Safety of control systems under uncertainty and time delays

Part 1

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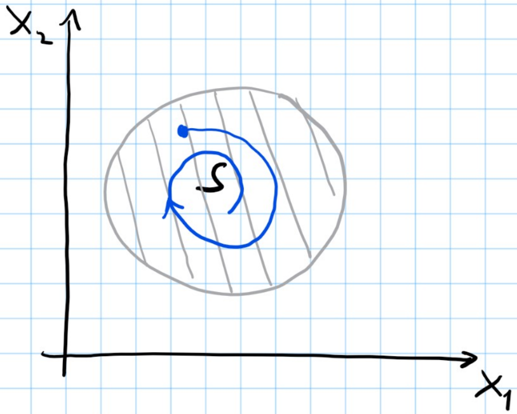
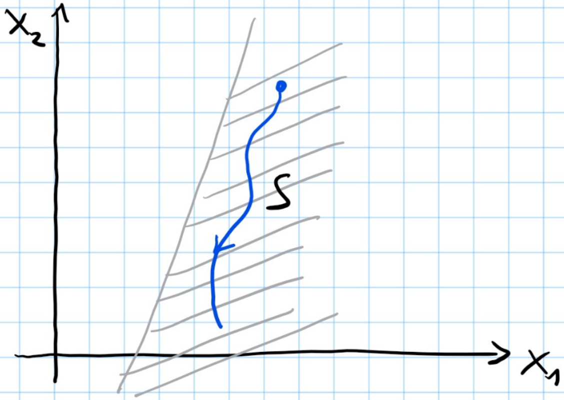
Barrier functions

dynamical system $\boxed{\dot{\underline{x}} = f(\underline{x})}$ $\quad \underline{x} \in \mathbb{R}^n$ $\quad$ (later we will deal with control system $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x})\,\underline{u}$)

$f: \mathbb{R}^n \to \mathbb{R}^n$

Def: the set $S \in \mathbb{R}^n$ is forward invariant if $\boxed{\forall\ \underline{x}(0) \in S \implies \underline{x}(t) \in S,\ t > 0}$





then we say that $\boxed{S \text{ is safe}}$

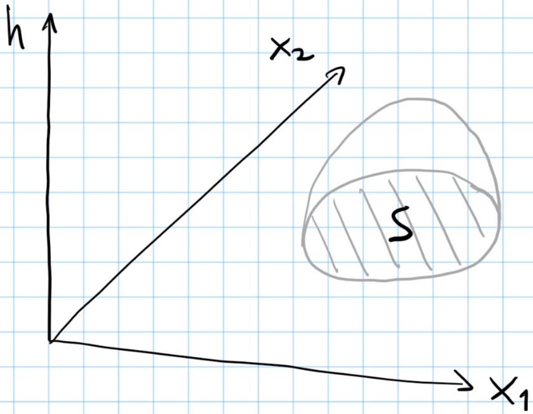
Def: the set S is a 0-superlevel set of $h: \mathbb{R}^n \to \mathbb{R}$ if

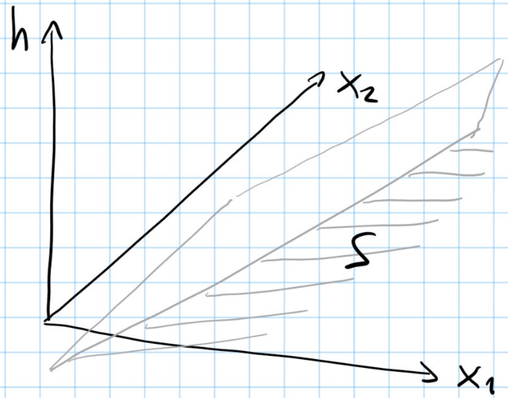
$$S = \{ x \in \mathbb{R}^n \mid h(\underline{x}) \geq 0 \}$$

boundary $\quad \partial S = \{ x \in \mathbb{R}^n \mid h(\underline{x}) = 0 \}$

interior $\quad \text{int}\, S = \{ x \in \mathbb{R}^n \mid h(\underline{x}) > 0 \}$

$\boxed{h \text{ - barrier function}}$



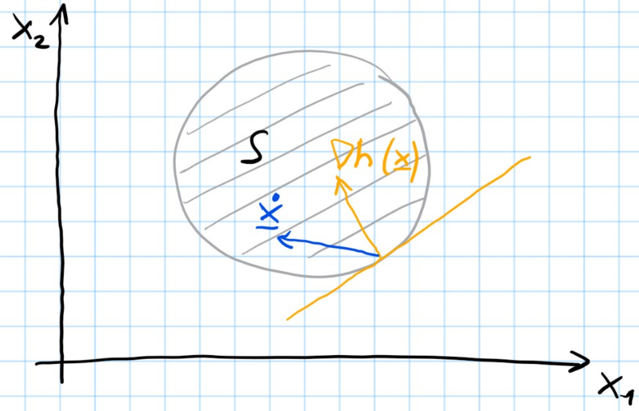


Theorem (Nagumo 1942)

Consider a continuously differentiable function $h: \mathbb{R}^n \to \mathbb{R}$ which satisfies $h(\underline{x}) = 0 \Rightarrow \nabla h(\underline{x}) \neq \underline{0}$

