(In-)Stability of Differential Inclusions

— Notions, Equivalences & Lyapunov-like Characterizations —

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Content

Mathematical Setting & Motivation

- Differential inclusions
- (In)stability characterizations for ordinary differential equations
- The Dini derivative

Strong (in)stability of differential inclusions & Lyapunov characterizations

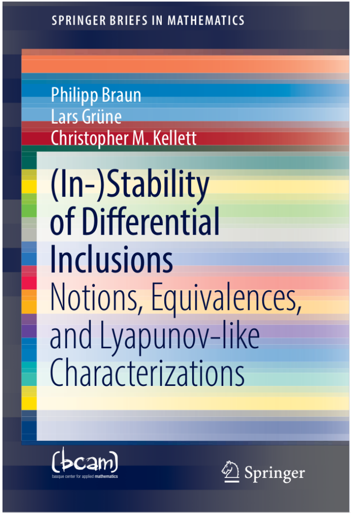
- Strong $\mathcal{KL}$-stability and Lyapunov functions
- $\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions
- Relations between Chetaev functions, Lyapunov functions & scaling
- $\mathcal{KL}$-stability with respect to (two) measures

Weak (in)stability of differential inclusions & Lyapunov characterizations

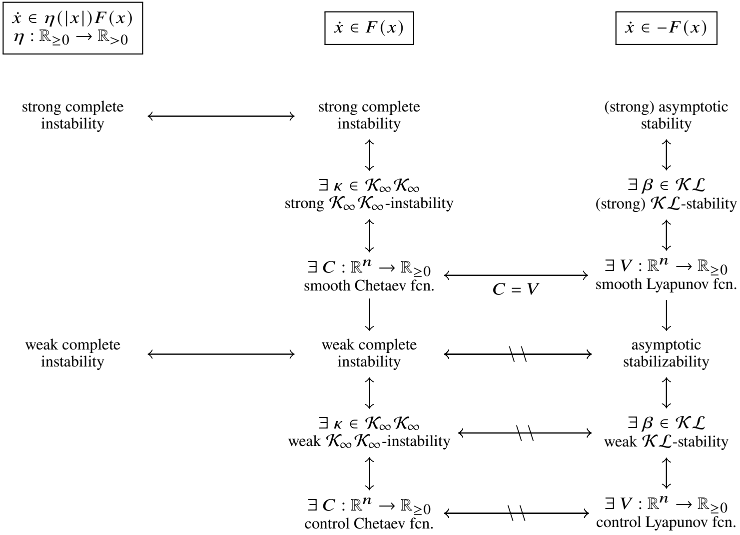
- Weak $\mathcal{KL}$-stability and control Lyapunov functions
- Weak $\mathcal{K}_\infty\mathcal{K}_\infty$-instability and control Chetaev functions
- Relations between control Chetaev functions, control Lyapunov functions and scaling
- Comparison to control barrier function results

Outlook & Further Topics

- Complete control Lyapunov functions
- Combined stabilizing and destabilizing controller design using hybrid systems



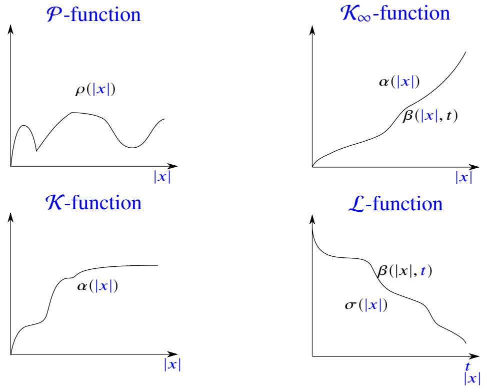
Overview



$\dot{x} \in \eta(|x|)F(x)$
$\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$
$\dot{x} \in F(x)$
$\dot{x} \in -F(x)$
strong complete instability
strong complete instability
(strong) asymptotic stability
$\exists\, \kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$
strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability
$\exists\, \beta \in \mathcal{KL}$
(strong) $\mathcal{KL}$-stability
$\exists\, C : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
smooth Chetaev fcn.
$C = V$
$\exists\, V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
smooth Lyapunov fcn.
weak complete instability
weak complete instability
asymptotic stabilizability
$\exists\, \kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$
weak $\mathcal{K}_\infty\mathcal{K}_\infty$-instability
$\exists\, \beta \in \mathcal{KL}$
weak $\mathcal{KL}$-stability
$\exists\, C : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
control Chetaev fcn.
$\exists\, V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
control Lyapunov fcn.

Notation: Comparison functions

- A continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{P}$ $(\rho \in \mathcal{P})$ if $\rho(0) = 0$, and $\rho(s) > 0$ for all $s > 0$.
- A function $\alpha \in \mathcal{P}$ is said to be of class $\mathcal{K}$ $(\alpha \in \mathcal{K})$ if it is strictly increasing.
- A function $\alpha \in \mathcal{K}$ is said to be of class $\mathcal{K}_\infty$ $(\alpha \in \mathcal{K}_\infty)$ if $\lim_{s\to\infty} \alpha(s) = \infty$.
- A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{L}$ $(\sigma \in \mathcal{L})$, if it is strictly decreasing, and $\lim_{s\to\infty} \sigma(s) = 0$.
- A continuous function $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ $(\beta \in \mathcal{KL})$ if $\beta(\cdot, s) \in \mathcal{K}_\infty$ for all $s \in \mathbb{R}_{\geq 0}$ and $\beta(s, \cdot) \in \mathcal{L}$ for all $s \in \mathbb{R}_{\geq 0}$.



Differential inclusions

Setting:

- Differential inclusion

$$\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$$

- defined through set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$
- we are interested in stability properties of the origin, i.e., $0 \in F(0)$ without loss of generality.

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Assumption (Basic conditions)

The set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $0 \in F(0)$ has nonempty, compact, and convex values on $\mathbb{R}^n$, and it is upper semicont.

Upper semicontinuity:

- For each $x \in \mathbb{R}^n$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\xi \in B_\delta(x)$ we have $F(\xi) \subset F(x) + B_\varepsilon(0)$.
- Example:

$$F(x) = \begin{cases} [0, 1], & x = 0 \\ 1, & x \neq 0 \end{cases}$$

Assumption (Lipschitz continuity)

The set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $0 \in F(0)$ is locally Lipschitz continuous on $\mathbb{R}^n \backslash \{0\}$.

Lipschitz continuity:

- If there exists a constant $L > 0$ and a neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of $x \in \mathbb{R}^n \backslash \{0\}$ such that

$$F(x_1) \subset F(x_2) + B_{L|x_1 - x_2|}(0) \qquad \forall\ x_1, x_2 \in \mathcal{O}$$

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Why do we care about differential inclusions?

- Consider the control system
$$\dot{x} = f(x,u), \qquad x_0 \in \mathbb{R}^n, \qquad u \in \mathcal{U}(x) \subset \mathbb{R}^m$$
- Define the set-valued map
$$F(x) = \overline{\text{conv}}\{f(x,u) \in \mathbb{R}^n | u \in \mathcal{U}(x)\}$$
- Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and continuous in u and $\mathcal{U} = \mathcal{U}(x)$ for all $x \in \mathbb{R}^n$ is compact or that $\mathcal{U}(x) = B_{c|x|}(0)$ for $c > 0$. Then F satisfies the basic condition and F is Lipschitz.
- Here, u can represent a disturbance or an input.

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Note that:

- Solutions of the differential inclusion:

  Absolutely continuous functions $\phi(\cdot; x_0) : [0, T) \to \mathbb{R}^n$, ($T \in \mathbb{R}_{>0} \cup \{\infty\}$) with $\dot{\phi}(\cdot; x_0) \in F(\phi(\cdot; x_0))$ for almost all $t \in [0, T)$.

$\rightsquigarrow$ Solutions exist for any initial value $x_0 \in \mathbb{R}^n$ under the basic condition.

