Avoidance and Stabilization for Linear Systems

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Research focus: Obstacle avoidance & target set stabilization

Setting: Linear System $\dot{x} = Ax + Bu$

Goal: Reactive controller design for obstacle avoidance & target set stabilization.

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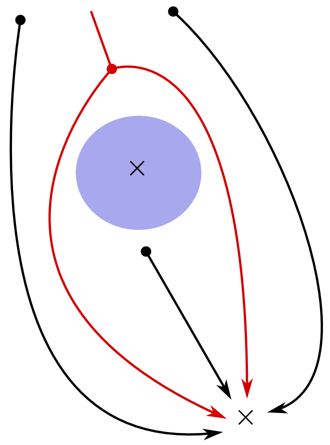

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- “We construct Lipschitz-continuous feedback laws which guarantee avoidance and stabilization.”
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- “The avoidance controller and the stabilizing control law can be designed independently.”

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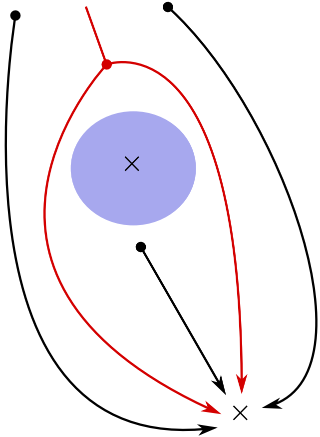

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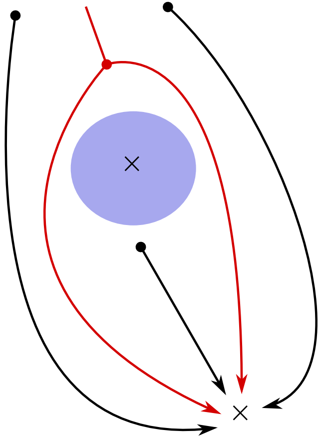

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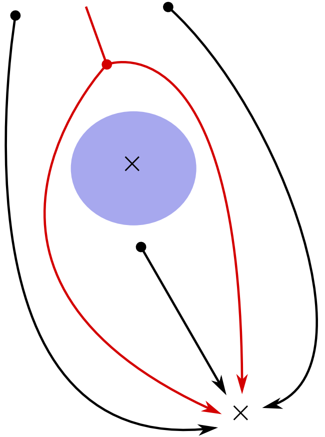

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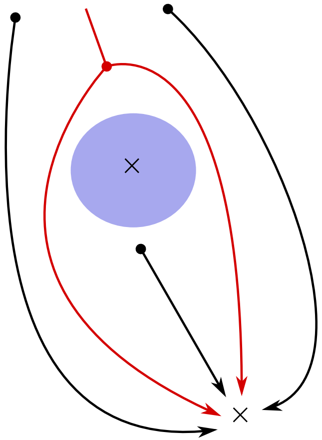

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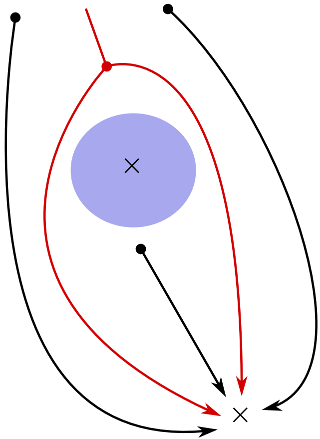
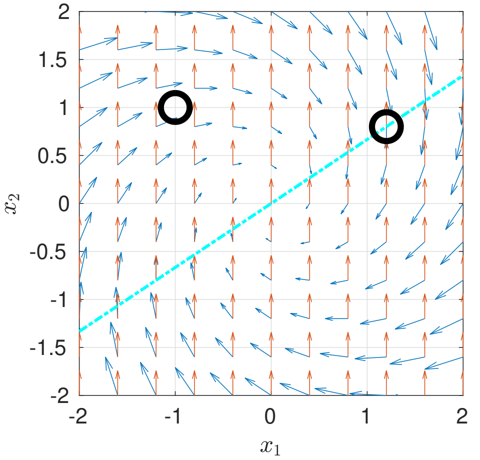
- "The avoidance controller and the stabilizing control law can be designed independently."
  - $\rightsquigarrow$ The combination of repelling & stabilizing controllers may introduce equilibria (stable or unstable) and limit cycles in the closed-loop dynamics.

Combined avoidance & stabilization:



(Underactuated) Systems with Nontrivial Drift (Position of the Obstacle)



The location of the obstacle:

- Consider

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

  (The system is controllable)
- Subspace of induced equilibria: ($B \in \mathbb{R}^n$)

$$\mathcal{E} = \{y \in \mathbb{R}^n : 0 = Ay + B\nu,\ \nu \in \mathbb{R}\}$$

- Obstacle $\mathcal{D}$ with $\mathcal{D} \cap \mathcal{E} = 0$
  - Use the natural drift Ax to 'leave the obstacle behind' and use Bu to avoid the obstacle
- Obstacle $\mathcal{D}$ with $\mathcal{D} \cap \mathcal{E} \neq 0$
  - Use u to destabilize a point $\hat{x} \in \mathcal{D} \cap \mathcal{E}$ to avoid the obstacle

Problem formulation & hybrid controller framework

Setting:

$$\dot{x} = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n$$

Obstacle:

$$\hat{x} \in \mathbb{R}^n \backslash \{0\}$$

Problem

Consider the linear system and a ***stabilizing feedback law***

$$u_s = K_s x.$$

For given $\varepsilon_2 > \varepsilon_1 > 0$, construct an avoidance (safety) controller $\gamma(x)$ such that

