Basic algorithms and recent advances on safe control synthesis with reachability and invariance

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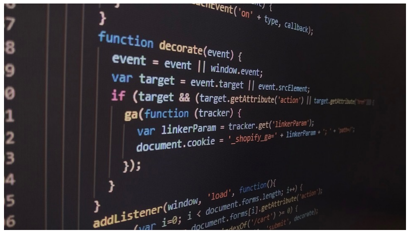
Mostly based on joint work with **Liren Yang** (Huazhong University of Science and Technology)

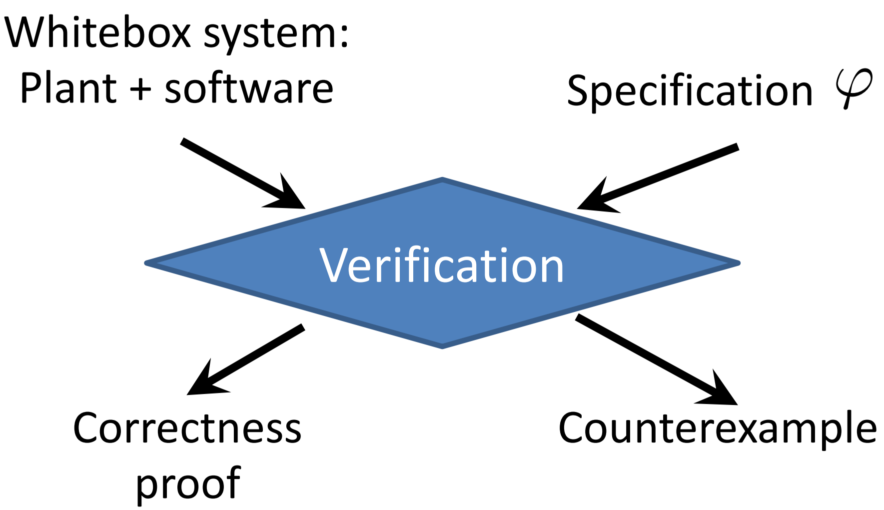
Research funded in part by

V&V landscape

Plant
(with a model)



Autonomy
software



PROS:

- Strong guarantees

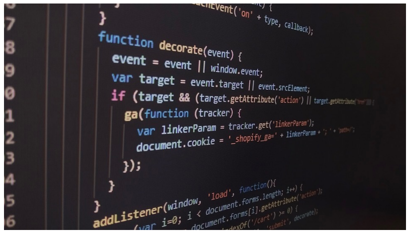
CONS:

- Hybrid system verification is computationally very hard
- Autonomy software ➔ up to millions of lines of code (loc):
  - hard to model
  - hard to scale

Mars curiosity rover: 5M loc
Boeing 787 flight software: 15M loc
F 35-fighter jet: 25M loc
Average modern high-end car: 100M loc

V&V landscape

+

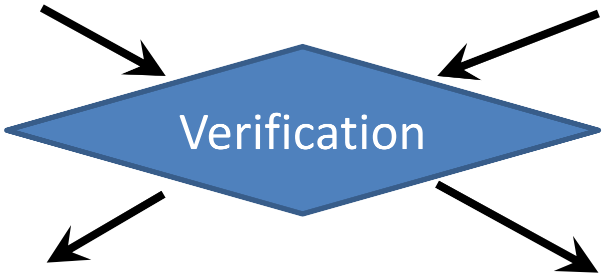


Plant
(with a model)

Autonomy
software

Whitebox system:
Plant + software

Specification φ



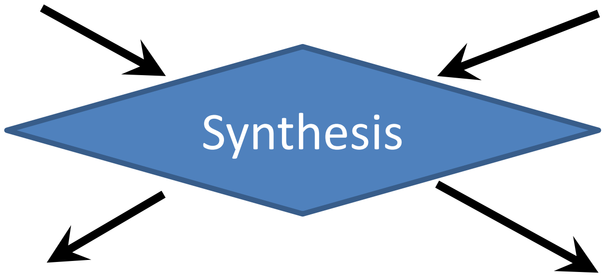
Correctness
proof

Counterexample

Alternative #1: Correct-by-construction (control) synthesis

Partial whitebox system:
Plant (∃? Software)

Specification



Software with
correctness guarantee
+ validity domain

Proof of
impossibility

PROS:

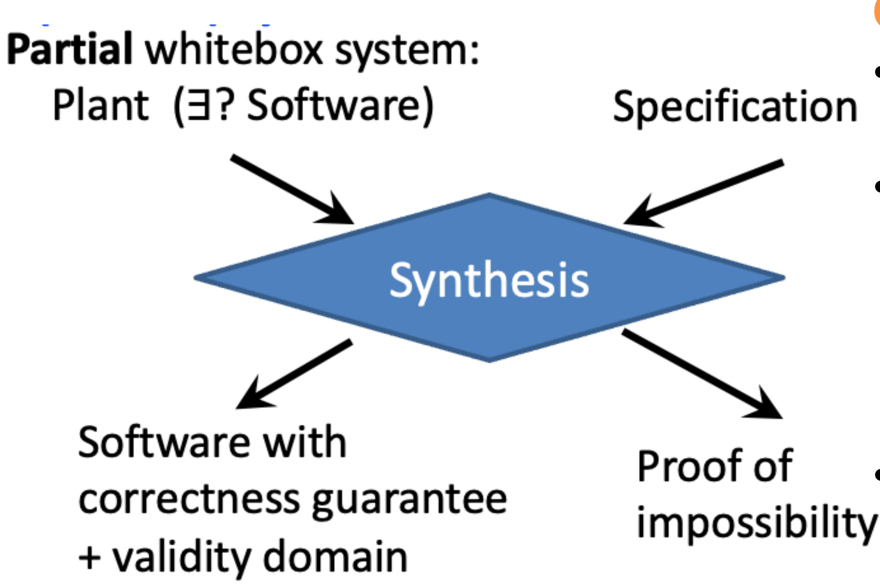
- Strong guarantees
- Avoids the complexity induced by software
- “Explains” fundamental limits (impossibility)

CONS:

- Correct-by-construction synthesis is computationally even harder
- Limited specs; spec correctness & completeness is more crucial
- Almost no synthesis approach for perception or learning components

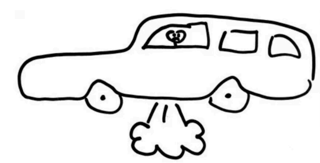
Deployment of synthesized controllers in Mcity

Alternative #1: Correct-by-construction (control) synthesis



What we learned from early deployments?

- Putting "correct" and automatically synthesized software on a car is feasible
- There were failures but having mathematical models and formal assumptions help detect failures
  - We realized that the model was missing actuator delays
- Conservativeness due to not looking ahead



- Motivated future work on safety with learned models, delays, and predictions

Alternative #1: Correct-by-construction (control) synthesis

$$\Sigma\text{: } x(t+1) = Ax(t) + Bu(t-\boldsymbol{\tau_d}) + Fd(t)$$

$$\Sigma_{\text{aug}}\text{: } \begin{cases} x(t+1) = Ax(t) + Bu_1(t) + Fd(t) \\ u_1(t+1) = u_2(t) \\ u_2(t+1) = u_3(t) \\ \quad\quad \dots \\ u_{\boldsymbol{\tau_d}}(t+1) = u(t) \end{cases}$$

$$S_{aug} = S \times U^{\tau_d}$$

- Goal: Find a set $C_{aug} \in S_{aug}$ (ideally the maximal such C_{aug}) such that if $x_{aug} \in C_{aug}$ then there exist $u \in U$ such that $x_{aug}^+ \in C_{aug}$ (C_{aug} is a controlled invariant set).

Alternative #1: Correct-by-construction (control) synthesis

$$\Sigma:\ x(t+1) = Ax(t) + Bu(t-\tau_d) + Fd(t)$$

$$\boxed{\begin{aligned} &T(x, u_1, \ldots, u_{\tau_d}) \\ &= A^{\tau_d}x + \Sigma_{i=1}^{\tau_d} A^{i-1}Bu_{\tau_d - i+1} \end{aligned}}$$

$$\Sigma_{\text{aug}}: \begin{cases} x(t+1) = Ax(t) + Bu_1(t) + Fd(t) \\ u_1(t+1) = u_2(t) \\ u_2(t+1) = u_3(t) \\ \quad\quad \ldots \\ u_{\tau_d}(t+1) = u(t) \end{cases}$$

$$S_{aug} = S \times U^{\tau_d}$$

Σ_{aug}, S_{aug} Dim= $n + m \cdot \tau_d$ → Direct method ✗ → C_{aug}

T

1. Dimension reduction

$$\boxed{\Sigma_{aux}:\ \hat{x}_{\tau_d}(t+1) = A\hat{x}_{\tau_d}(t) + Bu(t) + A^{\tau}Fd(t).}$$

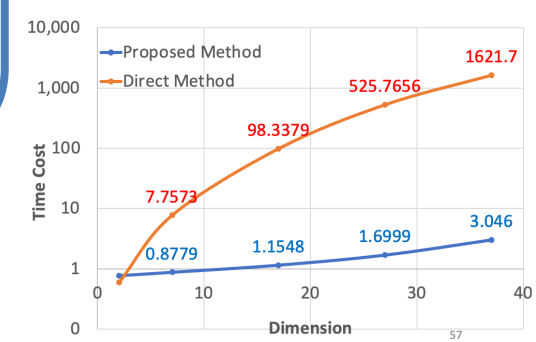
Σ_{aux}, S_{aux} Dim= n → C_{aux} Dim= n $\xrightarrow{T^{-1}}$ C_{lift} Dim= $n + m \cdot \tau_d$

$C_{aug} = C_{lift}$

2. Compute maximal invariant set of Σ_{aux}

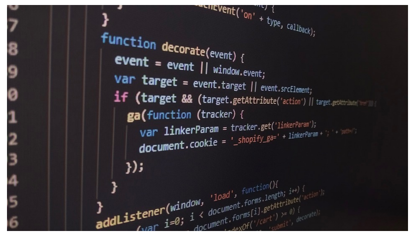
3. Lifting

Theorem: *Any controlled invariant set of the $n + m \cdot \tau_d$ dimensional system Σ_{aug} can be obtained from that of the n dimensional auxiliary system Σ_{aux}.*



