Augmented controller design for reference tracking and obstacle avoidance

Philipp Braun

School of Engineering,

Australian National University, Canberra, Australia

In Collaboration with:

L. Grüne: University of Bayreuth, Bayreuth, Germany

C. M. Kellett: School of Engineering, Australian National University, Canberra, Australia

L. Zaccarian: Dipartimento di Ingegneria Industriale, University of Trento, Italy, and LAAS-CNRS, Université de Toulouse, France

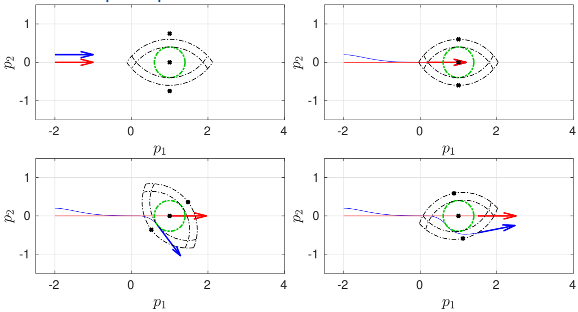
Augmented controller design for reference tracking & obstacle avoidance

Setting: Standard unicycle model

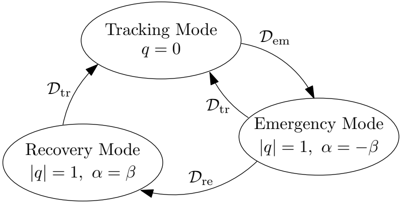
$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} v\cos(\phi) \\ v\sin(\phi) \\ w \end{bmatrix}, \qquad x = \begin{bmatrix} p_1 \\ p_2 \\ \phi \end{bmatrix}, \quad u = \begin{bmatrix} v \\ w \end{bmatrix}$$

with state $x \in \mathbb{R}^3$ and input $u \in [-2,2]^2 \subset \mathbb{R}^2$

Closed-loop snapshots:



Obstacle avoidance and reference tracking:

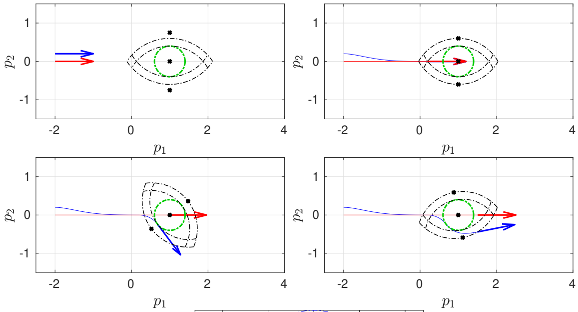


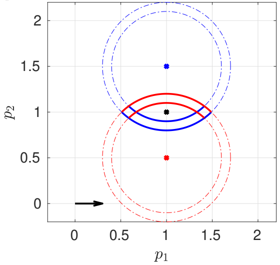
• reference trajectory, • closed-loop solution, • obstacle

Discussion on the avoidance neighborhood

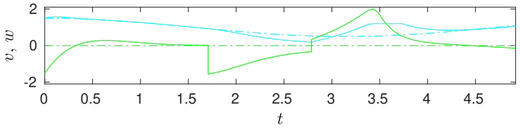
Assumption:

- Obstacle is contained in the green circle with fixed center and fixed radius





Closed loop input w and v:



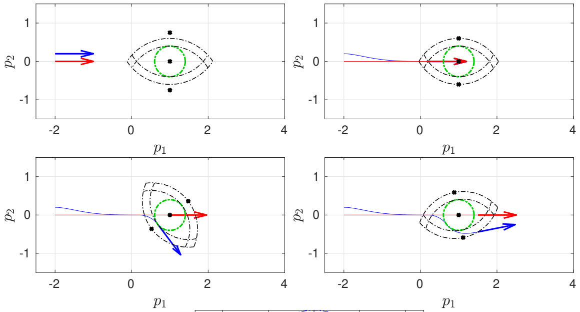
Discussion on the avoidance neighborhood

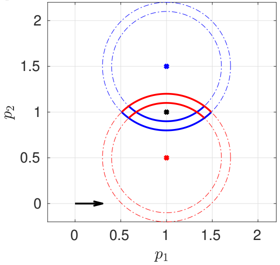
Assumption:

- Obstacle is contained in the green circle with fixed center and fixed radius

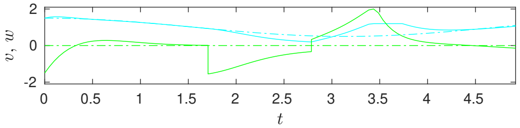
Construction of the (double) shell:

- Shell orientation depends on robot orientation (ϕ)
- Shell size depends on the velocity v of the robot
- Second shell: hysteresis region (and to avoid Zeno behavior)





Closed loop input w and v:



Discussion on the avoidance neighborhood

Assumption:

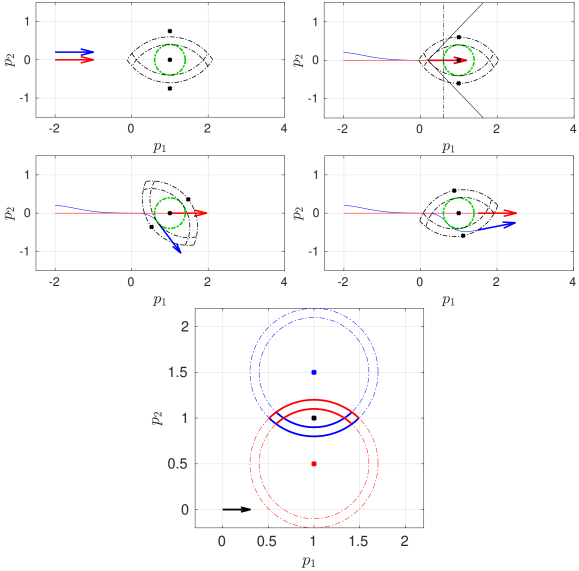
- Obstacle is contained in the green circle with fixed center and fixed radius

Construction of the (double) shell:

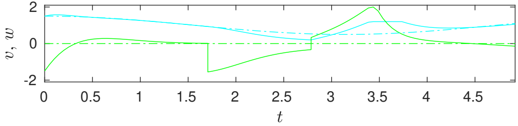
- Shell orientation depends on robot orientation (ϕ)
- Shell size depends on the velocity v of the robot
- Second shell: hysteresis region (and to avoid Zeno behavior)