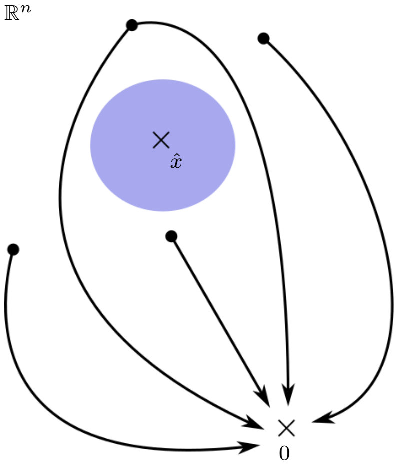
(i) *the origin $x = 0$ is globally asymptotically stable*

(ii) $\gamma(x)$ *satisfies*

$$\gamma(x) = K_s x \quad \forall x \in \mathbb{R}^n \setminus \mathcal{B}_{\varepsilon_2}(\hat{x})$$

$\Longrightarrow$ *Semiglobal Preservation*

(iii) *the closed loop solution $x(\cdot;\gamma)$ satisfies*
$x(t;x_0) \notin \mathcal{B}_{\varepsilon_1}(\hat{x}) \forall t \in \mathbb{R}_{\geq 0}, \forall x_0 \notin \mathcal{B}_{\varepsilon_2}(\hat{x})$
$\Longrightarrow$ *Semiglobal $\hat{x}$-avoidance*



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Hybrid controller design

Given: Controller selection

$$\gamma(x, q) = u_q(x), \quad q \in \{1, \ldots, p\}, \quad p \in \mathbb{N}$$

Orchestrate the controller selection through the flow map:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Ax + B\gamma(x, q) \\ 0 \end{bmatrix}, \qquad \xi \in \mathcal{C}$$

and the jump map

$$\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ \{i \in \mathbb{N} | \xi \in \mathcal{D}_i\} \end{bmatrix}, \qquad \xi \in \mathcal{D}$$

where

- $\mathcal{D} = \bigcup_{i=1}^{p} \mathcal{D}_i \subset \mathbb{R}^n \times \{1, \ldots, p\}$ (Jump set)
- $\mathcal{C} \subset \mathbb{R}^n \times \{1, \ldots, p\}$ (Flow set)

Obstacle avoidance ($\hat{x} \notin \mathcal{E}$): Problem formulation and assumptions

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Obstacle:

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Set of induced equilibria:

$$\mathcal{E} = \{y \in \mathbb{R}^n : 0 = Ay + B\nu, \ \nu \in \mathbb{R}\}$$

Basic Assumption

(a) Matrix $A_s := A + BK_s$ is Hurwitz. ✓

(b) $|B| = 1$. ✓

(c) The norm $x \mapsto |x|^2$ is contractive under the stabilizer $u_s = K_s x$ (i.e., $V(x) = x^T x$ is a Lyapunov function.) ✓

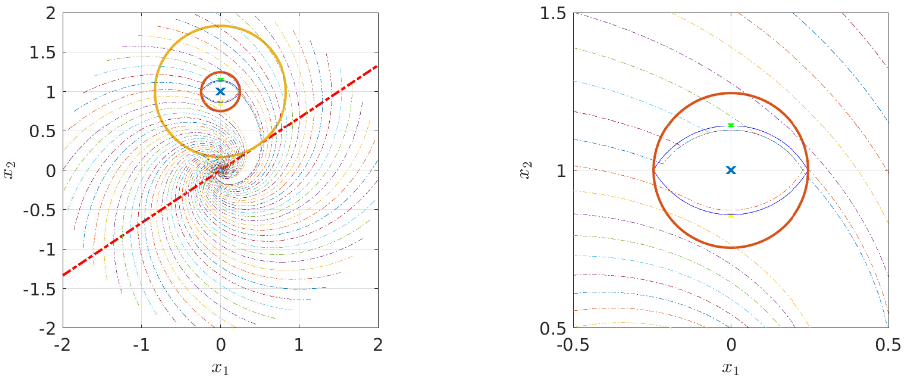
Discussion:

(a) (A, B) stabilizable
(Controllability is not necessary)

(b) Coordinate transformation: $B_\circ = B/|B|$, $u_\circ = |B|u$.

(c) Lyapunov function: $V_\circ(x) = x^T S_\circ^T S_\circ x$.
Coordinate transformation: $x_\circ = S_\circ x$.

Numerical example



System parameters:

$$\dot{x} = \begin{bmatrix} -1.0 & 1.5 \\ -1.5 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad \begin{array}{rcl} \sigma(A) & = & \{-1+1.5i, -1-1.5i\} \\ \sigma(A+A^T) & = & \{-2, -2\} \end{array}$$

$$u_s = 0, \quad \mu = 1.15, \quad \eta = 0.8321, \quad \zeta = 1.8028, \quad \delta^* = 0.2455$$

Controller design: 1. The "wipeout" property

- Distance to induced equilibria:

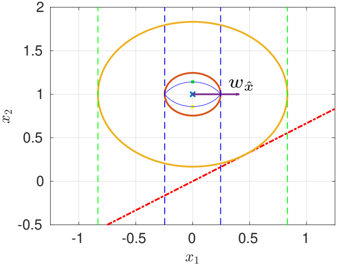
$$\eta^2 := \min_{y \in \{y \mid \exists u, Ay+Bu=0\}} |\hat{x} - y|^2$$

- Linear "wipeout" function/direction:

$$H(x) = \hat{x}^T A_B^T x = \hat{x}^T A^T (I - BB^T) x$$

$$w_{\hat{x}} = \frac{\nabla H(x)}{|\nabla H(x)|} = \frac{A_B \hat{x}}{\sqrt{\hat{x}^T A_B^T A_B \hat{x}}}$$

- Visualization:



Controller design: 1. The "wipeout" property

Remark

- For each $x \in \mathcal{B}_\eta(\hat{x})$ we have

$$\dot{H}(x) = \langle \nabla H(x), Ax + Bu \rangle \geq 0, \quad \forall u \in \mathbb{R}$$

- For each $\bar{\eta} < \eta$, there exists $\underline{h} > 0$ such that

$$\langle \nabla H(x), Ax + Bu \rangle \geq \underline{h}, \quad \forall u \in \mathbb{R}, \forall x \in \mathcal{B}_{\bar{\eta}}(\hat{x})$$