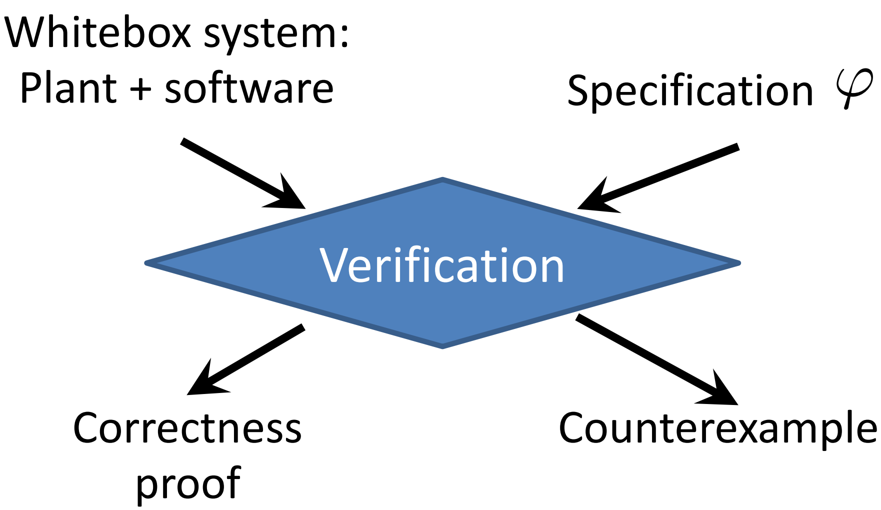
V&V landscape

+

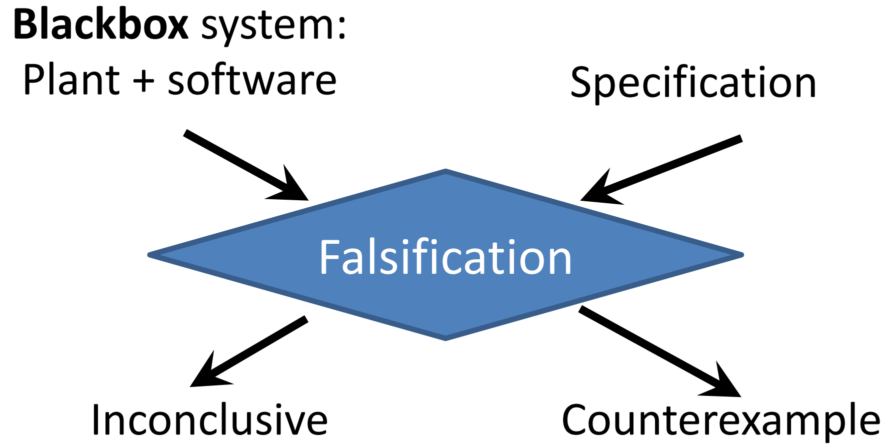


Plant
(with a model)

Autonomy
software



Alternative #2: Falsification



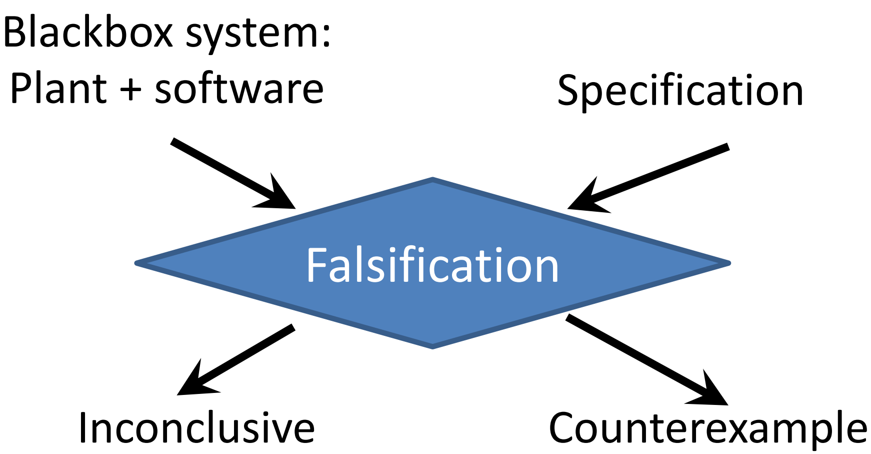
PROS:

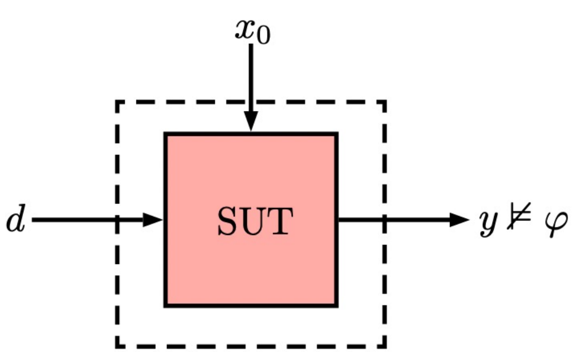
- Can handle arbitrarily complex models (plant + software) including learning-based components
- Industry-adopted tools (e.g., Breach, S-Taliro)

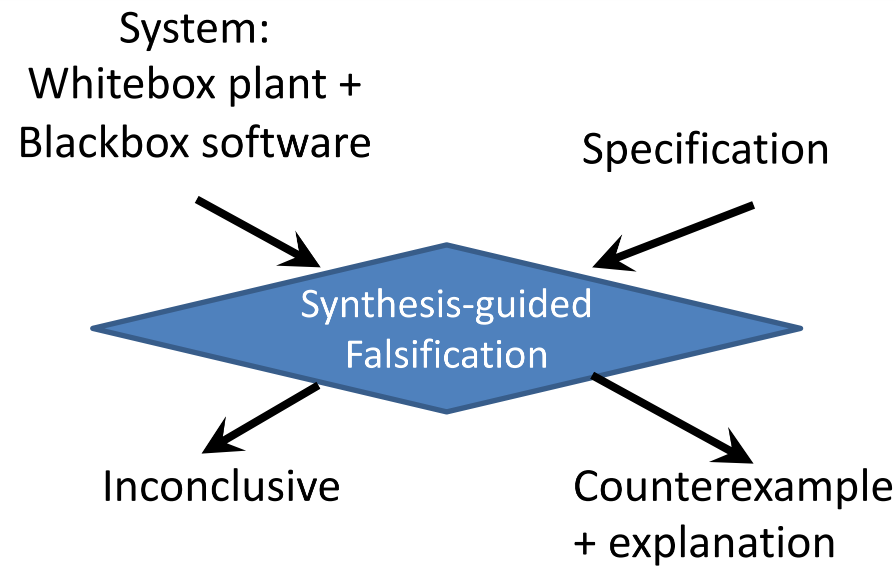
CONS:

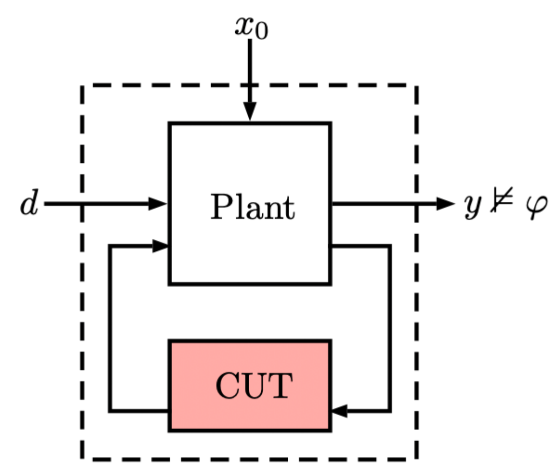
- Weaker conclusions
- No explanation of the counterexamples:
  - can give "trivial" counterexamples if assumptions are not modeled carefully
  - is it a hardware (plant) limitation or software bug?