- Set of solutions (with $\phi(0; x_0) = x_0$): $S(x_0)$.
- Solutions as extended real valued functions $\phi(\cdot; x_0)$:
  - If $\phi_i(T; x_0) = \pm\infty$ for $T > 0$ and $i \in \{1, \dots, n\}$, then $\phi_i(t; x_0) = \pm\infty$ for all $t \geq T$.
  - If $\phi_i(T; x_0) = \pm\infty$ for $T < 0$ and $i \in \{1, \dots, n\}$, then $\phi_i(t; x_0) = \pm\infty$ for all $t \leq T$.
- Solutions which satisfy $|\phi(t; x_0)| < \infty$ for all $t \in \mathbb{R}_{\geq 0}$ are called forward complete.

Differential inclusions (Time Scaling)

Consider

$$\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$$

- Set of solutions $\mathcal{S}(x_0)$
- If $\phi(\cdot; x_0) \in \mathcal{S}(x_0)$, $\phi(\cdot; x_0) : \mathbb{R} \to \mathbb{R}^n \cup \{\pm\infty\}^n$, then

  $$\psi(t; x_0) = \phi(-t; x_0)$$

  is a solution of (time reversed inclusion)

  $$\dot{x} \in -F(x) \qquad x_0 \in \mathbb{R}^n$$

- For a positive continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, consider the scaled differential inclusion

  $$\dot{x} \in F_\eta(x) = \eta(|x|)F(x), \qquad x_0 \in \mathbb{R}^n. \qquad (1)$$

  with set of solutions $\mathcal{S}_\eta(\cdot)$.
  (Note that $\eta(0) > 0$.)
- F satisfies basic assumpt. $\iff$ F_η satisfies basic assumpt.

Differential inclusions (Time Scaling)

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Theorem (Positive scaling of differential inclusions)

Consider $\dot{x} \in F(x)$ satisfying the basic assumption. Consider the scaled differential inclusion (1).

For all $x_0 \in \mathbb{R}^n$ and for all $\phi(\cdot; x_0) \in \mathcal{S}(x_0)$ with

$$|\phi(t; x_0)| < \infty, \quad \forall\, t < T \quad \textit{and} \quad |\phi(t; x_0)| = \infty \quad \forall\, t \geq T,$$

$T \in \mathbb{R}_{>0} \cup \{\infty\}$, there exist a continuous strictly increasing function $\alpha : [0, T) \to [0, M)$ and $M \in \mathbb{R}_{>0} \cup \{\infty\}$ with $\alpha(0) = 0$ such that

$$\phi_\eta(\cdot; x_0) = \phi(\alpha(\cdot); x_0) \in \mathcal{S}_\eta(x_0).$$

Conversely, if $\phi_\eta(\cdot; x_0) \in \mathcal{S}_\eta(x_0)$ then

$$\phi_\eta(\alpha^{-1}(\cdot); x_0) \in \mathcal{S}(x_0)$$

is satisfied. Moreover, in the limit, the solutions satisfy

$$\lim_{t \to T} |\phi(t; x_0)| = \lim_{t \to M} |\phi_\eta(t; x_0)|.$$

$\rightsquigarrow$ In particular, stability properties are preserved.
$\rightsquigarrow$ If $T = M = \infty$ both solutions are forward complete ($\alpha \in \mathcal{K}_\infty$)

Differential inclusions (Time Scaling, 2)

Corollary

Consider $\dot{x} \in F(x)$ satisfying the basic assumption. Then there exists a continuous positive function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that

$$\eta(|x|)F(x) \subset \overline{B}_1(0) \qquad \forall\, x \in \mathbb{R}^n$$

Moreover $\eta(|\cdot|)F(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the basic assumption and all solutions of the scaled differential equation are forward complete.

In particular, we can define

$$\eta(r) = \frac{1}{\nu(r) + 1}$$

where ν is continuous and

$$\nu(r) \geq \tilde{\nu}(r) = \max_{y \in F(x), |x| = r} |y|$$

Key takeaway:

- If we want to establish asymptotic stability properties of the origin of $\dot{x} \in F(x)$ we can assume forward completeness of solutions without loss of generality by considering an appropriate scaling.

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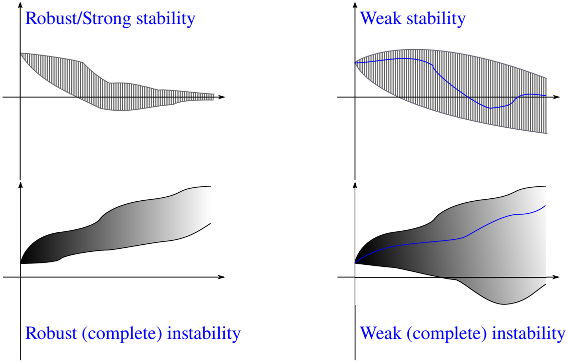

(In)stability characterizations for ordinary differential equations

We start with differential equations

$$\dot{x} = f(x), \qquad x_0 \in \mathbb{R}^n$$

- $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz
- $f(0) = 0$
- for each $x_0 \in \mathbb{R}^n$, $\mathcal{S}(x_0)$ contains a single element

Definition ((Global) stability)

The origin is (Lyapunov) stable if there exists $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon \geq 0$,

$$|\phi(t; x_0)| \leq \varepsilon \qquad \text{whenever } |x_0| \leq \delta(\varepsilon) \text{ and } t \geq 0.$$

Theorem (Lyapunov stability theorem)

Given $\dot{x} = f(x)$, suppose there exist a smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, $\forall\ x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$
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Then the origin is (globally) stable.

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$$|\phi(t; x_0)| \to 0 \qquad \text{for } t \to \infty.$$

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Given $\dot{x} = f(x)$ suppose there exist a smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\rho \in \mathcal{P}$ such that, $\forall x \in \mathbb{R}^n$

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Definition (Instability)

The origin is unstable for the system if it is not stable.

- ⇝ There are many different types of instability
- ⇝ Here, we focus on complete instability

(In)stability characterizations for ordinary differential equations (2)

We start with differential equations

$$\dot{x} = f(x), \qquad x_0 \in \mathbb{R}^n$$

- $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, $f(0) = 0$

Definition ((Global) complete instability)

The origin is completely unstable if there exists $\alpha \in \mathcal{K}_\infty$ such that for all $\delta > 0$ the condition $x_0 \in \mathbb{R}^n \backslash B_{\alpha(\delta)}(0)$ implies

$$|\phi(t; x_0)| \geq \delta \qquad \forall\ t \in \mathbb{R}_{\geq 0},$$
$$|\phi(t; x_0)| \to \infty \qquad \text{for } t \to \infty.$$

Theorem (Lyapunov complete instability theorem)

Suppose there exist a smooth Chetaev function $C : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *and* $\rho \in \mathcal{P}$ *such that,* $\forall\ x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq C(x) \leq \alpha_2(|x|),$$
$$\langle \nabla C(x), f(x) \rangle \geq \rho(|x|).$$

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Theorem (Chetaev's theorem)

Assume there exists a smooth Chetaev function $C : \mathbb{R}^n \to \mathbb{R}$ with $C(0) = 0$ and

$$\mathcal{O}_r = \{x \in B_r(0) : C(x) > 0\} \neq \emptyset \qquad \forall\, r > 0.$$

If for certain $r > 0$,

$$\langle \nabla C(x), f(x) \rangle > 0 \qquad \forall\, x \in \mathcal{O}_r$$

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then the origin is unstable.

Remark

Note that, as stated, the definition and characterizations are essentially global as they are stated for all all $x \in \mathbb{R}^n$ and for all $\varepsilon > 0$. Local versions are easily obtained by restricting ε and by restricting the attention to a domain around the origin.