Remark

- A solution $x(\cdot)$ such that $x(t) \in \mathcal{B}_{\bar{\eta}}(\hat{x}) \; \forall \, t \in [0, T]$, satisfies

$$\langle w_{\hat{x}}, x(T) - x(0) \rangle \geq T \frac{\underline{h}}{|\nabla H(x)|}.$$

- A solution $x(\cdot)$ such that $x(t) \in \mathcal{B}_\eta(\hat{x}) \; \forall \, t \in [0, T]$, satisfies

$$\langle w_{\hat{x}}, x(t_2) - x(t_1) \rangle \geq 0 \qquad \forall 0 \leq t_1 \leq t_2 \leq T.$$

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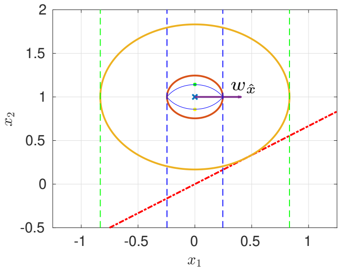

Controller design: 2. Definition of the shell $\mathcal{S}(\delta)$

- Design parameters of the shell $\mathcal{S}(\delta)$:

$$\delta \in \mathbb{R}_{>0}$$
$$\mu \in (0, 2)$$

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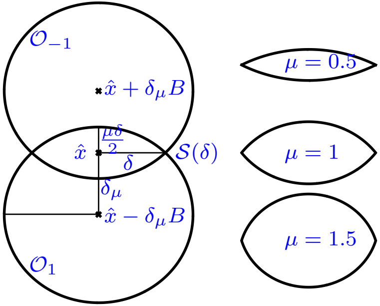

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- Definitions: $(q \in \{1, -1\})$

$$\delta_\mu := \delta \left(\frac{1}{\mu} - \frac{\mu}{4} \right)$$
$$\mathcal{O}_q := \mathcal{B}_{\left(\frac{\mu\delta}{2} + \delta_\mu\right)}(\hat{x} - q\delta_\mu B)$$
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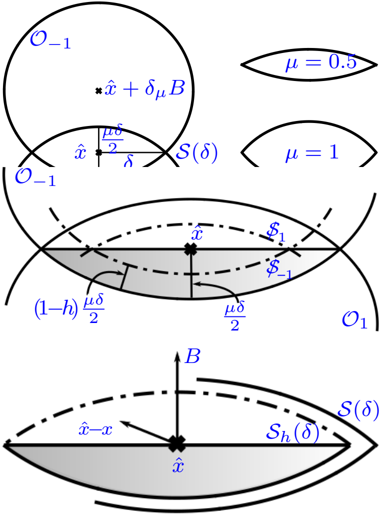

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- Hysteresis parameter: $h \in (0,1)$

$$\mathcal{O}_{h,q} = \mathcal{B}_{h\frac{\mu\delta}{2}+\delta_\mu}(\hat{x} - q\delta_\mu B)$$
$$\mathcal{S}_h(\delta) = \mathcal{O}_{h,1} \cap \mathcal{O}_{h,-1}$$
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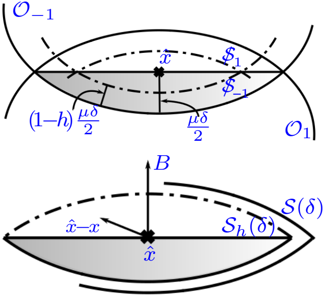
Controller design: 3. The avoidance controller



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- Avoidance control law: $(q \in \{1, -1\})$

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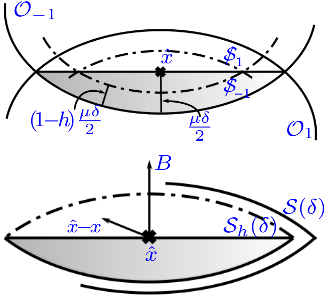

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$$u_a(x,q) = -\frac{\langle x-(\hat{x}-q\delta_\mu B), Ax\rangle}{\langle x-(\hat{x}-q\delta_\mu B), B\rangle}$$

Lemma

Let $\mu \in (0, 2/\sqrt{3})$, $\delta > 0$ and $h \in (0,1)$.

- *For $q \in \{-1,1\}$ and $x_0 \in \mathcal{S}_h(\delta) \subset \mathcal{S}(\delta)$, the controller $u = u_a(x_0,q)$ is well defined.*

Idea of the proof:

- We show that: $\langle x_0 - (\hat{x} - q\delta_\mu B), B\rangle \neq 0$ $\forall x_0 \in \mathcal{S}(\delta)$

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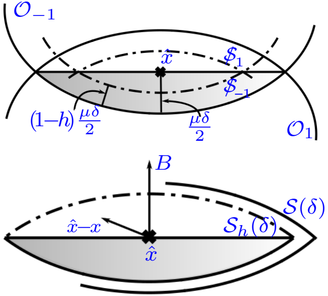

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- *The solution $x(\cdot, x_0, u_a)$ remains at a constant (non-negative) distance from the center*

$$c_q := \hat{x} - q\delta_\mu B$$

of the ball $\mathcal{O}_q$ until it remains in $\mathcal{S}(\delta)$.