System $\dot{\underline{x}} = f(\underline{x})$ is safe with respect to S if and only if

$$\boxed{h(\underline{x}) = 0 \quad \Rightarrow \quad \dot{h}(\underline{x}) \geqslant 0}$$



where $\dot{h}(\underline{x}) = \nabla h(\underline{x}) \cdot \dot{\underline{x}} = \nabla h(\underline{x}) \cdot f(\underline{x}) \geqslant 0$

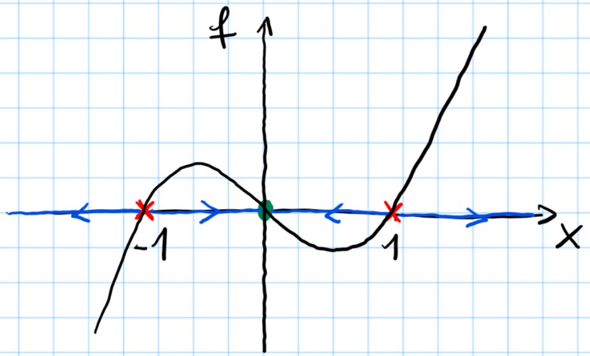
$$\nabla h(\underline{x}) = \frac{\partial h}{\partial \underline{x}}$$

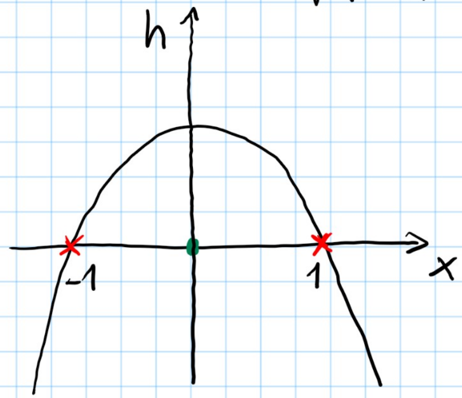
Example (Orm-Ames 2019)

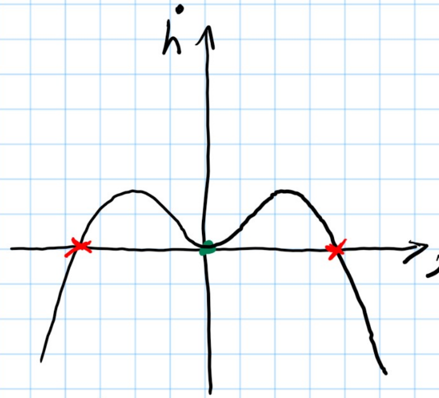
$$\dot{x} = \underbrace{-x + x^3}_{f(x)} \qquad x \in \mathbb{R}$$

Solution

$$x(t) = \pm \frac{1}{\sqrt{\left(\frac{1}{x^2(0)} - 1\right) e^{2t} + 1}}$$



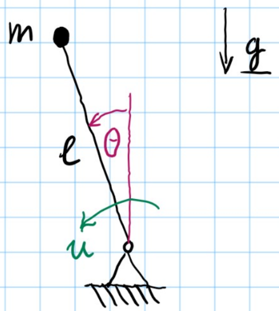




$$h(x) = \frac{1}{2} - \frac{x^2}{2} \qquad h(x) = 0 \Rightarrow x = \pm 1 \Rightarrow \frac{\partial h}{\partial x} = -x \qquad \begin{cases} \frac{\partial h}{\partial x}(1) = -1 \\ \frac{\partial h}{\partial x}(-1) = 1 \end{cases} \checkmark$$

$$\dot{h}(x) = \frac{\partial h}{\partial x} \dot{x} = \frac{\partial h}{\partial x} f(x) = -x(-x + x^3) = x^2 - x^4 \qquad \begin{cases} \dot{h}(1) = 0 \\ \dot{h}(-1) = 0 \end{cases} \checkmark$$

Example inverted pendulum



$$\dot{\underline{x}} = f(\underline{x}) + g(\underline{x})\,u = \begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} u$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

we want $\quad x_2 \leq \omega_{max}$

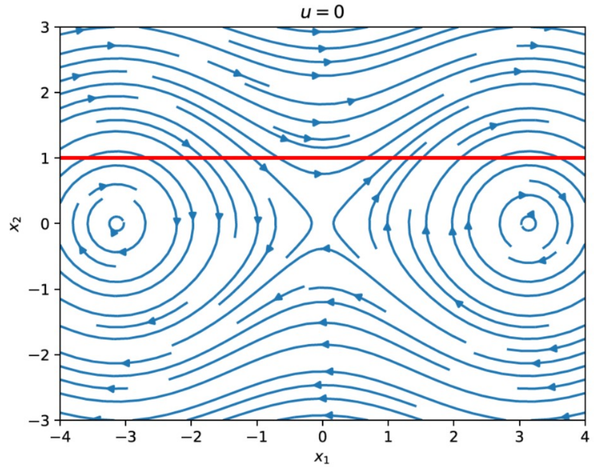
we choose $\quad u = -mg\ell\sin(x_1) + m\ell^2\gamma(\omega_{max} - x_2) \longrightarrow \dot{\underline{x}} = f_{CL}(\underline{x}) = \begin{bmatrix} x_2 \\ \gamma(\omega_{max} - x_2) \end{bmatrix}$