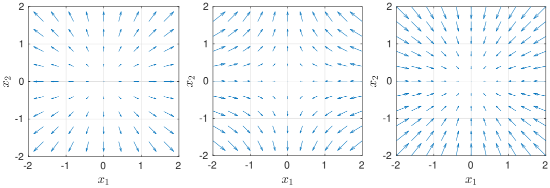
(In)stability characterizations for ordinary differential equations (A simple example)

Consider the three linear differential equations and their solutions

$$f_1(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \phi_1(t;x_0) = \begin{bmatrix} x_{1,0}e^{t} \\ x_{2,0}e^{t} \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}, \qquad \phi_2(t;x_0) = \begin{bmatrix} x_{1,0}e^{-t} \\ x_{2,0}e^{t} \end{bmatrix},$$

$$f_3(x) = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, \qquad \phi_3(t;x_0) = \begin{bmatrix} x_{1,0}e^{-t} \\ x_{2,0}e^{-t} \end{bmatrix}.$$



- Chetaev function for complete instability: $C_1(x) = x^T x$

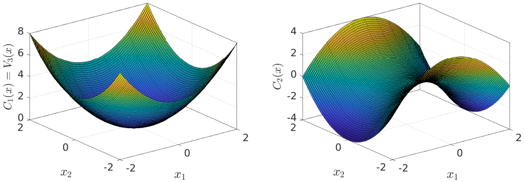
$$\langle \nabla C_1, f_1(x) \rangle = 2x^T x$$

- Chetaev function for instability: $C_2(x) = -x_1^2 + x_2^2$

$$\langle \nabla C_2, f_2(x) \rangle = 2x^T x$$

- Lyapunov function for asymptotic stability: $V_3(x) = x^T x$

$$\langle \nabla V_3, f_3(x) \rangle = -2x^T x$$



(In)stability characterizations for ordinary differential equations (A simple example)

Consider the three linear differential equations and their solutions

$$f_1(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \phi_1(t;x_0) = \begin{bmatrix} x_{1,0}e^{t} \\ x_{2,0}e^{t} \end{bmatrix},$$

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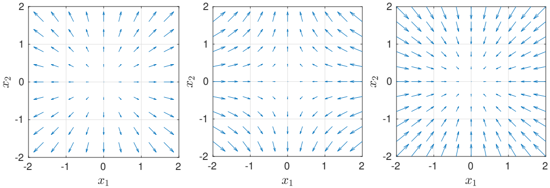

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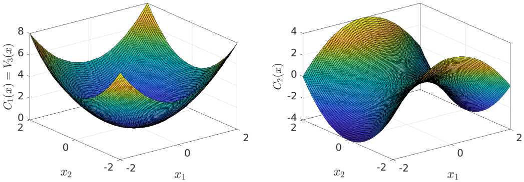

Simple observation:

$$\dot{x} = f(x), \quad 0 \text{ is asymptotically stable} \quad \Longleftrightarrow \quad \dot{x} = -f(x), \quad 0 \text{ is completely unstable}$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) \quad \overset{V=C}{\Longleftrightarrow} \quad \langle \nabla C(x), -f(x) \rangle \geq \rho(|x|)$$

(In)stability characterizations for ordinary differential equations (Local complete instability)

Recall the definition:

Definition ((Global) complete instability)

The origin is completely unstable if there exists $\alpha \in \mathcal{K}_\infty$ such that for all $\delta > 0$ the condition $x_0 \in \mathbb{R}^n \backslash B_{\alpha(\delta)}(0)$ implies

$$|\phi(t;x_0)| \geq \delta \qquad \forall\, t \in \mathbb{R}_{\geq 0}, \tag{2}$$
$$|\phi(t;x_0)| \to \infty \qquad \text{for } t \to \infty.$$

$\rightsquigarrow$ Is the condition (2) necessary?

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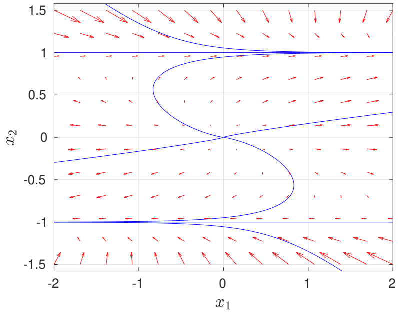
$$

$\rightsquigarrow$ Is the condition (2) necessary?



Example

Consider the two dimensional dynamics

$$
\begin{aligned}
\dot{x}_1 &= (c^2 - x_2^2)x_1 + x_2 \\
\dot{x}_2 &= (c^2 - x_2^2)x_2
\end{aligned}
$$

with parameter $c \in \mathbb{R}_{>0}$.

- For $x_2^2 = c^2$ the dynamics reduce to $\dot{x}_1 = x_2$ and $\dot{x}_2 = 0$.

(In)stability characterizations for ordinary differential equations (Local complete instability)

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Consider the two dimensional dynamics

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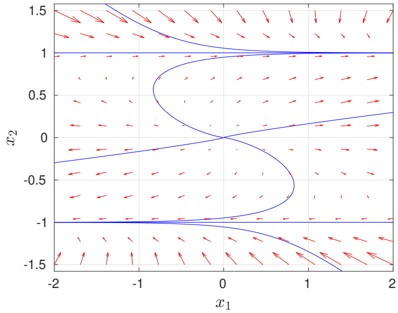


Note that:

- $\alpha \in \mathcal{K}_\infty$ is necessary to ensure that solutions starting arbitrarily far away from 0 stay arbitrarily far away from 0 $\forall t \in \mathbb{R}_{\geq 0}$ for global complete instability.
- If we restrict our analysis of complete instability of 0 to $B_{\frac{1}{2}c}(0)$, then 0 is locally completely unstable.

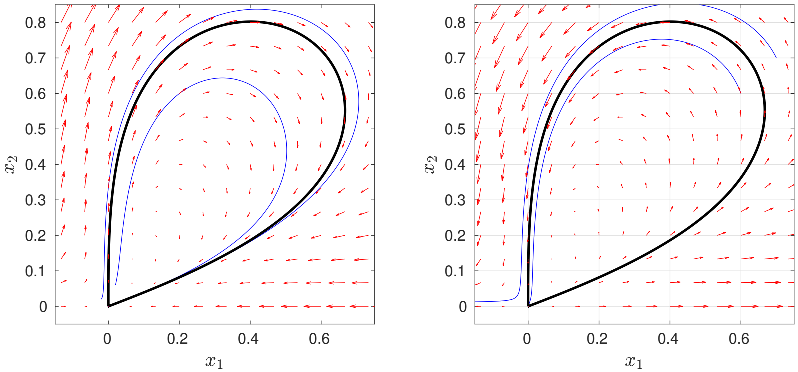
$\rightsquigarrow$ Is the condition (2) necessary for local complete instability? (I don't know.)

(In)stability characterizations for ordinary differential equations (Attractive but not stable)

Example (Vinograd's example)

$$\dot{x} = f(x) = \frac{1}{|x|_2^2(1+|x|_2^4)} \begin{bmatrix} x_1^2(x_2 - x_1) + x_2^5 \\ x_2^2(x_2 - 2x_1) \end{bmatrix}$$

- Classical example of a system with globally attractive origin (but not stable), i.e., the origin is not asymptotically stable.
- The origin of time reversal dynamics $\dot{x} = -f(x)$ is not completely unstable



(In)stability characterizations for ordinary differential equations (The Dini derivative)

Consider $\varphi : \mathbb{R}^n \to \mathbb{R}$

The Dini derivative at x in direction $w \in \mathbb{R}^n$ are defined as:

$$D^+\varphi(x;w) = \limsup_{v\to w;\ t\searrow 0} \tfrac{1}{t}\left(\varphi(x+tv) - \varphi(x)\right),$$

$$D_+\varphi(x;w) = \liminf_{v\to w;\ t\searrow 0} \tfrac{1}{t}\left(\varphi(x+tv) - \varphi(x)\right),$$

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(Upper right, lower right, upper left, and lower left Dini derivative)

The Dini derivatives for Lipschitz functions φ:

- The upper right Dini derivative simplifies to

$$D^+\varphi(x;w) = \limsup_{t\searrow 0} \tfrac{1}{t}\left(\varphi(x+tw) - \varphi(x)\right).$$

  (The remaining Dini derivatives simplify in the same way.)
- The Dini derivative is finite
- The Dini derivatives can all be different

If φ is differentiable in $x \in \mathbb{R}^n$, then

$$\langle \nabla\varphi(x), w\rangle = D^+\varphi(x;w)$$

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$$\langle \nabla\varphi(x), w\rangle = D^+\varphi(x;w)$$

For $\phi(\cdot;x_0) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ smooth and $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ smooth,

$$\dot{V}(\phi(t;x_0)) = \langle \nabla V(\phi(t;x_0)), \dot{\phi}(t;x_0)\rangle. \qquad (3)$$

indicates the derivative of V along the function ϕ. If ϕ is absolutely continuous and V is Lipschitz continuous, then (3) holds for almost all $t \in \mathbb{R}$.