Idea of the proof:

- We show that: $\langle x_0 - (\hat{x} - q\delta_\mu B), B\rangle \neq 0$ $\forall x_0 \in \mathcal{S}(\delta)$
- We show that: $\frac{d}{dt}|x(t;x_0,u_a) - c_q|^2 = 0$ $\forall x(t;x_0,u_a) \in \mathcal{S}(\delta)$

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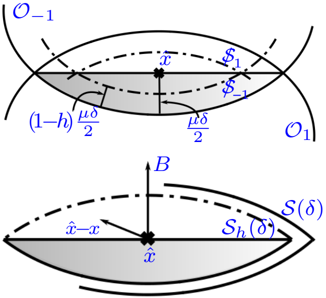

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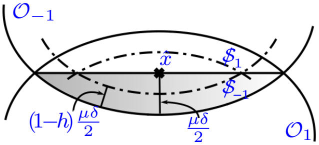

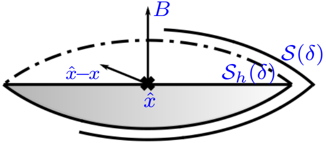

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- Controller design: $(q = \{-1,0,1\})$

$$\gamma(x,q) := (1-|q|)u_s(x) + |q|u_a(x,q)$$

- Jump set: $(q = \{-1,1\})$

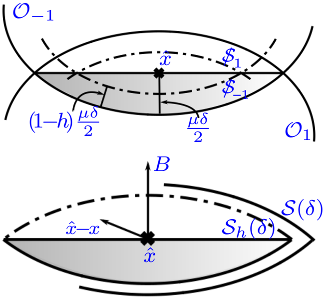
$$\mathcal{D}_q := \left(\mathcal{S}_h(\delta) \cap \$_q\right) \times \{0\},$$
$$\mathcal{D}_0 := \overline{\mathbb{R}^n \setminus \mathcal{S}(\delta)} \times \{1,-1\}$$
$$\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_{-1} \cup \mathcal{D}_0$$

- Jump map:

$$\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_q(\xi) \end{bmatrix}, \quad \xi \in \mathcal{D}$$

$$G_q(\xi) = \begin{cases} 1, & \xi \in \mathcal{D}_1 \setminus \mathcal{D}_{-1} \\ -1, & \xi \in \mathcal{D}_{-1} \setminus \mathcal{D}_1 \\ \{1,-1\}, & \xi \in \mathcal{D}_1 \cap \mathcal{D}_{-1} \\ 0, & \xi \in \mathcal{D}_0, \end{cases}$$

Controller design: 3. The avoidance controller



- Hysteresis parameter: $h \in (0,1)$

$$\mathcal{O}_{h,q} = \mathcal{B}_{h\frac{\mu\delta}{2}+\delta_\mu}(\hat{x} - q\delta_\mu B)$$

$$\mathcal{S}_h(\delta) = \mathcal{O}_{h,1} \cap \mathcal{O}_{h,-1}$$

$$\$_q = \mathcal{S}(\delta) \cap \{x \in \mathbb{R}^n : \ qB^T(x - \hat{x}) \geq 0\}$$

- Avoidance control law: $(q \in \{1,-1\})$

$$u_a(x,q) = -\frac{\langle x-(\hat{x} - q\delta_\mu B), Ax\rangle}{\langle x - (\hat{x} - q\delta_\mu B), B\rangle}$$

- Controller design: $(q = \{-1,0,1\})$

$$\gamma(x,q) := (1-|q|)u_s(x) + |q|u_a(x,q)$$

- Jump set: $(q = \{-1,1\})$

$$\mathcal{D}_q := \left(\mathcal{S}_h(\delta) \cap \$_q\right) \times \{0\},$$

$$\mathcal{D}_0 := \overline{\mathbb{R}^n \setminus \mathcal{S}(\delta)} \times \{1,-1\}$$

$$\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_{-1} \cup \mathcal{D}_0$$

- Flow set: $\mathcal{C} = \overline{(\mathbb{R}^n \times \{-1,0,1\}) \setminus \mathcal{D}}$
- Jump map:

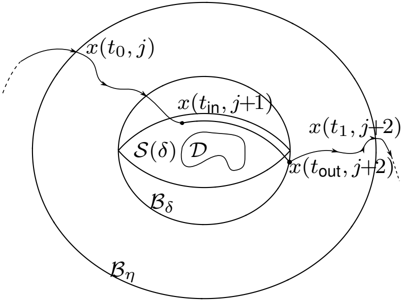
$$\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_q(\xi) \end{bmatrix}, \quad \xi \in \mathcal{D}$$

$$G_q(\xi) = \begin{cases} 1, & \xi \in \mathcal{D}_1 \setminus \mathcal{D}_{-1} \\ -1, & \xi \in \mathcal{D}_{-1} \setminus \mathcal{D}_1 \\ \{1,-1\}, & \xi \in \mathcal{D}_1 \cap \mathcal{D}_{-1} \\ 0, & \xi \in \mathcal{D}_0, \end{cases}$$

- Flow map:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Ax + B\gamma(x,q) \\ 0 \end{bmatrix}, \quad \xi \in \mathcal{C}$$

Controller design: 4. Global stability and obstacle avoidance



$$\eta^2 := \min_{y \in \{y \mid \exists u, Ay+Bu=0\}} |\hat{x} - y|^2.$$

$$\zeta = -\frac{2|A + BK_s|}{\lambda_{\text{max}}((A+BK_s)^T + (A+BK_s))} > 0$$

$$\delta^* = \frac{1}{2}\left(|\hat{x}| + \eta + \zeta - \sqrt{(|\hat{x}| + \eta + \zeta)^2 - 4|\hat{x}|\eta}\right) > 0$$

Theorem

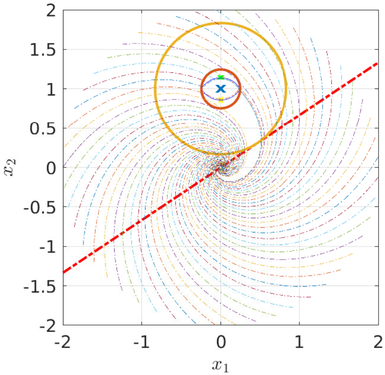
Let the basic assumption be satisfied.
Let $\delta \in (0, \min\{\delta^*, \frac{\eta}{1+\zeta}\})$, $\mu \in (0, 2/\sqrt{3})$, *and* $h \in (0,1)$.
Then the **hybrid controller guarantees** *that*

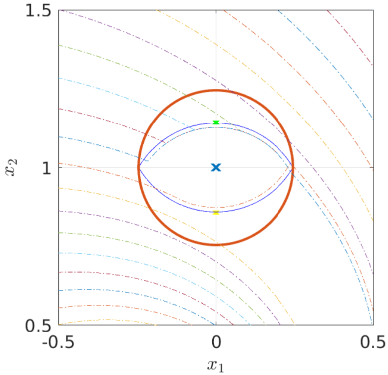
(i) *the origin* $\xi = (x, q) = (0, 0)$ *is* **uniformly globally asymptotically stable** *from* $\mathbb{R}^n \times \{-1, 0, 1\}$.

