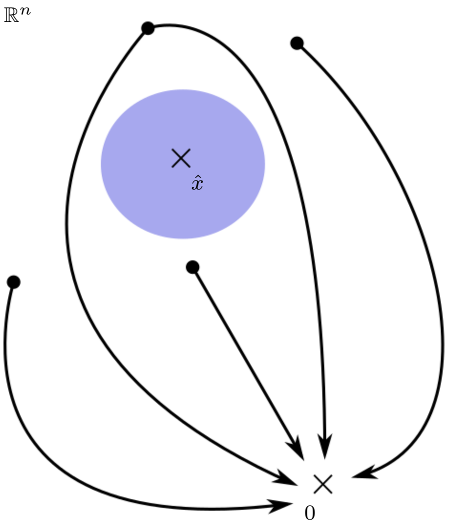
Assume for simplicity $\dot{x} = Ax + Bu$

- (A, B) controllable, i.e.,

  $\forall x_1, x_2 \in \mathbb{R}^n, \forall \varepsilon > 0 \;\; \exists \, u : [0, \varepsilon] \to \mathbb{R}^m :$
  $x(0; u(t)) = x_1$ & $x(\varepsilon; u(t)) = x_2$.



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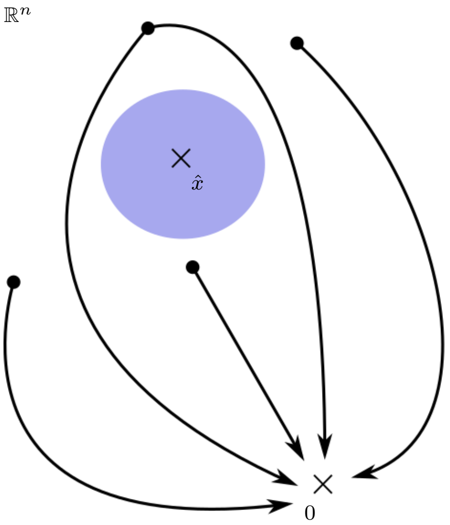
- it is easy to address 1. & 2. separately. But, how to ensure 1. & 2. simultaneously?



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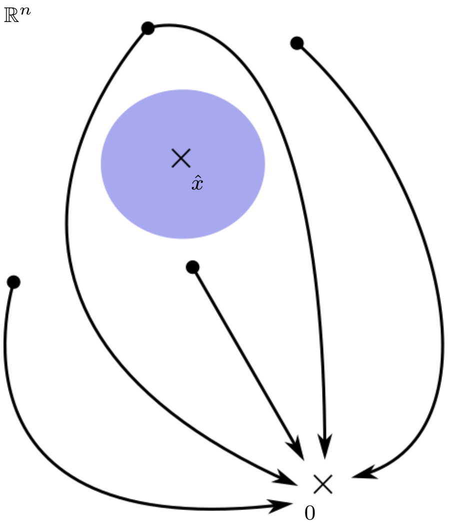
However (at least for linear systems)

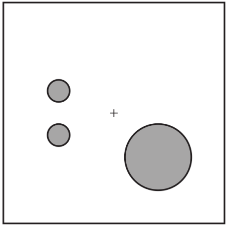
- it is easy to address 1. & 2. separately. But, how to ensure 1. & 2. simultaneously?
- How to define a (state dependent) feedback law (i.e., $u(x(t))$ instead of $u(t)$)?