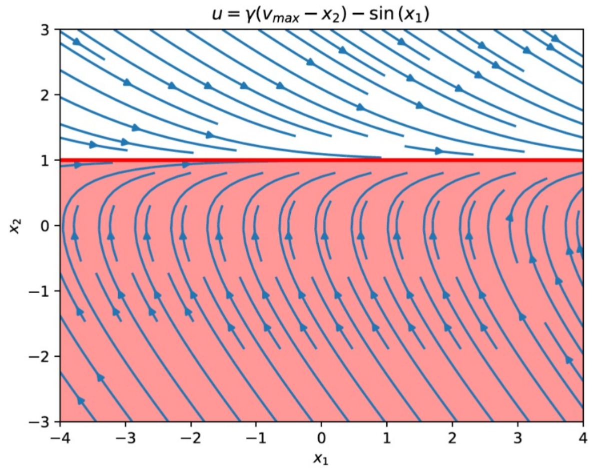
barrier function

$$h(\underline{x}) = \omega_{max} - x_2$$

$$\dot{h}(\underline{x}) = \frac{\partial h}{\partial \underline{x}} \cdot \dot{\underline{x}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ \gamma(\omega_{max} - x_2) \end{bmatrix} = -\gamma(\omega_{max} - x_2)$$

$$\dot{h}(\underline{x})\Big|_{h(\underline{x})=0} = 0 \quad \checkmark$$





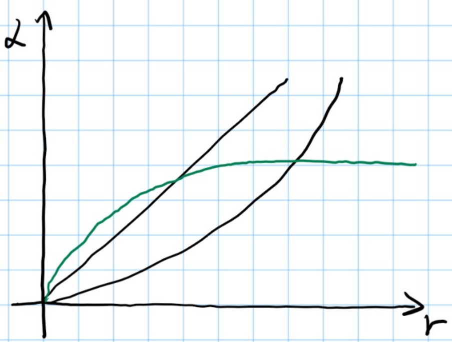
Def: <u>Class $\mathcal{K}$ function</u> $(\alpha \in \mathcal{K})$

$$\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad \alpha(0) = 0, \text{ continuous, strictly monotonically increasing}$$

Def: <u>Class $\mathcal{K}_\infty$ function</u> $(\alpha \in \mathcal{K}_\infty)$

$$\alpha \in \mathcal{K} \quad \text{and} \quad \lim_{r\to\infty} \alpha(r) = \infty \quad \text{(radially unbounded)}$$

examples



$\alpha(r) = k r^c \quad k, c > 0 \quad \in \mathcal{K}_\infty$

$\alpha(r) = 1 - e^{-r} \quad \in \mathcal{K} \quad \notin \mathcal{K}_\infty$

<u>Nice properties:</u> - invertability $\alpha \in \mathcal{K} \Rightarrow \alpha^{-1} \in \mathcal{K}$ $\left(\text{e.g. } \alpha^{-1}(r) = \frac{1}{k^{1/c}} r^{1/c}\right)$

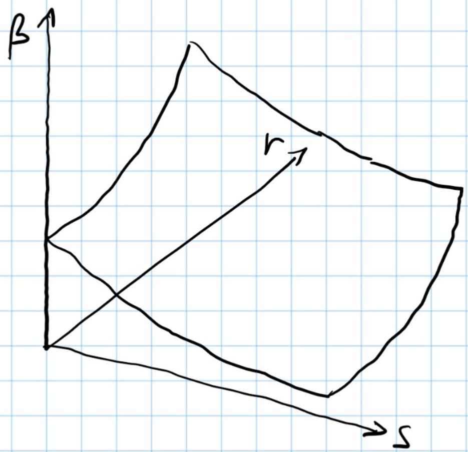
- composability $\alpha_1, \alpha_2 \in \mathcal{K} \Rightarrow \alpha_1 \circ \alpha_2 \in \mathcal{K}$

$\alpha_1(\alpha_2(r))$

Def: <u>class $\mathcal{KL}$ function</u> $(\beta \in \mathcal{KL})$

$$\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

- class $\mathcal{K}$ in first variable: $s \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, s) \in \mathcal{K}$
- $s \in \mathbb{R}_{\geq 0}$ $\lim_{s\to\infty} \beta(r, s) = 0$



Def: <u>Class $\mathcal{KL}_\infty$ function</u> $(\beta \in \mathcal{KL}_\infty)$

$$\beta \in \mathcal{KL} \quad \text{and} \quad \lim_{r\to\infty} \beta(r, s) = \infty$$

Comparison lemma (Sec 3.1, pages 39, 40, 46)

end of proof

Let $\alpha \in K$ which is locally Lipschitz

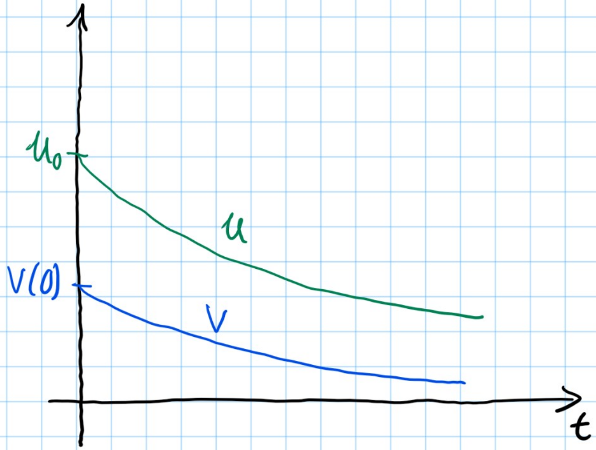
$\dot{u} = -\alpha(u) \quad u(0) = u_0$ has a unique solution for $t \in [0, a]$