Alternative #2.5: Synthesis-guided falsification



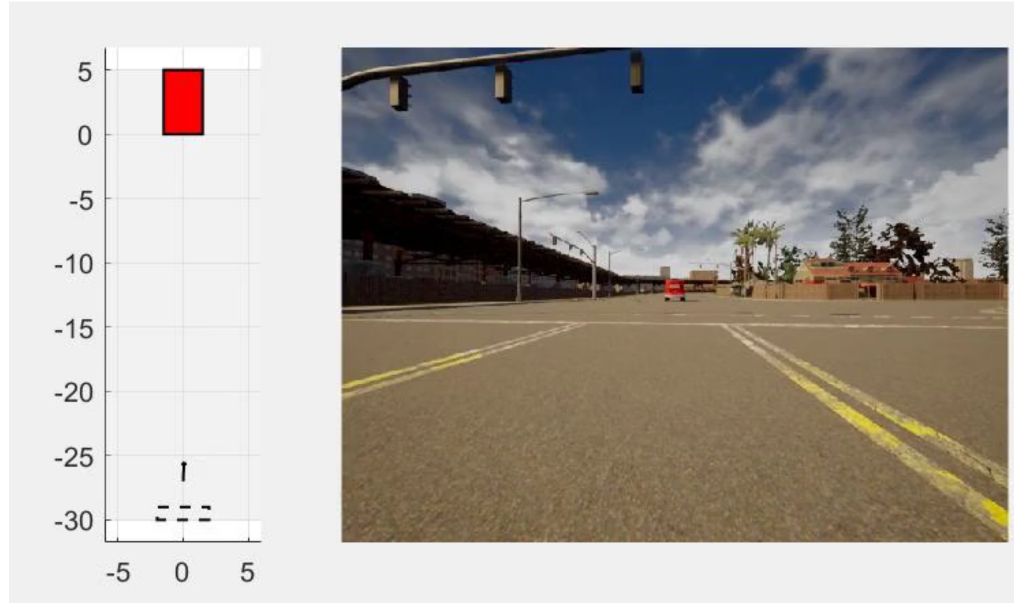


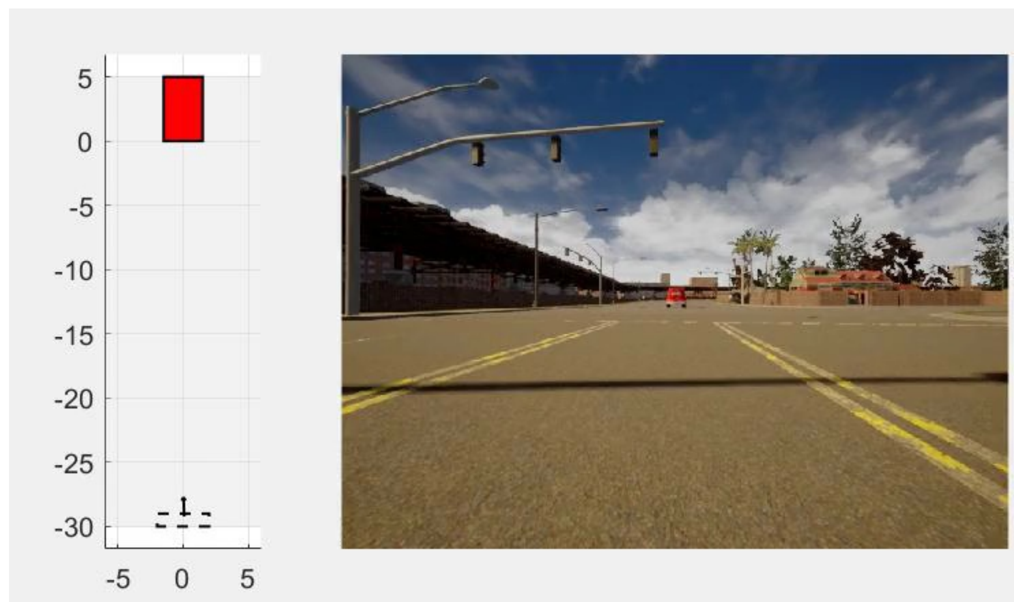




* SUT: system under test; CUT: controller (autonomy software) under test

a typical result with our falsification algorithm





Underlying tool in verification:

Forward reachable sets (FRS)

Underlying tool in synthesis and synthesis-guided falsification:

Backward reachable sets (BRS)

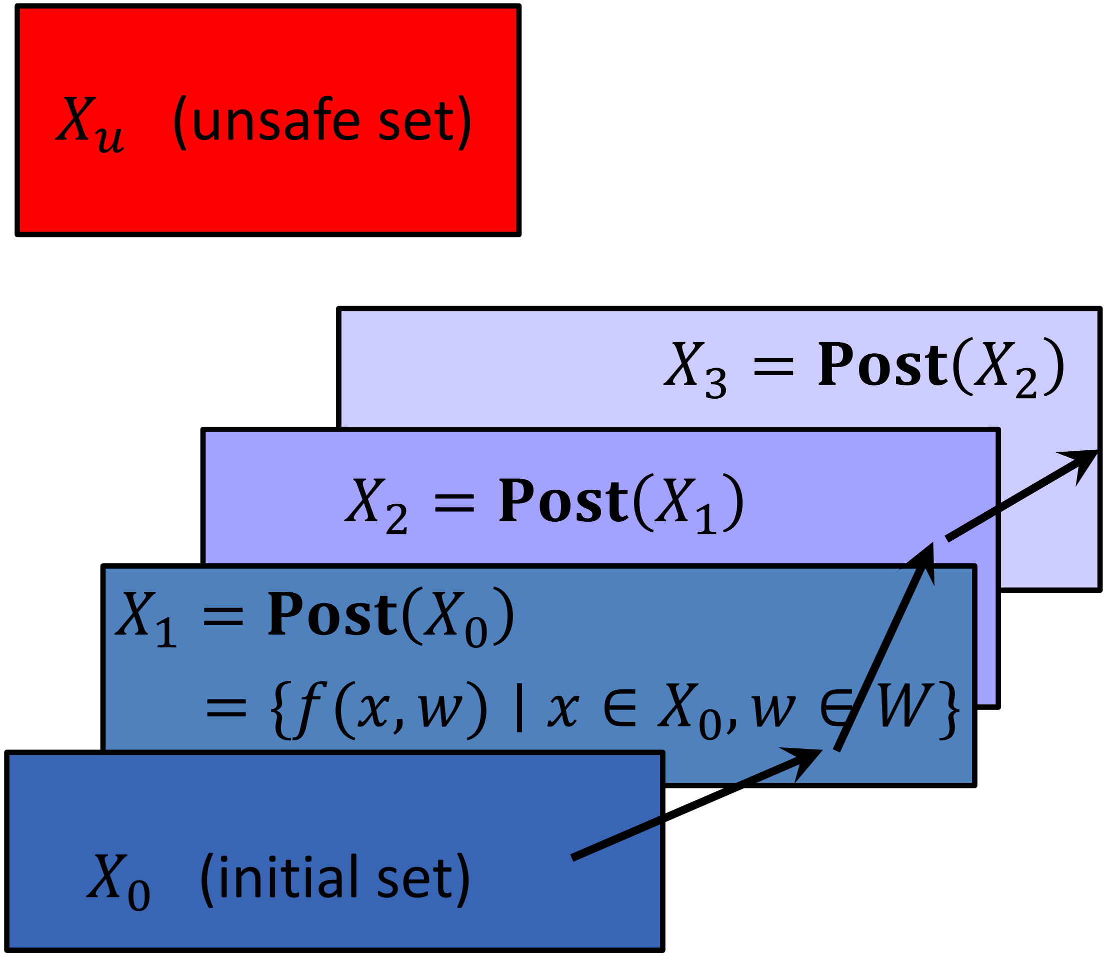
Forward reachable sets

Closed-loop system:

$$x_{t+1} = f(x_t, w_t)$$
$$w_t \in W$$

Typical **verification** problem:

Given a set of initial states X_0, an unsafe set X_u, and a time horizon T, prove or disprove that for all $x_0 \in X_0$, for all $t \in [0, T]$ and for all $w_{0:T-1} \in W^T$, we have $x_t \notin X_u$.



Reachability [Girard 2005, Kurzhansky & Varaiya 2011]
Also used in constructing symbolic models (abstractions)

Forward reachable sets

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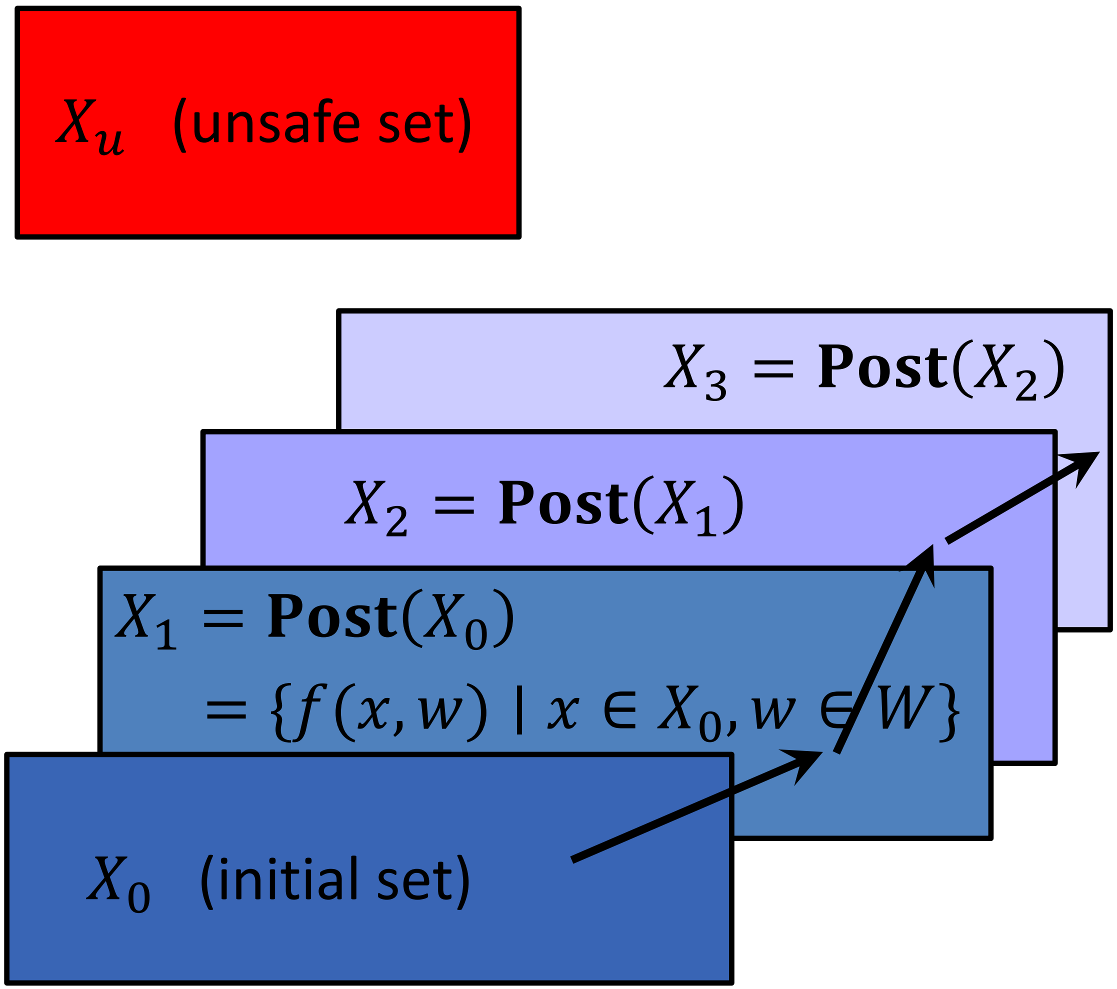
Outer-approximations can be used to prove safety