Related Settings, Applications and Solutions

Setting:

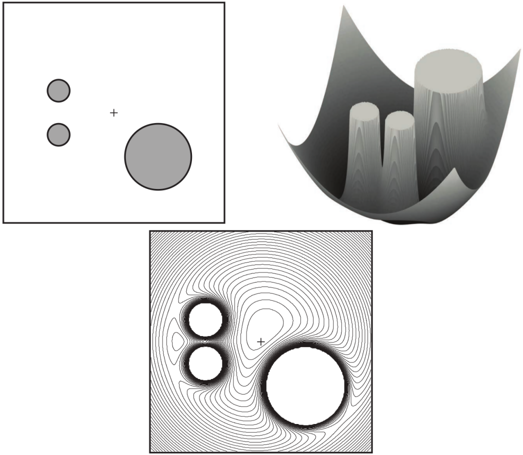
- Obstacle avoidance & target set stabilization
- A special case of constrained control
- Focus on obstacles leading to topological obstructions (i.e., the state space is not a simply connected domain)



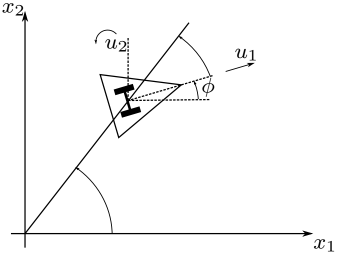
Control Solutions:

- Artificial potential fields and navigation functions
- Model predictive control
  - (Motion planning and reference tracking)
- (Control) Lyapunov functions and (control) barrier functions
- Control using logic based switching
  - (Orchestrate local control laws)

Artificial potential fields & navigation functions



Figures borrowed from: K. M. Lynch, F. C. Park, *Modern Robotics: Mechanics, planning, and control*, Cambridge University Press, 2017



Mobile robot (nonholonomic integrator):

$$\dot{x}_1 = u_1 \cos(\phi),$$
$$\dot{x}_2 = u_1 \sin(\phi),$$
$$\dot{\phi} = u_2.$$

Simplified mobile robot: $\dot{x} = u$

Artificial potential fields:

- Use gradient to guarantee a decrease with respect to the target set
- Local minima? ($\leadsto$ Navigation functions)
- Potential fields necessarily have saddle points

Model Predictive Control & Obstacle Avoidance

Given: Dynamical system

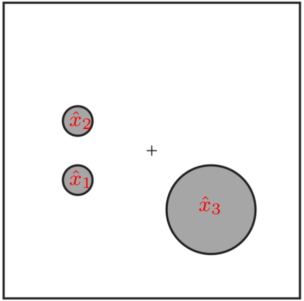
$$x_{k+1} = f(x_k, u_k), \qquad x \in \mathcal{X} \subset \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}^m$$
$$|x - \hat{x}_j| \geq c_j \quad \leftarrow \text{“obstacle constraints”}$$

Model predictive control: For $k \in \mathbb{N}$

1. Solve the optimization problem:

$$\min_{u_0,\ldots,u_{N-1}} \sum_{i=0}^{N-1} \ell(x_i, u_i)$$
$$\text{s.t.} \quad x_0 = x_k$$
$$x_{i+1} = f(x_i, u_i)$$
$$|x_i - \hat{x}_j| \geq c_j$$
$$(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$$
$$\forall i \in \{0, \ldots, N-1\}$$

2. Optimal solution $u_0^\star, \ldots, u_{N-1}^\star$
3. Define feedback law $\mu(x_k) = u_0^\star$
4. Define $x_{k+1} = f(x_k, \mu(x_k))$, set k to $k+1$ and go to step 1.



Note that

- model predictive control is able to handle “obstacle constraints”

But

- “obstacle constraints” naturally lead to non-convex optimization problems (either through constraints or cost function)
- closed-loop properties (i.e., performance, asymptotic stability, recursive feasibility) are more difficult to verify

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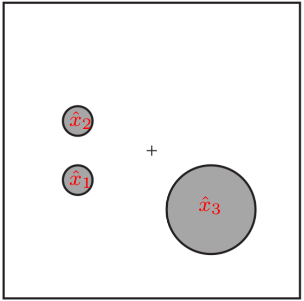

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(Control) Lyapunov and (control) barrier functions

Nonlinear system: $\dot{x} = f(x,u), \quad (x \in \mathbb{R}^n, u \in \mathbb{R}^m)$ Obstacle: $\mathcal{D} \subset \mathbb{R}^n$.