Properties by design:

- Reduced angle between surface of avoidance neighborhood and robot orientation
- Continuous velocity input



Closed loop input w and v:



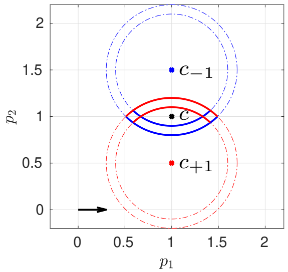
Constructive controller design: The avoidance neighborhood

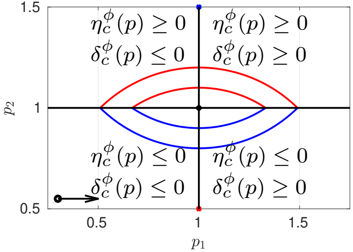
Definitions and assumptions:

- Center of the obstacle $c \in \mathbb{R}^2$
- Shifted center $c_q = c_q(\phi, \ell)$:

$$c_q := c - q\ell(v_{\text{ts}})R(\phi)\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad q \in \{\pm 1\}, \qquad R(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

  $\ell = \ell(v_{\text{ts}}) > 0$ depends on the magnitude of the reference velocity v_{ts}
- By design $|c - c_q| = \ell(v_{\text{ts}})$





$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$

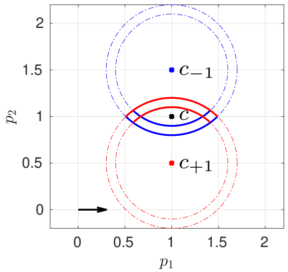
Constructive controller design: The avoidance neighborhood

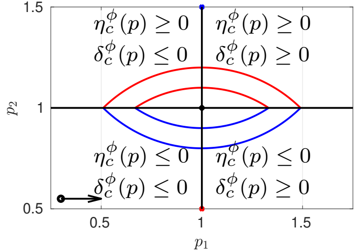
Definitions and assumptions:

- Center of the obstacle $c \in \mathbb{R}^2$
- Shifted center $c_q = c_q(\phi, \ell)$:

$$c_q := c - q\ell(v_{\text{ts}})R(\phi)\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad q \in \{\pm 1\}, \qquad R(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

  $\ell = \ell(v_{\text{ts}}) > 0$ depends on the magnitude of the reference velocity v_{ts}
- By design $|c - c_q| = \ell(v_{\text{ts}})$
- Penalize distance of the robot (p) to c_q through barrier function
  $B_q(x) := \frac{1}{2}|p - c_q|^2, \qquad q \in \{\pm 1\}$





$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$

Constructive controller design: The avoidance neighborhood

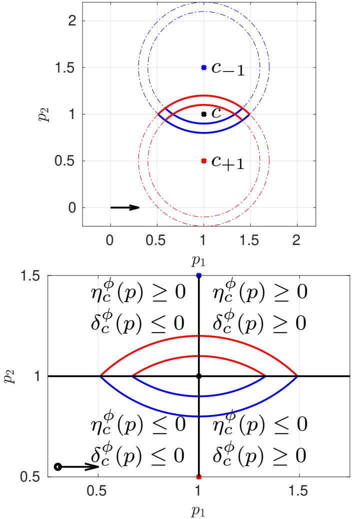
Definitions and assumptions:

- Center of the obstacle $c \in \mathbb{R}^2$
- Shifted center $c_q = c_q(\phi, \ell)$:

$$c_q := c - q\ell(v_{\text{ts}})R(\phi)\begin{bmatrix}0\\1\end{bmatrix}, \qquad q \in \{\pm 1\}, \qquad R(\phi) = \begin{bmatrix}\cos(\phi) & -\sin(\phi)\\ \sin(\phi) & \cos(\phi)\end{bmatrix}$$

  $\ell = \ell(v_{\text{ts}}) > 0$ depends on the magnitude of the reference velocity v_{ts}
- By design $|c - c_q| = \ell(v_{\text{ts}})$
- Penalize distance of the robot (p) to c_q through barrier function $B_q(x) := \frac{1}{2}|p - c_q|^2, \qquad q \in \{\pm 1\}$
- Define inner and outer shell $\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$ through $0 < r < s$ and

$$\mathcal{S}_s^\phi(\ell) := \left\{p \in \mathbb{R}^2 \,\middle|\, B_q(x) \le \tfrac{1}{2}(\ell + s)^2 \;\; \forall\, q \in \{\pm 1\}\right\}.$$



$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$

Constructive controller design: The avoidance neighborhood

Definitions and assumptions:

- Center of the obstacle $c \in \mathbb{R}^2$
- Shifted center $c_q = c_q(\phi, \ell)$:

$$c_q := c - q\ell(v_{\text{ts}})R(\phi)\begin{bmatrix}0\\1\end{bmatrix}, \qquad q \in \{\pm 1\}, \qquad R(\phi) = \begin{bmatrix}\cos(\phi) & -\sin(\phi)\\ \sin(\phi) & \cos(\phi)\end{bmatrix}$$

$\ell = \ell(v_{\text{ts}}) > 0$ depends on the magnitude of the reference velocity v_{ts}

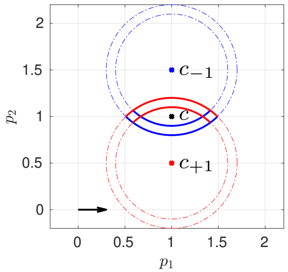
- By design $|c - c_q| = \ell(v_{\text{ts}})$

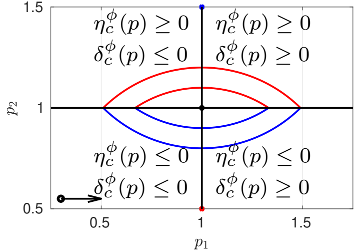
- Penalize distance of the robot (p) to c_q through barrier function
$B_q(x) := \frac{1}{2}|p - c_q|^2, \qquad q \in \{\pm 1\}$
- Define inner and outer shell $\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$ through $0 < r < s$ and

$$\mathcal{S}_s^\phi(\ell) := \left\{p \in \mathbb{R}^2 \,\middle|\, B_q(x) \leq \tfrac{1}{2}(\ell + s)^2 \ \ \forall\, q \in \{\pm 1\}\right\}.$$