(ii) $\forall\, \xi(0,0) \in \mathbb{R}^n \setminus \mathcal{S}(\delta) \times \{-1, 0, 1\}$,

$$|x(t,j)|_{\hat{x}} \geq h\frac{\mu\delta}{2} \quad \forall\, (t,j) \in \textit{dom}(\xi).$$

Numerical example



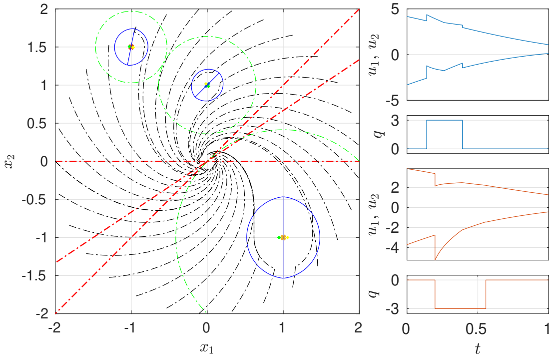


System parameters:

$$\dot{x} = \begin{bmatrix} -1.0 & 1.5 \\ -1.5 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad \begin{aligned} \sigma(A) &= \{-1 + 1.5i, -1 - 1.5i\} \\ \sigma(A + A^T) &= \{-2, -2\} \end{aligned}$$

$$u_s = 0, \qquad \mu = 1.15, \qquad \eta = 0.8321, \qquad \zeta = 1.8028, \qquad \delta^* = 0.2455$$

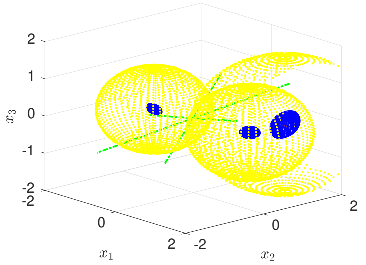
Extensions

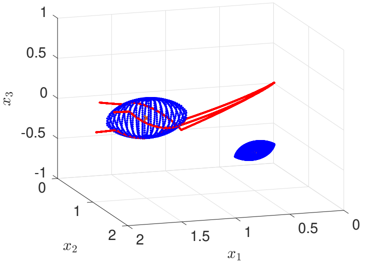


Extensions:

- Multiple obstacles
- In the multidimensional input case the method is applicable for all $\hat{x} \neq 0$

Extensions





Extensions:

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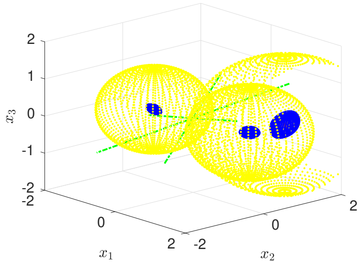
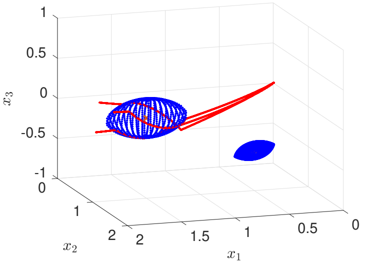
- In the multidimensional input case the method is applicable for all $\hat{x} \neq 0$
- The construction is independent of the state dimension $x \in \mathbb{R}^n$

Extensions



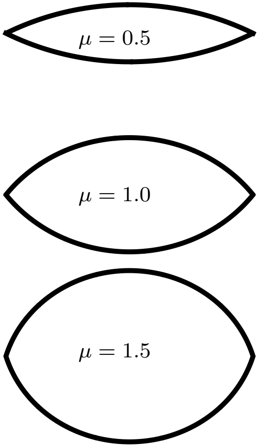


Extensions:

- Multiple obstacles
- In the multidimensional input case the method is applicable for all $\hat{x} \neq 0$
- The construction is independent of the state dimension $x \in \mathbb{R}^n$
- Extensions to robust stabilization and robust avoidance:

$$\begin{aligned} \dot{x} &= Ax + Bu + w_x \\ y &= x + w_y \end{aligned}$$

Extensions



Extensions:

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- Extensions to robust stabilization and robust avoidance:

$$\begin{aligned} \dot{x} &= Ax + Bu + w_x \\ y &= x + w_y \end{aligned}$$

- Bounded inputs

Obstacle avoidance ($\hat{x} \in \mathcal{E}$): Problem formulation and assumptions

Setting:

- (Linear) Dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n, \ \ (u \in \mathbb{R}^1)$$

- (A, B) controllable (stabilizability is not enough)

Subspace of induced equilibria: ($B \in \mathbb{R}^n$)

- $\mathcal{E} = \{y \in \mathbb{R}^n | 0 = Ay + B\nu, \ \nu \in \mathbb{R}\}$
- (W.l.o.g.) $\mathcal{E} = \operatorname{span}(A^{-1}B)$

Remember:

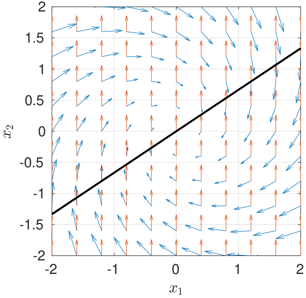
Let $\hat{x} \in \mathcal{E}$ and $0 = A\hat{x} + B\nu_{\hat{x}}$. Then

$$\begin{aligned} \dot{z} &= \overbrace{x - \hat{x}}^{\cdot} = A(x - \hat{x}) + B(u - \nu_{\hat{x}}) \\ &= Az + Bv \end{aligned}$$

where $z = x - \hat{x}$ and $v = u - \nu_{\hat{x}}$.

