Safety of control systems under uncertainty and time delays

Part 2

Gábor Orosz

University of Michigan, Ann Arbor

44th International Summer School of Automatic Control
Grenoble, France
August 28 - September 01, 2023

Material Covered

IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY 1

Control Barrier Functions and Input-to-State Safety With Application to Automated Vehicles

Anil Alan, *Graduate Student Member, IEEE*, Andrew J. Taylor, Chaozhe R. He, Aaron D. Ames, *Fellow, IEEE*, and Gábor Orosz, *Senior Member, IEEE*

IEEE TRANSACTIONS ON INTELLIGENT VEHICLES,

Integrating Safety With Performance in Connected Automated Truck Control: Experimental Validation

Anil Alan, *Graduate Student Member, IEEE*, Chaozhe R. He, Tamas G. Molnar, *Member, IEEE*, Johaan Chacko Mathew, A. Harvey Bell, and Gábor Orosz, *Senior Member, IEEE*

IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, VOL. 31, NO. 4, JULY 2023 1507

Safety-Critical Control With Input Delay in Dynamic Environment

Tamas G. Molnar, *Member, IEEE*, Adam K. Kiss, Aaron D. Ames, *Fellow, IEEE*, and Gábor Orosz, *Senior Member, IEEE*

Received: 16 June 2022 | Revised: 8 March 2023 | Accepted: 23 April 2023

DOI: 10.1002/rnc.6751

RESEARCH ARTICLE

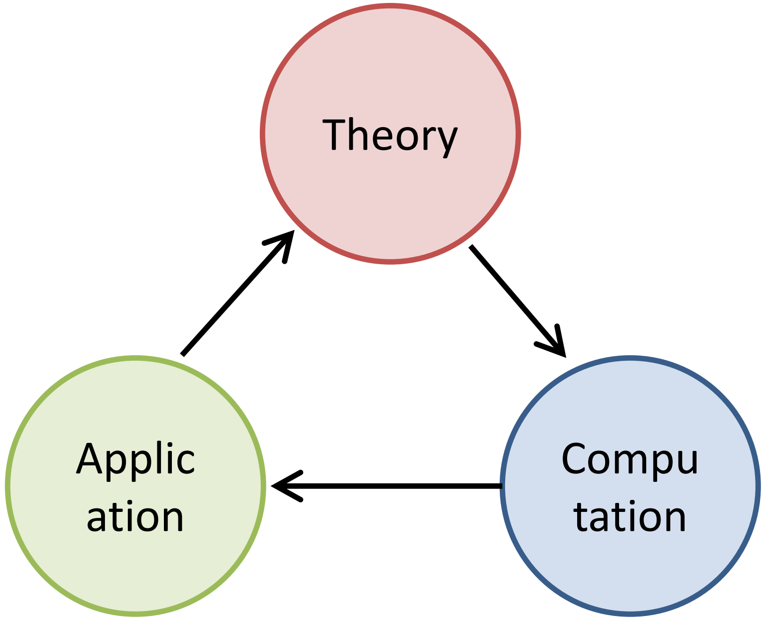
Control barrier functionals: Safety-critical control for time delay systems

Adam K. Kiss[1] | Tamas G. Molnar[2] | Aaron D. Ames[2] | Gabor Orosz[3]

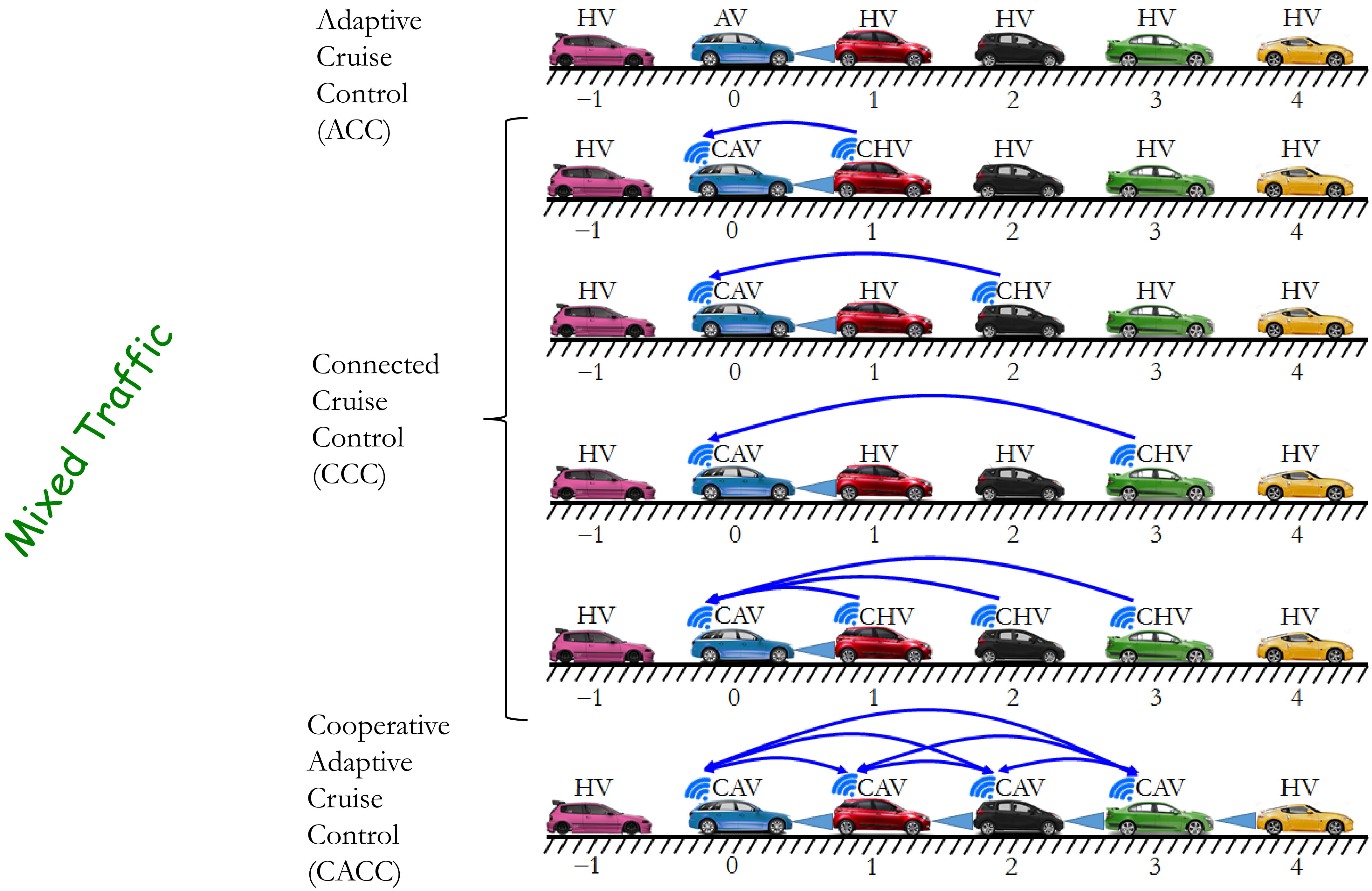
Motivation

5 advices from **Richard Murray**
(CSS Award Speech, 2017)

- be increasingly multilingual
- spread the gospel
- embrace diversity
- master the TCA cycle
- get out the box



Motivation

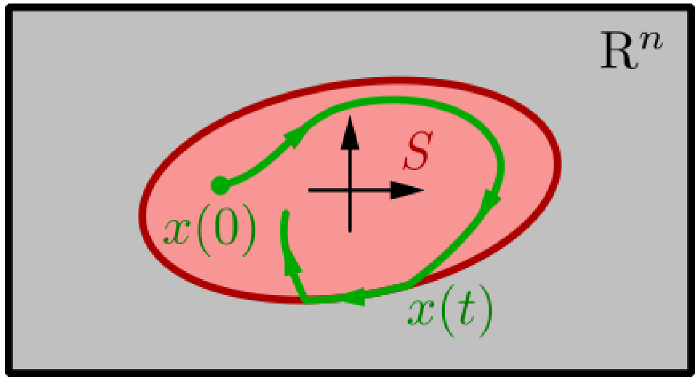

Mixed Traffic
Adaptive Cruise Control (ACC)
HV AV HV HV HV HV
−1 0 1 2 3 4
Connected Cruise Control (CCC)
HV CAV CHV HV HV HV
−1 0 1 2 3 4
HV CAV HV CHV HV HV
−1 0 1 2 3 4
HV CAV HV HV CHV HV
−1 0 1 2 3 4
HV CAV CHV CHV CHV HV
−1 0 1 2 3 4
Cooperative Adaptive Cruise Control (CACC)
HV CAV CAV CAV CAV HV
−1 0 1 2 3 4

Control Barrier Function 101

Control system:

$$\dot{x} = f(x) + g(x)u \qquad x \in \mathbb{R}^n$$
$$u \in \mathbb{R}^m$$

Safe set: $S \subset \mathbb{R}^n$

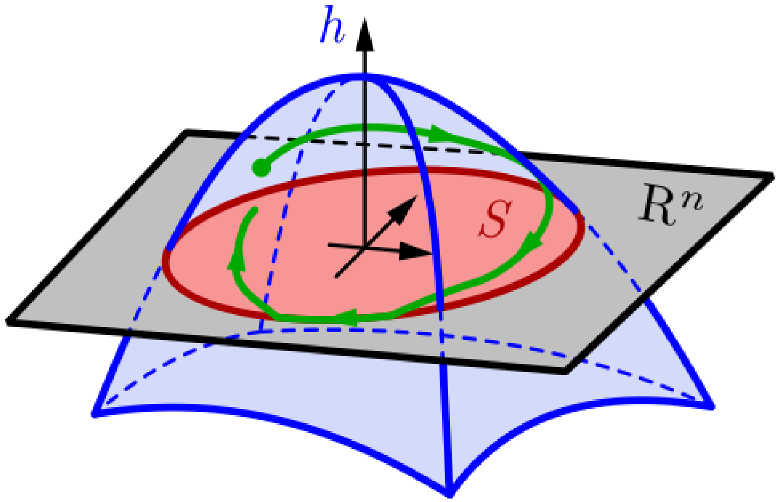


Safety as forward invariance:

$$x(0) \in S \;\Rightarrow\; x(t) \in S, \;\forall t \geq 0$$

Safe set constructed super-level set:

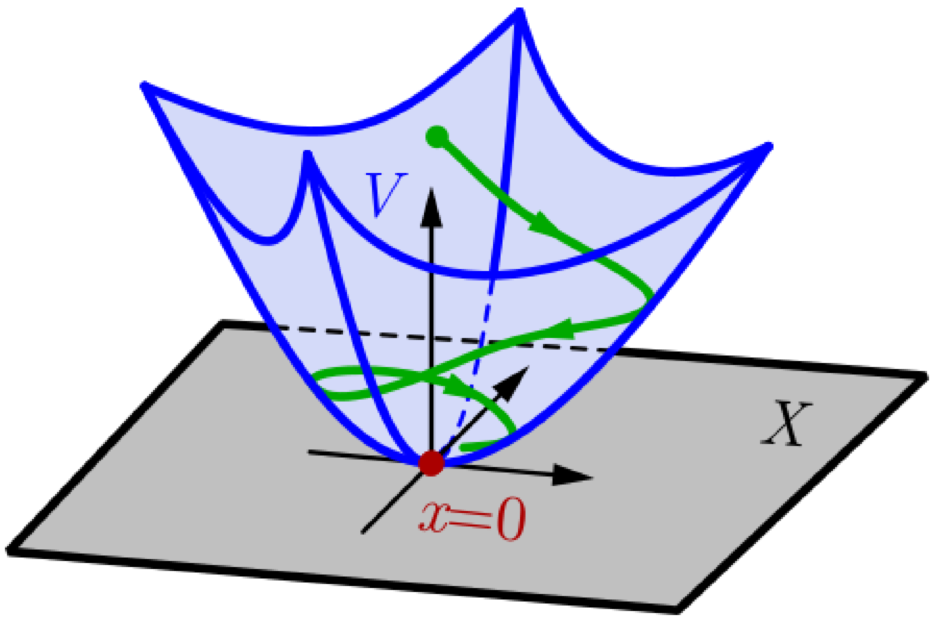
$$S = \{x \in X : h(x) \geq 0\}$$



Goal: synthesize u so that $h(x(t))$ stays nonnegative

Control Barrier Function 101

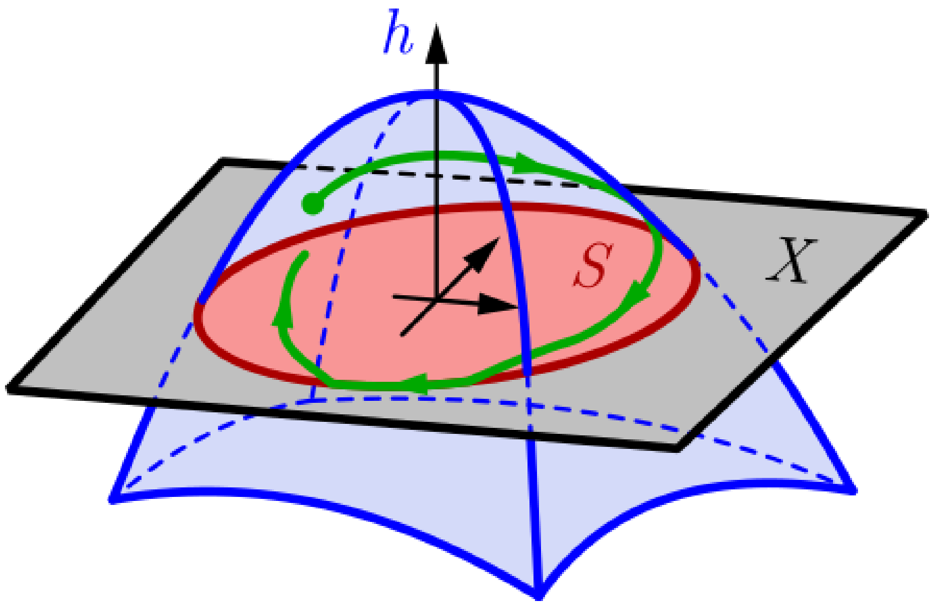
Control Lyapunov functions (CLFs)



$$k_1\|x\|^c \leq V(x) \leq k_2\|x\|^c$$
$$\dot{V}(x,u) \leq -\lambda V(x)$$

Approach equilibrium exponentially
with some minimum rate

Control barrier functions (CBFs)



$$S = \{x \in X : h(x) \geq 0\}$$
$$\dot{h}(x,u) \geq -\alpha h(x)$$

Approach safe set boundary
with some maximum rate

Control Barrier Function 101

Control how fast the safe set boundary is approached

$$\dot{h}(x,u) \geq -\alpha h(x) \qquad \dot{h}(x,u) = \nabla h(x)(f(x) + g(x)u)$$

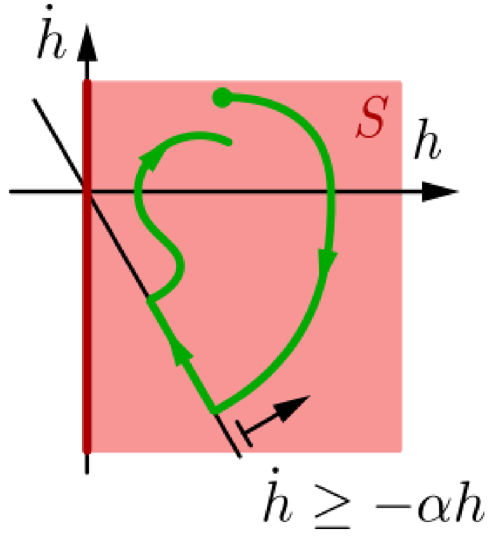
$$\alpha > 0$$

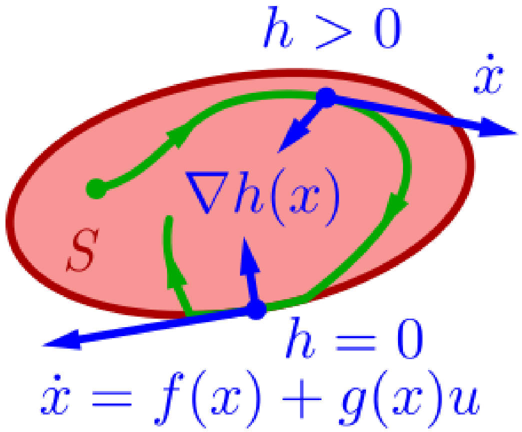
Theorem:

controllers $u = k(x)$

that satisfy $\dot{h}(x, k(x)) \geq -\alpha h(x)$

ensure safety: $x(0) \in S \;\Rightarrow\; x(t) \in S, \;\forall t \geq 0$



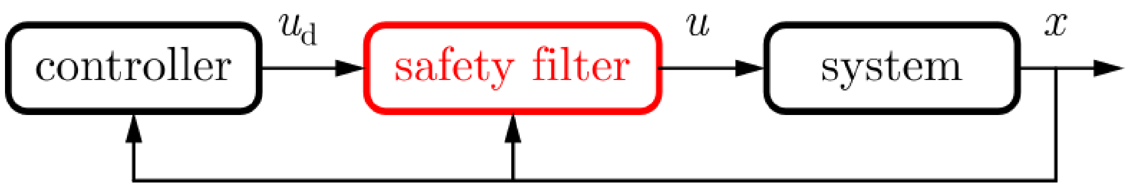


Control synthesis via optimization:

$$k_{\mathrm{QP}}(x) = \underset{u \in U}{\operatorname{argmin}} \; \|u - k_{\mathrm{n}}(x)\|^2$$

$$\text{s.t. } \dot{h}(x,u) \geq -\alpha h(x)$$

$\Longrightarrow$



Analytical solution (single input):

$$k_{\mathrm{QP}}(x) = \max\left\{k_{\mathrm{n}}(x), -\frac{\nabla h(x) f(x) + \alpha h(x)}{\nabla h(x) g(x)}\right\}$$