- For a fixed $\phi \in \mathbb{R}$, define the functions $\eta_c^\phi, \delta_c^\phi : \mathbb{R}^2 \to \mathbb{R}$,

$$\eta_c^\phi(p) = [\,0\ 1\,]\,R(\phi)^\top(p - c) = -\sin(\phi)(p_1 - c_1) + \cos(\phi)(p_2 - c_2)$$
$$\delta_c^\phi(p) = [\,1\ 0\,]\,R(\phi)^\top(p - c) = \cos(\phi)(p_1 - c_1) + \sin(\phi)(p_2 - c_2)$$





$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$

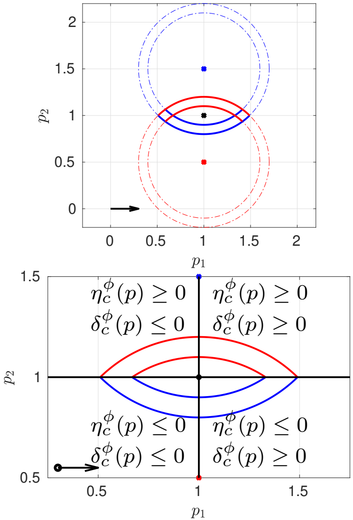
Constructive controller design: The role of the barrier function

Increase condition:

- If $p(0) \notin \mathcal{S}_r^{\phi(0)}(\ell)$, $\phi(0) \in \mathbb{R}$, then

$$\langle \nabla B_q(x(t)), \dot{x}(t) \rangle \geq 0 \qquad \text{for suitable } q \in \{\pm 1\},$$

implies that $\quad p(t) \notin \mathcal{S}_r^{\phi}(\ell) \quad \forall \quad t \in \mathbb{R}_{\geq 0}.$



$\mathcal{S}_r^{\phi}(\ell) \subset \mathcal{S}_s^{\phi}(\ell)$

Constructive controller design: The role of the barrier function

Increase condition:

- If $p(0) \notin \mathcal{S}_r^{\phi(0)}(\ell)$, $\phi(0) \in \mathbb{R}$, then

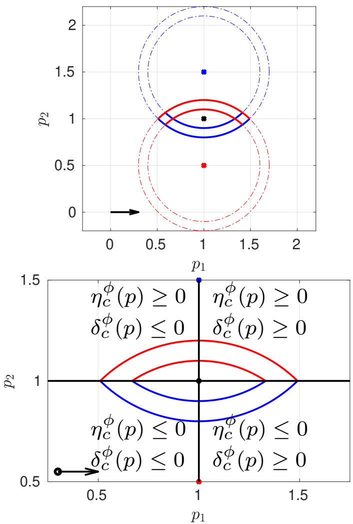
$$\langle \nabla B_q(x(t)), \dot{x}(t) \rangle \geq 0 \qquad \text{for suitable } q \in \{\pm 1\},$$

  implies that $\quad p(t) \notin \mathcal{S}_r^{\phi}(\ell) \quad \forall \quad t \in \mathbb{R}_{\geq 0}.$

- Assume that $\ell = \bar{\ell} \in \mathbb{R}_{>0}$ is constant. Then,

$$\langle \nabla B_q(x), \dot{x} \rangle = (v - q\bar{\ell}w)\delta_c^{\phi}(p) \qquad \text{for } \ q \in \{\pm 1\},$$

  (i.e., if $v(t) - q\bar{\ell}w(t) = 0$ then $|p(t) - c_q|$ is constant).



$\mathcal{S}_r^{\phi}(\ell) \subset \mathcal{S}_s^{\phi}(\ell)$

Constructive controller design: The role of the barrier function

Increase condition:

- If $p(0) \notin \mathcal{S}_r^{\phi(0)}(\ell)$, $\phi(0) \in \mathbb{R}$, then

$$\langle \nabla B_q(x(t)), \dot{x}(t) \rangle \geq 0 \qquad \text{for suitable } q \in \{\pm 1\},$$

  implies that $\quad p(t) \notin \mathcal{S}_r^{\phi}(\ell) \quad \forall \quad t \in \mathbb{R}_{\geq 0}.$

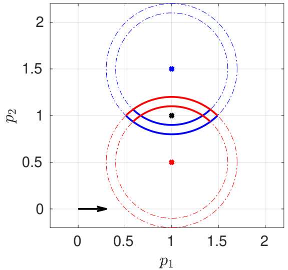
- Assume that $\ell = \bar{\ell} \in \mathbb{R}_{>0}$ is constant. Then,

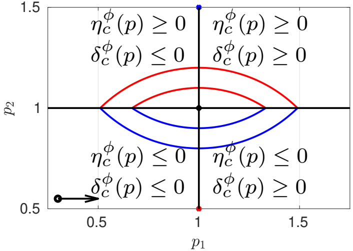
$$\langle \nabla B_q(x), \dot{x} \rangle = (v - q\bar{\ell}w)\delta_c^{\phi}(p) \qquad \text{for} \;\; q \in \{\pm 1\},$$

  (i.e., if $v(t) - q\bar{\ell}w(t) = 0$ then $|p(t) - c_q|$ is constant).

- Depending on the quadrant (i.e., the sign of $\delta_c^{\phi}(p)$) obstacle avoidance depends on the sign of

$$v - q\bar{\ell}w$$



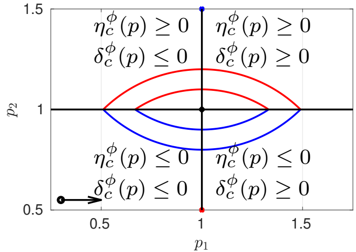


$\mathcal{S}_r^{\phi}(\ell) \subset \mathcal{S}_s^{\phi}(\ell)$

Constructive controller design: Intuitive controller design

- Assumption: $\dot{x} = f(x,u)$, $v(t) \geq 0$ and $\ell = \bar{\ell} \in \mathbb{R}_{>0}$ constant. Then,

$$\langle \nabla B_q(x), f(x,u) \rangle = (v - q\bar{\ell}w)\delta_c^{\phi}(p) \qquad \text{for } q \in \{\pm 1\}$$