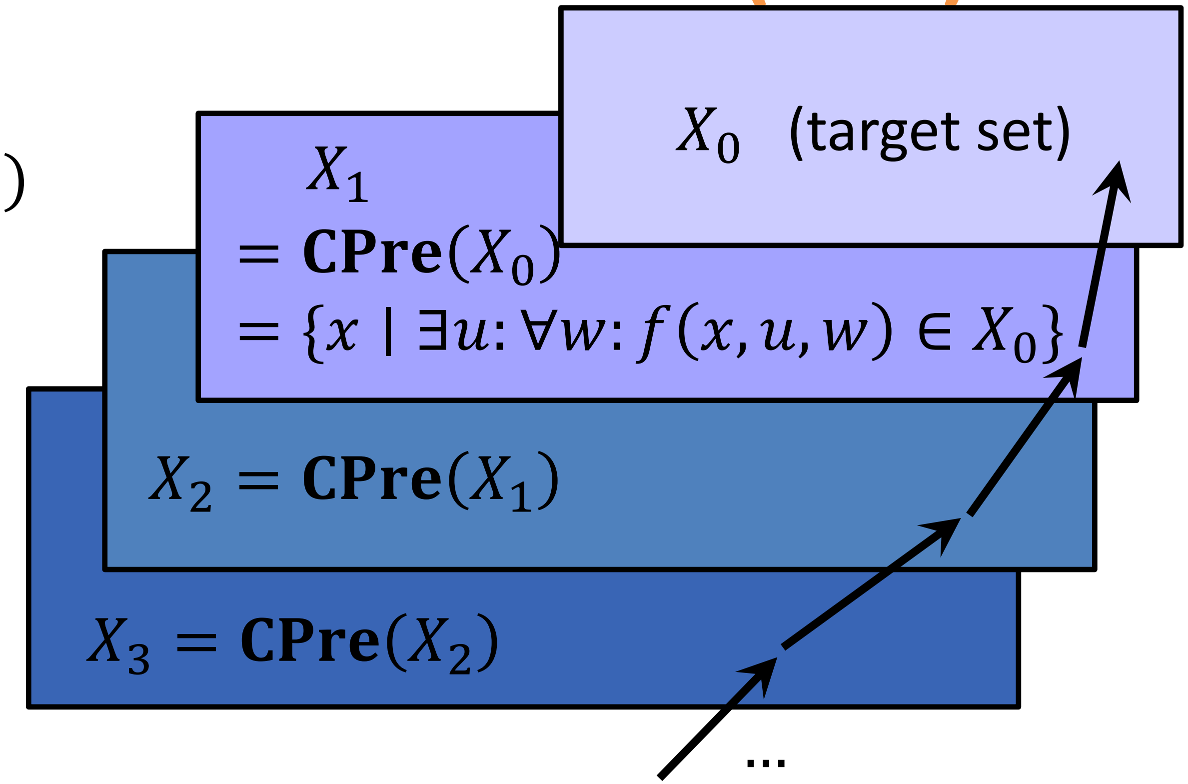
Backward Reachable Sets (BRS)

Control system:

$$x_{t+1} = f(x_t, u_t, w_t)$$

$$u_t \in U$$

$$w_t \in W$$



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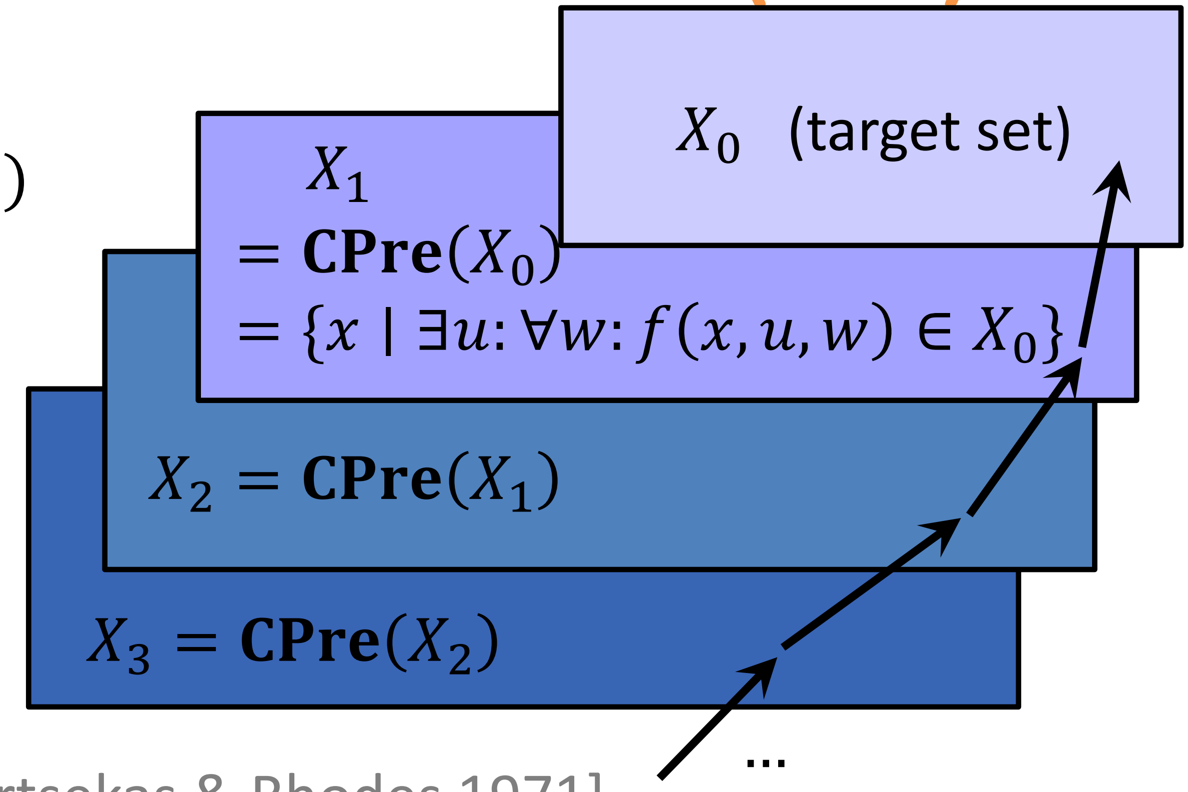


- Specification:
  - Reachability [Bertsekas & Rhodes 1971]
  - Safety [Bertsekas 1972]
  - Temporal logic spec [Chen et al. 2018]

Building block: BRS computation

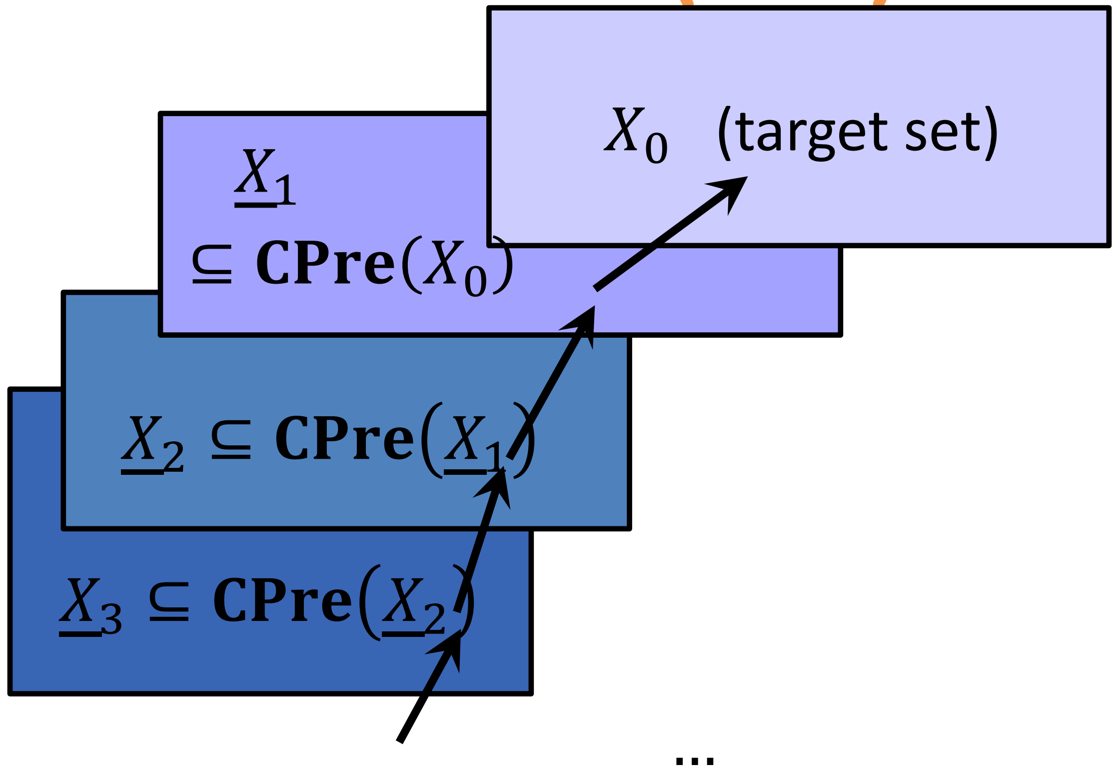
Backward Reachable Sets (BRS)

Control system:

$$x_{t+1} = f(x_t, u_t, w_t)$$

$$u_t \in U$$

$$w_t \in W$$



- Specification:
  - Reachability [Bertsekas & Rhodes 1971]
  - Safety [Bertsekas 1972]
  - Temporal logic spec [Chen et al. 2018]

Inner-approximations can be used for correct-by-construction control

For linear systems

Closed-loop system:

$$x_{t+1} = Ax_t + Bw_t$$
$$w_t \in W$$

$$\mathbf{Post}(X) = \{\, Ax + Bw \mid x \in X, w \in W \,\}$$
$$= AX \oplus BW$$

Control system:

$$x_{t+1} = Ax_t + Bu_t + Ew_t$$
$$u_t \in U$$
$$w_t \in W$$

$$\mathbf{CPre}(X) = \{x \mid \exists u : \forall w : Ax + Bu + Ew \in X\}$$
$$= \mathbf{Proj}_x \{(x, u) \mid Ax + Bu \oplus EW \subseteq X\}$$

If A is invertible:

$$= A^{-1}(X \ominus EW \oplus -BU)$$

What is needed for reachability?