Definition (Control Lyapunov function (CLF))

A **continuously differentiable** function $V : \mathbb{R}^n \to \mathbb{R}$ is called **Control Lyapunov function** (CLF) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\forall x \in \mathbb{R}^n \backslash \{0\} \; \exists u \in \mathbb{R}^m \text{ such that } \quad \langle \nabla V(x), f(x,u) \rangle < 0$$

$\rightsquigarrow$ Guarantees global asymptotic stability of the origin

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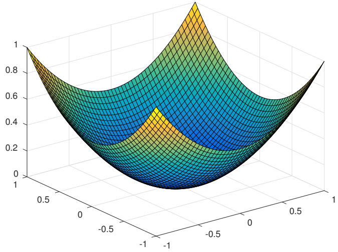
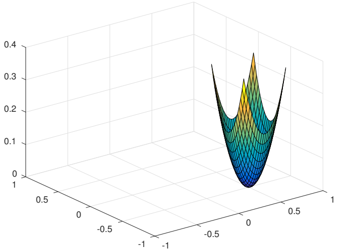
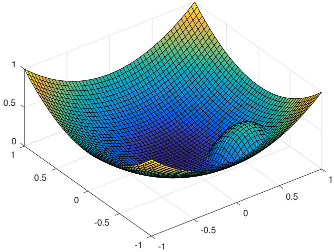
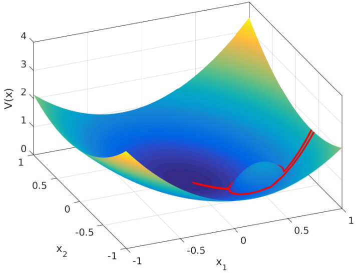
How to combine control Lyapunov and control barrier function results?
How to obtain robust and global results?

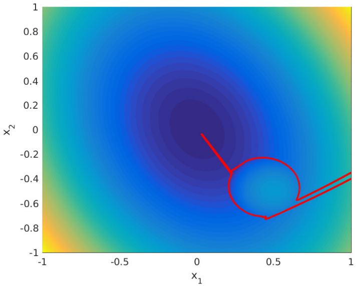
Linear combination of CLFs and CBFs



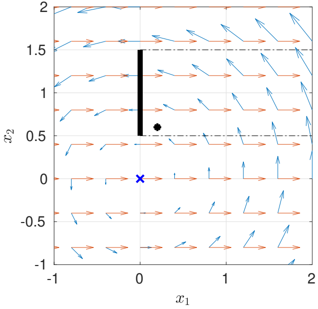








(Underactuated) Systems with Nontrivial Drift



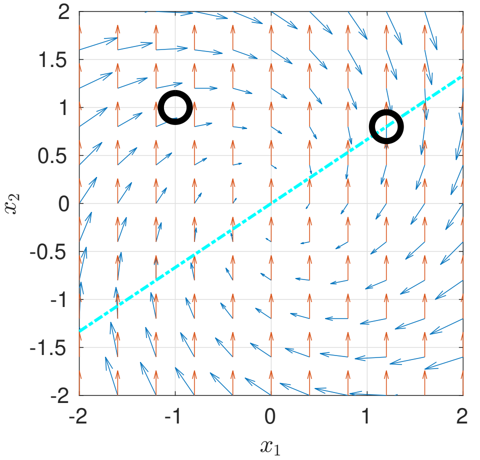
Systems with nontrivial drift

- Consider

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

  - The system is controllable
  - The influence of u is limited
    ($\rightsquigarrow$ Behind the obstacle, u can only be used to stall time)

(Underactuated) Systems with Nontrivial Drift (Position of the Obstacle)



The location of the obstacle:

- Consider

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(The system is controllable)