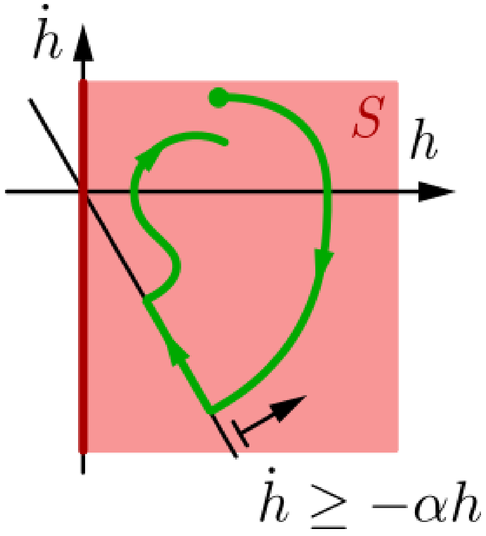
$$\nabla h(x) g(x) > 0$$

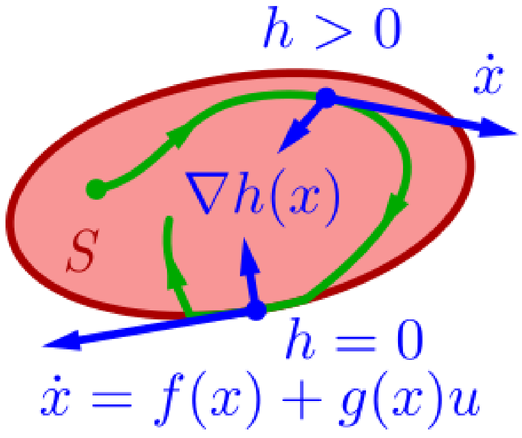
Control Barrier Function 101

Control how fast the safe set boundary is approached

$$\dot{h}(x,u) \geq -\alpha h(x) \qquad \dot{h}(x,u) = \nabla h(x)(f(x) + g(x)u)$$

$$\alpha > 0$$





Theorem:

controllers $\quad u = k(x)$

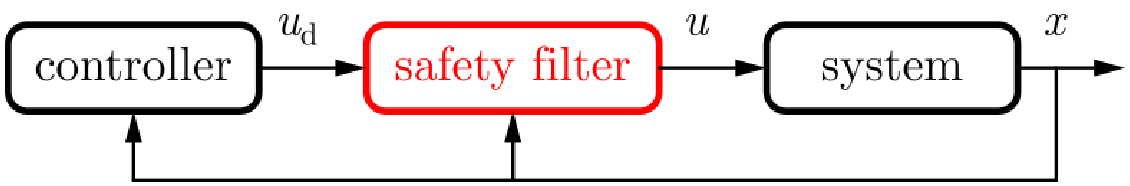
that satisfy $\quad \dot{h}(x, k(x)) \geq -\alpha h(x)$

ensure safety: $\quad x(0) \in S \;\Rightarrow\; x(t) \in S, \;\forall t \geq 0$

Control synthesis via optimization:

$$k_{\mathrm{QP}}(x) = \underset{u \in U}{\operatorname{argmin}} \; \|u - k_{\mathrm{n}}(x)\|^2$$
$$\text{s.t. } \dot{h}(x,u) \geq -\alpha h(x)$$

$\Longrightarrow$

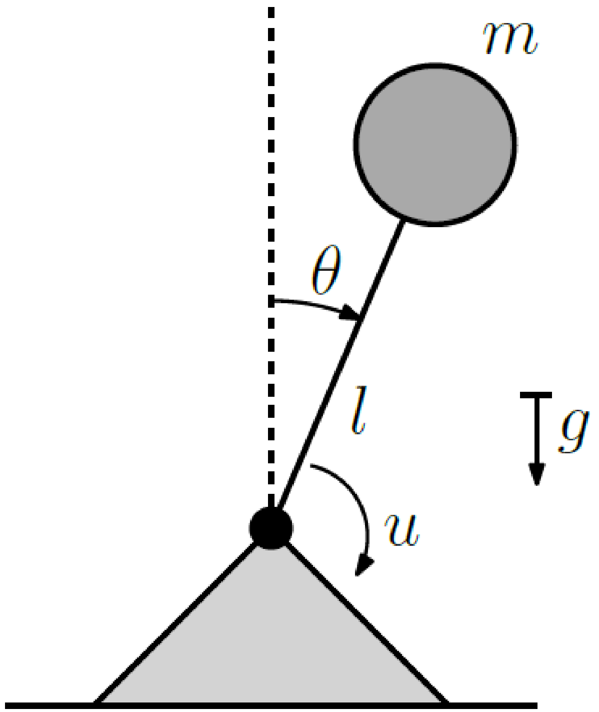


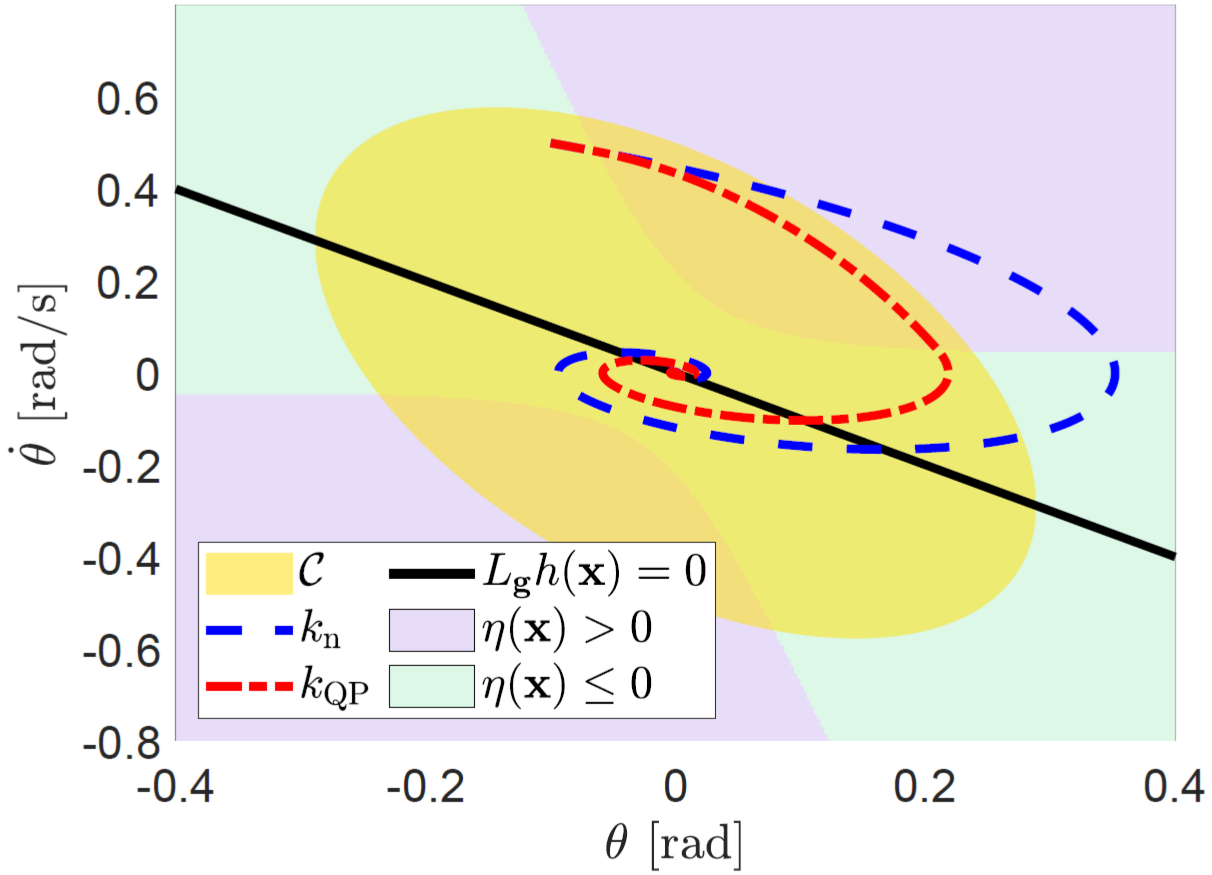
Analytical solution (single input):

$$k_{\mathrm{QP}}(x) = \min\left\{k_{\mathrm{n}}(x), -\frac{\nabla h(x) f(x) + \alpha h(x)}{\nabla h(x) g(x)}\right\}$$

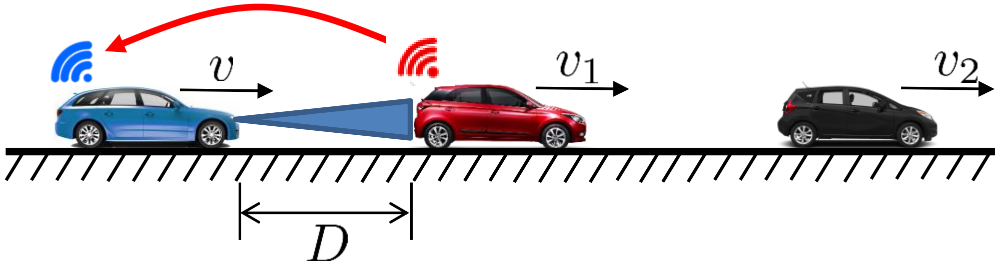
$$\nabla h(x) g(x) < 0$$

Control Barrier Function 101





CBF Applied to Vehicle Control



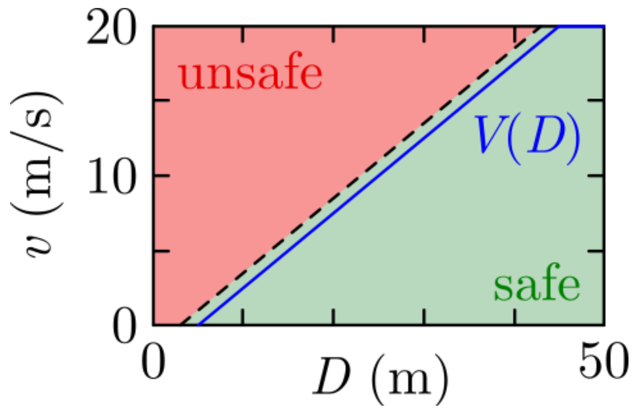
Connected Cruise Control

Dynamical model:

$$\left.\begin{aligned}\dot{D} &= v_{\rm L} - v \\ \dot{v} &= u \\ \dot{v}_{\rm L} &= a_{\rm L}(t)\end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)u$$

Safety measure:

$$h(x) = D - D_{\rm sf} - Tv$$



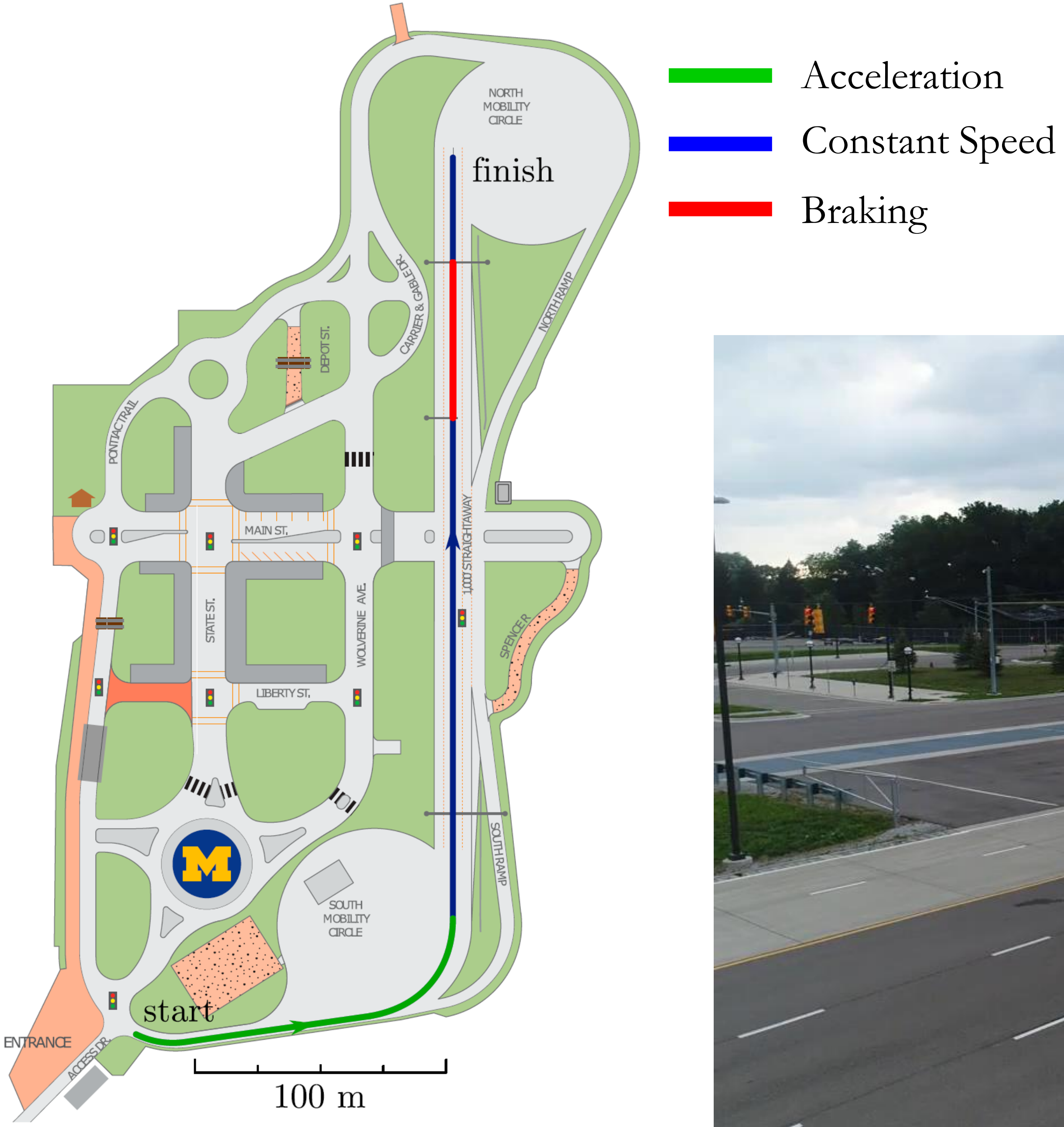
Safety critical control:

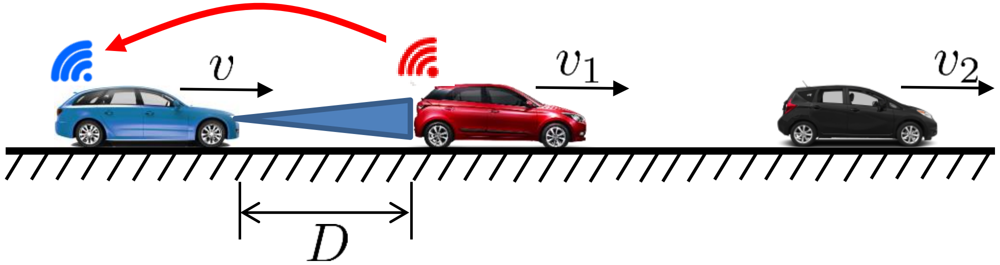
$$\dot{h}(t,x,u) \geq -\alpha h(x)$$

$$u \leq \alpha\left(\frac{1}{T}(D - D_{\rm sf}) - v\right) + \frac{1}{T}(v_{\rm L} - v)$$