(e.g. $\alpha(r) = \gamma r \Rightarrow \dot{u} = \gamma u \Rightarrow u(t) = u_0 e^{-\gamma t}$)

If we have a differentiable function $v : [0, a] \to \mathbb{R}$ such that

$\dot{v}(t) \leq -\alpha(v(t))$ and $v(0) < u_0 \Rightarrow v(t) \leq u(t)$

(e.g. $u(t) = u_0 e^{-\gamma t} \Rightarrow v(t) \leq v_0 e^{-\gamma t}$)



Lemma 2

Consider the initial value problem (IVP)

$$\dot{y} = -\alpha(y) \qquad y(0) = y_0 \qquad y \in \mathbb{R}$$

this has the unique solution

$$y(t) = \beta(y(0), t) \qquad t \geq 0$$

(e.g. $\dot{y} = -\gamma y \quad y(0) = y_0 \quad y(t) = y_0 e^{-\gamma t}$)

Theorem (Ames 2014)

Given $S \subset \mathbb{R}^n$ is a 0-superlevel set of the continuously differentiable function $h: \mathbb{R}^n \to \mathbb{R}$ which satisfies $h(\underline{x}) = 0 \Rightarrow \nabla h(\underline{x}) \neq \underline{0}$ $(*)$

then S is forward invariant (i.e. safe) if there exist $\alpha \in \mathcal{K}$ such that

$$\boxed{\dot{h}(\underline{x}) \geqslant -\alpha(h(\underline{x})) \quad \text{for all } \underline{x} \in S}$$

sufficient condition of safety

Remark 1 Ames' theorem implies Nagumo's theorem since at $h(\underline{x}) = 0$ we have $\alpha(h(\underline{x})) = 0$

Remark 2 Condition $(*)$ may be dropped but in that case we may need to have $\alpha \in \mathcal{K}^e$ – extended class $\mathcal{K}$

Proof:

Consider the IVP $\dot{y} = -\alpha(y)$ $\quad y(0) = h(\underline{x}(0))$

with unique solution $y(t) = \beta(y(0), t) = \beta(h(\underline{x}(0), t))$

Using the comparison lemma

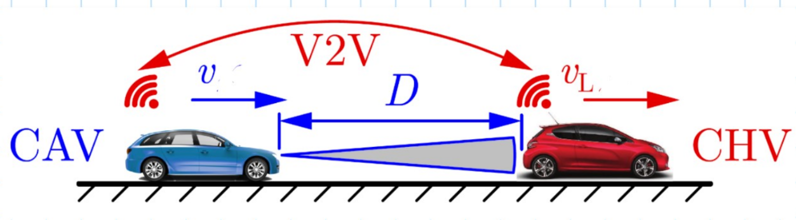
$$h(\underline{x}(t)) \geqslant \beta(h(\underline{x}(0)), t) \qquad t \geqslant 0$$

This implies

$$h(\underline{x}(t)) \geqslant 0 \qquad t \geqslant 0$$

and thus S is forward invariant.

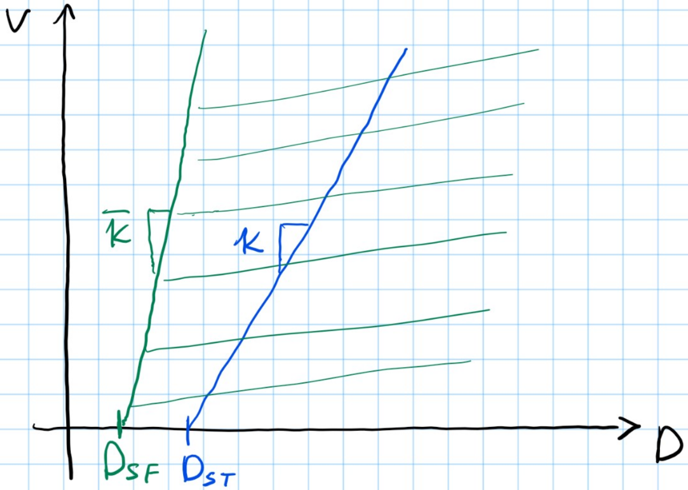
Example: Connected Cruise Control (CCC)



$$\dot{\underline{x}} = f(\underline{x}) \qquad \begin{cases} \dot{D} = V_L - V \\ \dot{V} = A\big(\kappa(D - D_{ST}) - V\big) + B(V_L - V) \end{cases}$$

$$\underline{x} = \begin{bmatrix} D \\ V \end{bmatrix}$$

A, B feedback gains
$\frac{1}{\kappa}$ time headway
D_{ST} stopping distance
} positive parameters



safe set

$$S = \{\underline{x} \in \mathbb{R}^3 \mid \underbrace{\bar{\kappa}(D - D_{SF}) - V}_{h(\underline{x})} \geq 0\}$$

$\frac{1}{\bar{\kappa}}$ minimum time headway
D_{SF} safety distance
} positive parameters

$$\dot{h}(\underline{x}) = \nabla h(\underline{x}) \cdot f(\underline{x}) = \begin{bmatrix} \bar{\kappa} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} V_L - V \\ A\big(\kappa(D - D_{SF}) - V\big) + \beta(V_L - V) \end{bmatrix}$$

Nagumo: $\dot{h}(\underline{x}) \geq 0$ at $\underline{x} \in \partial S$ (where $h(\underline{x}) = 0$)