$\mathcal{S}_r^{\phi}(\ell) \subset \mathcal{S}_s^{\phi}(\ell)$

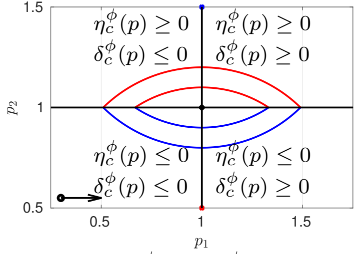
Constructive controller design: Intuitive controller design

- Assumption: $\dot{x} = f(x,u), \quad v(t) \geq 0$ and $\ell = \bar{\ell} \in \mathbb{R}_{>0}$ constant.
  Then,

$$\langle \nabla B_q(x), f(x,u) \rangle = (v - q\bar{\ell}w)\delta_c^{\phi}(p) \qquad \text{for } q \in \{\pm 1\}$$

- Tracking mode: If $p \in \overline{\mathbb{R}^2 \backslash \mathcal{S}_s^{\phi}(\ell(v_{\text{ts}}))}$ then, use tracking controller

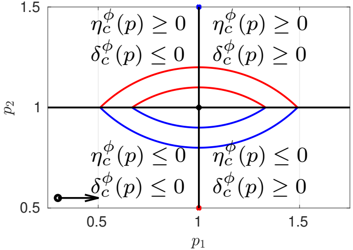
$$u_{\text{ts}}(x) = [v_{\text{ts}}(x), w_{\text{ts}}(x)]^{\top}, \qquad v_{\text{ts}}(x) \in [-\bar{v}, \bar{v}], \quad w_{\text{ts}}(x) \in [-\bar{w}, \bar{w}],$$



$\mathcal{S}_r^{\phi}(\ell) \subset \mathcal{S}_s^{\phi}(\ell)$

Constructive controller design: Intuitive controller design

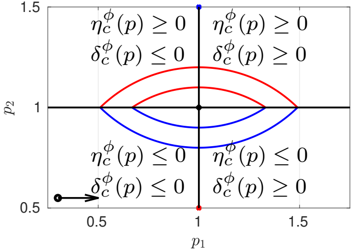
- Assumption: $\dot{x} = f(x,u), \quad v(t) \geq 0 \quad$ and $\quad \ell = \bar{\ell} \in \mathbb{R}_{>0}$ constant.
  Then,
  $$\langle \nabla B_q(x), f(x,u) \rangle = (v - q\bar{\ell}w)\delta_c^{\phi}(p) \qquad \text{for } q \in \{\pm 1\}$$
- Tracking mode: If $p \in \overline{\mathbb{R}^2 \backslash \mathcal{S}_s^{\phi}(\ell(v_{\rm ts}))}$ then, use tracking controller
  $$u_{\rm ts}(x) = [v_{\rm ts}(x), w_{\rm ts}(x)]^\top, \qquad v_{\rm ts}(x) \in [-\bar{v}, \bar{v}], \quad w_{\rm ts}(x) \in [-\bar{w}, \bar{w}],$$
- Emergency mode: If $p \in \mathcal{S}_r^{\phi}(\ell(v_{\rm ts}(\tilde{t})))$ keep $\bar{\ell} = \ell(v_{\rm ts}(\tilde{t}))$ constant and define
  $$u_{\rm em} = \begin{bmatrix} v_{\rm av} \\ q v_{\rm av}/\bar{\ell} \end{bmatrix} \quad \text{where} \quad v_{\rm av} := \mathrm{sat}^{\bar{v}_w}(v_{\rm ts}), \qquad \bar{v}_w := \min\{\bar{v}, \bar{\ell}\bar{w}\}$$
  until $p \notin \{p \in \mathcal{S}_s^{\phi}(\bar{\ell}) |\ \delta_c^{\phi}(p) \leq 0\}$ is satisfied.
  (Note that $\langle \nabla B_q(x), f(x, u_{\rm em}) \rangle = 0$ by construction.)



$\mathcal{S}_r^{\phi}(\ell) \subset \mathcal{S}_s^{\phi}(\ell)$

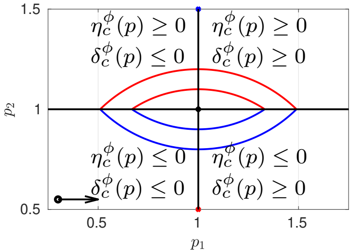
Constructive controller design: Intuitive controller design

- Assumption: $\dot{x} = f(x,u)$, $\quad v(t) \geq 0$ and $\ell = \bar{\ell} \in \mathbb{R}_{>0}$ constant.
  Then,
  $$\langle \nabla B_q(x), f(x,u) \rangle = (v - q\bar{\ell}w)\delta_c^\phi(p) \qquad \text{for } q \in \{\pm 1\}$$
- Tracking mode: If $p \in \overline{\mathbb{R}^2 \backslash \mathcal{S}_s^\phi(\ell(v_{\text{ts}}))}$ then, use tracking controller
  $$u_{\text{ts}}(x) = [v_{\text{ts}}(x), w_{\text{ts}}(x)]^\top, \qquad v_{\text{ts}}(x) \in [-\bar{v}, \bar{v}], \quad w_{\text{ts}}(x) \in [-\bar{w}, \bar{w}],$$
- Emergency mode: If $p \in \mathcal{S}_r^\phi(\ell(v_{\text{ts}}(\tilde{t})))$ keep $\bar{\ell} = \ell(v_{\text{ts}}(\tilde{t}))$ constant and define
  $$u_{\text{em}} = \begin{bmatrix} v_{\text{av}} \\ q v_{\text{av}}/\bar{\ell} \end{bmatrix} \quad \text{where} \quad v_{\text{av}} := \text{sat}^{\bar{v}_w}(v_{\text{ts}}), \qquad \bar{v}_w := \min\{\bar{v}, \bar{\ell}\bar{w}\}$$
  until $p \notin \{p \in \mathcal{S}_s^\phi(\bar{\ell}) |\ \delta_c^\phi(p) \leq 0\}$ is satisfied.
  (Note that $\langle \nabla B_q(x), f(x, u_{\text{em}}) \rangle = 0$ by construction.)
- Recovery mode: If $p \in \{p \in \mathcal{S}_r^\phi(\bar{\ell}) |\ \delta_c^\phi(p) \geq 0\}$, then
  $$u_{\text{re}} = \begin{bmatrix} v_{\text{av}} \\ \text{sat}^{v_{\text{av}}/\bar{\ell}}(w_{\text{ts}}) \end{bmatrix} \qquad \text{until} \quad p \notin \mathcal{S}_s^\phi(\bar{\ell})$$
  (Note that $\langle \nabla B_q(x), f(x, u_{\text{re}}) \rangle \geq 0$ by construction.)