- Subspace of induced equilibria: ($B \in \mathbb{R}^n$)

$$\mathcal{E} = \{y \in \mathbb{R}^n : 0 = Ay + B\nu,\ \nu \in \mathbb{R}\}$$

- Obstacle $\mathcal{D}$ with $\mathcal{D} \cap \mathcal{E} = 0$
  - Use the natural drift Ax to 'leave the obstacle behind' and use Bu to avoid the obstacle
- Obstacle $\mathcal{D}$ with $\mathcal{D} \cap \mathcal{E} \neq 0$
  - Use u to destabilize a point $\hat{x} \in \mathcal{D} \cap \mathcal{E}$ to avoid the obstacle

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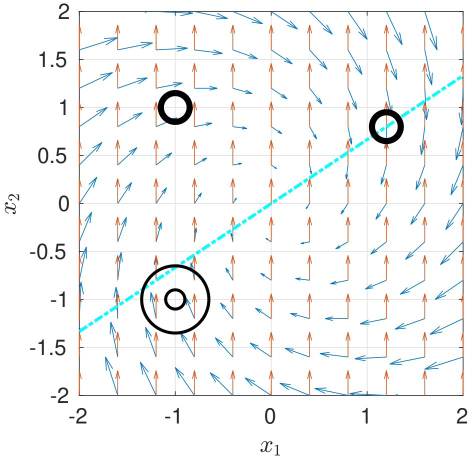

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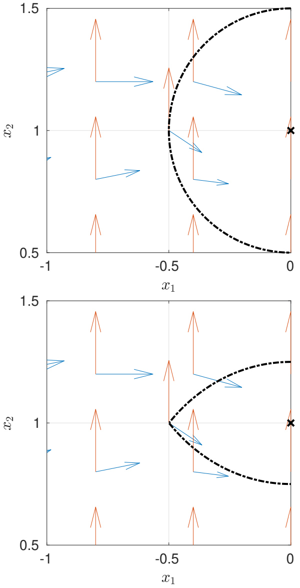
(Underactuated) Systems with Nontrivial Drift (Shape of the Obstacle)



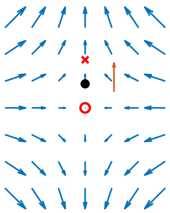
The shape of the obstacle

- Consider again ($\dot{x} = Ax + Bu$)

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- Consider an obstacle $\mathcal{D} \subset \mathbb{R}^n$ with a smooth boundary
  - ⇝ There exists a point $x \in \partial\mathcal{D}$ such that
    - ★ B and the tangent $T(x)$ of $\partial\mathcal{D}$ are linear dependent
    - ★ Ax points inside $\mathcal{D}$

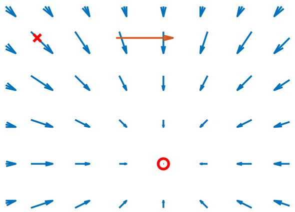
(Underactuated) Systems with Nontrivial Drift (Necessity of Controllability)



- Consider

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

  - The system is stabilizable but not controllable (consider $u(x) = [0 \ -2]x$, for example).
  - Any obstacle on the x_2-axis can be easily avoided.
  - For any obstacle touching the x_2-axis the combined control problem is not solvable



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  - The system is stabilizable but not controllable.
  - The shape and the location of the obstacle are important.

Outline

- Stability and instability characterizations for dynamical systems using Lyapunov arguments

- Controller designs for stability & avoidance (relying on Lyapunov methods, barrier arguments and hybrid systems)

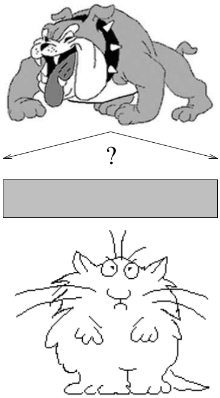

Figure borrowed from: E. D. Sontag, *Nonlinear Feedback Stabilization Revisited*, volume 25 of Progress in Systems and Control Theory, pages 223-262. Birkhäuser, 1999