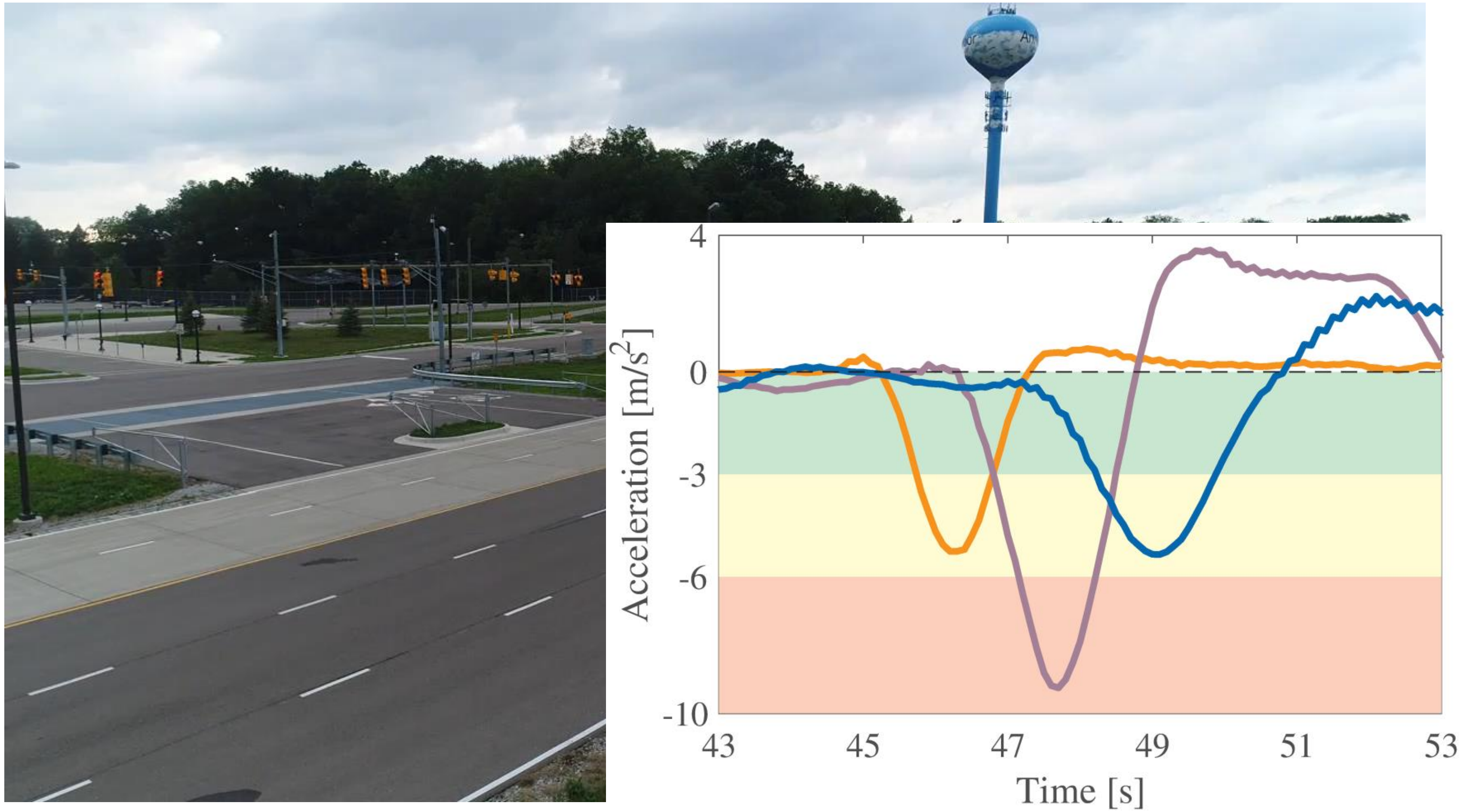
$$u = \alpha\Big(\underbrace{\frac{1}{T}(D - D_{\rm st})}_{V(D)} - v\Big) + \frac{1}{T}(v_{\rm L} - v)$$

Connected Cruise Control Experiments

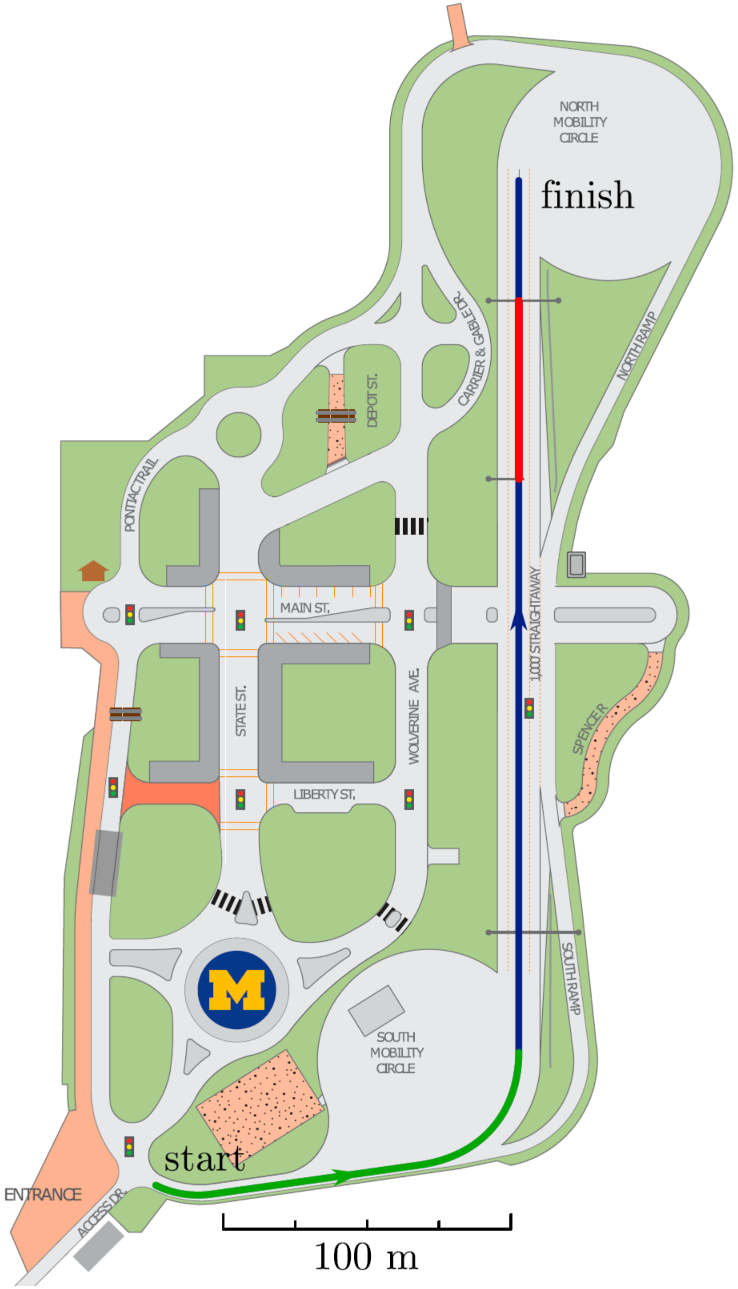




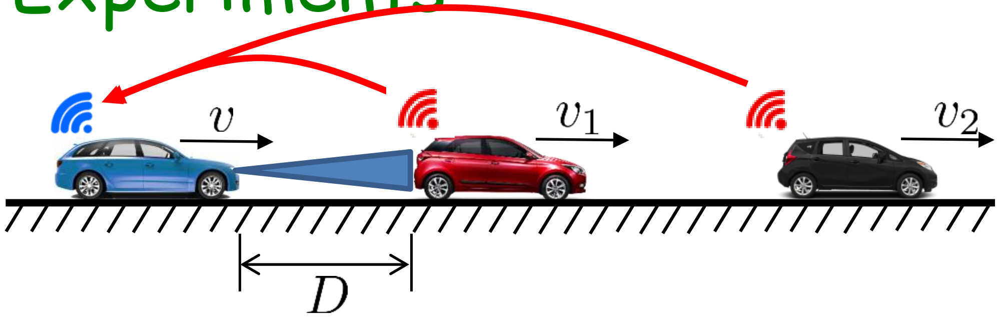
Connected Cruise Control



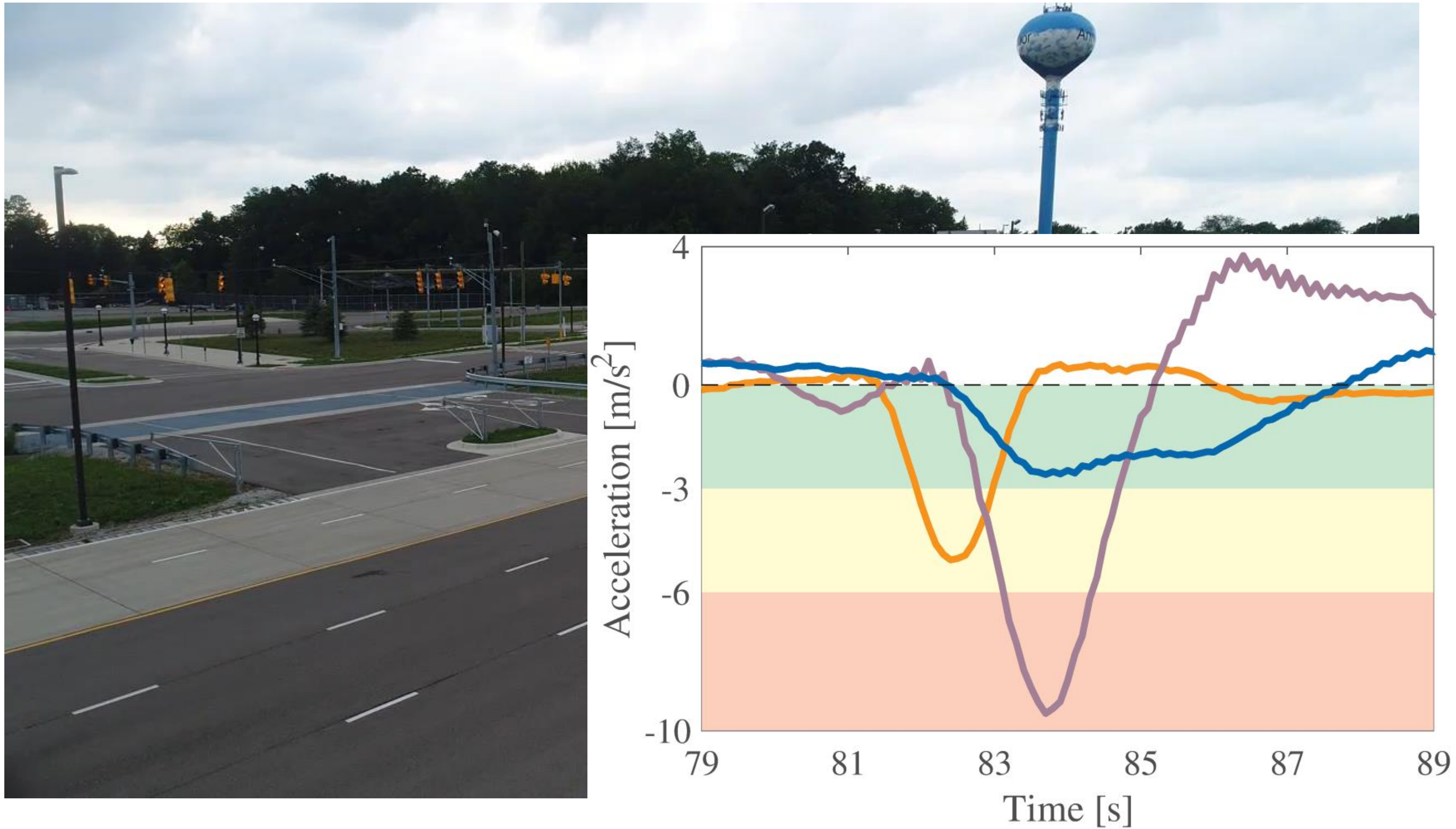
Connected Cruise Control Experiments



Acceleration
Constant Speed
Braking



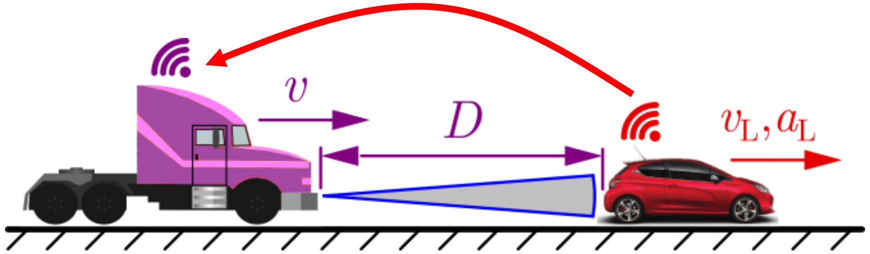
Connected Cruise Control



CBF Applied to Truck Control

Dynamical model:

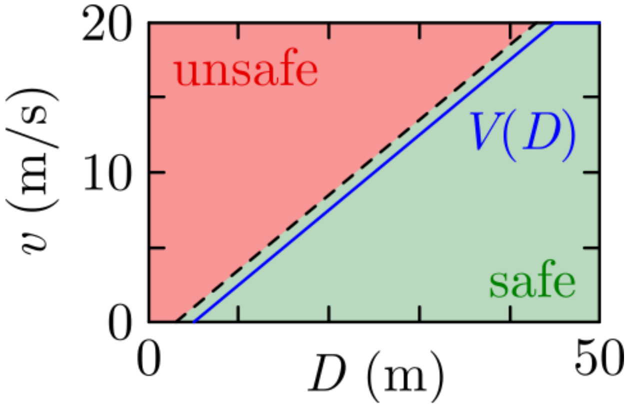
$$\left.\begin{aligned} \dot{D} &= v_{\mathrm{L}} - v \\ \dot{v} &= u \\ \dot{v}_{\mathrm{L}} &= a_{\mathrm{L}}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)u$$



Connected Cruise Control

Safety measure:

$$h(x) = D - D_{\mathrm{sf}} - Tv$$



Safety critical control:

$$\dot{h}(t,x,u) \geq -\alpha h(x)$$

$$u \leq \alpha\left(\frac{1}{T}(D - D_{\mathrm{sf}}) - v\right) + \frac{1}{T}(v_{\mathrm{L}} - v)$$

$$u = \alpha\Big(\underbrace{\frac{1}{T}(D - D_{\mathrm{st}})}_{V(D)} - v\Big) + \frac{1}{T}(v_{\mathrm{L}} - v)$$


NAVISTAR INC.
USDOT


10
5
0
-5
-10
CBF (m)
0
2
4
6
8
10
t (s)

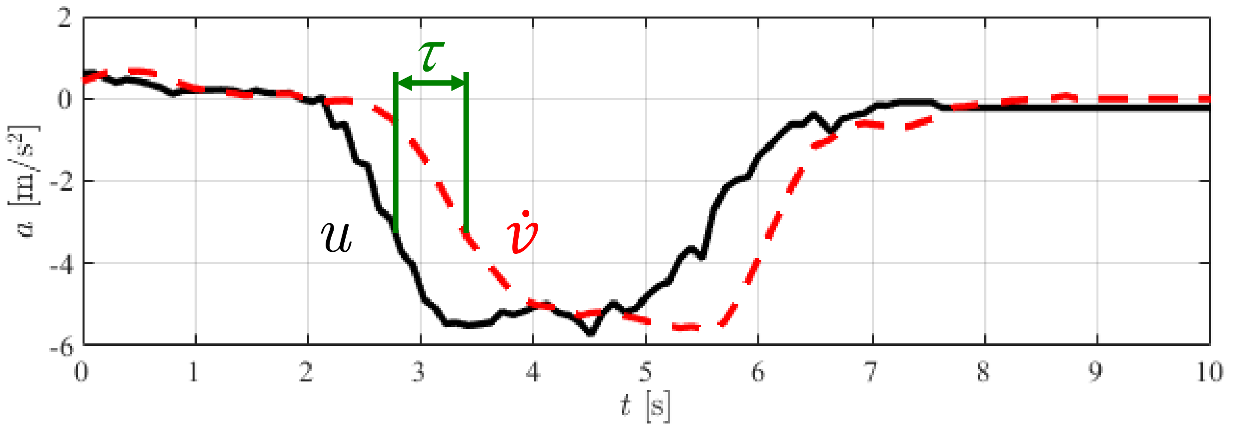
CBF (m)
10
5
0
-5
-10
Desired controller
0
2
4
6
8
10
t (s)

Delays and Disturbances Destroy Safety

Dynamical model:

$$\left.\begin{aligned} \dot{D} &= v_{\mathrm{L}} - v \\ \dot{v} &= u \\ \dot{v}_{\mathrm{L}} &= a_{\mathrm{L}}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)u$$

Why did the experiment fail?