Ames: $\dot{h}(\underline{x}) \geq -\alpha(h(\underline{x}))$ at $\underline{x} \in S$

Class κ function

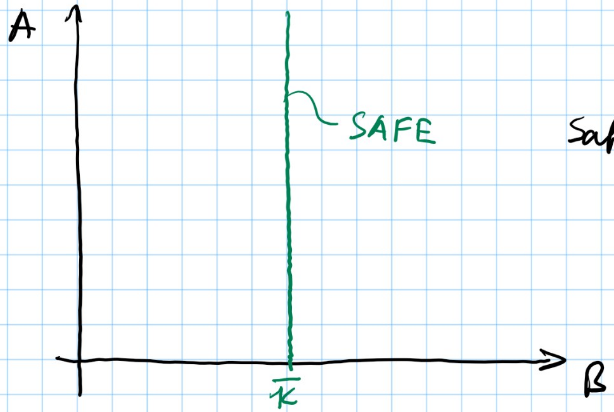
Let us use Nagumo

$$\bar{\kappa}(V_L - V) - A\big(\kappa(D - D_{ST}) - V\big) - \beta(V_L - V) \geq 0 \quad \text{at} \quad \bar{\kappa}(D - D_{SF}) - V = 0$$

add $A\left(\bar{\kappa}(D - D_{SF}) - v\right) = 0$ to both sides

$$\underbrace{A(\bar{\kappa} - \kappa)}_{\geq 0}\underbrace{(D - D_{SF})}_{\geq \text{ within } S} + \underbrace{A\kappa(D_{ST} - D_{SF})}_{\geq 0} + \underbrace{(\bar{\kappa} - B)(v_L - v)}_{= 0} \geq 0$$

$\boxed{\bar{\kappa} \geq \kappa}$ $\boxed{D_{ST} \geq D_{SF}}$ $\boxed{\bar{\kappa} = B}$

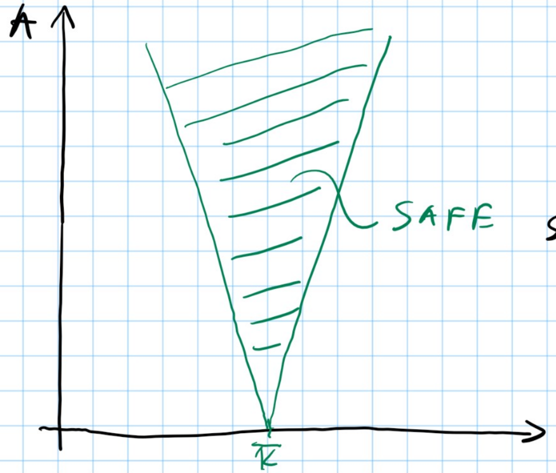


Safety chart

assume $|v_L - v| \leq \Delta v$ then look at the last two terms

$$A\kappa(D_{ST} - D_{SF}) + (\bar{\kappa} - B)(v_L - v) \geq A\kappa(D_{ST} - D_{SF}) - |\bar{\kappa} - B|\Delta v \geq 0$$

$$\Rightarrow \boxed{A \geq \frac{|\bar{\kappa} - B|\Delta v}{\kappa(D_{ST} - D_{SF})}}$$



Safety chart

Summary – Safety of dynamical systems

Consider $\boxed{\dot{\underline{x}} = f(\underline{x})}$

Def: The set $S \in \mathbb{R}^n$ is forward invariant if $\boxed{\forall\, \underline{x}(0) \in S \Rightarrow \underline{x}(t) \in S \quad t > 0}$

then we say that S is safe

Def: The set S is a 0-superlevel set of $h : \mathbb{R}^n \to \mathbb{R}$ if $\boxed{S = \{\underline{x} \in \mathbb{R}^n \mid h(\underline{x}) \geq 0\}}$

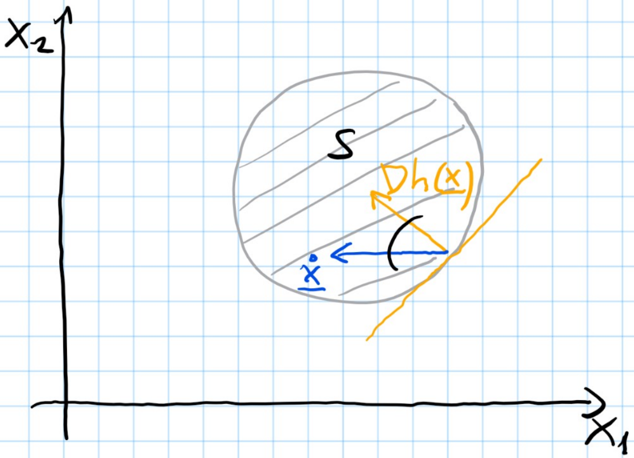
we call h a barrier function

Theorem (Nagumo 1942)

Consider the continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$
which satisfies $h(\underline{x}) = 0 \Rightarrow \nabla h(\underline{x}) \neq \underline{0}$
System $\dot{\underline{x}} = f(\underline{x})$ is safe with respect to S if and only if

$$\boxed{h(\underline{x}) = 0 \quad \Rightarrow \quad \dot{h}(\underline{x}) \geq 0}$$

where $\dot{h}(\underline{x}) = \nabla h(\underline{x}) \cdot \dot{\underline{x}} = \nabla h(\underline{x}) \cdot f(\underline{x})$



Theorem (Ames 2014)