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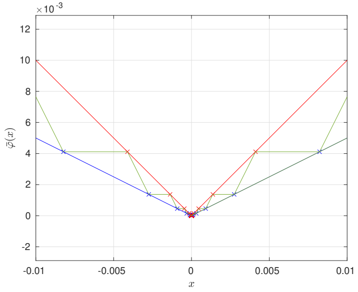

Strong $\mathcal{KL}$-stability and Lyapunov functions

Consider: $\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$

- Assume F satisfies the basic conditions

Definition (Global asymptotic stability)

The differential inclusion is uniformly globally asymptotically stable with respect to $0 \in \mathbb{R}^n$ if the following properties are satisfied. There exists a function $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon \geq 0$ and for all $\phi \in \mathcal{S}(x_0)$,

$$|\phi(t; x_0)| \leq \varepsilon \qquad \text{whenever } |x_0| \leq \delta(\varepsilon) \text{ and } t \geq 0,$$
$$|\phi(t; x_0)| \to 0 \qquad \text{for } t \to \infty.$$

Definition ((Strong) $\mathcal{KL}$-stability)

The differential inclusion is *strongly $\mathcal{KL}$-stable* with respect to $0 \in \mathbb{R}^n$ if there exists $\beta \in \mathcal{KL}$, such that for all $x_0 \in \mathbb{R}^n$ every solution $\phi \in \mathcal{S}(x_0)$ satisfies

$$|\phi(t; x_0)| \leq \beta(|x_0|, t), \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Theorem

The differential inclusion is uniformly globally asymptotically stable with respect to 0 *if and only if it is (strongly)* $\mathcal{KL}$*-stable.*

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Theorem

The differential inclusion is uniformly globally asymptotically stable with respect to 0 *if and only if it is (strongly) $\mathcal{KL}$-stable.*

Definition ((Robust) Lyapunov function)

A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is called a (robust) Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad \forall x \in \mathbb{R}^n$$
$$\max_{w \in F(x)} D^+V(x;w) \leq -\rho(|x|) \qquad \forall x \in \mathbb{R}^n$$

Theorem (Stability characterization)

The following are equivalent.

- *The differential inclusion is strongly $\mathcal{KL}$-stable with respect to the origin.*
- *There exists a smooth Lyapunov function*

$\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions

Consider: $\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$

- Assume F satisfies the basic conditions

Definition (Strong complete instability)

The differential inclusion is strongly completely unstable with respect to $0 \in \mathbb{R}^n$ if the following properties are satisfied. There exists a function $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon > 0$ and for all solutions $\phi \in \mathcal{S}(x_0)$,

$$
\begin{aligned}
|\phi(t; x_0)| &\geq \varepsilon && \text{for all } t \geq 0, \\
|\phi(t; x_0)| &\to \infty && \text{for } t \to \infty,
\end{aligned}
$$

whenever $|x_0| \geq \delta(\varepsilon)$.

$\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions

Consider: $\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$

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whenever $|x_0| \geq \delta(\varepsilon)$.

Definition ($\mathcal{K}_\infty\mathcal{K}_\infty$-functions)

Consider the continuous function $\kappa : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$.

- κ is said to be of class $\mathcal{K}_\infty\mathcal{K}_\infty$ ($\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$) if $\kappa(\cdot, s) \in \mathcal{K}_\infty \ \forall\, s \in \mathbb{R}_{\geq 0}$ and $\kappa(s, \cdot) - \kappa(s, 0) \in \mathcal{K}_\infty$ $\forall\, s \in \mathbb{R}_{>0}$.

Example:

- $\kappa(s, t) = c e^{\lambda t} s \in \mathcal{K}_\infty\mathcal{K}_\infty$ if $\lambda > 0$, $c > 0$
- $\kappa(s, t) = (t + 1)s \in \mathcal{K}_\infty\mathcal{K}_\infty$

Definition (Strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

The differential inclusion is strongly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable with respect to $0 \in \mathbb{R}^n$ if there exists $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ every solution $\phi \in \mathcal{S}(x_0)$ satisfies

$$|\phi(t; x_0)| \geq \kappa(|x_0|, t), \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

$\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions (2)

Consider: $\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$

- Assume F satisfies the basic conditions

Definition (Strong complete instability)

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Definition (Strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

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$$|\phi(t;x_0)| \geq \kappa(|x_0|, t), \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Theorem

The differential inclusion is strongly completely unstable with respect to 0 if and only if the origin is strongly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable.

Definition ((Robust) Chetaev function)

A continuous function $C : \mathbb{R}^n \to \mathbb{R}$ is called a Chetaev function for the differential inclusion if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that

$$\alpha_1(|x|) \leq C(x) \leq \alpha_2(|x|) \qquad \forall\, x \in \mathbb{R}^n$$
$$\min_{w \in F(x)} D_+C(x;w) \geq \rho(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

Theorem (Instability characterization)

The following are equivalent.

- *The differential inclusion is strongly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable.*
- *There exists a smooth Chetaev function.*

Relations between Chetaev and Lyapunov functions & scaling

Lemma

Consider $\dot{x} \in F(x)$ *satisfying the basic condition and* $\dot{x} \in \eta(|x|)F(x)$ *for a Lipschitz* $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$.

- *Assume* V *is a smooth Lyapunov function for* $\dot{x} \in F(x)$. *Then* V *is a smooth Lyapunov function of* $\dot{x} \in \eta(|x|)F(x)$.
- *Assume* C *is a smooth Chetaev function for* $\dot{x} \in F(x)$. *Then* C *is a smooth Chetaev function of* $\dot{x} \in \eta(|x|)F(x)$.

Proof.

Let V denote a smooth Lyapunov function. Then there exists $\rho \in \mathcal{P}$ such that

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \leq -\rho(|x|) \qquad x \in \mathbb{R}^n.$$

$$\begin{aligned} \max_{w \in \eta(|x|)F(x)} \langle \nabla V(x), w \rangle &= \max_{w \in F(x)} \langle \nabla V(x), \eta(|x|)w \rangle \\ &\leq -\eta(|x|)\rho(|x|) = \tilde{\rho}(|x|) \end{aligned}$$

□

$\leadsto$ Solutions are forward complete w.l.o.g.

Relations between Chetaev and Lyapunov functions & scaling

Lemma

Consider $\dot{x} \in F(x)$ satisfying the basic condition and $\dot{x} \in \eta(|x|)F(x)$ for a Lipschitz $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$.