Example:

$$\dot{x} = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



Obstacle centered around

$$\hat{x} \in \mathcal{E}$$

Target set $0 \in \mathbb{R}^n$

Controller design ($\hat{x} \in \mathcal{E} = \text{span}(A^{-1}B)$)

Setting:

- Linear Dynamical system

$$\dot{x} = Ax + Bu, \quad x(0) \in \mathbb{R}^n, \quad (u \in \mathbb{R}^1)$$

- Obstacle & target set:

$$\hat{x} \in \mathcal{E} = \text{span}(A^{-1}B), \qquad 0 \in \mathbb{R}^n$$

- (A, B) controllable

(Intuitive) Controller design:

Use pole placement to compute K_s, $\in \mathbb{R}^{1\times n}$ such that

$A_s = (A + BK_s)$ are Hurwitz, i.e.,

- $u_s(x) = K_s x$ stabilizes 0

Lyapunov equation:

$$A_s^T P_s + P_s A_s = -I,$$

Lyapunov functions

$$V_s(x) = x^T P_s x$$

Lyapunov decrease

$$\dot{V}(x(t)) = \langle \nabla V_s(x), Ax + Bu_s(x) \rangle < 0$$

$$x \neq 0$$

Controller design ($\hat{x} \in \mathcal{E} = \mathrm{span}(A^{-1}B)$)

Setting:

- Linear Dynamical system

$$\dot{x} = Ax + Bu, \quad x(0) \in \mathbb{R}^n, \quad (u \in \mathbb{R}^1)$$

- Obstacle & target set:

$$\hat{x} \in \mathcal{E} = \mathrm{span}(A^{-1}B), \qquad 0 \in \mathbb{R}^n$$

- (A, B) controllable
- Shifted system

$$\dot{z} = Az + Bv, \quad z = x - \hat{x}, \quad v = u - \nu_{\hat{x}}$$

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Use pole placement to compute K_s, $\in \mathbb{R}^{1\times n}$ such that $A_s = (A + BK_s)$ are Hurwitz, i.e.,

- $u_s(x) = K_s x$ stabilizes 0
- $u_{sa}(x) = K_s(x - \hat{x}) + \nu_{\hat{x}}$ stabilizes $\hat{x}$

Lyapunov equation:

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Lyapunov functions

$$V_s(x) = x^T P_s x$$
$$V_{sa}(x) = (x - \hat{x})^T P_s (x - \hat{x})$$

Lyapunov decrease

$$\dot{V}(x(t)) = \langle \nabla V_s(x), Ax + Bu_s(x) \rangle < 0$$
$$\langle \nabla V_{sa}(x), Ax + Bu_{sa}(x) \rangle < 0$$

$$x \neq 0, x \neq \hat{x}$$

Controller design ($\hat{x} \in \mathcal{E} = \mathrm{span}(A^{-1}B)$)

Setting:

- Linear Dynamical system

$$\dot{x} = Ax + Bu, \quad x(0) \in \mathbb{R}^n, \quad (u \in \mathbb{R}^1)$$

- Obstacle & target set:

$$\hat{x} \in \mathcal{E} = \mathrm{span}(A^{-1}B), \qquad 0 \in \mathbb{R}^n$$

- (A, B) controllable
- Shifted system

$$\dot{z} = Az + Bv, \quad z = x - \hat{x}, \quad v = u - \nu_{\hat{x}}$$

- Time reversal system

$$\dot{y} = -Ay - Bv, \quad y(t) = z(-t)$$

(Intuitive) Controller design:

Use pole placement to compute $K_s, K_d \in \mathbb{R}^{1\times n}$ such that $A_s = (A + BK_s)$ and $A_d = -(A + BK_d)$ are Hurwitz, i.e.,

- $u_s(x) = K_s x$ — stabilizes 0
- $u_{sa}(x) = K_s(x - \hat{x}) + \nu_{\hat{x}}$ — stabilizes $\hat{x}$
- $u_d(x) = K_d(x - \hat{x}) + \nu_{\hat{x}}$ — destabilizes $\hat{x}$

Lyapunov equation:

$$A_s^T P_s + P_s A_s = -I,$$
$$A_d^T P_d + P_d A_d = -I$$

Lyapunov functions & Chetaev functions:

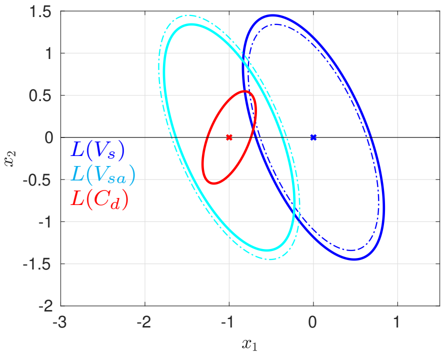
$$V_s(x) = x^T P_s x$$
$$V_{sa}(x) = (x - \hat{x})^T P_s (x - \hat{x})$$
$$C_d(x) = (x - \hat{x})^T P_d (x - \hat{x})$$

Lyapunov decrease/Chetaev increase

$$\dot{V}(x(t)) = \langle \nabla V_s(x), Ax + Bu_s(x) \rangle < 0$$
$$\langle \nabla V_{sa}(x), Ax + Bu_{sa}(x) \rangle < 0$$
$$\langle \nabla C_d(x), Ax + Bu_d(x) \rangle > 0$$
$$x \neq 0, x \neq \hat{x}$$

Controller design ($\hat{x} \in \mathcal{E} = \mathrm{span}(A^{-1}B)$)

Setting:



$\mathbb{R}^1)$

n

$- \nu_{\hat{x}}$

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Lyapunov decrease/Chetaev increase