- Set representations (and their complexity)
- Operations on the sets:
  - Affine transformation
  - Projection
  - Intersection
  - Minkowski sum
  - Emptiness check
  - Membership check

Some convex set representations

hyperplane: $P_{hp} = \{x \mid a^T x = b\} \subseteq \mathbb{R}^n$, where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$

halfspace: $P_{hs} = \{x \mid a^T x \leq b\} \subseteq \mathbb{R}^n$, where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$

polyhedron: $P = \{x \mid a_j^T x \leq b_j,\ j = 1, \ldots, m,\ c_i^T x = d_i,\ i = 1, \ldots, p\}$, alternatively, in matrix form $P = \{x \mid Ax \leq b,\ Cx = d\}$, or, equivalently $P = \{x \mid \bar{A}x \leq \bar{b}\}$,

Some convex set representations

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zonotope: $Z = \{x \in \mathbb{R}^n \mid x = c + \sum_{i=1}^p \alpha_i g_i,\ -1 \leq \alpha_i \leq 1\}$, where $c, g_1, \ldots, g_p \in \mathbb{R}^n$. The point c is called the *center* of Z; $g_1, \ldots, g_p$ are called the generators of Z. We denote a zonotope as $Z = (c, < g_1, \ldots, g_p >)$.

hyperbox: $H = \{x \in \mathbb{R}^n \mid x_i \in [l_i, u_i],\ i = 1, \ldots, n\}$, where $l_1, \ldots, l_n$ and $u_1, \ldots, u_n$ are real numbers corresponding to lower and upper limits for each coordinate. Hyperboxes are usually denoted as cross-products of intervals, i.e. $H = [l_1, u_1] \times \ldots \times [l_n, u_n]$.

Operations on convex sets

	Intersection $C = C^{(1)} \cap C^{(2)}$	Minkowski sum $C = C^{(1)} \oplus C^{(2)}$	Linear (affine) transformation
Hyperbox $[l_1,u_1] \times \ldots \times [l_n,u_n]$			Not a box!
Zonotope $(c, <g_1, \ldots, g_p>)$	Not a zonotope!	$(c^{(1)} + c^{(2)}, < g_1^{(1)}, .., g_{p_1}^{(1)}, g_1^{(2)}, .., g_{p_2}^{(2)} >)$	$(Lc, <Lg_1, \ldots, Lg_p>)$
Polytope V-rep $\mathrm{Conv}(\{v_1,\ldots,v_k\})$			
Polytope H-rep $Ax \leq b$			

Operations on convex sets

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Hyperbox $[l_1,u_1] \times \ldots \times [l_n,u_n]$	simple min-max: $l_i = \max(l_i^{(1)}, l_i^{(2)})$ $u_i = \min(u_i^{(1)}, u_i^{(2)})$	simple algebra: $l_i = l_i^{(1)} + l_i^{(2)}$ $u_i = u_i^{(1)} + u_i^{(2)}$	Not a box!
Zonotope $(c, <g_1, \ldots, g_p>)$	Not a zonotope!	$(c^{(1)} + c^{(2)}, < g_1^{(1)}, .., g_{p_1}^{(1)}, g_1^{(2)}, .., g_{p_2}^{(2)} >)$	$(Lc, <Lg_1, \ldots, Lg_p>)$
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Zonotope $(c, <g_1, \ldots, g_p>)$	Not a zonotope!	$(c^{(1)} + c^{(2)}, < g_1^{(1)}, .., g_{p_1}^{(1)}, g_1^{(2)}, .., g_{p_2}^{(2)} >)$	$(Lc, <Lg_1, \ldots, Lg_p>)$
Polytope V-rep $\mathrm{Conv}(\{v_1,\ldots,v_k\})$		$\mathrm{Conv}(\{v_i^{(1)} + v_j^{(2)}\}_{i,j})$ ★	$\mathrm{Conv}(\{Lv_1,\ldots,Lv_k\})$
Polytope H-rep $Ax \leq b$	concatenation: ★ $\begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} x \leq \begin{bmatrix} b^{(1)} \\ b^{(2)} \end{bmatrix}$	Outer-approx. when $C^{(2)}$ is ∞-norm ball: $Ax \leq b + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \lvert\lvert A \rvert\rvert_\infty \epsilon$	

Radius of the ball

★ Representations might be redundant, reductions are possible.

In theory do not scale well with dimension n, in practice it is OK.

Operations on convex sets

	Membership check Is point x in C?	Emptiness check Is C the empty set?
Hyperbox $[l_1,u_1] \times \ldots \times [l_n,u_n]$	simple comparisons: $l_i \le x_i \le u_i$ for all i?	Non-empty iff $l_i \le u_i$ for all i
Zonotope $(c, \langle g_1, \ldots, g_p \rangle)$	linear program (LP) poly-time	Can't represent empty sets
Polytope V-rep $\mathrm{Conv}(\{v_1,\ldots,v_k\})$	linear program (LP) poly-time	Can't represent empty sets (trivial empty vertex set)
Polytope H-rep $Ax \le b$	simple algebra: $Ax \le b$?	linear program (LP) poly-time

Additional notes

- Other important operations:
  - Containment check (see Sadraddini & Tedrake'19), Minkowski difference, complexity reduction
- Approximate (inner or outer) set computations when exact operations are hard
- Many other set representations:
  - Constrained zonotopes, hybrid zonotopes, polynomial zonotopes, AH-polytopes, star sets, ellipsoids, …
- Software packages for manipulating sets:
  - Matlab MPT3, Python polytope, Julia JuMP
- Reachability software:
  - FRS: CORA, JuliaReach, SpaceX, dReach, etc. (see https://ieeecss.org/tc/hybrid-systems/tools)
  - BRS: HJB, MPT3

Backward Reachable Set Computation

- Methods/tools (far from being complete):
  - HJB [Mitchell et al. 2007], interval analysis [Li & Liu 2017], polynomial optimization [Lasserre 2015], linear optimization [Blanchini & Miani 2008], etc.

- Challenge: **scalability** (even for linear systems):

$$
\begin{aligned}
x_{t+1} &= Ax_t + Bu_t + w_t \\
u_t \in U &= \{u \mid H_u u \le h_u\} \\
w_t \in W &= \{w \mid H_w w \le h_w\} \\
X_0 &= \{x \mid H_x x \le h_x\}
\end{aligned}
$$

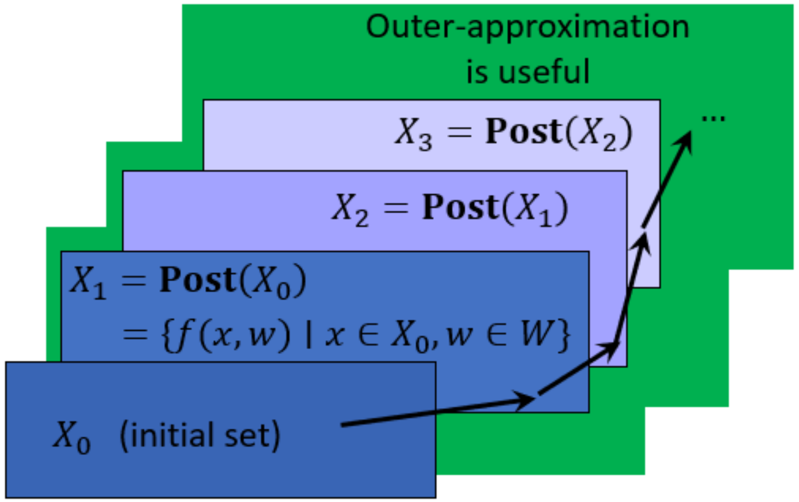
Polytopes in half-space representations (H-Reps)

Projection is difficult for H-Reps

$$
X_{k+1} = \boxed{\mathbf{Proj}_x} \{(x,u) \mid Ax + Bu + EW \subseteq X_k\}
$$

Can we use zonotopes to represent X_k?

- Zonotope: $\{ G\theta + c \mid \theta \in [-1,1]^N \} = \boxed{\langle G, c \rangle}$ Generator representation (G-Rep)
- Advantages:
  - Affine transformation (projection), Minkowski sum are easy
  - Order reduction (for outer-approximations)



$$X_{k+1} = AX_k \oplus BW$$

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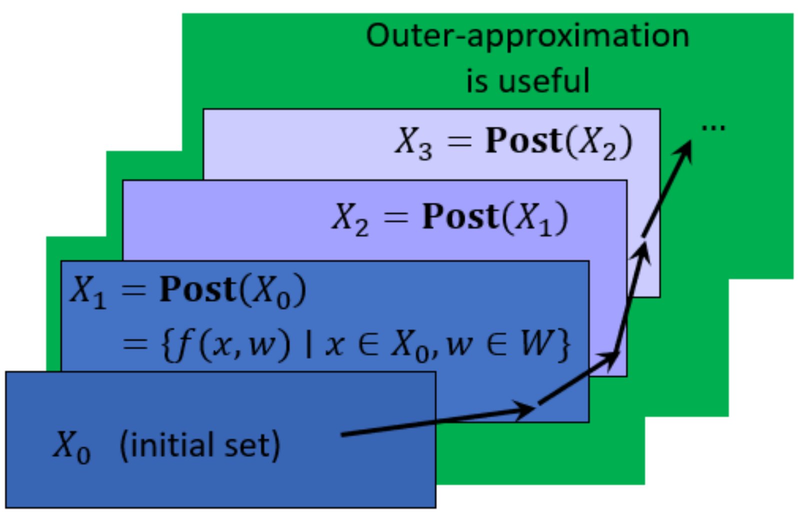
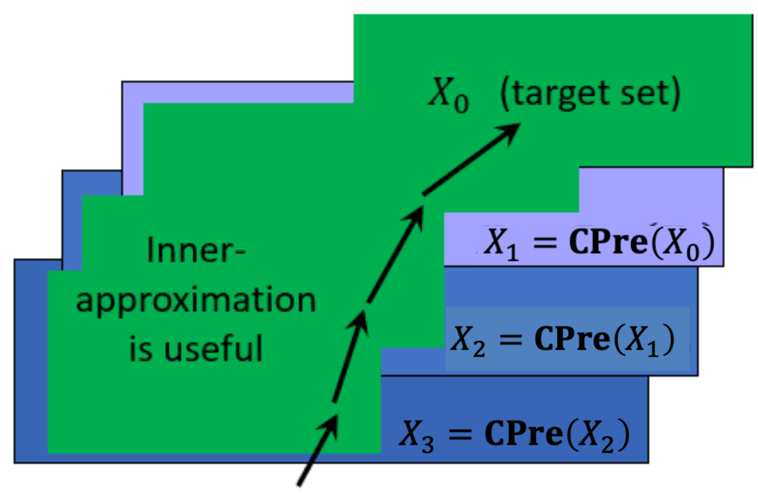
- Advantages:
  - Affine transformation (projection), Minkowski sum are easy
  - Order reduction (for outer-approximations)
- For backward reachability, there lack efficient algos for:
  - Minkowski difference
  - Order reduction (for inner-approximations)



$$X_{k+1} = AX_k \oplus BW$$



$$X_{k+1} = A^{-1}(X_k \ominus EW \oplus -BU)$$

Main Results

- Efficient inner/outer-approximations of the Minkowski difference when the minuend is a zonotope – LCSS’22