Given $S \in \mathbb{R}^n$ is a 0-superlevel set of the continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$
which satisfies $h(\underline{x}) = 0 \Rightarrow \nabla h(\underline{x}) \neq \underline{0}$
S is forward invariant (i.e. safe) if there exist $\alpha \in \mathcal{K}$ such that

$$\boxed{\dot{h}(\underline{x}) \geq -\alpha(h(\underline{x})) \quad \text{for all} \quad \underline{x} \in S}$$

Safety of control systems

control affine system $\boxed{\dot{\underline{x}} = f(\underline{x}) + g(\underline{x})\, u}$ $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$ (single input for simplicity)

$f, g : \mathbb{R}^n \to \mathbb{R}^n$

safe set $\boxed{S = \{ \underline{x} \in \mathbb{R}^n \mid h(\underline{x}) \geqslant 0 \}}$

using $u = k(\underline{x})$ $\Rightarrow$ $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x})\, u = f_{CL}(\underline{x})$

$h : \mathbb{R}^n \to \mathbb{R}$

so we can verify whether a controller is safe

now we want to synthetize safe controllers

$$\dot{h}(\underline{x}, u) = \nabla h(\underline{x}) \cdot \dot{\underline{x}} = \underbrace{\nabla h(\underline{x}) \cdot f(\underline{x})}_{L_f h(\underline{x})} + \underbrace{\nabla h(\underline{x}) \cdot g(\underline{x})}_{L_g h(\underline{x})} u \geqslant -\alpha(h(\underline{x}))$$

we will choose u such that $\geqslant$ holds

Def: the continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$
which satisfies $h(\underline{x}) = \underline{0} \Rightarrow \nabla h(\underline{x}) \neq \underline{0}$
is a control barrier function (CBF) on S
if there exist $\alpha \in \mathcal{K}$ such that

$$\boxed{\sup_{u \in \mathbb{R}} \dot{h}(\underline{x}, u) > -\alpha(h(\underline{x}))}$$

Note: we have $>$ and not $\geqslant$

Def: set of safe controllers $\boxed{K_{CBF}(\underline{x}) = \{ u \in \mathbb{R} \mid \dot{h}(\underline{x}, u) \geqslant -\alpha(h(\underline{x})) \}}$

Note: we have $\geqslant$ and not $>$

Note: sup gives $u \to \pm\infty$ except when $\nabla h(\underline{x}) \cdot g(\underline{x}) = 0$, in which case it gives $\nabla h(\underline{x}) \cdot f(\underline{x})$

that is the Def can be rewritten as $\boxed{\nabla h(\underline{x}) \cdot g(\underline{x}) = 0 \Rightarrow \nabla h(\underline{x}) \cdot f(\underline{x}) > -\alpha(h(\underline{x}))}$

Theorem (Ames 2014)

If h is CBF for $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}) u$ in S

then any locally Lipschitz controller $k(\underline{x})$

that satisfies

$$\boxed{\dot{h}(\underline{x}, k(\underline{x})) \geqslant -\alpha(h(\underline{x}))} \quad \text{for all } \underline{x} \in S$$

renders the system safe

To synthetize a controller assume we have a desired controller $k_d(\underline{x})$, which may not be safe

$$\boxed{\begin{aligned} k(\underline{x}) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \ & (u - k_d(\underline{x}))^2 \\ \text{such that } \ & \dot{h}(\underline{x}, u) \geqslant -\alpha(h(\underline{x})) \end{aligned}}$$

This has a unique solution (can be proven using the KKT condition)

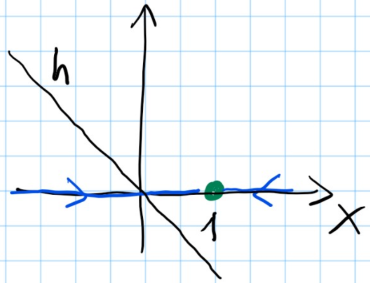
$$\boxed{k(\underline{x}) = \begin{cases} \min\{k_d(\underline{x}), k_s(\underline{x})\} & \text{if } \nabla h(\underline{x}) \cdot g(\underline{x}) < 0 \\ k_d(\underline{x}) & \text{if } \nabla h(\underline{x}) \cdot g(x) = 0 \\ \max\{k_d(\underline{x}), k_s(\underline{x})\} & \text{if } \nabla h(\underline{x}) \cdot g(x) > 0 \end{cases}}$$

where

$$\boxed{k_s(\underline{x}) = -\frac{\nabla h(\underline{x}) \cdot f(\underline{x}) + \alpha(h(\underline{x}))}{\nabla h(\underline{x}) \cdot g(\underline{x})}}$$

Example

$$\dot{x} = u \qquad x \in \mathbb{R}, \quad u \in \mathbb{R}$$



$$f(x) = 0 \qquad g(x) = 1$$

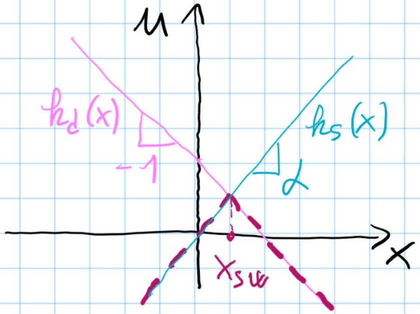
$$u = \boxed{k_d(x) = 1 - x} \qquad \text{stabilizes } x(t) \equiv 1$$