- *Assume V is a smooth Lyapunov function for $\dot{x} \in F(x)$. Then V is a smooth Lyapunov function of $\dot{x} \in \eta(|x|)F(x)$.*
- *Assume C is a smooth Chetaev function for $\dot{x} \in F(x)$. Then C is a smooth Chetaev function of $\dot{x} \in \eta(|x|)F(x)$.*

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Let V denote a smooth Lyapunov function. Then there exists $\rho \in \mathcal{P}$ such that

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$$\begin{aligned} \max_{w \in \eta(|x|)F(x)} \langle \nabla V(x), w \rangle &= \max_{w \in F(x)} \langle \nabla V(x), \eta(|x|) w \rangle \\ &\leq -\eta(|x|)\rho(|x|) = \tilde{\rho}(|x|) \end{aligned}$$

□

Corollary

Consider $\dot{x} \in F(x)$ satisfying basic conditions together with $\dot{x} \in -F(x)$

- *Let V be a smooth Lyapunov function for $\dot{x} \in F(x)$. Then $C = V$ is a smooth Chetaev function for $\dot{x} \in -F(x)$.*
- *Let C be a smooth Chetaev function for $\dot{x} \in F(x)$. Then $V = C$ is a smooth Lyapunov function for $\dot{x} \in -F(x)$.*

Proof.

Let V denote a smooth Lyapunov function for $\dot{x} \in F(x)$. Then there exists $\rho \in \mathcal{P}$ such that

$$-\rho(|x|) \geq \max_{w \in F(x)} \langle \nabla V(x), w \rangle = - \min_{w \in F(x)} -\langle \nabla V(x), w \rangle$$

for all $x \in \mathbb{R}^n$. Equivalently

$$\rho(|x|) \geq \min_{w \in F(x)} -\langle \nabla V(x), w \rangle = \min_{w \in -F(x)} \langle \nabla V(x), w \rangle$$

i.e., $C = V$ is a Chetaev function for $\dot{x} \in -F(x)$. □

$\rightsquigarrow$ Solutions are forward complete w.l.o.g.

Weak (in)stability of differential inclusions & Lyapunov characterizations

Weak $\mathcal{KL}$-stability and control Lyapunov functions

Weak (in)stability of differential inclusions & Lyapunov characterizations

Weak $\mathcal{KL}$-stability and control Lyapunov functions

Definition (Global asymptotic stabilizability)

$\dot{x} \in F(x)$ is uniformly globally asymptotically stabilizable with respect to 0 if the following are satisfied. There exists a function $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon \geq 0$ and all $x_0 \in \mathbb{R}^n$ with $|x_0| \leq \delta(\varepsilon)$ there exists $\phi \in \mathcal{S}(x_0)$ with

$$|\phi(t;x_0)| \leq \varepsilon \qquad \text{for all } t \geq 0 \quad \text{and}$$
$$|\phi(t;x_0)| \to 0 \qquad \text{for } t \to \infty.$$

Definition (Weak $\mathcal{KL}$-stability)

$\dot{x} \in F(x)$ is *weakly* $\mathcal{KL}$-stable with respect to the equilibrium 0 if there exists $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathbb{R}^n$ there exists $\phi \in \mathcal{S}(x_0)$ with

$$|\phi(t;x_0)| \leq \beta(|x_0|,t), \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Corollary

Consider $\dot{x} \in F(x)$ satisfying the basic conditions. $\dot{x} \in F(x)$ is globally asymptotically stabilizable with respect to 0 if and only if it is is weakly $\mathcal{KL}$-stable.

Definition (Control Lyapunov function)

A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is called control Lyapunov function for $\dot{x} \in F(x)$ if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ and

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad \forall x \in \mathbb{R}^n$$
$$\min_{w \in F(x)} D_+V(x;w) \leq -\rho(|x|) \qquad \forall x \in \mathbb{R}^n$$

Theorem

Suppose F satisfies the basic conditions and is Lipschitz. Then the following are equivalent.

- *$\dot{x} \in F(x)$ is weakly $\mathcal{KL}$-stable.*
- *There exists a Lipschitz control Lyapunov function.*

Weak $\mathcal{K}_\infty\mathcal{K}_\infty$-instability and control Chetaev functions

Definition (Weak complete instability)

$\dot{x} \in F(x)$ is weakly completely unstable with respect to 0 if the following properties are satisfied. There exists a function $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon > 0$ and all $x_0 \in \mathbb{R}^n$ with $|x_0| \geq \delta(\varepsilon)$ there exists $\phi \in \mathcal{S}(x_0)$ with

$$|\phi(t; x_0)| \geq \varepsilon \qquad \text{for all } t \geq 0 \quad \text{and}$$
$$|\phi(t; x_0)| \to \infty \qquad \text{for } t \to \infty.$$

Definition (Weak $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

$\dot{x} \in F(x)$ is weakly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable with respect to 0 if there exists $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ there exists $\phi \in \mathcal{S}(x_0)$ so that

$$|\phi(t; x_0)| \geq \kappa(|x_0|, t) \quad \text{for all } t \geq 0.$$

Corollary

Consider $\dot{x} \in F(x)$ satisfying the basic conditions. $\dot{x} \in F(x)$ is weakly completely unstable with respect to 0 if and only if it is is weakly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable.

Definition (Control Chetaev function)

A continuous function $C : \mathbb{R}^n \to \mathbb{R}$ is called control Chetaev function for $\dot{x} \in F(x)$ if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that

$$\alpha_1(|x|) \leq C(x) \leq \alpha_2(|x|) \qquad \forall x \in \mathbb{R}^n$$
$$\max_{w \in F(x)} D^+C(x; w) \geq \rho(|x|) \qquad \forall x \in \mathbb{R}^n$$

Theorem

Suppose F satisfies the basic conditions and is Lipschitz. Then the following are equivalent.

- *The origin of $\dot{x} \in F(x)$ is weakly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable.*
- *There exists a continuous control Chetaev function.*

When are nonsmooth control Lyapunov/Chetaev functions necessary? (Examples)

Consider the differential inclusion

$$\dot{x} \in F(x) = \overline{\text{conv}}\{f(x,u) | u \in \mathcal{U}(x)\}$$

where $f(x,u)$ and $\mathcal{U}$ are defined as

$$f(x,u) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad \text{and}$$

$$\mathcal{U}(x) = [-2|x|, 2|x|] \,.$$

Assume there exists a smooth control Chetaev function C.

- Then, $V = C$ is a CLF for $\dot{x} = -f(x,u)$:

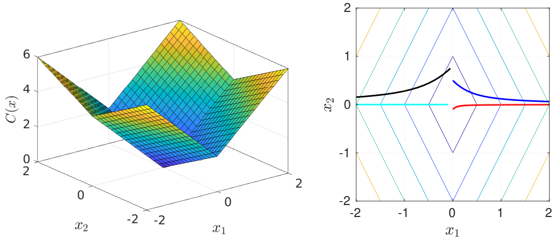
$$\sup_{u \in \mathcal{U}(x)} \langle \nabla C(x), f(x,u) \rangle \geq \rho(|x|) \quad \Longleftrightarrow$$

$$\min_{u \in \mathcal{U}(x)} \langle \nabla C(x), -f(x,u) \rangle \leq -\rho(|x|).$$

- The second component x_2 of $-f$, is not stabilizable to the origin, i.e., a smooth CLF cannot exist and thus a smooth CCF cannot exist
- However, intuitively it should be clear that the origin is weakly completely unstable

Nonsmooth control Chetaev function:

$$C(x) = 2|x_1| + |x_2|$$



Corollary

There are differential inclusions satisfying basic conditions and F locally Lipschitz which are weakly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable and which do not admit smooth control Chetaev functions.

Relations between control Chetaev functions, control Lyapunov functions, and scaling

Note that

- Results on the positive scaling $\dot{x} \in \eta(|x|)F(x)$ remain valid in the weak setting
- The connections between $\dot{x} \in F(x)$ and $\dot{x} \in -F(x)$ established in the strong setting are in general not satisfied in the weak setting

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In particular, let V be a control Lyapunov function for $\dot{x} \in F(x)$, i.e., for $\rho \in \mathcal{P}$ for all $x \in \mathbb{R}^n$

$$-\rho(|x|) \geq \min_{w \in F(x)} D_+ V(x; w)$$

This implies that

$$\begin{aligned}
\rho(|x|) &\leq \max_{w \in F(x)} -D_+ V(x; w) \\
&= \max_{w \in F(x)} \left(- \liminf_{v \to w;\, t \searrow 0} \tfrac{1}{t}(V(x + tv) - V(x)) \right) \\
&= \max_{w \in F(x)} \limsup_{v \to w;\, t \searrow 0} -\tfrac{1}{t}(V(x + tv) - V(x)) \\
&= \max_{w \in F(x)} \limsup_{v \to w;\, t \nearrow 0} \tfrac{1}{t}(V(x - tv) - V(x)) \\
&= \max_{w \in -F(x)} \limsup_{v \to w;\, t \nearrow 0} \tfrac{1}{t}(V(x + tw) - V(x)) \\
&= \max_{w \in -F(x)} D^- V(x; w).
\end{aligned}$$

$\rightsquigarrow$ The left Dini derivative cannot be used to define a CCF for $\dot{x} \in -F(x)$.