- A zonotope order reduction technique (for inner-approximation) – LCSS’22

- A scalable BRS under-approximation algorithm – LCSS’22

- Extensions to constrained zonotopes and nonlinear systems (with some results on complexity and approximability) – EMSOFT’22

Under-Approximating $X_k \ominus EW$

- $X_k = \langle G, c \rangle, \;\; W = \mathrm{cvxh}(V_W)$

- Step I: over-approximate EW by $\langle G\ \mathrm{diag}(\alpha), c' \rangle$

  α, c' can be found by solving a linear program

  #variables $= \mathcal{O}(MN + n)$

  #constraints $= \mathcal{O}(MN + Mn)$

n: dimension of X_k
M: #vertices in V_W
N: #columns in G

- Step II: $X_k \ominus EW \subseteq \langle G, c \rangle \ominus \langle G\ \mathrm{diag}(\alpha), c' \rangle$

  $= \langle G\ \mathrm{diag}(1 - \alpha), c - c' \rangle$

Step II is just G-Rep manipulation

Under-Approximating $X_k \ominus EW$

- $X_k = \langle G, c \rangle, \quad W = \mathrm{cvxh}(V_W)$

- Step I: over-approximate EW by $\langle G\ \mathrm{diag}(\alpha), c' \rangle$
  α, c' can be found by solving a linear program

$$
\begin{aligned}
\min_{\theta,\,\alpha,\,c} \quad & \textstyle\sum_{i=1}^{N} b_i \alpha_i \\
\text{s.t.} \quad & \forall w_j \in V : c + \textstyle\sum_{i=1}^{N} \theta_{ij} g_i = E w_j \\
& |\theta_{ij}| \le \alpha_i \le 1, \;\; i = 0, 1, \ldots N
\end{aligned}
$$

n: dimension of X_k
M: #vertices in V_W
N: #columns in G

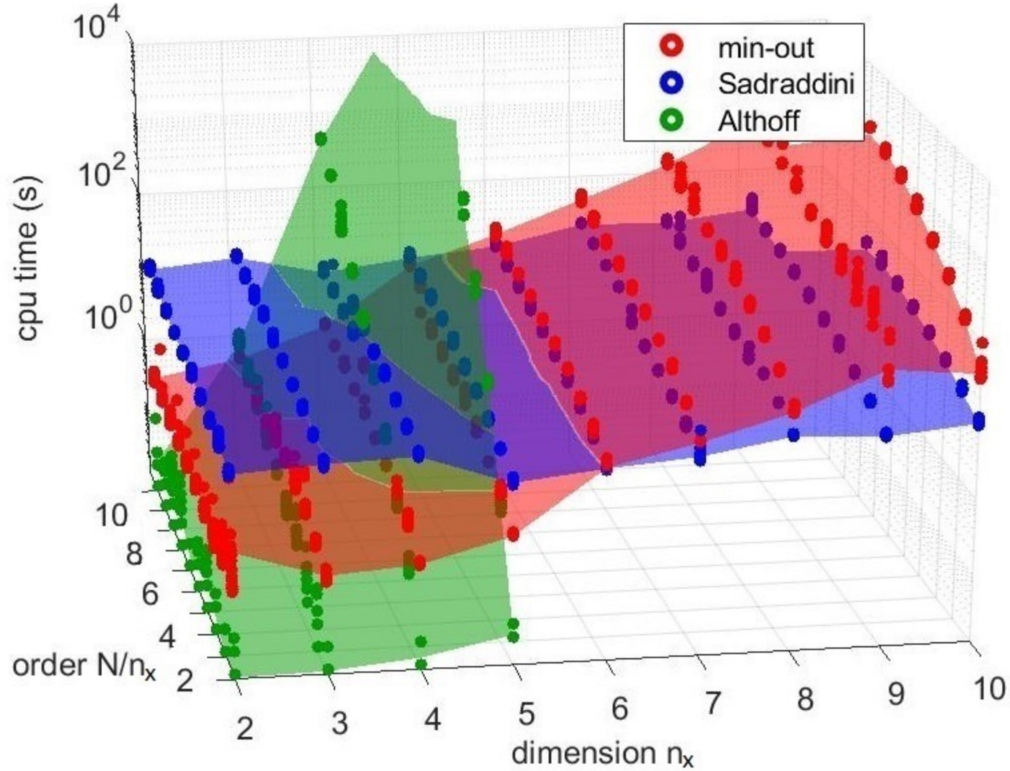
- Step II: $X_k \ominus EW \subseteq \langle G, c \rangle \ominus \langle G\ \mathrm{diag}(\alpha), c' \rangle$

$$= \langle G\ \mathrm{diag}(1-\alpha), c - c' \rangle$$

Step II is just G-Rep manipulation

Evaluation with Random Instances

- Comparison
  - H-Rep manipulation [Althoff 2015]
  - Zonotope containment [Sadraddini & Tedrake 2019], [Raghuraman & Koeln 2022] if W has a G-Rep



	mean	std.	min	max
$(V/V)^{1/n}$	1.0017	0.0577	0.9900	1.3856
$(V/V)^{1/n}$	0.9678	0.1891	0.8372	1.7498

Order Reduction (for Inner-Approximation)

- Idea:

$$\langle [g_1, g_2, \dots, g_N], c \rangle \xrightarrow[]{\text{replace } [g_i, g_j] \text{ by } g_i + g_j \text{ or } g_i - g_j} \text{\#generators reduces by one}$$

Order Reduction

- Idea:

$$\langle [g_1, g_2, \dots, g_N], c \rangle \xrightarrow[]{\text{replace } [g_i, g_j] \text{ by } g_i + g_j \text{ or } g_i - g_j} \begin{array}{c}\text{\#generators} \\ \text{reduces by one}\end{array}$$

- Two questions:
  - Which $[g_i, g_j]$ to "combine"?
  - Replace $[g_i, g_j]$ with $g_i + g_j$ or $g_i - g_j$?

Order Reduction

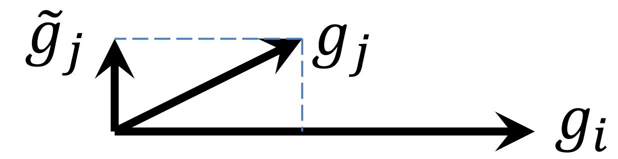
- Idea:

$$\langle [g_1, g_2, \dots, g_N], c \rangle \xrightarrow{\text{replace } [g_i, g_j] \text{ by } g_i + g_j \text{ or } g_i - g_j}$$

#generators reduces by one

- Two questions:
  - Which $[g_i, g_j]$ to "combine"?

Pick **small** or **closely-aligned** generators



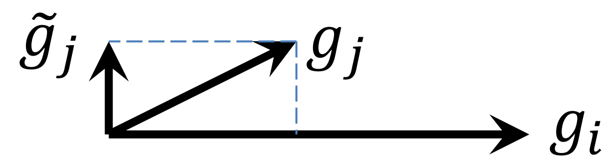
  - Replace $[g_i, g_j]$ with $g_i + g_j$ or $g_i - g_j$?

Order Reduction

- Idea:

$$\langle [g_1, g_2, \dots, g_N], c\rangle \xrightarrow{\substack{\text{replace } [g_i, g_j] \text{ by} \\ g_i + g_j \text{ or } g_i - g_j}} \begin{array}{c}\#\text{generators} \\ \text{reduces by one}\end{array}$$

- Two questions:
  - Which $[g_i, g_j]$ to "combine"?
    - $(i, j) = \operatorname{argmin} \|g_i\|_2 \|\tilde{g}_j\|_2$
    - $(i, j) = \operatorname{argmin} \|g_i\|_2 \|g_j\|_2 \frac{\sigma_{\min}}{\sigma_{\max}}$



  - Replace $[g_i, g_j]$ with $g_i + g_j$ or $g_i - g_j$?

Order Reduction

- Idea:

$$\langle [g_1, g_2, \dots, g_N], c \rangle \xrightarrow{\text{replace } [g_i, g_j] \text{ by } g_i + g_j \text{ or } g_i - g_j} \begin{array}{c}\#\text{generators} \\ \text{reduces by one}\end{array}$$

- Two questions:
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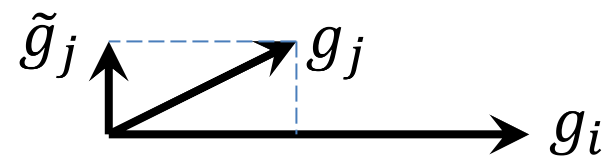
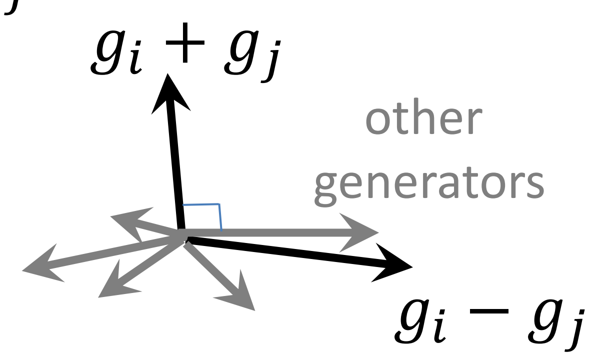


  - Replace $[g_i, g_j]$ with $g_i + g_j$ or $g_i - g_j$?