Disturbances:

Truck's dynamics are uncertain
(engine, transmission, brakes, tires…)

Delay:

Truck is massive, with large response time

Delays and Disturbances Destroy Safety

Dynamical model:

$$\left.\begin{aligned} \dot{D} &= v_{\mathrm{L}} - v \\ \dot{v} &= u(t-\tau) + d(t) \\ \dot{v}_{\mathrm{L}} &= a_{\mathrm{L}}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)\big(u(t-\tau) + d(t)\big)$$

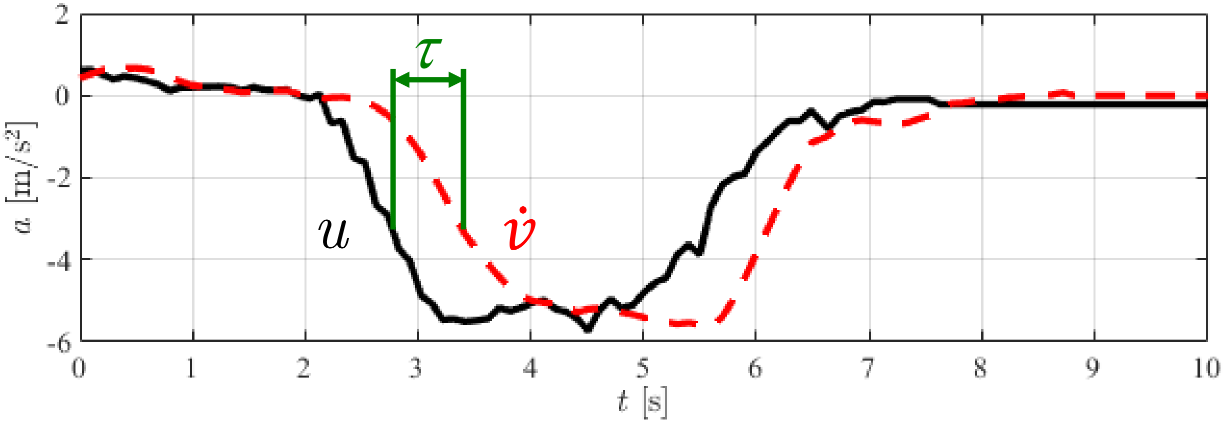
Why did the experiment fail?

Disturbances:

Truck's dynamics are uncertain
(engine, transmission, brakes, tires…)

Delay:

Truck is massive, with large response time



Content

Disturbances/Environment

Input-to-state safe control barrier functions (ISSf-CBF)

Tunable Input-to-state safe control barrier functions (TISSf-CBF)

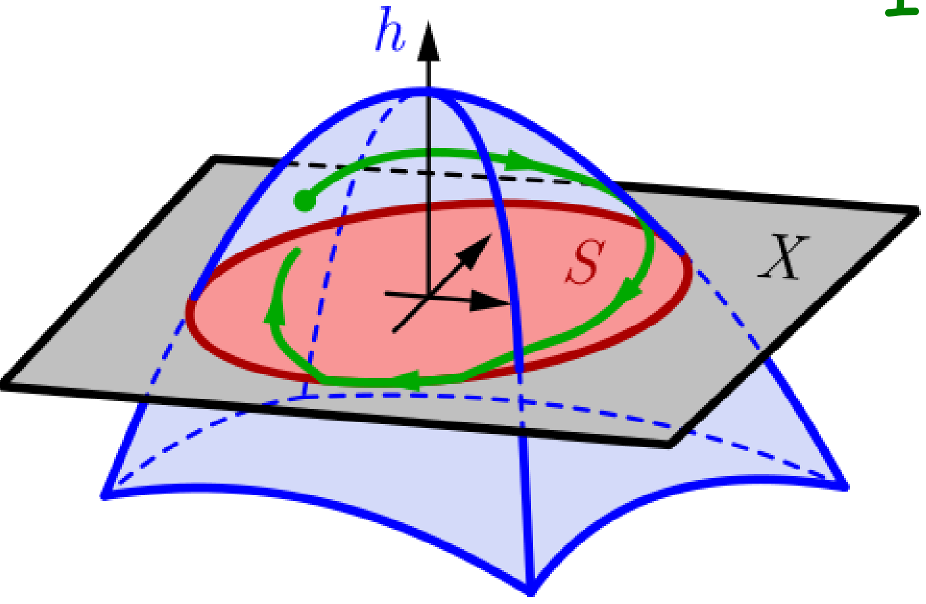
Time delays

Input delay $\rightarrow$ Predictors

Application of safety filters in the real world

Moving from test track to the real world

Input-to-State Safety



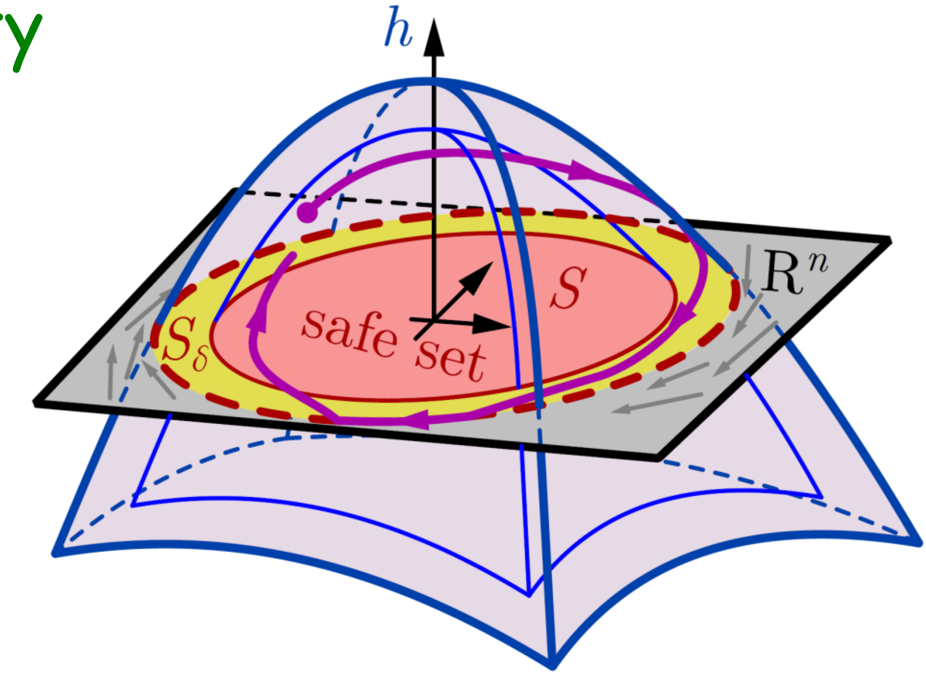
$$S = \{x \in X : h(x) \geq 0\}$$

$$\dot{x} = f(x) + g(x)(u + d)$$

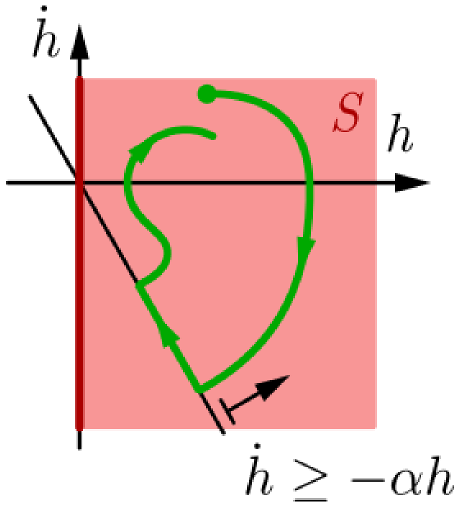
$$\|d\|_\infty \leq \delta$$

$$\dot{h}(x,u) \geq -\alpha h(x) + ???$$

$$\alpha > 0$$



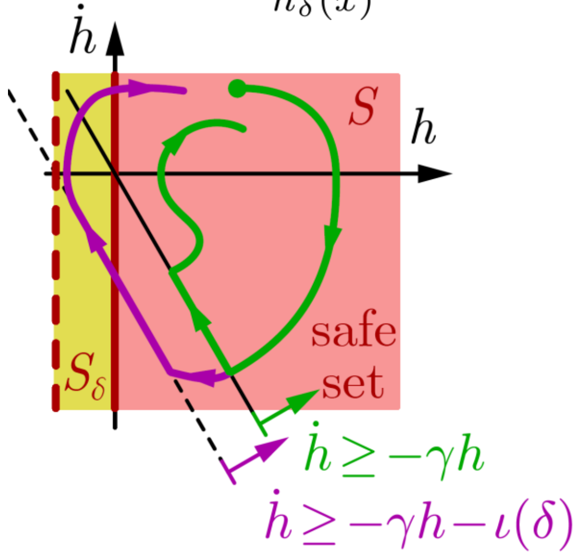
$$S_\delta = \{x \in X : \underbrace{h(x) + \gamma(\delta)}_{h_\delta(x)} \geq 0\}$$



Theorem:

$$\dot{h}(x,u) \geq -\alpha h(x) + \sigma \|\nabla h(x) g(x)\|^2$$

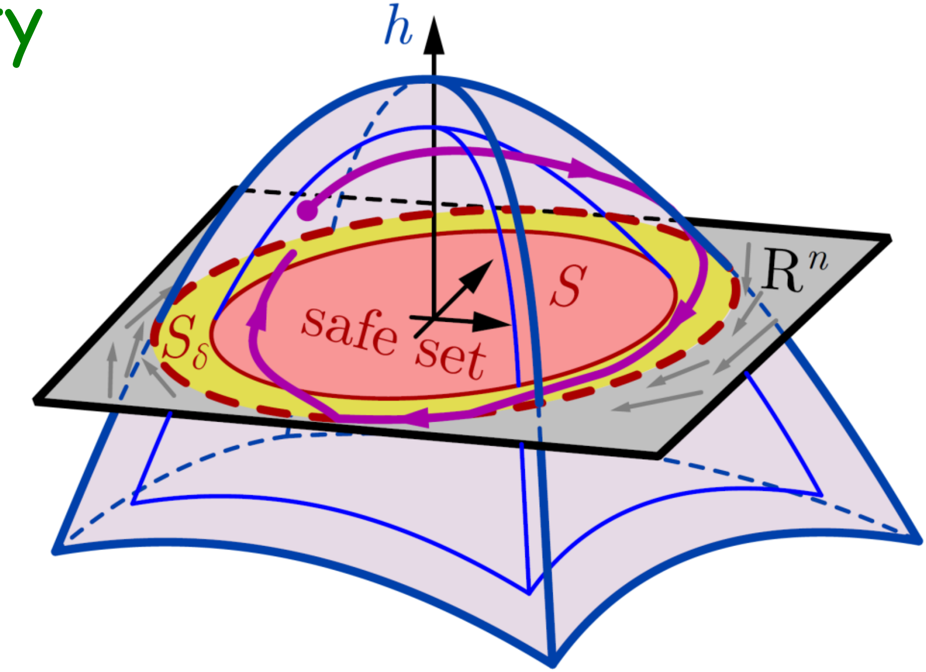
+ formulae linking $\sigma, \alpha, \iota, \gamma$



Input-to-State Safety

Theorem:

$$\dot{h}(x,u) \geq -\alpha h(x) + \sigma \|\nabla h(x) g(x)\|^2$$



If $k(x)$ safe without disturbance:

$$\overline{k}(x) = k(x) + \sigma \big(\nabla h(x) g(x)\big)^\top$$

$$S_\delta = \{x \in X : \underbrace{h(x) + \gamma(\delta)}_{h_\delta(x)} \geq 0\}$$

Otherwise use QP:

$$\overline{k}_{\mathrm{QP}}(x) = \underset{u \in U}{\operatorname{argmin}} \ \|u - k_{\mathrm{n}}(x)\|^2$$

$$\text{s.t. } \dot{h}(x,u) \geq -\alpha h(x) + \sigma \|\nabla h(x) g(x)\|^2$$

$\implies$ **Analytical solution** (single input):

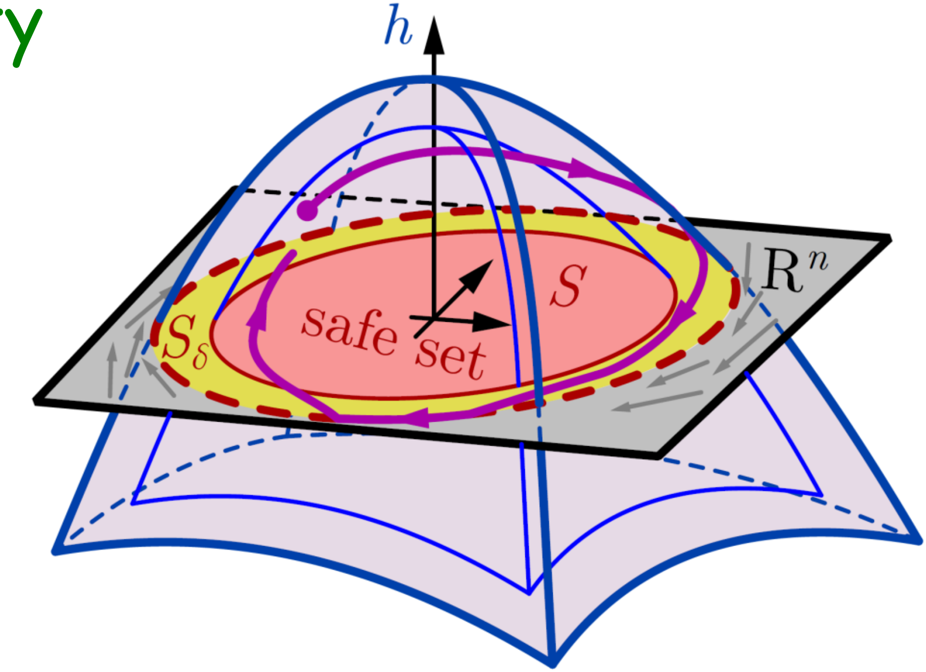
$$\overline{k}_{\mathrm{QP}}(x) = \max\left\{ k_{\mathrm{n}}(x), -\frac{\nabla h(x) f(x) + \alpha h(x)}{\nabla h(x) g(x)} + \sigma \nabla h(x) g(x) \right\}$$

$$\nabla h(x) g(x) > 0$$

Input-to-State Safety

Theorem:

$$\dot{h}(x,u) \geq -\alpha h(x) + \sigma \|\nabla h(x) g(x)\|^2$$



$$S_\delta = \{x \in X : \underbrace{h(x) + \gamma(\delta)}_{h_\delta(x)} \geq 0\}$$

If $k(x)$ safe without disturbance:

$$\overline{k}(x) = k(x) + \sigma \big(\nabla h(x) g(x)\big)^\top$$

Otherwise use QP:

$$\overline{k}_{\mathrm{QP}}(x) = \underset{u \in U}{\operatorname{argmin}} \ \|u - k_{\mathrm{n}}(x)\|^2$$

$$\text{s.t. } \dot{h}(x,u) \geq -\alpha h(x) + \sigma \|\nabla h(x) g(x)\|^2$$

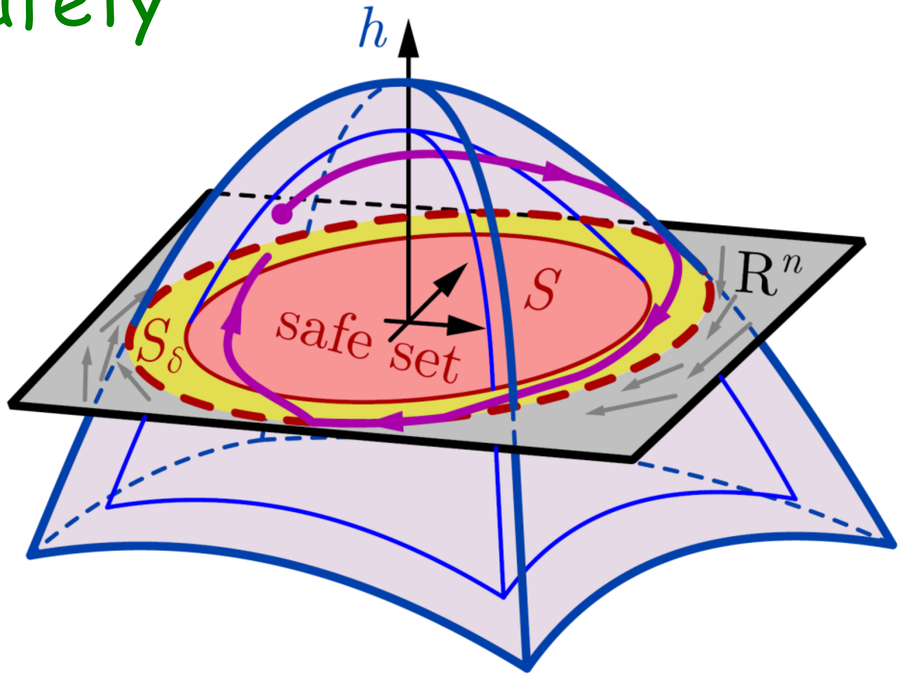
$\Longrightarrow$ **Analytical solution** (single input):

$$\overline{k}_{\mathrm{QP}}(x) = \min\left\{ k_{\mathrm{n}}(x), -\frac{\nabla h(x) f(x) + \alpha h(x)}{\nabla h(x) g(x)} + \sigma \nabla h(x) g(x) \right\}$$

$$\nabla h(x) g(x) < 0$$

Keeps the truck safe but it keeps very large distance all the time!