$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$

Constructive controller design: A switched/hybrid control law

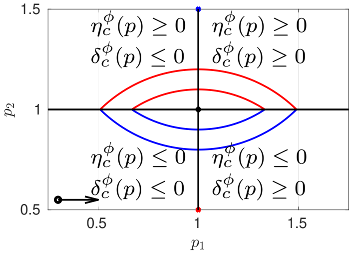


$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$

Constructive controller design: A switched/hybrid control law

Introduce discrete state variables $\kappa = [q, \alpha, \beta]^\top \in \{\pm 1\}^3$

$$\kappa^+ = \begin{bmatrix} q^+ \\ \alpha^+ \\ \beta^+ \end{bmatrix} \in \begin{bmatrix} s(\eta_c^\phi(p)) \\ s(\delta_c^\phi(p)) \\ s(v_{\mathrm{ts}}) \end{bmatrix} \begin{array}{l} \leftarrow \text{ obstacle left/right} \\ \leftarrow \text{ obstacle ahead/behind} \\ \leftarrow \text{ forward/backward} \end{array}$$



Set valued sign function

$$s(r) = \begin{cases} \mathrm{sign}(r), & r \neq 0 \\ \{-1, 1\}, & r = 0 \end{cases}$$

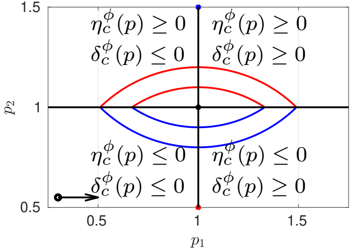
$$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$$

Constructive controller design: A switched/hybrid control law

Introduce discrete state variables $\kappa = [q, \alpha, \beta]^\top \in \{\pm 1\}^3$

$$\kappa^+ = \begin{bmatrix} q^+ \\ \alpha^+ \\ \beta^+ \end{bmatrix} \in \begin{bmatrix} s(\eta_c^\phi(p)) \\ s(\delta_c^\phi(p)) \\ s(v_{\text{ts}}) \end{bmatrix} \begin{matrix} \leftarrow \text{ obstacle left/right} \\ \leftarrow \text{ obstacle ahead/behind} \\ \leftarrow \text{ forward/backward} \end{matrix}$$

Additionally enforce $\beta \frac{d}{dt} \delta_c^\phi(p) = \eta_c^\phi(p) w + v \geq 0$



Set valued sign function

$$s(r) = \begin{cases} \text{sign}(r), & r \neq 0 \\ \{-1, 1\}, & r = 0 \end{cases}$$

$$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$$

Constructive controller design: A switched/hybrid control law

Introduce discrete state variables $\kappa = [q, \alpha, \beta]^\top \in \{\pm 1\}^3$

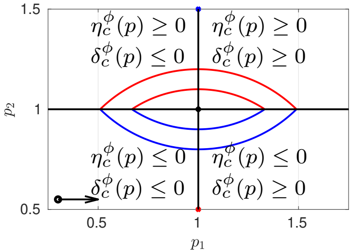
$$\kappa^+ = \begin{bmatrix} q^+ \\ \alpha^+ \\ \beta^+ \end{bmatrix} \in \begin{bmatrix} s(\eta_c^\phi(p)) \\ s(\delta_c^\phi(p)) \\ s(v_{\text{ts}}) \end{bmatrix} \begin{matrix} \leftarrow \text{ obstacle left/right} \\ \leftarrow \text{ obstacle ahead/behind} \\ \leftarrow \text{ forward/backward} \end{matrix}$$

Additionally enforce $\beta \frac{d}{dt}\delta_c^\phi(p) = \eta_c^\phi(p) w + v \geq 0$

Controller design based on: ($q, \alpha, \beta \in \{\pm 1\}$ fixed)

$$u^\star = \underset{u \in [-\bar{v}, \bar{v}] \times [-\bar{w}, \bar{w}]}{\operatorname{argmin}} \ \tfrac{1}{2}(v - \beta |v_{\text{ts}}|)^2 + \tfrac{k}{2}(w - w_{\text{ts}})^2$$

$$\text{s.t. } (v - q\bar{\ell} w)\alpha \geq 0, \quad (\eta_c^\phi(p) w + v)\beta \geq 0$$



Set valued sign function

$$s(r) = \begin{cases} \operatorname{sign}(r), & r \neq 0 \\ \{-1, 1\}, & r = 0 \end{cases}$$

$$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$$

Constructive controller design: A switched/hybrid control law

Introduce discrete state variables $\kappa = [q, \alpha, \beta]^\top \in \{\pm 1\}^3$

$$\kappa^+ = \begin{bmatrix} q^+ \\ \alpha^+ \\ \beta^+ \end{bmatrix} \in \begin{bmatrix} s(\eta_c^\phi(p)) \\ s(\delta_c^\phi(p)) \\ s(v_{\text{ts}}) \end{bmatrix} \begin{matrix} \leftarrow \text{ obstacle left/right} \\ \leftarrow \text{ obstacle ahead/behind} \\ \leftarrow \text{ forward/backward} \end{matrix}$$

Additionally enforce $\quad \beta \frac{d}{dt} \delta_c^\phi(p) = \eta_c^\phi(p) w + v \geq 0$

Controller design based on: ($q, \alpha, \beta \in \{\pm 1\}$ fixed)

$$u^\star = \operatorname*{argmin}_{u \in [-\bar{v}, \bar{v}] \times [-\bar{w}, \bar{w}]} \tfrac{1}{2}(v - \beta |v_{\text{ts}}|)^2 + \tfrac{k}{2}(w - w_{\text{ts}})^2$$

$$\text{s.t. } (v - q\bar{\ell} w)\alpha \geq 0, \quad (\eta_c^\phi(p) w + v)\beta \geq 0$$