Relations between control Chetaev functions, control Lyapunov functions (Artstein's Circles)

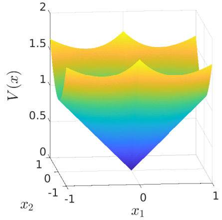
- Consider ($u \in [-1, 1] = \mathcal{U}$)

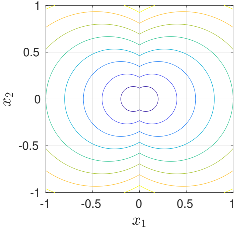
$$\dot{x}_1(t) = \left(-x_1(t)^2 + x_2(t)^2\right) u(t),$$
$$\dot{x}_2(t) = \left(-2x_1(t)x_2(t)\right) u(t)$$

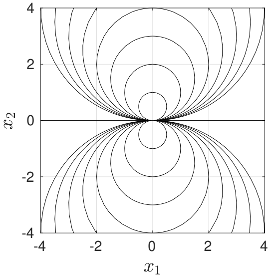
(the origin is weakly $\mathcal{KL}$-stable)

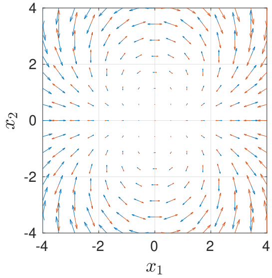
- Control Lyapunov function:

$$V(x) = \sqrt{4x_1^2 + 3x_2^2} - |x_1|$$









- All solutions corresponding to $x_0 \in \mathbb{R}^2 \backslash (\mathbb{R} \times \{0\})$ are bounded

$\rightsquigarrow$ The origin is not weakly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable.

Corollary

Weak $\mathcal{KL}$-stability of the origin for $\dot{x} \in F(x)$ is not equivalent to weak $\mathcal{K}_\infty\mathcal{K}_\infty$-instability of the origin for $\dot{x} \in -F(x)$.

Relations between control Chetaev functions, control Lyapunov functions (Brockett integrator)

Example

Consider the dynamics of the Brockett integrator,

$$F(x) = \overline{\text{conv}}\{f(x,u) | u \in \mathcal{U}\}$$

defined through

$$f(x,u) = \begin{bmatrix} u_1 \\ u_2 \\ x_1 u_2 - x_2 u_1 \end{bmatrix} \qquad \text{and} \qquad \mathcal{U} = [-1,1]^2 .$$

(Note that the dynamics in forward time are equivalent to the dynamics in backward time.)

- It can be shown that

$$V(x) = x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2}$$

  is CLF but not a CCF.

- It can be shown that

$$C(x) = |x_1| + |x_2| + |x_3|$$

  is a CCF but not a CLF

Comparison to control barrier function results

Consider the control affine system

$$\dot{x} = f(x) + g(x)u$$

- f, g locally Lipschitz
- $C \subset \mathbb{R}^n$ is called forward invariant if for every $x_0 \in C$,

$$\phi(t; x_0) \in C, \qquad \forall t \in \mathbb{R}_{\geq 0}$$

  - (in the strong sense) $\forall \phi \in \mathcal{S}(x_0)$
  - (in the weak sense) $\exists \phi \in \mathcal{S}(x_0)$
- For $u = k(x)$ Lipschitz, $\dot{x} = f(x) + g(x)k(x)$ is called safe with respect to C if C is forward invariant.

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Definition (Control barrier function (CBF))

Let $C \subset \mathbb{R}^n$ be the superlevel set

$$C = \{x \in \mathbb{R}^n | \ B(x) \geq 0\}.$$

of a smooth function $B : \mathbb{R}^n \to \mathbb{R}$. Then B is a CBF if there exists an extended class $\mathcal{K}_\infty$ function $\delta : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{u \in \mathcal{U}} \left(\langle \nabla B(x), f(x) \rangle + \langle \nabla B(x), g(x) \rangle u \right) \geq -\delta(B(x)) \quad (4)$$

- δ, extended $\mathcal{K}_\infty$ function if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that $\delta(r) = \alpha_1(r)$ and $\delta(-r) = -\alpha_2(r)$ for all $r \in \mathbb{R}_{\geq 0}$.
- If $B(x)$ is a control barrier function, then C is safe and asymptotically stable with respect to $\dot{x} = f(x) + g(x)u$ and a control law $u = k(x)$ satisfying inequality (4).
- Note that, if $B(x)$ is large, (4) is not restrictive.
- Note that, for $x \in \{x \in \mathbb{R}^n | \ B(x) = 0\}$, (4) is restrictive
- CBFs are usually used in the context of invariance (not (in)stability)

Comparison to control barrier function results

Consider the control affine system

$$\dot{x} = f(x) + g(x)u$$

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- $\mathcal{C} \subset \mathbb{R}^n$ is called forward invariant if for every $x_0 \in \mathcal{C}$,

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  - (in the weak sense) $\exists \phi \in \mathcal{S}(x_0)$
- For $u = k(x)$ Lipschitz, $\dot{x} = f(x) + g(x)k(x)$ is called safe with respect to $\mathcal{C}$ if $\mathcal{C}$ is forward invariant.

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- CBFs are usually used in the context of invariance (not (in)stability)

In combination with CLFs V:

$$u = k(x) = \operatorname{argmin}_{u \atop (u,\gamma) \in \mathcal{U} \times \mathbb{R}} \quad u^T u + \gamma^2$$

$$\text{subject to} \quad \langle \nabla V(x), f(x) + g(x)u \rangle \leq -\rho(|x|) + \gamma$$
$$\langle \nabla B(x), f(x) + g(x)u \rangle \geq -\delta(B(x)),$$

Complete control Lyapunov functions: Stability & Avoidance

Definition (Weak $\mathcal{KL}$-stab. with avoidance prop.)

Let $\mathcal{O} \subset \mathbb{R}^n$, $0 \notin \mathcal{O}$, be open. $\dot{x} \in F(x)$ is weakly $\mathcal{KL}$-stable with respect to 0 with avoidance property with respect to $\mathcal{O}$, if there exists $\beta \in \mathcal{KL}$ such that, for each $x_0 \in \mathbb{R}^n \backslash \mathcal{O}$, there exists $\phi(\cdot; x_0) \in \mathcal{S}(x_0)$ so that

$$|\phi(t; x_0)| \leq \beta(|x_0|, t) \qquad \text{and} \qquad \phi(t; x_0) \notin \mathcal{O} \qquad \forall\, t \geq 0.$$

Consider the special case: $\mathcal{O} = \bigcup_{i=1}^N \mathcal{O}_i$ for $\mathcal{O}_1, \ldots, \mathcal{O}_N$ open and for simplicity assume $N = 1$ in the following.