Tunable Input-to-State Safety



$$S_\delta = \{x \in X : \underbrace{h(x) + \gamma(\delta, h(x))}_{h_\delta(x)} \geq 0\}$$

Key idea:

$$\sigma \quad \rightarrow \quad \sigma(h(x))$$

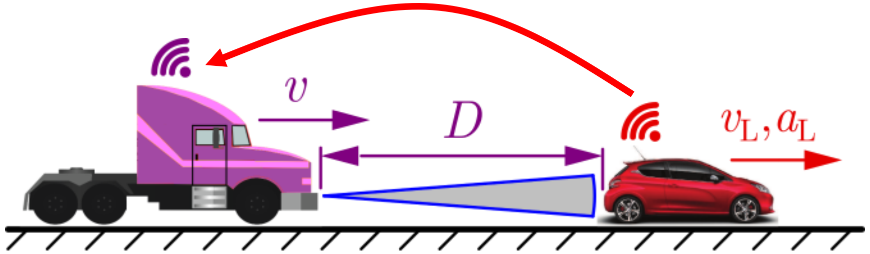
Theorem:

The above construction still works
if $\sigma(r)$ is strictly monotonically decreasing

Example:

$$\sigma(r) = \sigma_0 \mathrm{e}^{-\lambda r}$$

TISSf-CBF Applied to Truck Control



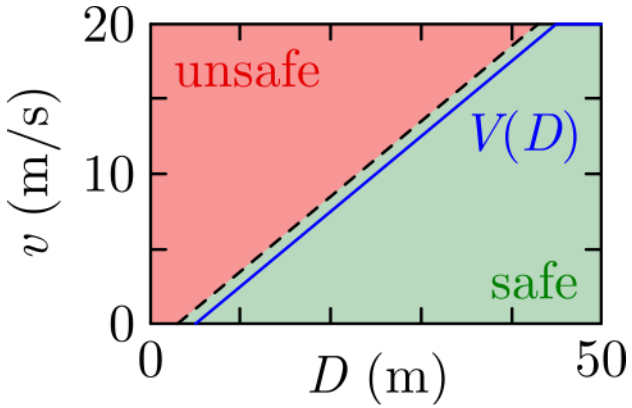
Connected Cruise Control

Dynamical model:

$$\left.\begin{aligned} \dot{D} &= v_{\mathrm{L}} - v \\ \dot{v} &= u \\ \dot{v}_{\mathrm{L}} &= a_{\mathrm{L}}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)u$$

Safety measure:

$$h(x) = D - D_{\mathrm{sf}} - Tv$$



Safety critical control:

$$\dot{h}(t,x,u) \geq -\alpha h(x) + \sigma\big(h(x)\big)\|\nabla h(x) g(x)\|^2 \qquad \sigma(r) = \sigma_0 \mathrm{e}^{-\lambda r}$$

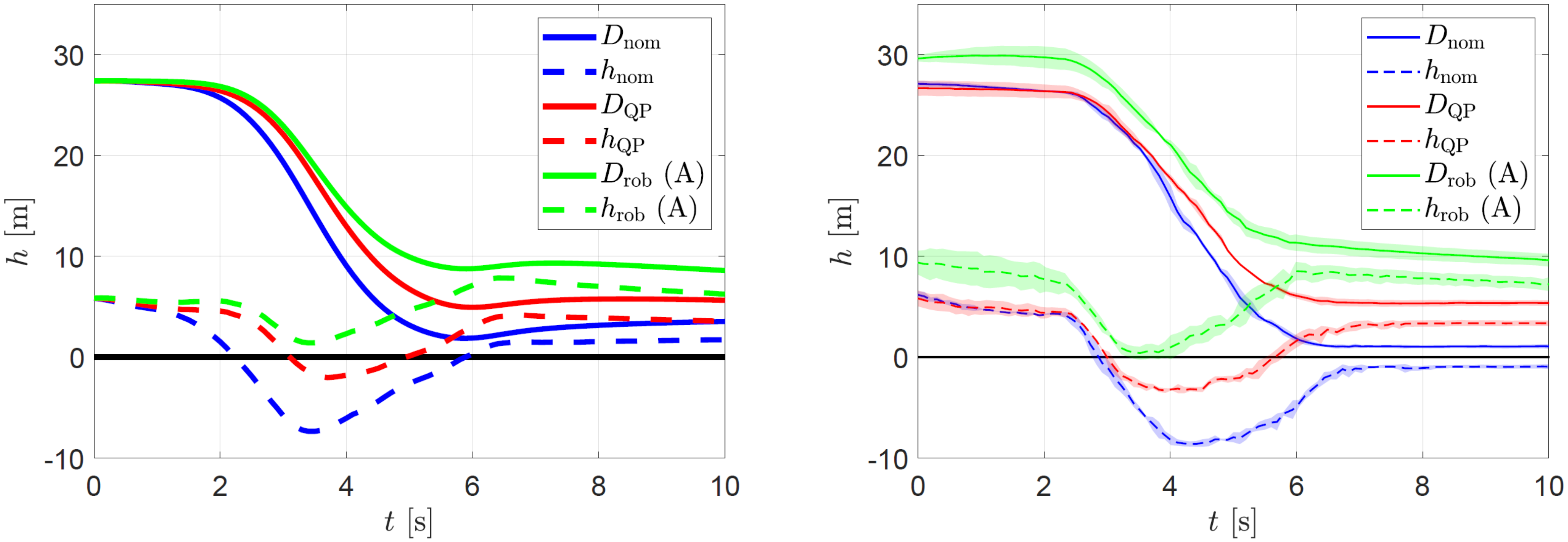
$$u \leq \alpha\left(\frac{1}{T}(D - D_{\mathrm{sf}}) - v\right) + \frac{1}{T}(v_{\mathrm{L}} - v) - T\sigma_0 \mathrm{e}^{-\lambda(D - D_{\mathrm{sf}} - Tv)}$$

$$u = \alpha\Bigg(\underbrace{\frac{1}{T}(D - D_{\mathrm{st}})}_{V(D)} - v\Bigg) + \frac{1}{T}(v_{\mathrm{L}} - v) - T\sigma_0 \mathrm{e}^{-\lambda(D - D_{\mathrm{sf}} - Tv)}$$

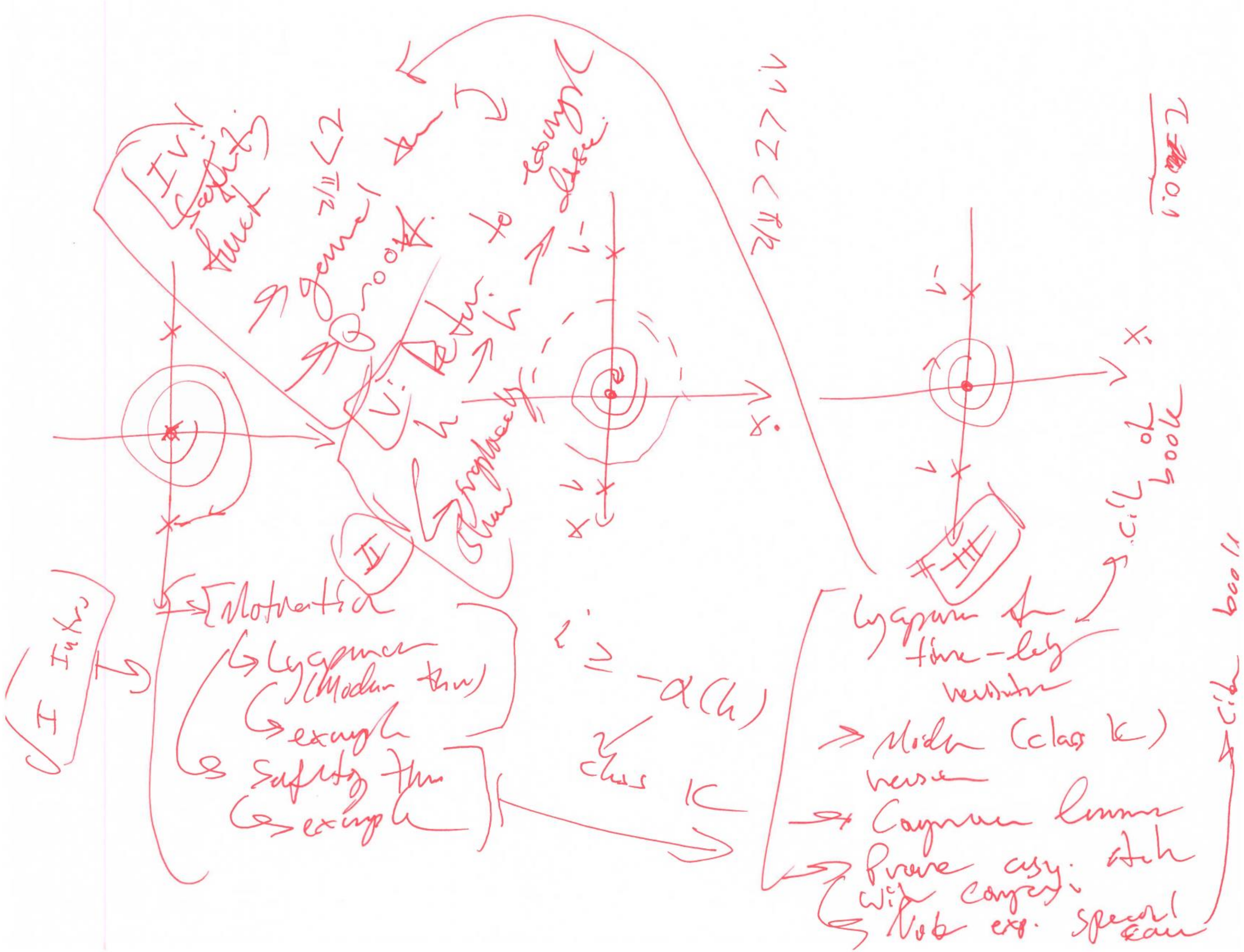

NAVISTAR INC.


10
5
0
-5
-10
CBF (m)
CBF-based
Desired controller
0
2
4
6
8
10
t (s)

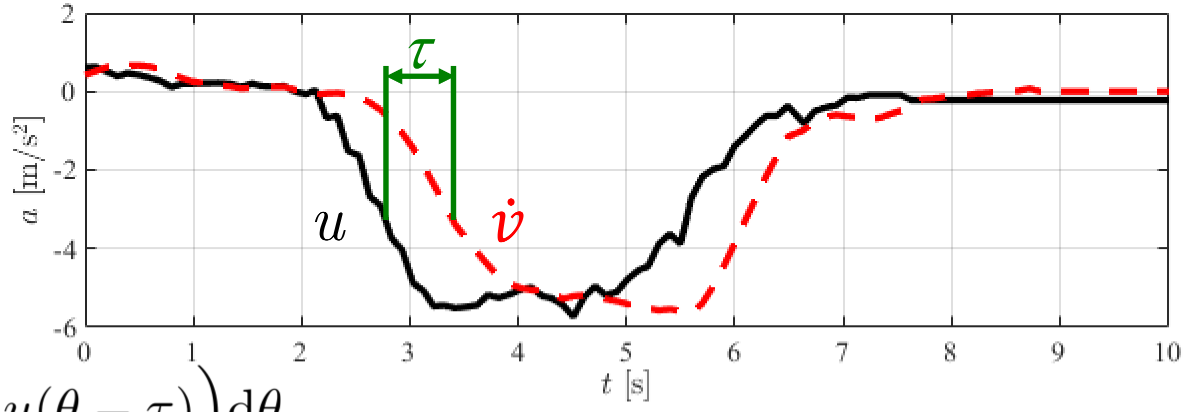
TISSf-CBF Applied to Truck Control



Safety with Time Delay



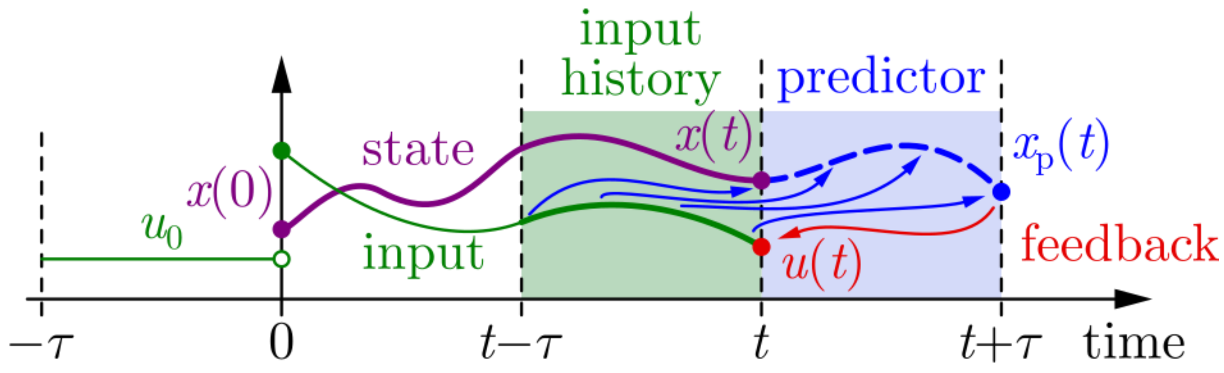
Input Delay



Predictor feedback for delay compensation:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t-\tau)$$

$$x_{\mathrm{p}}(t) = x(t+\tau) = x(t) + \int_t^{t+\tau} \Big(f\big(x(\theta)\big) + g\big(x(\theta)\big) u(\theta-\tau) \Big) \mathrm{d}\theta$$



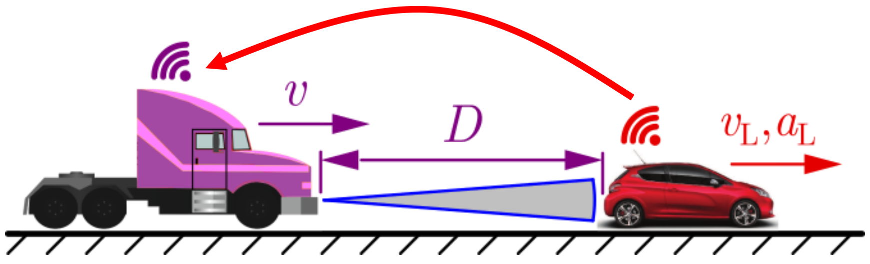
Theorem:

controllers $\quad u = k(x_{\mathrm{p}})$

that satisfy $\quad \dot{h}(x_{\mathrm{p}}, k(x_{\mathrm{p}})) \geq -\alpha h(x_{\mathrm{p}})$

ensure safety: $\quad x(\theta) \in S,\ \forall \theta \in [0, \tau] \ \Rightarrow\ x(t) \in S,\ \forall t \geq 0$

Predictors Recover Safety Guarantees



Connected Cruise Control

Dynamical model:

$$\left.\begin{aligned} \dot{D} &= v_{\rm L} - v \\ \dot{v} &= u(t-\tau) \\ \dot{v}_{\rm L} &= a_{\rm L}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)u(t-\tau)$$

Safety measure:

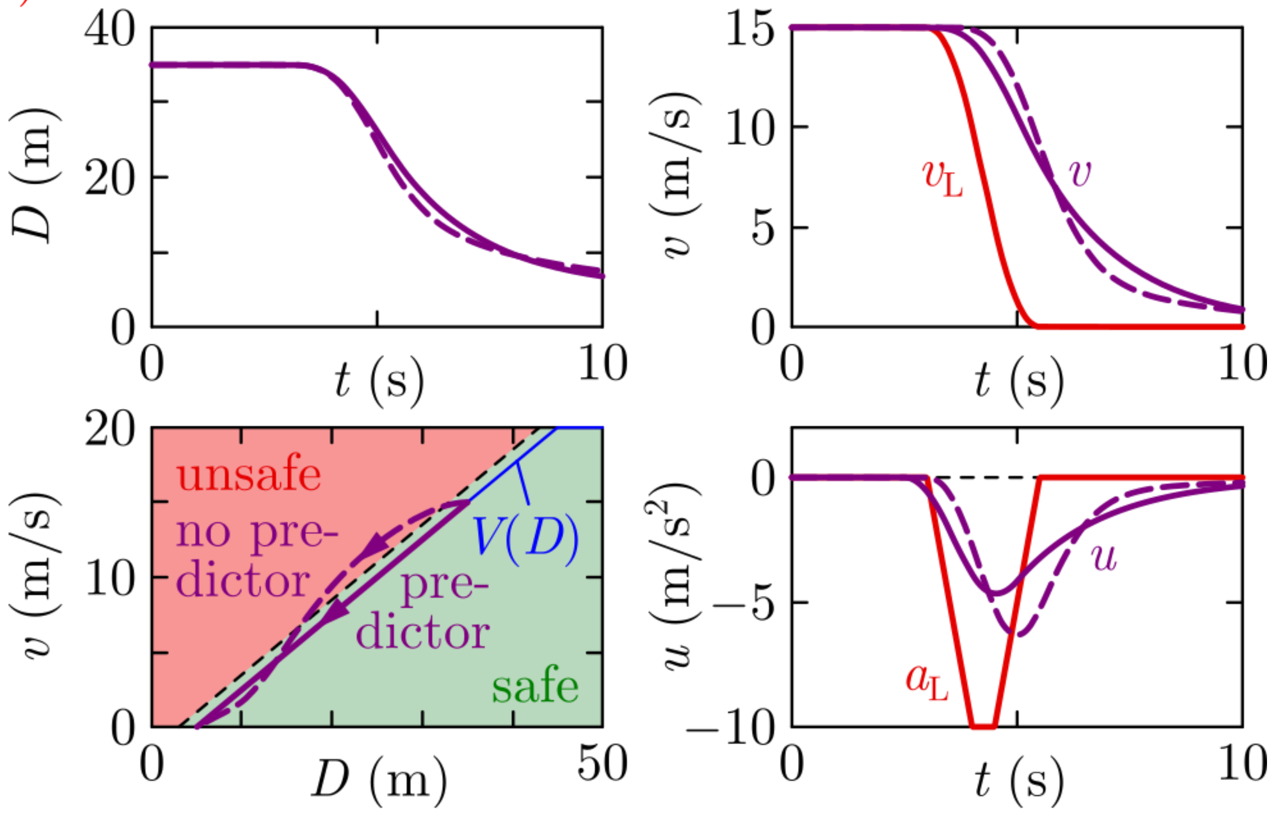
$$h(x) = D - D_{\rm sf} - Tv$$

Safety critical control:

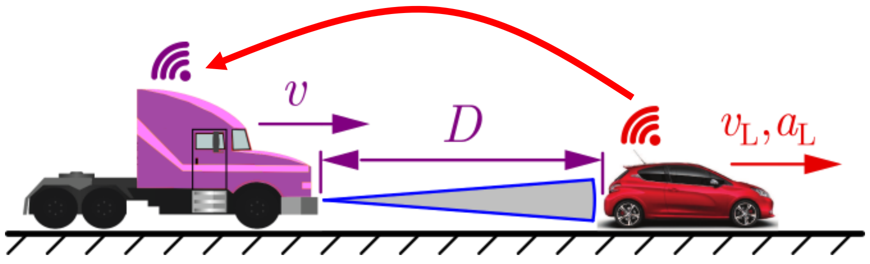
$$\dot{h}(t_{\rm p}, x_{\rm p}, u) \geq -\alpha h(x_{\rm p})$$

$$u = \alpha\Bigg(\underbrace{\frac{1}{T}(D_{\rm p} - D_{\rm st})}_{V(D_{\rm p})} - v_{\rm p}\Bigg) + \frac{1}{T}(v_{\rm L,p} - v_{\rm p})$$

intent of preceding vehicle



Prediction Errors Contribute to Disturbances



Connected Cruise Control

Dynamical model:

$$\left.\begin{aligned} \dot{D} &= v_{\rm L} - v \\ \dot{v} &= u(t-\tau) + d(t) \\ \dot{v}_{\rm L} &= a_{\rm L}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)\big(u(t-\tau) + d(t)\big)$$

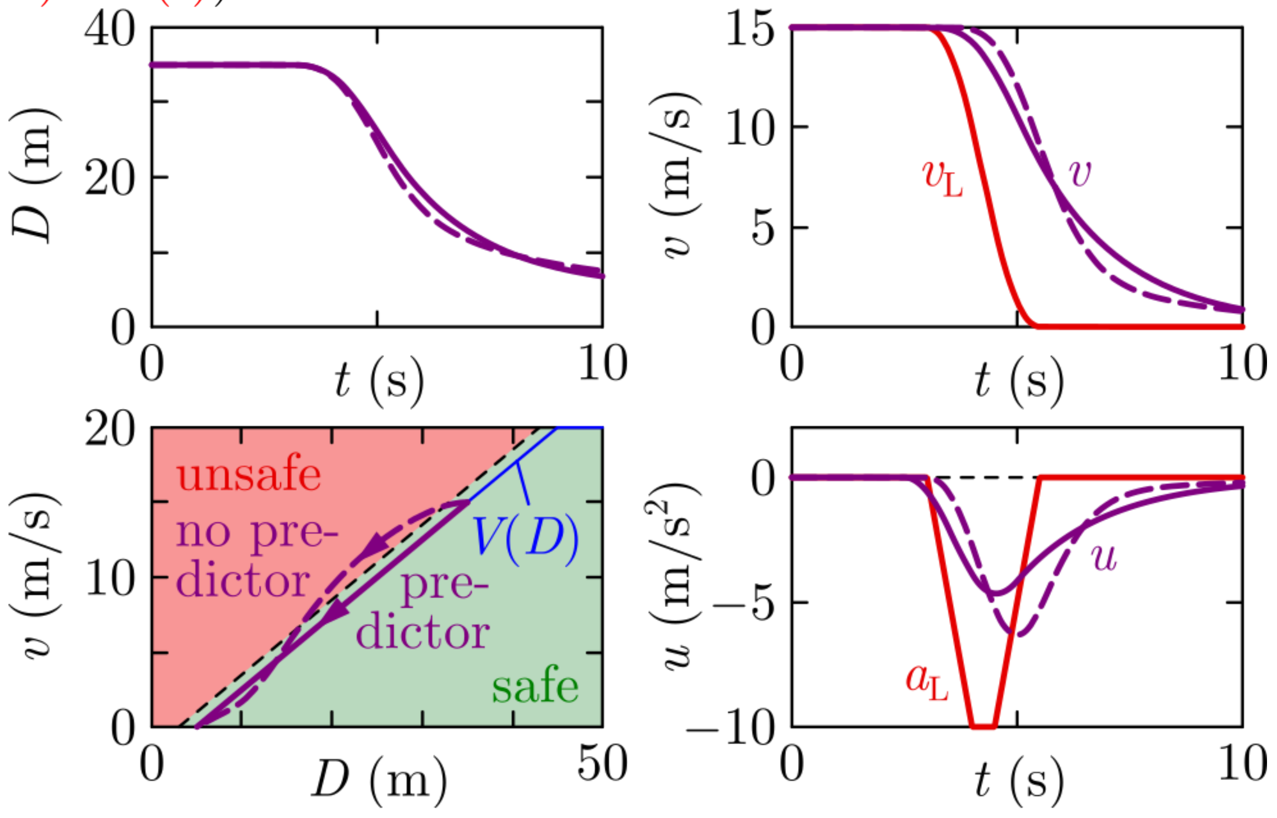
Safety measure:

$$h(x) = D - D_{\rm sf} - Tv$$

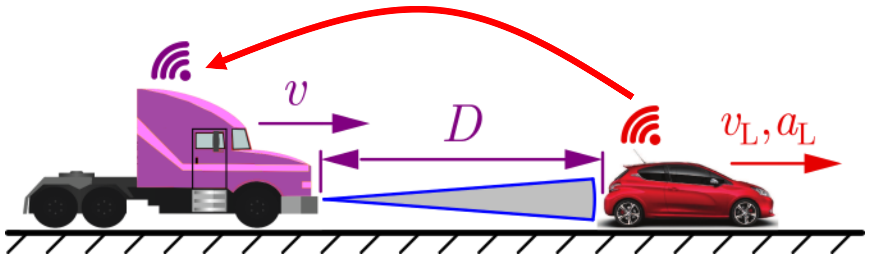
Safety critical control:

$$\dot{h}(t_{\rm p}, x_{\rm p}, u) \geq -\alpha h(x_{\rm p})$$

$$\hat{u} = \alpha\left(V(D_{\rm p}) - v_{\rm p}\right) + \frac{1}{T}(\hat{v}_{\rm L,p} - v_{\rm p}) = u + d$$



Robust Safety Critical Control is Achieved



Connected Cruise Control

Dynamical model:

$$\left.\begin{aligned} \dot{D} &= v_\mathrm{L} - v \\ \dot{v} &= u(t-\tau) + d(t) \\ \dot{v}_\mathrm{L} &= a_\mathrm{L}(t) \end{aligned}\right\} \quad \dot{x} = f(t,x) + g(x)\big(u(t-\tau) + d(t)\big)$$

Disturbance (unmodelled 1st order lag):

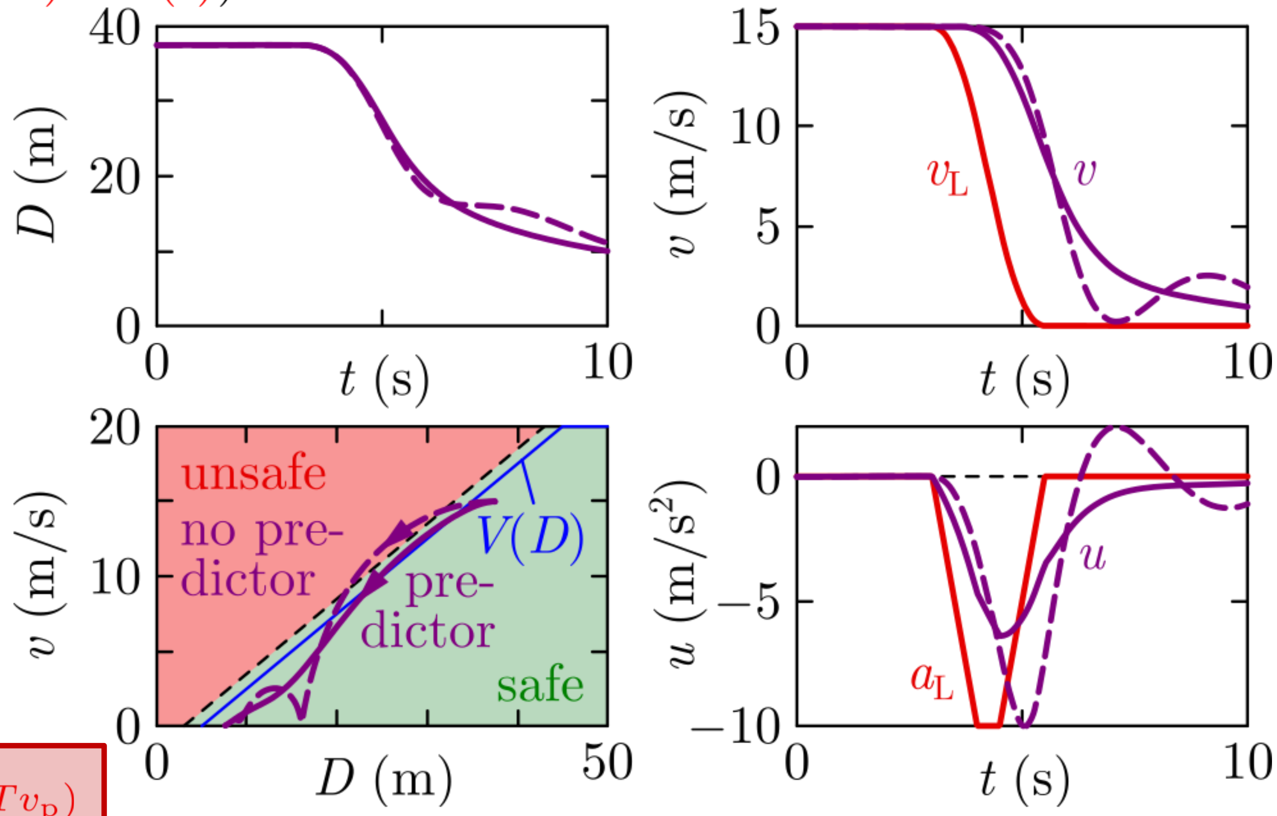
$$\dot{a}(t) = \frac{1}{\xi}\big(-a(t) + u(t-\tau)\big),$$

$$d(t) = a(t) - u(t-\tau)$$

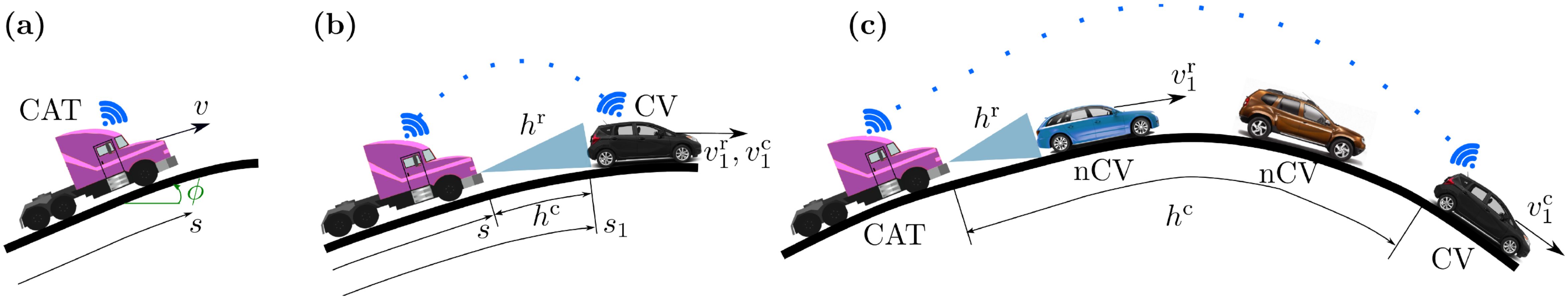
Safety critical control:

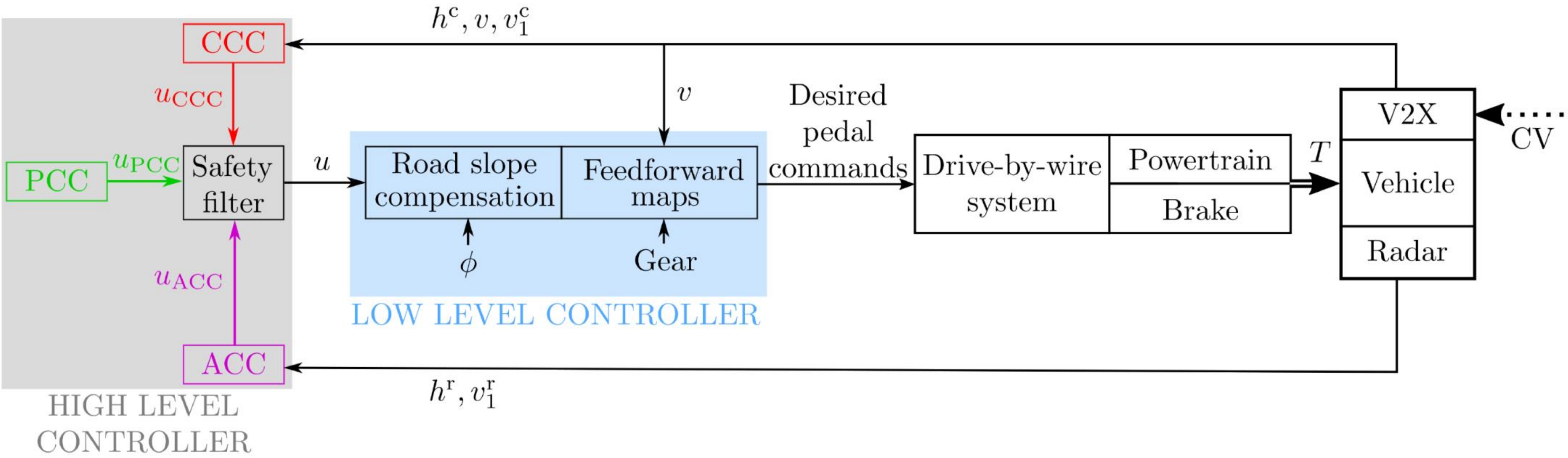
$$\dot{h}(t_\mathrm{p}, x_\mathrm{p}, u) \geq -\alpha h(x_\mathrm{p}) + \sigma\big(h(x_\mathrm{p})\big)\|\nabla h(x_\mathrm{p}) g(x_\mathrm{p})\|^2$$

$$u = \alpha\big(V(D_\mathrm{p}) - v_\mathrm{p}\big) + \frac{1}{T}(\hat{v}_\mathrm{L,p} - v_\mathrm{p}) - T\sigma_0 \mathrm{e}^{-\lambda(D_\mathrm{p} - D_\mathrm{sf} - T v_\mathrm{p})}$$



Real-world Experiments



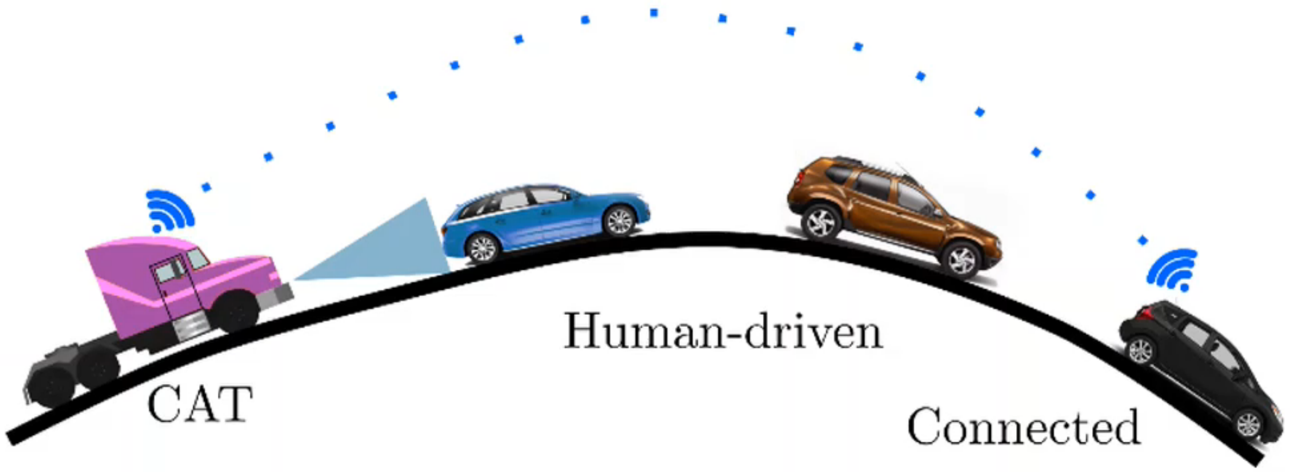


$$u = \min\{u_{\mathrm{PCC}}, u_{\mathrm{ACC}}, u_{\mathrm{CCC}}\}$$

Safe Controller Integration For a Connected Automated Truck (CAT)