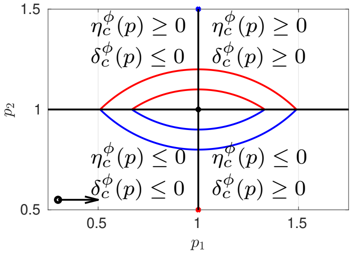
Suboptimal solution in 2 steps:
For v consider

$$v_{\text{av}} = \operatorname*{argmin}_{v \in [-\bar{v}, \bar{v}]} \tfrac{1}{2}(v - \beta |v_{\text{ts}}|)^2$$

$$\text{s.t. } (v + \bar{w}\bar{\ell}) \geq 0, \quad (v - \bar{w}\bar{\ell}) \leq 0 \quad \beta v \geq 0,$$

Optimal solution:

$$v_{\text{av}} = \text{sat}^{\bar{v}_w}(\beta |v_{\text{ts}}|), \quad \bar{v}_w := \min\{\bar{v}, \bar{w}\ell\},$$



Set valued sign function

$$s(r) = \begin{cases} \text{sign}(r), & r \neq 0 \\ \{-1, 1\}, & r = 0 \end{cases}$$

$$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$$

Constructive controller design: A switched/hybrid control law

Introduce discrete state variables $\kappa = [q, \alpha, \beta]^\top \in \{\pm 1\}^3$

$$\kappa^+ = \begin{bmatrix} q^+ \\ \alpha^+ \\ \beta^+ \end{bmatrix} \in \begin{bmatrix} s(\eta_c^\phi(p)) \\ s(\delta_c^\phi(p)) \\ s(v_{\text{ts}}) \end{bmatrix} \begin{matrix} \leftarrow \text{ obstacle left/right} \\ \leftarrow \text{ obstacle ahead/behind} \\ \leftarrow \text{ forward/backward} \end{matrix}$$

Additionally enforce $\quad \beta \frac{d}{dt}\delta_c^\phi(p) = \eta_c^\phi(p) w + v \geq 0$

Controller design based on: ($q, \alpha, \beta \in \{\pm 1\}$ fixed)

$$u^\star = \operatorname*{argmin}_{u \in [-\bar{v}, \bar{v}] \times [-\bar{w}, \bar{w}]} \tfrac{1}{2}(v - \beta |v_{\text{ts}}|)^2 + \tfrac{k}{2}(w - w_{\text{ts}})^2$$

$$\text{s.t. } (v - q\bar{\ell} w)\alpha \geq 0, \quad (\eta_c^\phi(p) w + v)\beta \geq 0$$

Suboptimal solution in 2 steps:
For v consider

$$v_{\text{av}} = \operatorname*{argmin}_{v \in [-\bar{v}, \bar{v}]} \tfrac{1}{2}(v - \beta |v_{\text{ts}}|)^2$$

$$\text{s.t. } (v + \bar{w}\bar{\ell}) \geq 0, \quad (v - \bar{w}\bar{\ell}) \leq 0 \quad \beta v \geq 0,$$

Optimal solution:

$$v_{\text{av}} = \text{sat}^{\bar{v}_w}(\beta |v_{\text{ts}}|), \quad \bar{v}_w := \min\{\bar{v}, \bar{w}\ell\},$$

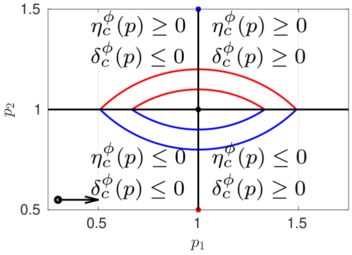
For w consider

$$w_{\text{av}} = \operatorname*{argmin}_{w \in [-\bar{w}, \bar{w}]} \tfrac{1}{2}(w - w_{\text{ts}})^2$$

$$\text{s.t. } (v_{\text{av}} - q w \bar{\ell})\alpha \geq 0, \quad \beta \eta_c^\phi(p) w \geq -|v_{\text{av}}|,$$

Suboptimal solution:

$$w_{\text{av}} = \begin{cases} \beta q |v_{\text{av}}|/\bar{\ell}, & \text{if } q \in \{\pm 1\}, \quad \alpha = -\beta, \\ \text{sat}^{|v_{\text{av}}|/\bar{\ell}}(w_{\text{ts}}), & \text{if } q \in \{\pm 1\}, \quad \alpha = \beta \end{cases}$$



Set valued sign function

$$s(r) = \begin{cases} \text{sign}(r), & r \neq 0 \\ \{-1, 1\}, & r = 0 \end{cases}$$

$$\mathcal{S}_r^\phi(\ell) \subset \mathcal{S}_s^\phi(\ell)$$

A hybrid system formulation

Augmented state: $x = [p_1, p_2, \phi]^\top, \quad \kappa = [q, \alpha, \beta]^\top$

$$\xi := \begin{bmatrix} x \\ \bar{\ell} \\ \kappa \end{bmatrix} \in \Xi := \mathbb{R}^3 \times [\ell_{\min}, \ell_{\max}] \times \{0, \pm 1\} \times \{\pm 1\}^2$$

Hybrid dynamics:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{\bar{\ell}} \\ \dot{\kappa} \end{bmatrix} = \begin{bmatrix} f(x, \gamma(\xi, u_{\mathrm{ts}})) \\ 0 \\ 0 \end{bmatrix}, \quad \xi \in \mathcal{C} := \overline{\Xi \setminus \mathcal{D}},$$

$$\xi^+ = \begin{bmatrix} x^+ \\ \bar{\ell}^+ \\ \kappa^+ \end{bmatrix} \in \begin{bmatrix} x \\ G(\xi, u_{\mathrm{ts}})) \end{bmatrix}, \qquad \xi \in \mathcal{D}.$$

Flow sets and jump sets:

$$\mathcal{C} := \overline{\Xi \setminus \mathcal{D}}, \qquad \mathcal{D} := \mathcal{D}_{\mathrm{tr}} \cup \mathcal{D}_{\mathrm{em}} \cup \mathcal{D}_{\mathrm{re}}$$

$$\mathcal{D}_{\mathrm{tr}} := \{\xi \in \Xi : |q| = 1, p \in \overline{\mathbb{R}^2 \setminus \mathcal{S}_s^{\phi}(\bar{\ell})}\},$$

$$\mathcal{D}_{\mathrm{em}} := \{\xi \in \Xi : q = 0, p \in \mathcal{S}_r^{\phi}(\ell(v_{\mathrm{ts}}))\}$$

$$\mathcal{D}_{\mathrm{re}} := \{\xi \in \Xi : |q| = 1, \alpha \delta_c^{\phi}(p) \leq 0, \alpha\beta = -1\}$$