Safety goal: keep $x \leq 0$

$$h(x) = -x$$

is h a CBF

$$\frac{\partial h}{\partial x} \cdot g(x) = -1 < 0 \quad \checkmark$$



switch across

$$k_d(x) = k_s(x)$$

$$1 - x = \alpha x$$

$$x_{sw} = \frac{1}{1+\alpha} < 1$$

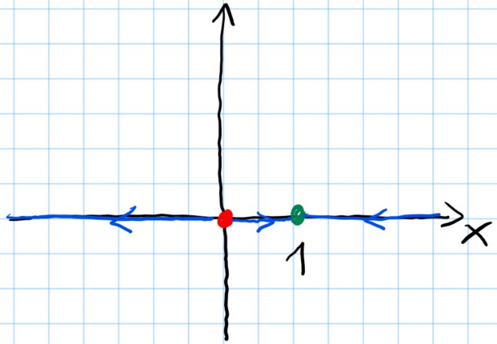
we also need

$$\frac{\partial h}{\partial x} \cdot f(x) = 0$$

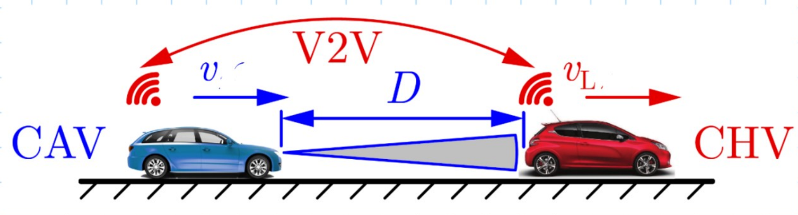
$$k_s(x) = - \frac{\frac{\partial h}{\partial x} \cdot f(x) + \alpha(h(x))}{\frac{\partial h}{\partial x} \cdot g(x)}$$

choose $\alpha(r) = \alpha r$

$$\boxed{k_s(x) = \alpha x} \qquad \boxed{u = \min\{k_d(x), k_s(x)\}}$$



Example: Connected Cruise Control (CCC)

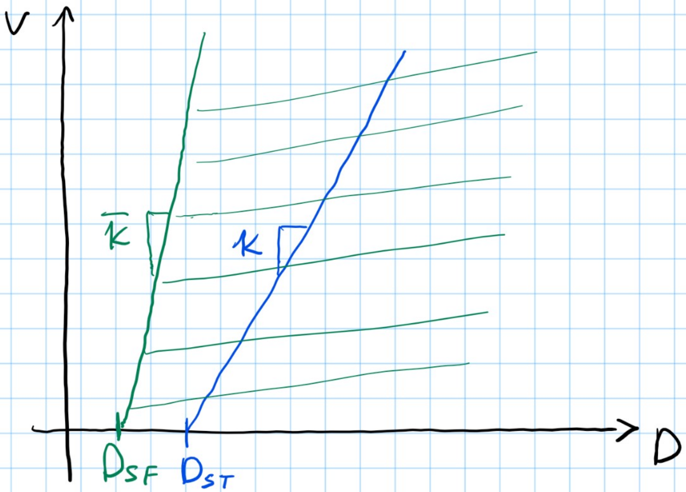


$$\dot{\underline{x}} = f(\underline{x}) + g(\underline{x})u \qquad \begin{cases} \dot{D} = v_L - v \\ \dot{v} = u \end{cases}$$

$$\underline{x} = \begin{bmatrix} D \\ v \end{bmatrix} \qquad f(\underline{x}) = \begin{bmatrix} v_L - v \\ 0 \end{bmatrix} \qquad g(\underline{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

safe set

$$S = \{\underline{x} \in \mathbb{R}^3 \mid h(\underline{x}) \geq 0\} \quad \text{where} \quad \boxed{h(\underline{x}) = \bar{\kappa}(D - D_{SF}) - v}$$



$\frac{1}{\bar{\kappa}}$ minimum time headway

D_{SF} safety distance

penalties parameters

$u = k_d(\underline{x})$ may not ensure safety, e.g., $\boxed{k_d(\underline{x}) = A\big(\kappa(D - D_{ST}) - v\big) + B(v_L - v)}$ with nonsafe gains

is h a CBF $\quad \nabla h(\underline{x}) \cdot g(\underline{x}) = \begin{bmatrix} \bar{\kappa} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 < 0 \quad ✓$

we also need $\quad \nabla h(\underline{x}) \cdot f(\underline{x}) = \begin{bmatrix} \bar{\kappa} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} v_L - v \\ 0 \end{bmatrix} = \bar{\kappa}(v_L - v)$

which occurs at $k_s(\underline{x}) = k_d(\underline{x})$

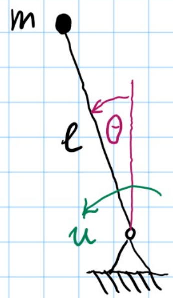
$$\boxed{u = \min\{k_d(\underline{x}), k_s(\underline{x})\}}$$

$$k_s(\underline{x}) = -\frac{\nabla h(\underline{x}) \cdot f(\underline{x}) + \alpha(h(\underline{x}))}{\nabla h(\underline{x}) \cdot g(\underline{x})}$$

choose $\alpha(r) = \alpha \cdot r$

$$\boxed{k_s(\underline{x}) = \alpha\big(\bar{\kappa}(D - D_{SF}) - v\big) + \bar{\kappa}(v_L - v)}$$