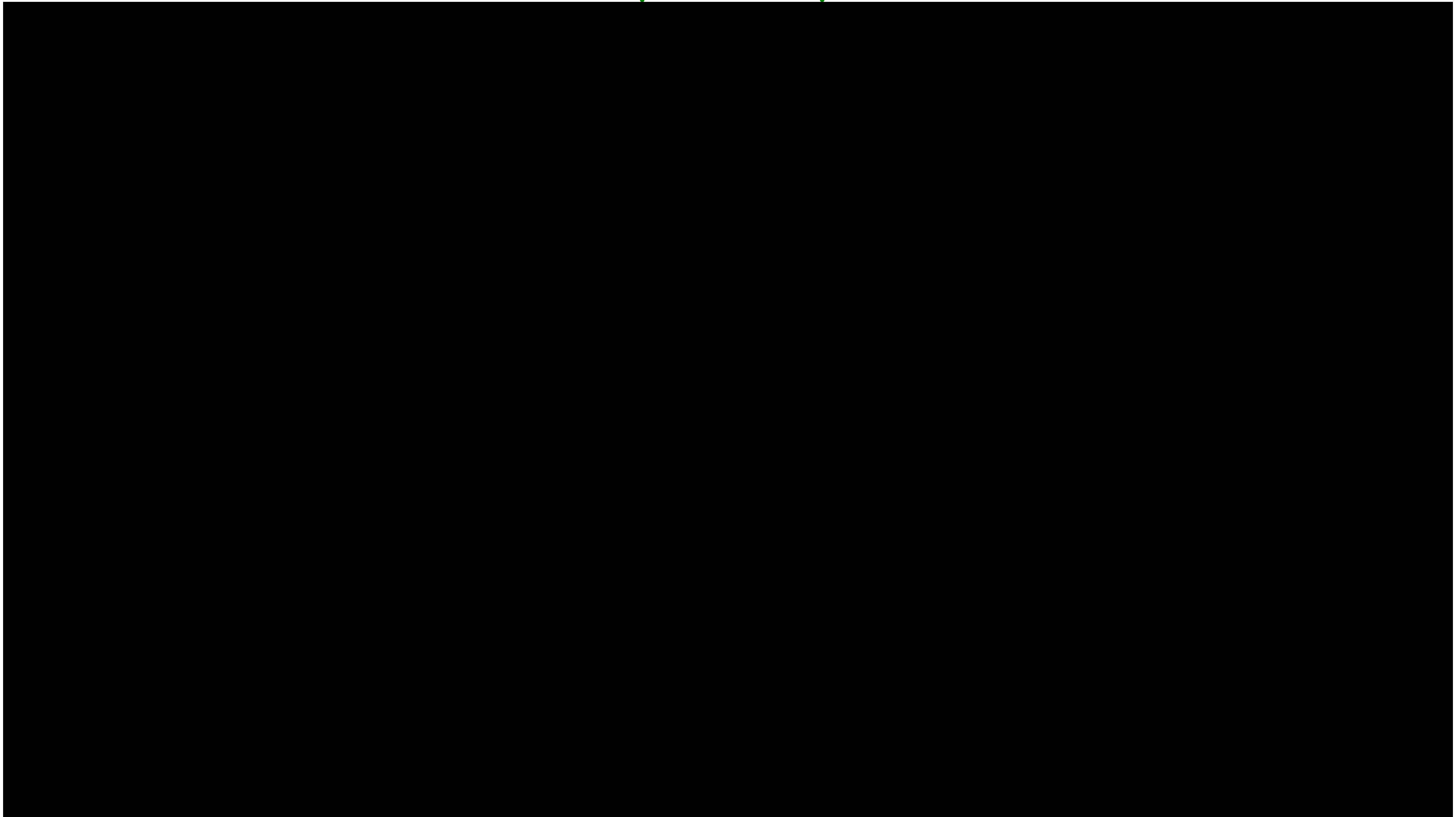
UNIVERSITY OF MICHIGAN

NAVISTAR



Because Traffic has a Memory

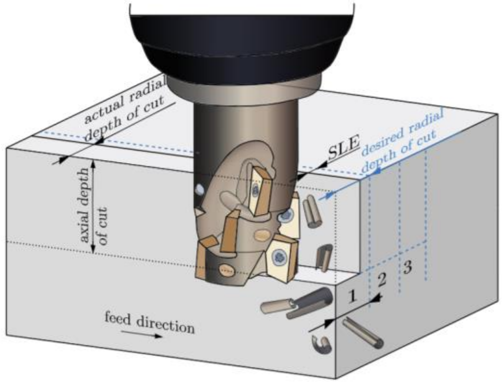
Input Delay



State Delay

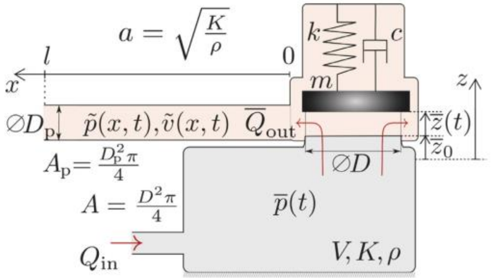
$$\dot{x}(t) = f\big(x(t), \boxed{x(t-\tau)}\big) + g\big(x(t), \boxed{x(t-\tau)}\big)u(t)$$

$$h(x(t), \boxed{x(t-\tau)}) \geq 0$$

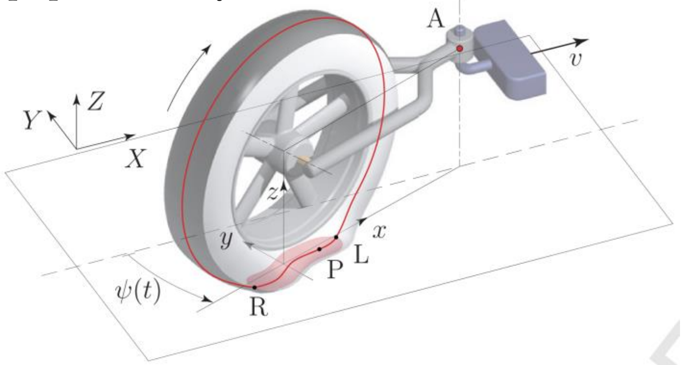


machine tool vibrations

population dynamics



hydraulic systems

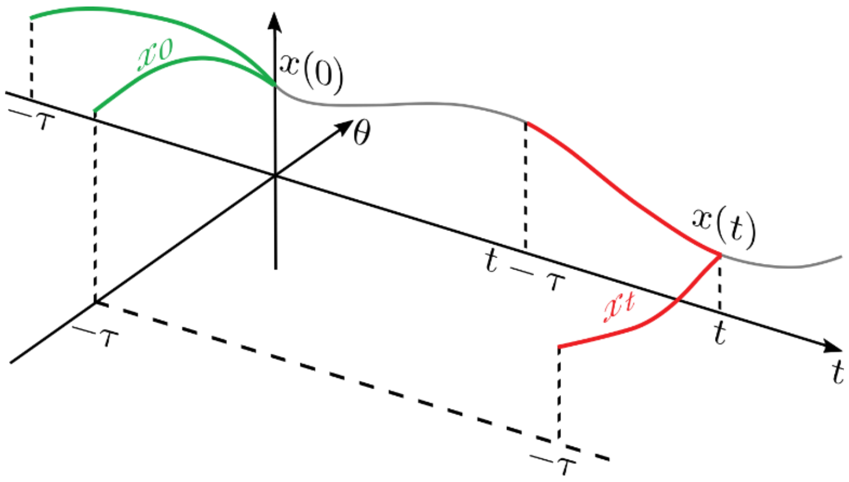


tire dynamics

State Delay

$$\dot{x}(t) = f\big(x(t), x(t-\tau)\big) + g\big(x(t), x(t-\tau)\big)u(t)$$

$$h(x(t), x(t-\tau)) \geq 0$$



Infinite dimensional state space

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0]$$

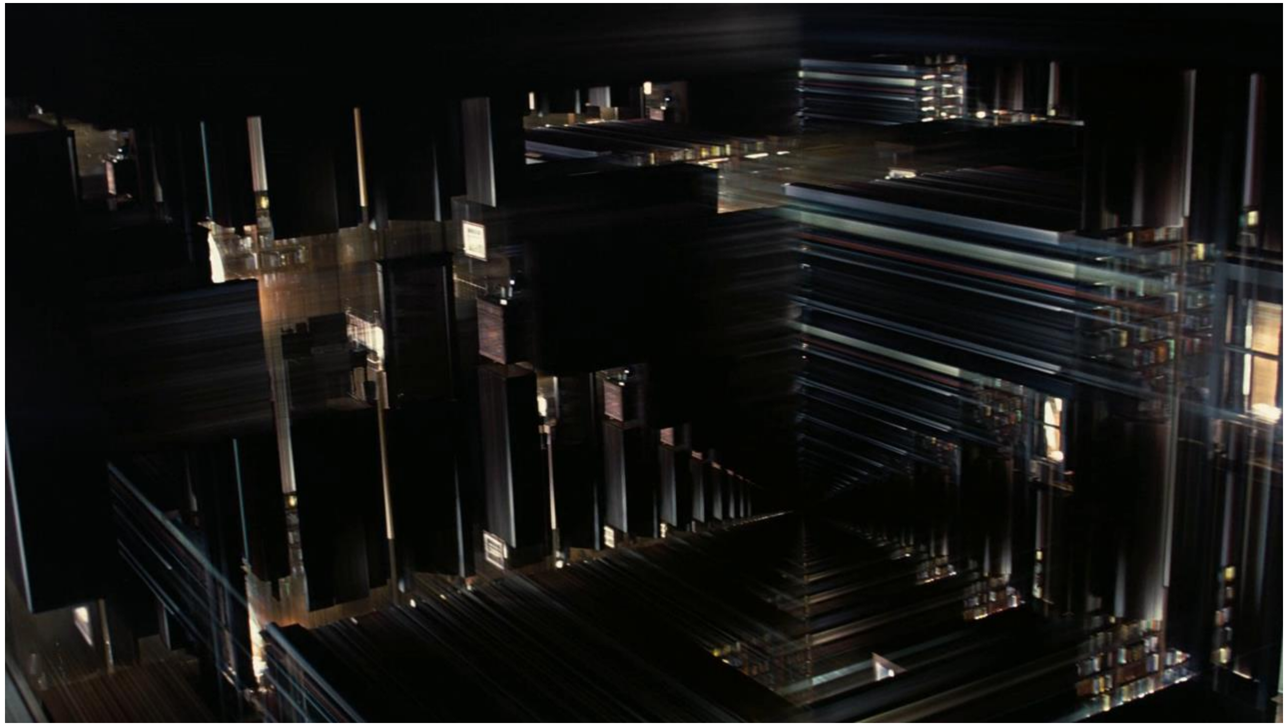
$$x_t \in \mathcal{B} = C([-\tau, 0], \mathbb{R}^n)$$

Functional differential equation

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t)$$

Control Safety Functional

$$\mathcal{S} = \{x_t \in \mathcal{B} : \mathcal{H}(x_t) \geq 0\}$$



Control Barrier Functionals

Nondelayed case

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

$$x \in \mathbb{R}^n$$
$$f : \mathbb{R}^n \to \mathbb{R}^n$$
$$g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

Safe set:

$$S = \{x \in \mathbb{R}^n : h(x) \geq 0\}$$

Theorem:

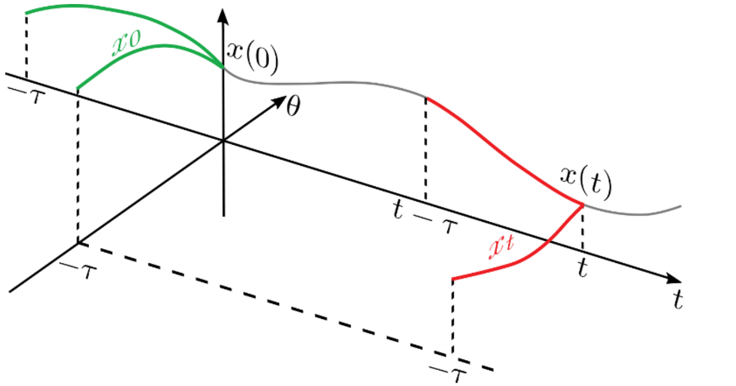
$$u = k(x)$$

$$\dot{h}(x, u) \geq -\alpha h(x)$$

Control synthesis:

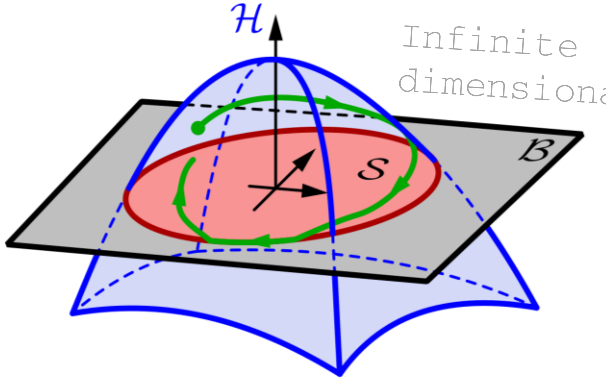
$$k(x) = \underset{u \in U}{\text{argmin}} \quad \|u - u_{\text{d}}\|^2$$
$$\text{s.t.} \quad \dot{h}(x, u) \geq -\alpha h(x)$$

Delayed case



$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t)$$

$$x_t \in \mathcal{B} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$$
$$\mathcal{F} : \mathcal{B} \to \mathbb{R}^n$$
$$\mathcal{G} : \mathcal{B} \to \mathbb{R}^{n \times m}$$



Safe set:

$$\mathcal{S} = \{x_t \in \mathcal{B} : \mathcal{H}(x_t) \geq 0\}$$

Theorem:

$$u = \mathcal{K}(x_t, \dot{x}_t)$$

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) \geq -\alpha \mathcal{H}(x_t)$$

Control synthesis:

$$\mathcal{K}(x_t, \dot{x}_t) = \underset{u \in U}{\text{argmin}} \quad \|u - u_{\text{d}}\|^2$$
$$\text{s.t.} \quad \dot{\mathcal{H}}(x_t, \dot{x}_t, u) \geq -\alpha \mathcal{H}(x_t)$$

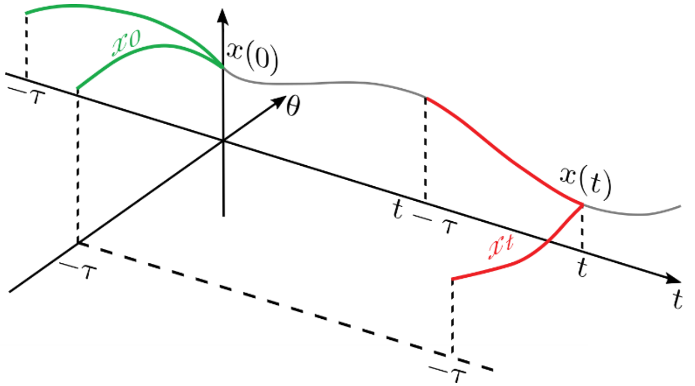
How to calculate $\dot{\mathcal{H}} = ?$

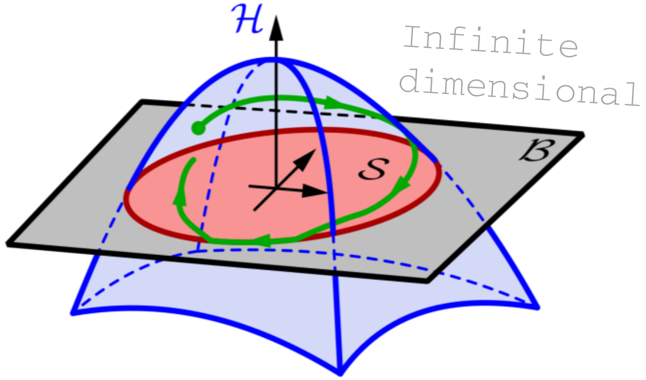
Control Barrier Functionals

Theorem

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) = \int_{-\tau}^{0} \mathrm{d}_\theta \eta(x_t, \theta) \cdot \dot{x}_t(\theta)$$