Use the one that is **larger and "more perpendicular"** to the other generators

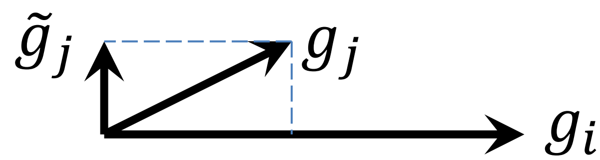


Order Reduction

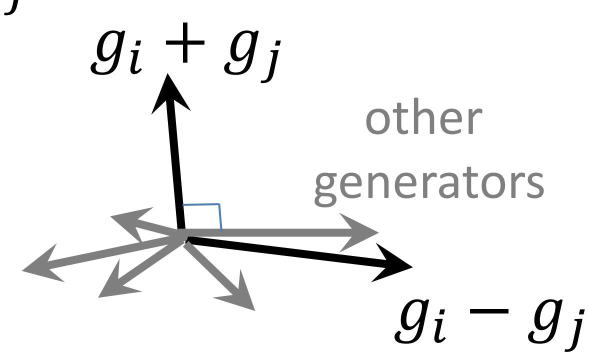
- Idea:

$$\langle [g_1, g_2, \dots, g_N], c \rangle \xrightarrow{\substack{\text{replace } [g_i, g_j] \text{ by} \\ g_i + g_j \text{ or } g_i - g_j}} \begin{matrix}\#\text{generators} \\ \text{reduces by one}\end{matrix}$$

- Two questions:
  - Which $[g_i, g_j]$ to "combine"?
    - $(i, j) = \operatorname{argmin} \|g_i\|_2 \|\tilde{g}_j\|_2$
    - $(i, j) = \operatorname{argmin} \|g_i\|_2 \|g_j\|_2 \dfrac{\sigma_{\min}}{\sigma_{\max}}$



  - Replace $[g_i, g_j]$ with $g_i + g_j$ or $g_i - g_j$?
    - Replace with $g_i + g_j$ if $\|G^\dagger (g_i + g_j)\|_2 \geq \|G^\dagger (g_i - g_j)\|_2$ and $g_i - g_j$ otherwise

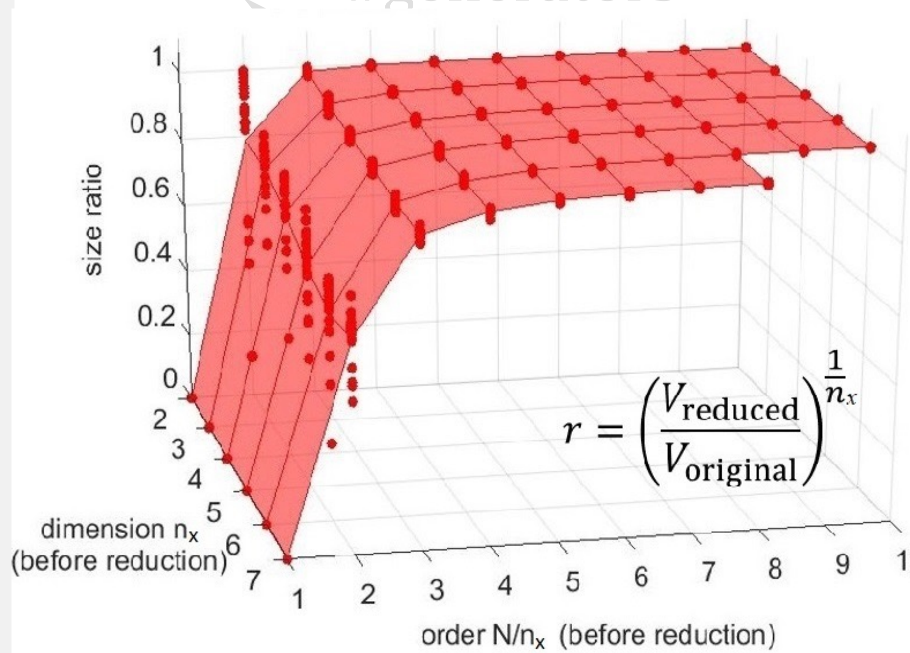


Order Reduction

- Idea:

replace $[g_i, g_j]$ by $g_i + g_j$ or $g_i - g_j$

#generators



- Replace with $g_i + g_j$ if $\|G^{\dagger}(g_i + g_j)\|_2 \geq \|G^{\dagger}(g_i - g_j)\|_2$ and $g_i - g_j$ otherwise

Example: Aircraft Position Control

- Longitudinal: x in 6D, u in 2D,
- Lateral: x in 6D, u in 2D

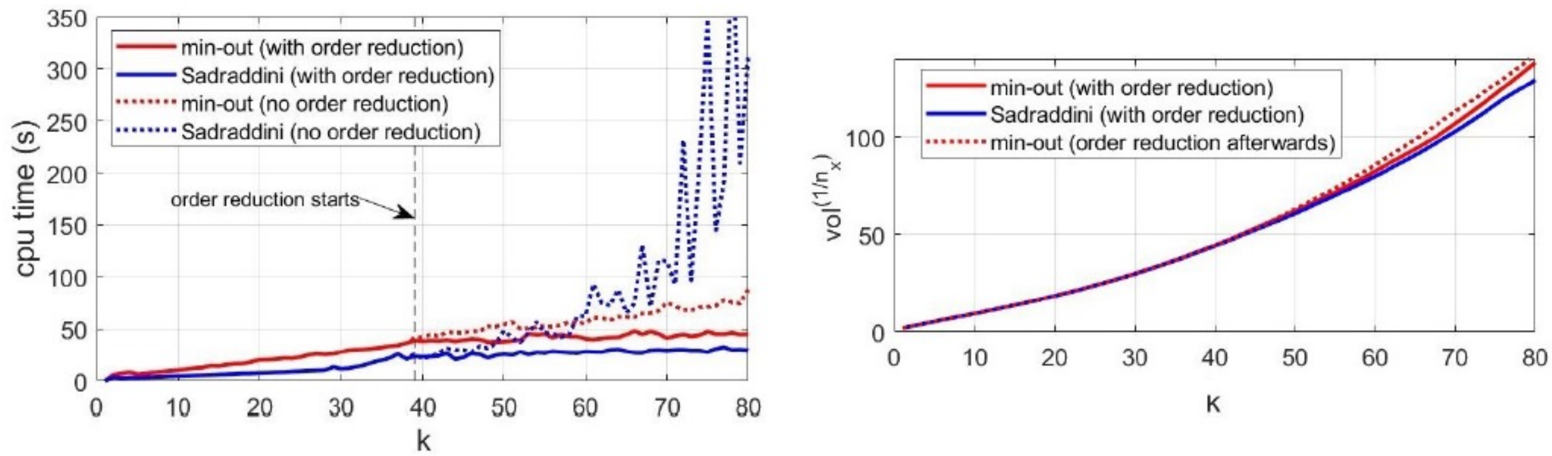
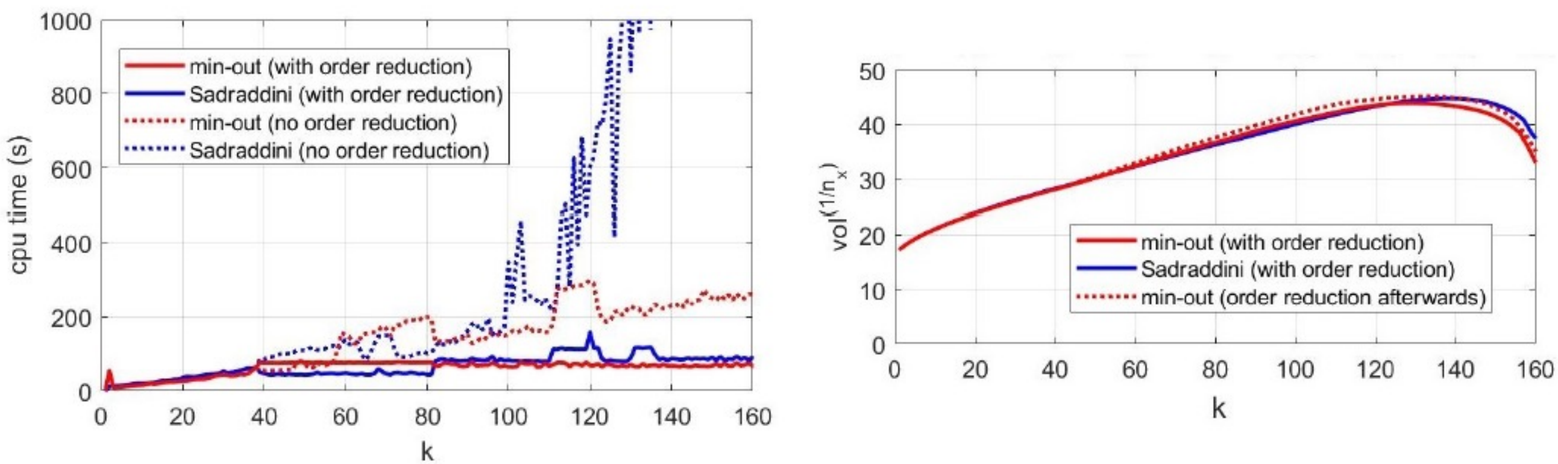


Fig. 3: Backward reachable set computation for lateral dynamics. Left: computation time. Right: set volume.



Limitations & extensions

- Zonotopes are not as rich as polytopes in terms of expressiveness
  - How about constrained zonotopes?
- So far, applicable to linear systems
  - How about nonlinear systems?