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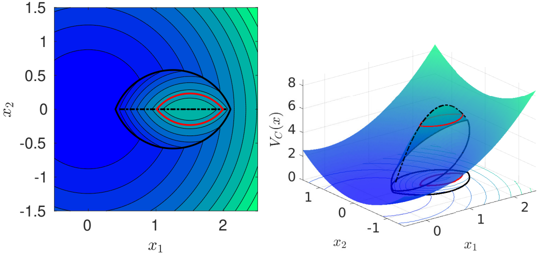

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Suppose F satisfies the basic condition and is Lipschitz. Let $\mathcal{O}_1 \subset \mathbb{R}^n$ define an open set and let $V_C : \mathbb{R}^n \to \mathbb{R}$ be a cont. function. Assume there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that the following are satisfied. There exists $c_1 \in \mathbb{R}_{>0}$ such that

$$V_C(x) = c_1 \quad \forall x \in \partial\mathcal{O}_1 \quad \text{and} \quad c_1 \le \inf_{x \in \mathcal{O}_1} V_C(x).$$

$$\alpha_1(|x|) \le V_C(x) \le \alpha_2(|x|), \qquad \forall\, x \in \mathbb{R}^n$$

$$\min_{w \in F(x)} D_+ V_C(x; w) \le -\rho(x), \qquad \forall\, x \in \mathbb{R}^n \backslash \mathcal{O}_1.$$

Then V_C is called complete control Lyapunov function.

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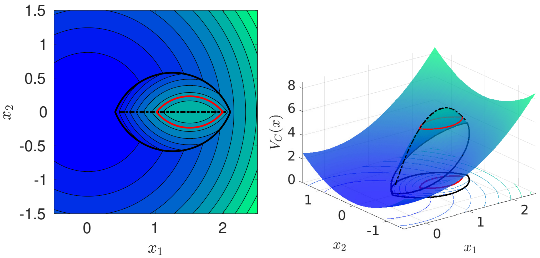

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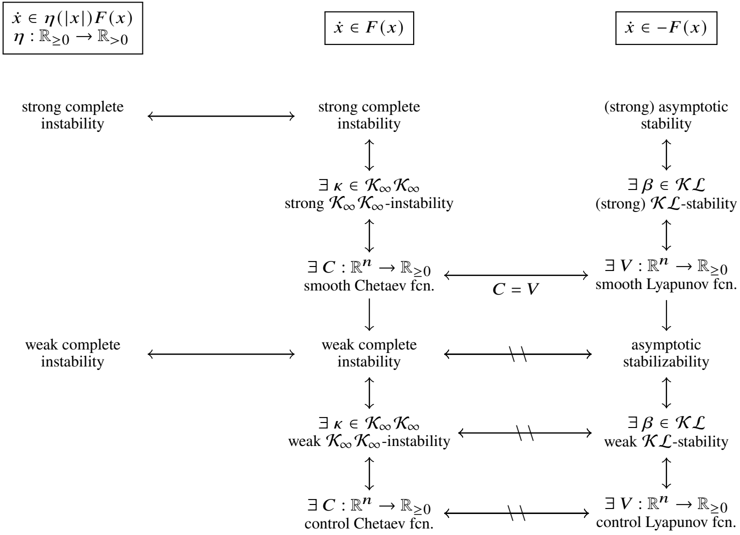
Then V_C is called complete control Lyapunov function.

Theorem

Consider $\dot{x} \in F(x)$ satisfying the basic conditions and assume F is Lipschitz. Let $\mathcal{O}_1$ be open and let $V_C : \mathbb{R}^n \to \mathbb{R}$ be a complete control Lyapunov function. Then $\dot{x} \in F(x)$ is weakly $\mathcal{KL}$-stable with respect to the origin and has the avoidance property with respect to $\mathcal{O}_1$.

$\rightsquigarrow$ If $\mathcal{O}_1$ is bounded, V_C is necessarily nonsmooth.

Overview



$\dot{x} \in \eta(|x|)F(x)$
$\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$
$\dot{x} \in F(x)$
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strong complete instability
strong complete instability
(strong) asymptotic stability
$\exists\, \kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$
strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability
$\exists\, \beta \in \mathcal{KL}$
(strong) $\mathcal{KL}$-stability
$\exists\, C : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
smooth Chetaev fcn.
$C = V$
$\exists\, V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
smooth Lyapunov fcn.
weak complete instability
weak complete instability
asymptotic stabilizability
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control Lyapunov fcn.

(In-)Stability of Differential Inclusions

— Notions, Equivalences & Lyapunov-like Characterizations —

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$\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions

Consider: $\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$

- Assume F satisfies the basic conditions

Definition (Strong complete instability)

The differential inclusion is strongly completely unstable with respect to $0 \in \mathbb{R}^n$ if the following properties are satisfied. There exists a function $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon > 0$ and for all solutions $\phi \in \mathcal{S}(x_0)$,

$$|\phi(t;x_0)| \geq \varepsilon \qquad \text{for all } t \geq 0,$$
$$|\phi(t;x_0)| \to \infty \qquad \text{for } t \to \infty,$$

whenever $|x_0| \geq \delta(\varepsilon)$.

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whenever $|x_0| \geq \delta(\varepsilon)$.

Definition ($\mathcal{K}_\infty\mathcal{K}$- and $\mathcal{K}_\infty\mathcal{K}_\infty$-functions)

Consider the continuous function $\kappa : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$.

- κ is said to be of class $\mathcal{K}_\infty\mathcal{K}$ ($\kappa \in \mathcal{K}_\infty\mathcal{K}$) if $\kappa(\cdot, s) \in \mathcal{K}_\infty$ $\forall\, s \in \mathbb{R}_{\geq 0}$ and $\kappa(s, \cdot) - \kappa(s, 0) \in \mathcal{K}$ $\forall\, s \in \mathbb{R}_{>0}$.
- κ is said to be of class $\mathcal{K}_\infty\mathcal{K}_\infty$ ($\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$) if $\kappa(\cdot, s) \in \mathcal{K}_\infty$ $\forall\, s \in \mathbb{R}_{\geq 0}$ and $\kappa(s, \cdot) - \kappa(s, 0) \in \mathcal{K}_\infty$ $\forall\, s \in \mathbb{R}_{>0}$.

Example:

- $\kappa(s, t) = ce^{\lambda t} s \in \mathcal{K}_\infty\mathcal{K}_\infty$ if $\lambda > 0$, $c > 0$
- $\kappa(s, t) = (t + 1)s \in \mathcal{K}_\infty\mathcal{K}_\infty$

Definition (Strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

The differential inclusion is strongly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable with respect to $0 \in \mathbb{R}^n$ if there exists $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ every solution $\phi \in \mathcal{S}(x_0)$ satisfies

$$|\phi(t; x_0)| \geq \kappa(|x_0|, t), \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

$\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions

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$$|\phi(t;x_0)| \geq \varepsilon \qquad \text{for all } t \geq 0,$$
$$|\phi(t;x_0)| \to \infty \qquad \text{for } t \to \infty,$$

whenever $|x_0| \geq \delta(\varepsilon)$.

Definition ($\mathcal{K}_\infty\mathcal{K}$- and $\mathcal{K}_\infty\mathcal{K}_\infty$-functions)

Consider the continuous function $\kappa : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$.

- κ is said to be of class $\mathcal{K}_\infty\mathcal{K}$ ($\kappa \in \mathcal{K}_\infty\mathcal{K}$) if $\kappa(\cdot, s) \in \mathcal{K}_\infty$ $\forall\ s \in \mathbb{R}_{\geq 0}$ and $\kappa(s,\cdot) - \kappa(s,0) \in \mathcal{K}$ $\forall\ s \in \mathbb{R}_{>0}$.
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Example:

- $\kappa(s,t) = ce^{\lambda t}s \in \mathcal{K}_\infty\mathcal{K}_\infty$ if $\lambda > 0$, $c > 0$
- $\kappa(s,t) = (t+1)s \in \mathcal{K}_\infty\mathcal{K}_\infty$

Definition (Strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

The differential inclusion is strongly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable with respect to $0 \in \mathbb{R}^n$ if there exists $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ every solution $\phi \in \mathcal{S}(x_0)$ satisfies

$$|\phi(t;x_0)| \geq \kappa(|x_0|, t), \quad \forall\ t \in \mathbb{R}_{\geq 0}.$$

Can $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ be replaced by $\kappa \in \mathcal{K}_\infty\mathcal{K}$ in the Definition?