Definition of the feedback law: $u = \gamma(\xi, u_{\mathrm{ts}})$

$$\gamma(\xi, u_{\mathrm{ts}}) := \begin{cases} u_{\mathrm{ts}}, & \text{if } q = 0, \\ u_{\mathrm{em}} = \begin{bmatrix} v_{\mathrm{av}} \\ \beta q |v_{\mathrm{av}}| / \bar{\ell} \end{bmatrix}, & \text{if } \begin{array}{l} |q| = 1, \\ \alpha\beta = -1 \end{array} \\ u_{\mathrm{re}} = \begin{bmatrix} v_{\mathrm{av}} \\ \mathrm{sat}^{|v_{\mathrm{av}}|/\bar{\ell}}(w_{\mathrm{ts}}) \end{bmatrix}, & \text{if } \begin{array}{l} |q| = 1, \\ \alpha\beta = 1 \end{array} \end{cases}$$

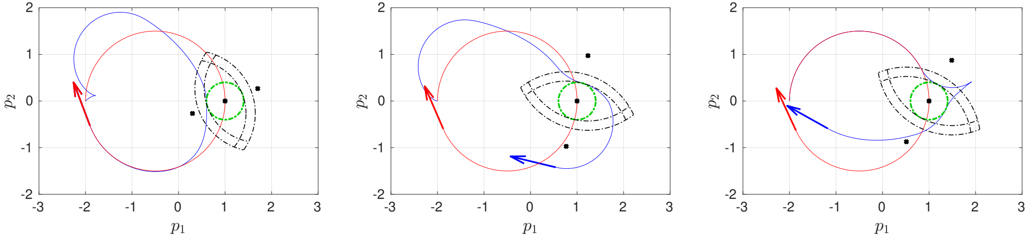
Discrete time controller selection:

$$G(\xi, u_{\mathrm{ts}}) = \bigcup_{i \in \{\mathrm{tr}, \mathrm{em}, \mathrm{re}\} : \xi \in \mathcal{D}_i} g_i(\xi, u_{\mathrm{ts}})$$

$$g_{\mathrm{tr}} := \begin{bmatrix} \bar{\ell} \\ 0 \\ \alpha \\ \beta \end{bmatrix}, \; g_{\mathrm{em}} := \begin{bmatrix} g_\ell(x, v_{\mathrm{ts}}) \\ s(\eta_c^{\phi}(p)) \\ -s(v_{\mathrm{ts}}) \\ s(v_{\mathrm{ts}}) \end{bmatrix}, \; g_{\mathrm{re}} := \begin{bmatrix} \bar{\ell} \\ q \\ \beta \\ \beta \end{bmatrix},$$

$$g_\ell(x, v_{\mathrm{ts}}) := \min\{|p - c_{s(\eta_c^{\phi}(p))}| - r, \ell(v_{\mathrm{ts}})\},$$

Numerical simulations: Examples with varying initial condition



Next steps

Under which conditions can avoidance AND tracking be guaranteed

$\rightsquigarrow$ Focus on the tracking controller

Moving obstacles

Next steps

Under which conditions can avoidance AND tracking be guaranteed

⇝ Focus on the tracking controller

Moving obstacles

Extended unicycle model:

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\phi} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} v\cos(\phi) \\ v\sin(\phi) \\ w \\ a_v \\ a_w \end{bmatrix}, \qquad x = \begin{bmatrix} p_1 \\ p_2 \\ \phi \\ v \\ w \end{bmatrix}, \quad u = \begin{bmatrix} a_v \\ a_w \end{bmatrix}$$

- How to define an appropriate barrier function?

⇝ Continuous velocity v and angular velocity w

Augmented controller design for reference tracking and obstacle avoidance

Philipp Braun

School of Engineering,

Australian National University, Canberra, Australia

In Collaboration with:

L. Grüne: University of Bayreuth, Bayreuth, Germany

C. M. Kellett: School of Engineering, Australian National University, Canberra, Australia

L. Zaccarian: Dipartimento di Ingegneria Industriale, University of Trento, Italy, and
LAAS-CNRS, Université de Toulouse, France

The tracking controller

Reference signal u_{ref} and reference system:

$$\dot{x}_{\text{ref}} = f(x_{\text{ref}}, u_{\text{ref}})$$

Error variables:

$$\tilde{x} = x - x_{\text{ref}}$$

Rotated error variables:

$$\tilde{p}_{\text{rot}} = R(\phi)^\top \tilde{p}, \qquad \tilde{\phi}_{\text{rot}} = \tilde{\phi}$$

The tracking controller:

$$u_{\text{tr}} = \begin{bmatrix} v_{\text{tr}} \\ w_{\text{tr}} \end{bmatrix} = \begin{bmatrix} -k_1 \tilde{p}_{1,\text{ref}} + v_{\text{ref}} \cos(\tilde{\phi}_{\text{rot}}) \\ -k_\phi \sin(\tilde{\phi}_{\text{rot}}) - k_2 v_{\text{ref}} \tilde{p}_{2,\text{rot}} + w_{\text{ref}} \end{bmatrix}, \qquad k_1, k_2, k_\phi \in \mathbb{R}_{>0}$$

Saturated tracking controller:

$$u_{\text{ts}} = \begin{bmatrix} v_{\text{ts}} \\ w_{\text{ts}} \end{bmatrix} = \begin{bmatrix} \text{sat}^{\bar{v}}(v_{\text{tr}}) \\ \text{sat}^{\bar{w}}(w_{\text{tr}}) \end{bmatrix}$$

Constructive controller design: Multiparametric programming

A minimally invasive avoidance control law:

Consider

$$u^\star = \underset{u\in[-\bar{v},\bar{v}]\times[-\bar{w},\bar{w}]}{\operatorname{argmin}} \ \tfrac{1}{2}(v - v_{\text{ts}})^2 + \tfrac{k}{2}(w - w_{\text{ts}})^2$$

$$\text{s. t. } (v - q\bar{\ell}w)\delta_c^\phi(p) \geq 0,$$

where

- $u^\star = [v^\star, w^\star]^\top$ depends on $(x, q, u_{\text{ts}}) \in \mathbb{R}^3 \times \{\pm 1\} \times \mathbb{R}^2$.
- Tracking controller: $u_{\text{ts}} = [v_{\text{ts}}, w_{\text{ts}}]^\top$
- Weighting factor $k \in \mathbb{R}_{>0}$