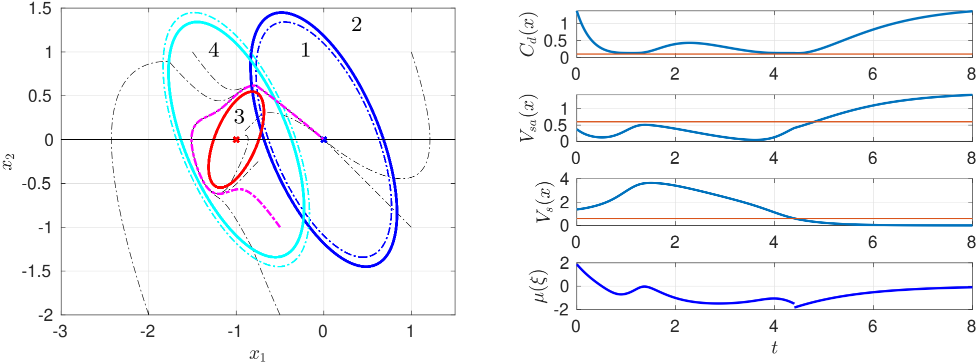
$$\dot{V}(x(t)) = \langle \nabla V_s(x), Ax + Bu_s(x)\rangle < 0$$
$$\langle \nabla V_{sa}(x), Ax + Bu_{sa}(x)\rangle < 0$$
$$\langle \nabla C_d(x), Ax + Bu_d(x)\rangle > 0$$
$$x \neq 0, x \neq \hat{x}$$

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\hat{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in \mathcal{E} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

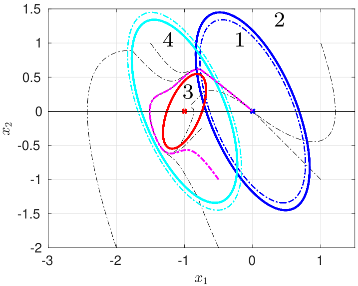
Intuitive controller design



Controller design:

- In 1 & 2: $\gamma(\xi) = u_s(x) = K_s(x)$ (asymptotically stabilize 0)
- In 3: $\gamma(\xi) = u_d(x) = K_d(x - \hat{x}) + \nu_{\hat{x}}$ (completely destabilize $\hat{x}$)
- In 4: $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x), \quad (\lambda(x) \in [0,1])$ (stay away from $\hat{x}$)
- (Dashed lines: avoid Zeno behavior)

Stability properties of the closed loop



Closed-loop properties:

- $\hat{x}$-avoidance? (Clear)
- Asymptotic stability of the origin?

Controller design:

- 1&2: $\gamma(\xi) = u_s(x) = K_s(x)$
- 3: $\gamma(\xi) = u_d(x) = K_d(x - \hat{x}) + \nu_{\hat{x}}$
- 4: $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x)$, $(\lambda(x) \in [0, 1])$
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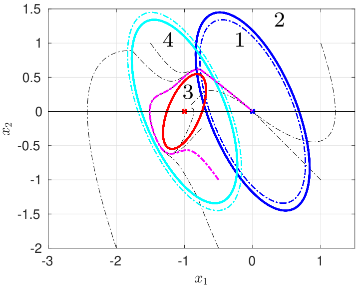

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- (Dashed lines: avoid Zeno behavior)

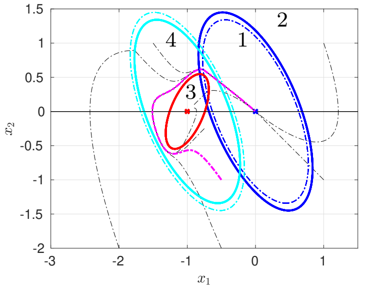
Closed-loop properties:

- $\hat{x}$-avoidance? (Clear)
- Asymptotic stability of the origin?
- Does the control law introduce equilibria? By construction
  - $0 = Ax + Bu_s(x) \iff x = 0$
  - $0 = Ax + Bu_d(x) \iff x = \hat{x}$

  Note that

  - $0 = Ax + B\nu \implies x \in \mathcal{E}$

Stability properties of the closed loop



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- 4: $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x)$, ($\lambda(x) \in [0, 1]$)
- (Dashed lines: avoid Zeno behavior)

Lemma

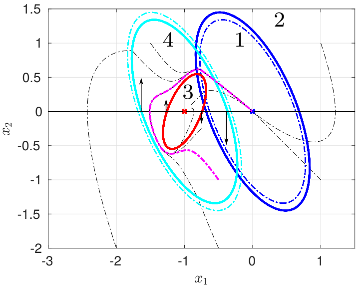
Let $x \in \mathcal{E}$. Then

$$Ax + Bu_{sa}(x) = \rho(Ax + Bu_d(x))$$

where

$$\rho = \frac{1 + K_{sa}A^{-1}B}{1 + K_d A^{-1}B} \in \mathbb{R}\backslash\{0\}.$$

Stability properties of the closed loop



Closed-loop properties:

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- Does the control law introduce equilibria? By construction
  - $0 = Ax + Bu_s(x) \iff x = 0$
  - $0 = Ax + Bu_d(x) \iff x = \hat{x}$

  Note that
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Controller design:

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- (Dashed lines: avoid Zeno behavior)

It holds

$$Ax + Bu_{sa}(x) = \rho(Ax + Bu_d(x)), \rho \in \mathbb{R}$$

- $\rho > 0$: No induced equilibria
- $\rho < 0$: Two induced equilibria

Lemma

Let $x \in \mathcal{E}$. Then

$$Ax + Bu_{sa}(x) = \rho(Ax + Bu_d(x))$$

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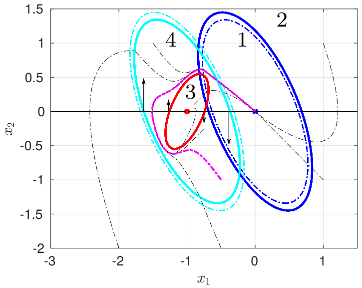
Stability properties of the closed loop