Example (Counterexample)

Consider $\dot{x} = 0$ which has 0 as a stable equilibrium. Assume that $\kappa \in \mathcal{K}_\infty\mathcal{K}$ is used to define complete instability and consider

$$\kappa(r,t) = \tfrac{1}{2}r(2 - e^{-t}) \in \mathcal{K}_\infty\mathcal{K} \setminus \mathcal{K}_\infty\mathcal{K}_\infty.$$

For all $x_0 \in \mathbb{R}^n$ and for all $t \in \mathbb{R}_{\geq 0}$ it holds that

$$|\phi(t;x_0)| = |x_0| \geq \tfrac{1}{2}|x_0|(2 - e^{-t}) = \kappa(|x_0|, t)$$

$\mathcal{K}_\infty\mathcal{K}_\infty$-instability and Chetaev functions

Consider: $\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n$

- Assume F satisfies the basic conditions

Definition (Strong complete instability)

The differential inclusion is strongly completely unstable with respect to $0 \in \mathbb{R}^n$ if the following properties are satisfied. There exists a function $\delta \in \mathcal{K}_\infty$ such that for all $\varepsilon > 0$ and for all solutions $\phi \in \mathcal{S}(x_0)$,

$$|\phi(t; x_0)| \geq \varepsilon \qquad \text{for all } t \geq 0,$$
$$|\phi(t; x_0)| \to \infty \qquad \text{for } t \to \infty,$$

whenever $|x_0| \geq \delta(\varepsilon)$.

Definition ($\mathcal{K}_\infty\mathcal{K}$- and $\mathcal{K}_\infty\mathcal{K}_\infty$-functions)

Consider the continuous function $\kappa : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$.

- κ is said to be of class $\mathcal{K}_\infty\mathcal{K}$ ($\kappa \in \mathcal{K}_\infty\mathcal{K}$) if $\kappa(\cdot, s) \in \mathcal{K}_\infty$ $\forall\, s \in \mathbb{R}_{\geq 0}$ and $\kappa(s, \cdot) - \kappa(s, 0) \in \mathcal{K}$ $\forall\, s \in \mathbb{R}_{>0}$.
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Example:

- $\kappa(s, t) = ce^{\lambda t}s \in \mathcal{K}_\infty\mathcal{K}_\infty$ if $\lambda > 0$, $c > 0$
- $\kappa(s, t) = (t + 1)s \in \mathcal{K}_\infty\mathcal{K}_\infty$

Definition (Strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

The differential inclusion is strongly $\mathcal{K}_\infty\mathcal{K}_\infty$-unstable with respect to $0 \in \mathbb{R}^n$ if there exists $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ every solution $\phi \in \mathcal{S}(x_0)$ satisfies

$$|\phi(t; x_0)| \geq \kappa(|x_0|, t), \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Definition (Local Strong $\mathcal{K}_\infty\mathcal{K}_\infty$-instability)

Let $0 \in \mathcal{O} \subset \mathbb{R}^n$ be an open neighborhood. $0 \in \mathbb{R}^n$ is locally strongly completely unstable with respect to the differential inclusion and $\mathcal{O}$ if there exists a $\kappa \in \mathcal{K}_\infty\mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{O}$ every solution $\phi \in \mathcal{S}(x_0)$ satisfies

$$|\phi(t; x_0)| \geq \kappa(|x_0|, t),$$

for all $t \in \mathbb{R}_{\geq 0}$ such that $\phi(t; x_0) \in \mathcal{O}$.

$\mathcal{KL}$-stability with respect to (two) measures

- Consider two measures $\omega_1, \omega_2 : \mathcal{G} \to \mathbb{R}_{\geq 0}$, i.e., two positive functions from an open set $\mathcal{G} \subset \mathbb{R}^n$ to the positive real numbers.
- Then $\dot{x} \in F(x)$ is called $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ if there exists a $\mathcal{KL}$-function β such that for all $x \in \mathcal{G}$,

$$\begin{aligned} & \omega_1(\phi(t;x_0)) \leq \beta(\omega_2(x_0), t) \qquad \forall\, t \geq 0 \\ \text{and} \qquad & \phi(t;x_0) \in \mathcal{G} \qquad \forall \phi \in \mathcal{S}(x_0) \qquad \forall\, t \geq 0. \end{aligned}$$

Note that:

- For $\mathcal{G} = \mathbb{R}^n$ and $\omega_1(x) = \omega_2(x) = |x|$, the definition of (string) $\mathcal{KL}$-stability of the origin is recovered.
- For $\mathcal{G} \subset \mathbb{R}^n \backslash \{0\}$ excluding the origin, the measures $\omega_1(x) = \omega_2(x) = \frac{1}{|x|}$ ensure certain instability properties. In particular, the bound

$$|\phi(t;x_0)| \geq \left(\beta\left(\left|\frac{1}{x_0}\right|, t\right)\right)^{-1}$$

is obtained.

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In the context of Lyapunov functions:

- A Lyapunov function characterizing $\mathcal{KL}$-stability with respect to (ω_1, ω_2), needs to satisfy

$$\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)).$$

- For $\omega_1(x) = \omega_2(x) = |x|^{-1}$ this implies

$$\frac{1}{|x|} \leq V(x) \leq \frac{1}{|x|}$$

and for $\omega_1(x) = \omega_2(x) = |x|$ this implies

$$|x| \leq V(x) \leq |x|$$

- As an example
  - $V(x) = x^2$ characterizes stability of $\dot{x} = -x$
  - $V(x) = x^{-2}$ characterizes instability of $\dot{x} = x$

$\rightsquigarrow$ V behaves different close to the origin

Relations between Chetaev and Lyapunov functions & scaling (2)

Scaling of Lyapunov/Chetaev functions:

- A Chetaev function satisfies:

$$\alpha_1(|x|) \leq C(x) \leq \alpha_2(|x|) \qquad \forall\, x \in \mathbb{R}^n$$
$$\min_{w \in F(x)} D_+C(x;w) \geq \rho(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

- For $\hat{\rho} = \rho \circ \alpha_2^{-1} \in \mathcal{P}$, it holds that

$$\min_{w \in F(x)} D_+C(x;w) \geq \rho(|x|) \geq \rho(\alpha_2^{-1}(C(x))) = \hat{\rho}(C(x)).$$

- Select $\hat{\alpha} \in \mathcal{K}_\infty$ continuously differentiable such that

$$\hat{\alpha}'(s) > 0 \quad \text{and} \quad \hat{\rho}(s)\hat{\alpha}'(s) \geq \hat{\alpha}(s) \qquad \forall\, s \in \mathbb{R}_{>0},$$

- Note that for $\widehat{C}(x) = \hat{\alpha}(C(x))$:

$$D_+\widehat{C}(x;w) = \hat{\alpha}'(C(x))D_+C(x;w) \qquad \forall\, w \in \mathbb{R}^n.$$

  (chain rule with respect to the Dini derivative) and thus

$$\min_{w \in F(x)} D_+\widehat{C}(x;w) \geq \hat{\alpha}'(C(x))\hat{\rho}(C(x)) \geq \hat{\alpha}(C(x)) = \widehat{C}(x)$$

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- As a last step define

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In particular the conditions

$$\alpha_1(|x|) \leq C(x) \leq \alpha_2(|x|) \qquad \forall\, x \in \mathbb{R}^n$$
$$\min_{w \in F(x)} D_+C(x;w) \geq \rho(|x|) \qquad \forall\, x \in \mathbb{R}^n$$

are equivalent to

$$\hat{\alpha}_1(|x|) \leq \widehat{C}(x) \leq \hat{\alpha}_2(|x|) \qquad \forall\, x \in \mathbb{R}^n$$
$$\min_{w \in F(x)} D_+\widehat{C}(x;w) \geq \widehat{C}(x) \qquad \forall\, x \in \mathbb{R}^n$$