Constructive controller design: Multiparametric programming

A minimally invasive avoidance control law:

Consider

$$u^\star = \underset{u\in[-\bar{v},\bar{v}]\times[-\bar{w},\bar{w}]}{\operatorname{argmin}} \; \tfrac{1}{2}(v - v_{\text{ts}})^2 + \tfrac{k}{2}(w - w_{\text{ts}})^2$$

$$\text{s.t. } (v - q\bar{\ell}w)\delta_c^\phi(p) \geq 0,$$

where

- $u^\star = [v^\star, w^\star]^\top$ depends on $(x, q, u_{\text{ts}}) \in \mathbb{R}^3 \times \{\pm 1\} \times \mathbb{R}^2$.
- Tracking controller: $u_{\text{ts}} = [v_{\text{ts}}, w_{\text{ts}}]^\top$
- Weighting factor $k \in \mathbb{R}_{>0}$

Note that:

- Multiparamtric optimization problem depending on (x, q, u_{ts})
- Feasibility: $\quad v = w = 0 \quad \checkmark$
- Objective function: strongly convex; Feasible domain: convex and compact $\rightsquigarrow u^\star$ is unique for all (x, q, u_{ts}).

Constructive controller design: Multiparametric programming

A minimally invasive avoidance control law:

Consider

$$u^\star = \operatorname*{argmin}_{u\in[-\bar{v},\bar{v}]\times[-\bar{w},\bar{w}]} \tfrac{1}{2}(v-v_{\text{ts}})^2 + \tfrac{k}{2}(w-w_{\text{ts}})^2$$
$$\text{s.t. } (v - q\bar{\ell}w)\delta_c^\phi(p) \geq 0,$$

where

- $u^\star = [v^\star, w^\star]^\top$ depends on $(x, q, u_{\text{ts}}) \in \mathbb{R}^3 \times \{\pm 1\} \times \mathbb{R}^2$.
- Tracking controller: $u_{\text{ts}} = [v_{\text{ts}}, w_{\text{ts}}]^\top$
- Weighting factor $k \in \mathbb{R}_{>0}$

Note that:

- Multiparamtric optimization problem depending on (x, q, u_{ts})
- Feasibility: $v = w = 0 \quad \checkmark$
- Objective function: strongly convex; Feasible domain: convex and compact $\rightsquigarrow u^\star$ is unique for all (x, q, u_{ts}).

For $\alpha = \operatorname{sign}(\delta_c^\phi(p)) \in \{0, \pm 1\}$, consider

$$u^\star_{q,\alpha} = \operatorname*{argmin}_{u\in[-\bar{v},\bar{v}]\times[-\bar{w},\bar{w}]} \tfrac{1}{2}(v-v_{\text{ts}})^2 + \tfrac{k}{2}(w-w_{\text{ts}})^2$$
$$\text{s.t. } \alpha v - (\alpha q\bar{\ell})w \geq 0$$

- Quadratic multiparametric program for α, q fixed, $(v_{\text{ts}}, w_{\text{ts}}) \in [-\bar{v}, \bar{v}] \times [-\bar{w}, \bar{w}]$, (can be solved offline)
- (To obtain an explicit solution the optimization problem needs to be solved six times: $\alpha \in \{0, \pm 1\}$, $q \in \{\pm 1\}$)

Constructive controller design: Multiparametric programming

A minimally invasive avoidance control law:

Consider

$$u^\star = \operatorname*{argmin}_{u\in[-\bar{v},\bar{v}]\times[-\bar{w},\bar{w}]} \tfrac{1}{2}(v-v_{\mathrm{ts}})^2 + \tfrac{k}{2}(w-w_{\mathrm{ts}})^2$$
$$\text{s. t. } (v - q\bar{\ell}w)\delta_c^\phi(p) \geq 0,$$

where

- $u^\star = [v^\star, w^\star]^\top$ depends on $(x, q, u_{\mathrm{ts}}) \in \mathbb{R}^3 \times \{\pm 1\} \times \mathbb{R}^2$.
- Tracking controller: $u_{\mathrm{ts}} = [v_{\mathrm{ts}}, w_{\mathrm{ts}}]^\top$
- Weighting factor $k \in \mathbb{R}_{>0}$

Note that:

- Multiparamtric optimization problem depending on (x, q, u_{ts})
- Feasibility: $v = w = 0$ ✓
- Objective function: strongly convex; Feasible domain: convex and compact $\rightsquigarrow u^\star$ is unique for all (x, q, u_{ts}).

For $\alpha = \mathrm{sign}(\delta_c^\phi(p)) \in \{0, \pm 1\}$, consider

$$u^\star_{q,\alpha} = \operatorname*{argmin}_{u\in[-\bar{v},\bar{v}]\times[-\bar{w},\bar{w}]} \tfrac{1}{2}(v-v_{\mathrm{ts}})^2 + \tfrac{k}{2}(w-w_{\mathrm{ts}})^2$$
$$\text{s. t. } \alpha v - (\alpha q\bar{\ell})w \geq 0$$

- Quadratic multiparametric program for α, q fixed, $(v_{\mathrm{ts}}, w_{\mathrm{ts}}) \in [-\bar{v}, \bar{v}] \times [-\bar{w}, \bar{w}]$, (can be solved offline)
- (To obtain an explicit solution the optimization problem needs to be solved six times: $\alpha \in \{0, \pm 1\}$, $q \in \{\pm 1\}$)

Then

- For fixed $q \in \{\pm 1\}$, $\alpha \in \{0, \pm 1\}$, $\bar{\ell} > 0$, the feedback law $u^\star_{q,\alpha} = u^\star_{q,\alpha}(x, u_{\mathrm{ts}})$ is a piecewise linear.
- Moreover, $u^\star_{q,\alpha}(x, u_{\mathrm{ts}})$ is continuous at any point (x, u_{ts}) satisfying $\delta_c^\phi(p) \neq 0$.

However

- Even though $\delta_c^\phi(p) \neq 0$ defines a set of measure zero, this set may be critical in the controller design