Nonlinear functional of x_t
and linear functional of $\dot{x}_t$





Example

$$\mathcal{H}(x_t) = h\left(x_t(0), x_t(-\tau), \int_{-\sigma_1}^{-\sigma_2} \rho(\vartheta)\, \kappa\big(x_t(\vartheta)\big)\, \mathrm{d}\vartheta \right)$$

headpoint　　point delay　　distributed delay

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) = \underbrace{\nabla_0 h(.)}_{w_0(x_t)} \cdot \big(\mathcal{F}(x_t) + \mathcal{G}(x_t) u\big) + \underbrace{\nabla_1 h(.)}_{w_1(x_t)} \cdot \dot{x}_t(-\tau) + \int_{-\sigma_1}^{-\sigma_2} \underbrace{\nabla_2 h(.) \rho(\vartheta) \nabla \kappa\big(x_t(\vartheta)\big)}_{w_\mathrm{d}(x_t, \vartheta)} \cdot \dot{x}_t(\vartheta)\, \mathrm{d}\vartheta$$

Closed Loop Safe System

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t)$$

No delay in safety condition

$$h(x(t)) \geq 0$$

$$\mathcal{S} = \{x_t \in \mathcal{B} : \mathcal{H}(x_0) \geq 0\}$$

Control law:

$$u = \mathcal{K}(x_t) = k(x(t), x(t-\tau))$$

Functional differential equation:

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)\mathcal{K}(x_t)$$

Delay in safety condition

$$h(x(t), x(t-\tau)) \geq 0$$

$$\mathcal{S} = \{x_t \in \mathcal{B} : \mathcal{H}(x_t) \geq 0\}$$

Control law:

$$u = \mathcal{K}(x_t, \dot{x}_t) = k(x(t), x(t-\tau), \dot{x}(t-\tau))$$

Neutral Functional differential equation:

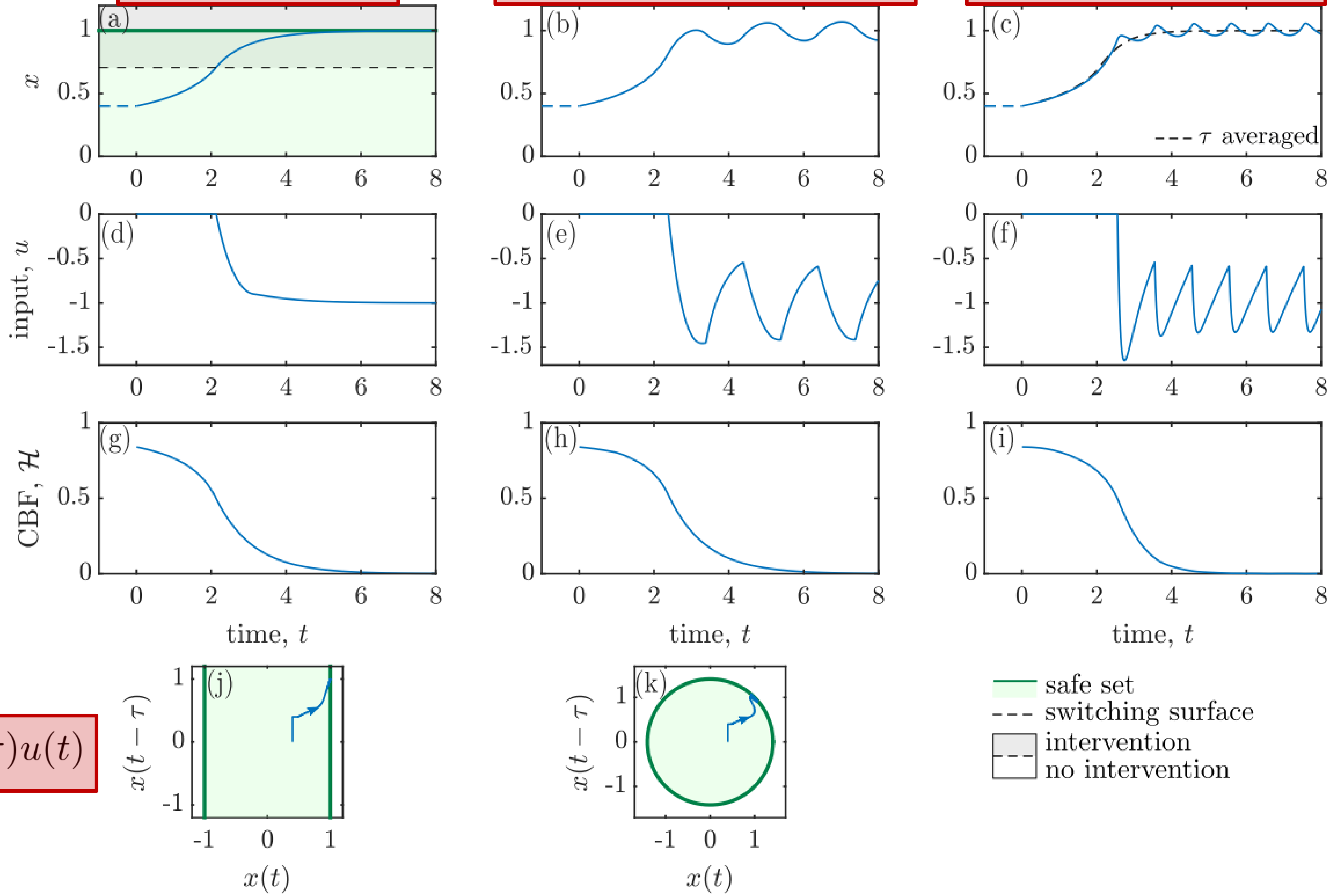
$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)\mathcal{K}(x_t, \dot{x}_t)$$

Example

$\mathcal{H}(x_t) = 1 - x^2(t)$

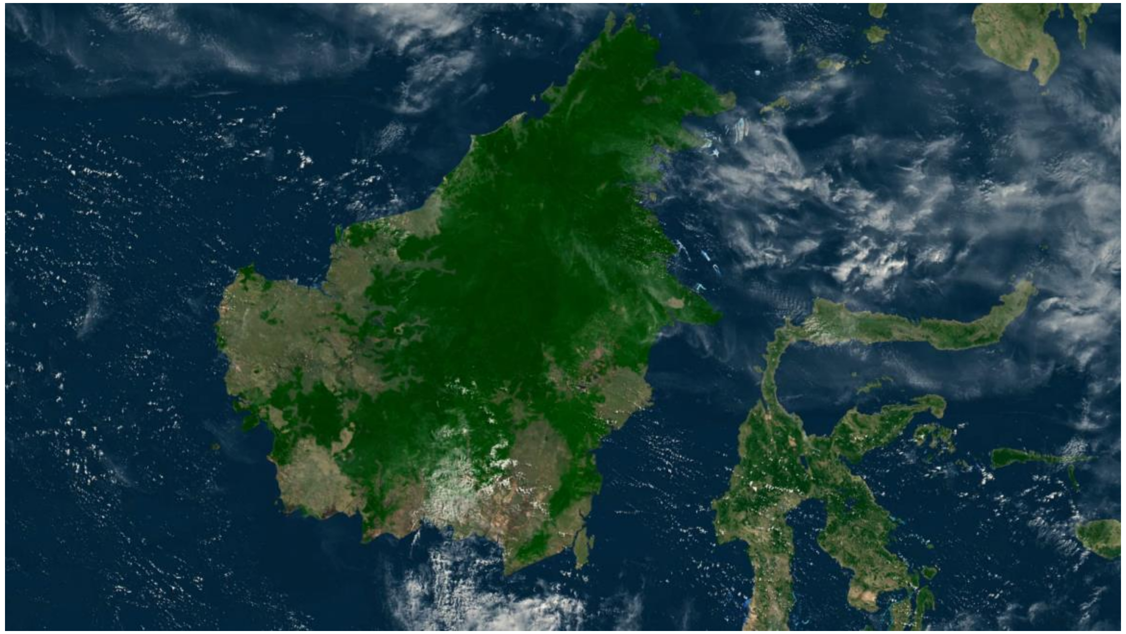
$\mathcal{H}(x_t) = 1 - \frac{1}{2}\left(x^2(t) + x^2(t-\tau)\right)$

$\mathcal{H}(x_t) = 1 - \frac{1}{\tau}\int_{-\tau}^{0} x_t^2(\vartheta)\,\mathrm{d}\vartheta$



$$\dot{x}(t) = x(t)^3 + x(t-\tau)u(t)$$

Many Ecosystems are at the Verge of Safety

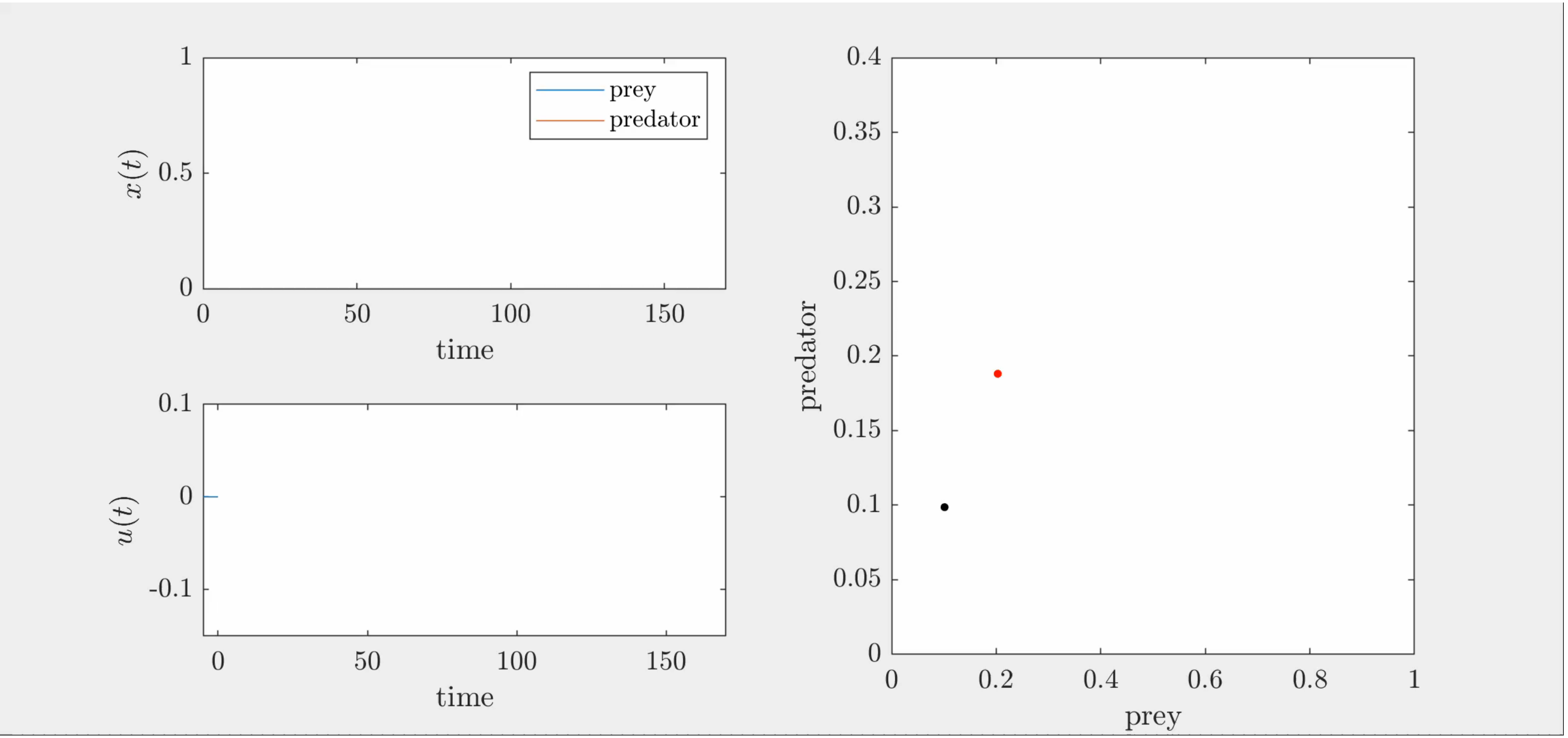


2019 David Attenborough: A Life on Our Planet

Population Dynamics

delayed predator-prey model

$$\underbrace{\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} ry(t) - ay^2(t) - py(t)z(t) \\ bpy(t-\tau)z(t-\tau) - dz(t) - mz^2(t) \end{bmatrix}}_{f(x(t),x(t-\tau))}$$



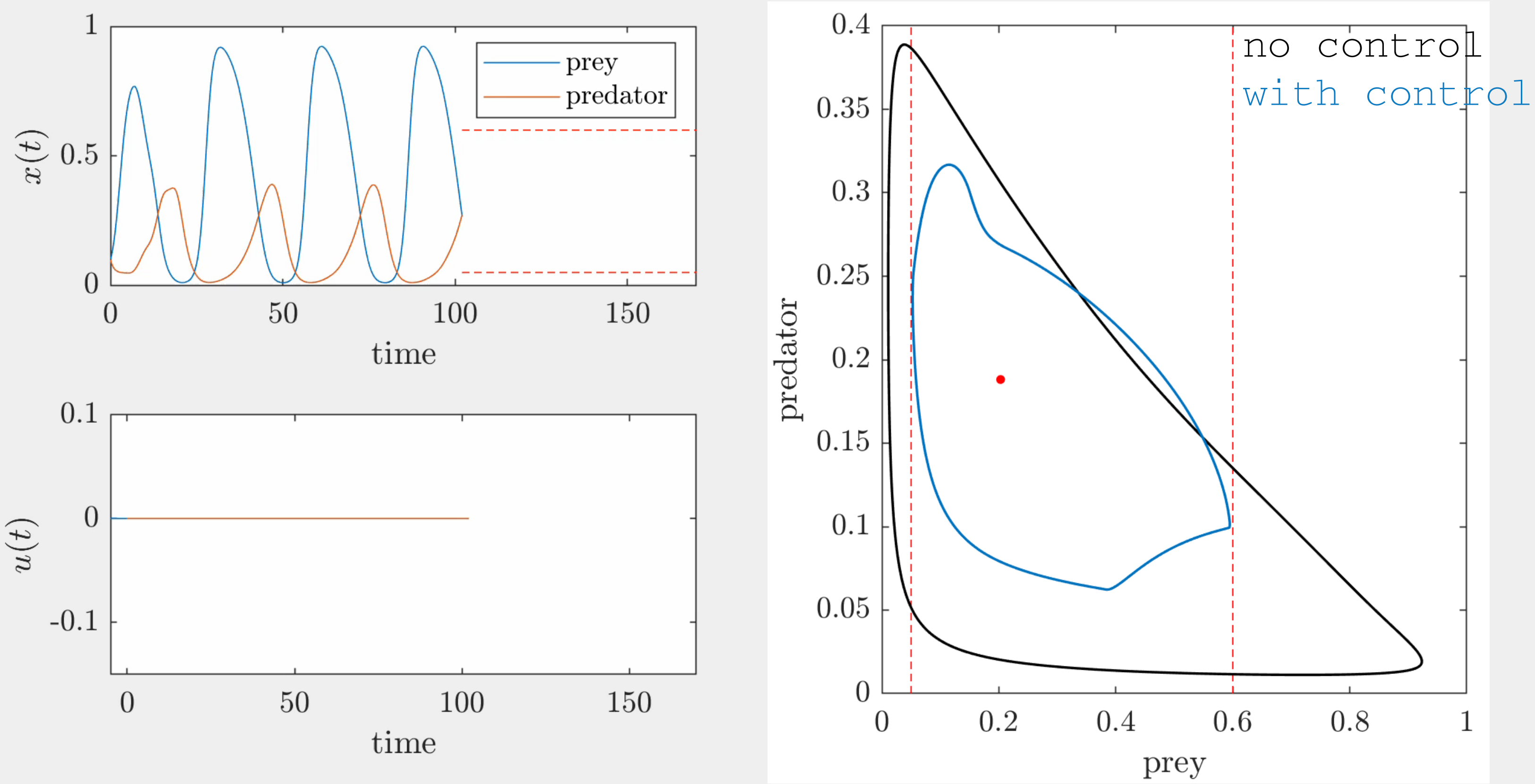
y	prey population
z	predator population
r	prey growth rate
a	self-regulation rate
p	predation rate
b	conversion rate
d	predator mortality
m	predators competition
τ	maturation time

Population Dynamics

delayed predator-prey model

$$\underbrace{\begin{bmatrix}\dot{y}(t)\\ \dot{z}(t)\end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} ry(t) - ay^2(t) - py(t)z(t) \\ bpy(t-\tau)z(t-\tau) - dz(t) - mz^2(t)\end{bmatrix}}_{f(x(t),x(t-\tau))} + \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{g} u(t)$$

$$u(t) = K(x(t), x(t-\tau))$$



y	prey population
z	predator population
r	prey growth rate
a	self-regulation rate
p	predation rate
b	conversion rate
d	predator mortality
m	predators competition
τ	maturation time

Collaborators


Anil Alan


Chaozhe He


Harvey Bell


Gábor Orosz

UNIVERSITY OF MICHIGAN


Andrew Taylor


Tamás Molnár


Aaron Ames

Caltech


Ádám Kiss

Budapest University of Technology