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- 4: $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x)$, $(\lambda(x) \in [0, 1])$
- (Dashed lines: avoid Zeno behavior)

It holds

$$Ax + Bu_{sa}(x) = \rho(Ax + Bu_d(x)), \rho \in \mathbb{R}$$

- $\rho > 0$: No induced equilibria (n even)
- $\rho < 0$: Two induced equilibria (n odd)

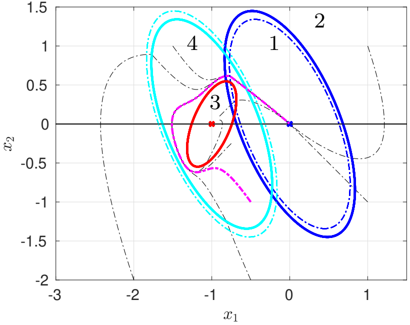
Lemma ($\operatorname{sign}(\rho) = (-1)^n \quad (x \in \mathbb{R}^n)$)

$$\prod_{i=1}^n \lambda_i^d = \det(A + BK_d) = \det(A)\det(1 + K_d A^{-1}B) > 0$$

$$\prod_{i=1}^n \lambda_i^{sa} = \det(A)\det(1 + K_{sa}A^{-1}B) \gtrless 0$$

and $\rho = \frac{1 + K_{sa}A^{-1}B}{1 + K_d A^{-1}B}$

Summary: Obstacle avoidance & target set stabilization ($\hat{x} \in \mathcal{E}$)



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- In 4: $\gamma(\xi) = (1 - \lambda(x))u_{sa}(x) + \lambda(x)u_d(x)$, ($\lambda(x) \in [0, 1]$)
- (Dashed lines: avoid Zeno behavior)

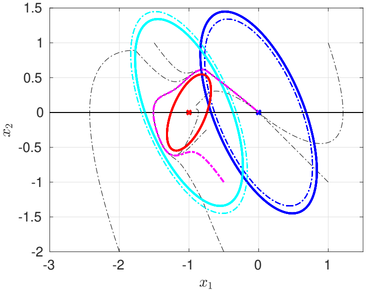
Assumptions:

- $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in R^1$
- (A, B) controllable ((A, B) stabilizable is not enough)
- $\hat{x} \in \mathcal{E} = \text{span}(A^{-1}B)$

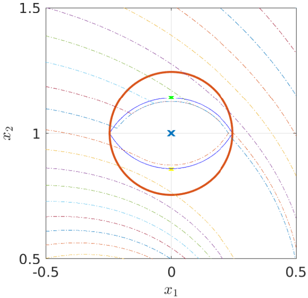
Results:

- $\forall\ n \geq 2$: Obstacle avoidance
- $n = 2$: Obstacle avoidance & global asymptotic stability
- $\forall\ n > 2$ odd: No global asymptotic stability
- $\forall\ n \geq 4$ even: Obstacle avoidance & maybe global asymptotic stability (We cannot exclude the existence of periodic orbits)

Conclusion & discussion



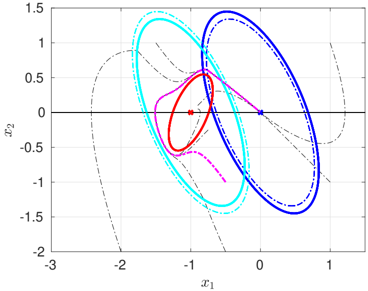
$\hat{x} \in \mathcal{E}$



$\hat{x} \notin \mathcal{E}$

Linear system: $\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$

Conclusion & discussion



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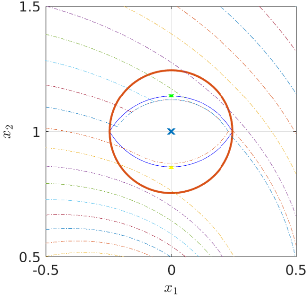

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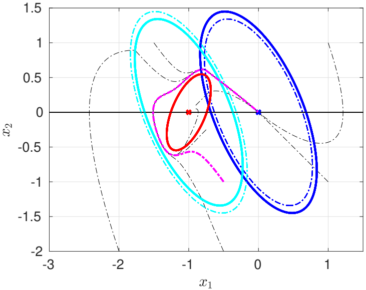
Assumption:

- (A, B) controllable
- (A, B) stabilizable

Conclusion & discussion



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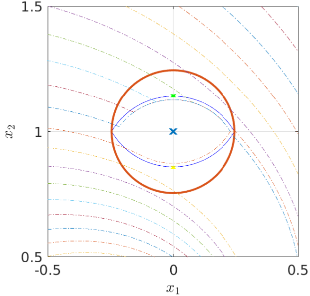

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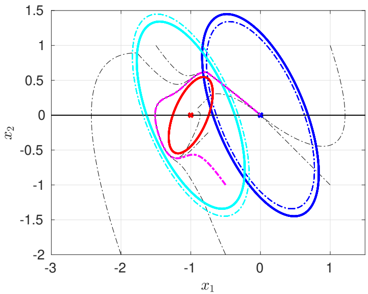
If $m > 1$:

- $\hat{x} \in \{y | 0 = Ay + Bu, u \in \mathbb{R}^m\} \backslash \{0\}$
- $\hat{x} \in \mathbb{R}^n \backslash \{0\}$

Conclusion & discussion



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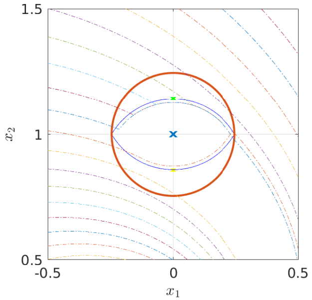

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Closed-loop properties:

- Only applicable if $n \geq 2$ is even
- Guarantees only for $n = 2$
- Independent of $n \in \mathbb{N}$, $n \geq 2$