Example inverted pendulum



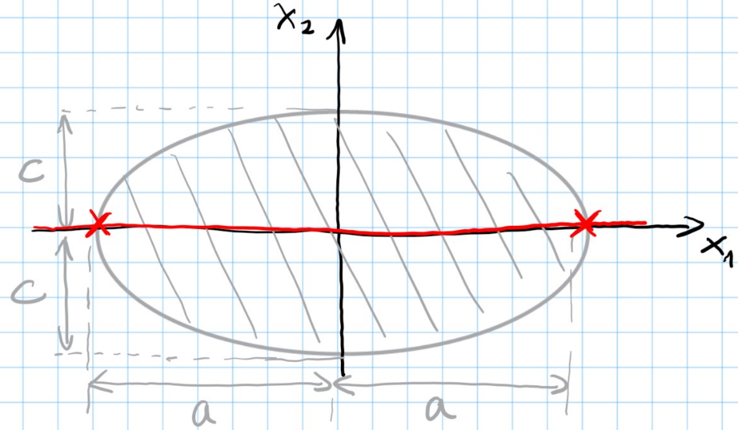
$\downarrow \underline{g}$

$$\dot{\underline{x}} = f(\underline{x}) + g(\underline{x})\,u = \begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} u$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

CBF candidate

$$h(\underline{x}) = 1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{c^2}$$

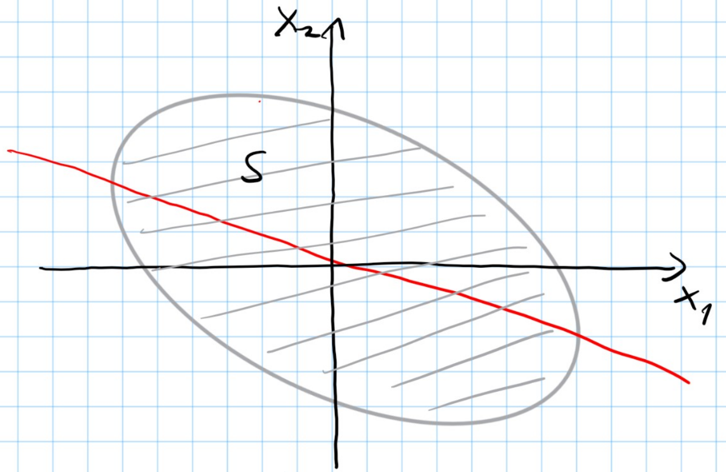


$$\nabla h(\underline{x}) \cdot g(\underline{x}) = \begin{bmatrix} -\frac{2x_1}{a} \\ -\frac{2x_2}{c} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} = -\frac{2x_2}{cm\ell^2} = 0 \Rightarrow \boxed{x_2 = 0} \; (*)$$

$$\nabla h(\underline{x}) \cdot f(\underline{x}) + \alpha(h(\underline{x}))\Big|_{(*)} = \underbrace{\begin{bmatrix} -\frac{2x_1}{a} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{g}{\ell}\sin(x_1) \end{bmatrix}}_{0} + \alpha\left(1 - \frac{x_1^2}{a^2}\right) \not> 0 \quad \text{if } |x_1| = a \quad \Rightarrow \text{h is } \underline{\text{NOT}} \text{ CBF !}$$

CBF candidate

$$h(\underline{x}) = 1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{c^2} - \frac{x_1 x_2}{ac}$$



$$\nabla h(\underline{x}) \cdot g(\underline{x}) = \begin{bmatrix} -\frac{2x_1}{a^2} - \frac{x_2}{ac} \\ -\frac{2x_2}{c^2} - \frac{x_1}{ac} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} = 0 \Rightarrow \boxed{x_2 = -\frac{c}{2a} x_1} \; (**)$$

$$\nabla h(\underline{x}) \cdot f(\underline{x}) + \alpha(h(\underline{x}))\Big|_{(**)} = \begin{bmatrix} -\frac{3}{2a^2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{c}{2a} x_1 \\ \frac{g}{\ell}\sin(x_1) \end{bmatrix} + \alpha\left(1 - \frac{3}{4a^2} x_1^2\right) = \alpha + \frac{3}{4a^2}\left(\frac{c}{a} - \alpha\right) x_1^2 > 0 \quad \text{if } 0 < \alpha \leq \frac{c}{a}$$

$\alpha(r) := \alpha r$

Recall

$$\boxed{k(\underline{x}) = \begin{cases} \min\{k_d(\underline{x}),\ k_s(\underline{x})\} & \text{if } \nabla h(\underline{x})\cdot g(\underline{x}) < 0 \\ k_d(\underline{x}) & \text{if } \nabla h(\underline{x})\cdot g(\underline{x}) = 0 \\ \max\{k_d(\underline{x}),\ k_s(\underline{x})\} & \text{if } \nabla h(\underline{x})\cdot g(\underline{x}) > 0 \end{cases}}$$

where

$$\boxed{k_s(\underline{x}) = -\frac{\nabla h(\underline{x})\cdot f(\underline{x}) + \alpha(h(\underline{x}))}{\nabla h(\underline{x})\cdot g(\underline{x})}}$$

e.g. $k_d(\underline{x}) = m\ell^2\left(-\frac{g}{\ell}\sin(x_1) - p x_1 - d x_2\right) \Rightarrow \dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -p & -d \end{bmatrix}\underline{x}$ stabilizes $\underline{x}(t)=\underline{0}$