Problem Formulation

- **System:**

$$\boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t) + \boldsymbol{w}_t$$
$$\boldsymbol{u}_t \in \mathcal{U}, \quad \boldsymbol{w}_t \in \mathcal{W}$$

Assumptions on sets

Target set, control set

$\mathcal{X}_0 = \langle \boldsymbol{G}_0, \boldsymbol{c}_0, \boldsymbol{A}_0, \boldsymbol{b}_0 \rangle$

$\mathcal{U} = \langle \boldsymbol{G}_u, \boldsymbol{c}_u, \boldsymbol{A}_u, \boldsymbol{b}_u \rangle$

are constrained zonotopes

Disturbance set

$\mathcal{W} = \langle \boldsymbol{G}_w, \boldsymbol{c}_w \rangle$ **is a zonotope**

Safe set

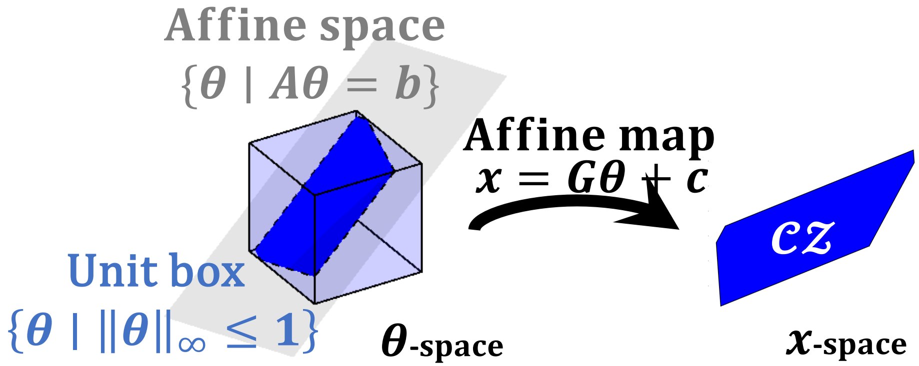
$\mathcal{X}_{\text{safe}} = \{ \boldsymbol{x} \mid \boldsymbol{H}\boldsymbol{x} \le \boldsymbol{a} \}$ **is a polyhedron**

Constrained zonotope

If no A, b the set is just a zonotope

$\mathcal{CZ} = \{ \boldsymbol{G}\boldsymbol{\theta} + \boldsymbol{c} \mid \|\boldsymbol{\theta}\|_\infty \le 1, \cancel{A\theta} = b \}$

CG-Rep of $\mathcal{CZ}$: $\langle \boldsymbol{G}, \boldsymbol{c}, A, b \rangle$



BRS Computation Using Constrained Zonotopes

- **Linear system:** $$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{u}_t + \boldsymbol{w}_t$$ $$\boldsymbol{u}_t \in \mathcal{U}, \quad \boldsymbol{w}_t \in \mathcal{W}$$
- **BRS computation:** $$\mathcal{X}_{k+1} = Pre(\mathcal{X}_k) = \boldsymbol{A}^{-1}(\mathcal{X}_k \ominus \mathcal{W} \oplus -\boldsymbol{B}\mathcal{U}) \cap \mathcal{X}_{\text{safe}}$$

For nonlinear systems, replace $\mathcal{W}$ by $\mathcal{W} \oplus \mathcal{L}$, where $\mathcal{L}$ contains all linearization error [Althoff 2008]

Assumptions on sets

Target set, BRS, control set

$\mathcal{X}_k = \langle \boldsymbol{G}_k, \boldsymbol{c}_k, \boldsymbol{A}_k, \boldsymbol{b}_k \rangle$

$\mathcal{U} = \langle \boldsymbol{G}_u, \boldsymbol{c}_u, \boldsymbol{A}_u, \boldsymbol{b}_u \rangle$

are constrained zonotopes

Disturbance set

$\mathcal{W} = \langle \boldsymbol{G}_w, \boldsymbol{c}_w \rangle$ **is a zonotope**

Safe set

$\mathcal{X}_{\text{safe}} = \{\boldsymbol{x} \mid \boldsymbol{H}\boldsymbol{x} \leq \boldsymbol{a}\}$ **is a polyhedron**

Set operations involved

Affine map

$A\mathcal{X} = \{Ax \mid x \in \mathcal{X}\}$

$\mathcal{Y}\}$

Easy to compute for **constrained zonotopes,** via **CG-Rep manipulation**

Minkowski difference

$\mathcal{X} \ominus \mathcal{Y} = \{x \mid \{x\} \oplus \mathcal{Y} \subseteq \mathcal{X}\}$?

Main Result: Under-Approximate $\mathcal{CZ} \ominus \mathcal{Z}$ Efficiently

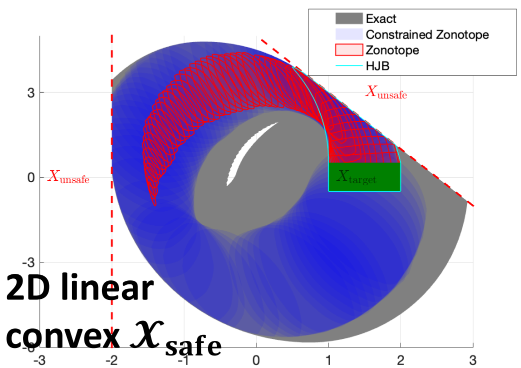
Theorem 1.
Given $\mathcal{CZ} = \langle \boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b} \rangle$ and $\mathcal{Z} = \langle \boldsymbol{G}', \boldsymbol{c}' \rangle$, no algorithm finds a polynomial-size CG-Rep of $\mathcal{CZ} \ominus \mathcal{Z}$ in polynomial-time (unless P=NP).

Theorem 2.
We can find a set $\mathcal{CZ}_{\mathrm{d}} \subseteq \mathcal{CZ} \ominus \mathcal{Z}$ by solving a linear program, whose # variables and # constraints are polynomial in the size of $\mathcal{CZ}$'s and $\mathcal{Z}$'s representations.

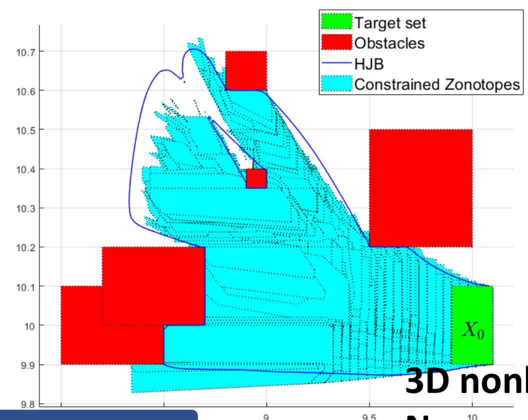
Theorem 3.
CG-Rep not unique
Every constrained zonotope $\mathcal{CZ}$ has a "rich" enough CG-Rep s.t. our under-approximation is exact, i.e., $\mathcal{CZ}_{\mathrm{d}} = \mathcal{CZ} \ominus \mathcal{Z}$.

There is a trade-off between accuracy & efficiency, which can be "tuned" via CG-Rep selection

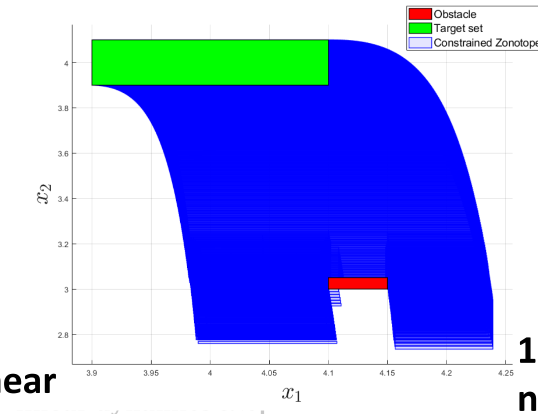
BRS Under-Approximation Algorithms



2D linear convex $\mathcal{X}_{\text{safe}}$



3D nonlinear Nonconvex $\mathcal{X}_{\text{safe}}$



10D nonlinear Nonconvex $\mathcal{X}_{\text{safe}}$

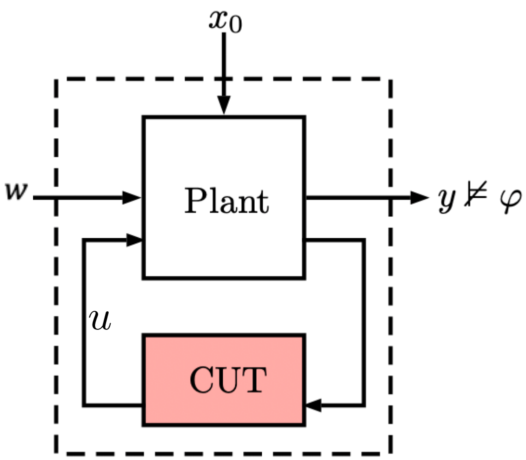
- **Less conservative then zonotope-based approach** [Yang & Ozay 2022]
- **Deal with constraints (convex or nonconvex)**
- **More scalable than HJB** [Bansal et al. 2017]

	n	k	Splitting	Scaling	HJB
Example 2	2	100	N/A	56.1s	45.2s
Example 3	10	10	226.7s	N/A	Memory error
Example 4: Convex constraints	3	—	478.9s ($k = 25$)	2821.3s ($k = 400$)	Memory error
Example 4: Nonconvex constraints	3	20	1564.1s	N/A	4521.6s
Example 5	10	340	951.6s	N/A	Memory error

HJB: hard to scale

How to use backward reachable sets (BRS) in falsification?

Synthesis-guided falsification



w disturbance (external input)
x_0: initial condition
u: control input
y: output
φ: specification

Falsification problem: Given a plant model, a (blackbox) controller and a specification, can we find an initial condition and an external input sequence (disturbance sequence) so that the specification is violated?

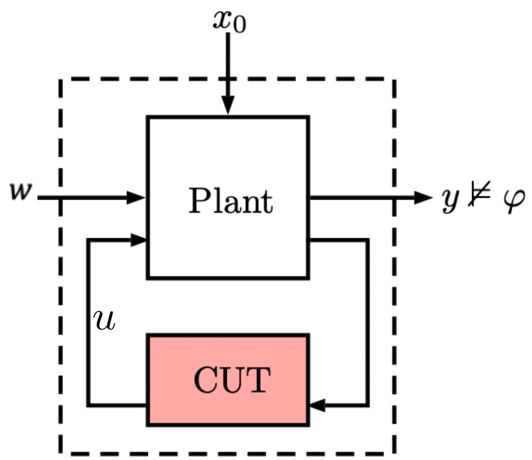
Assumptions:

- "simple enough" plant model
$$x_{t+1} = f(x_t, u_t, w_t)$$
- "simple enough" specification: we will focus on safety (invariance) and reachability specifications

"simple enough": almost anything for which you can compute the validity domain (i.e., winning set) of the synthesis problem

Chou et al. EMSOFT 18

Synthesis-guided falsification

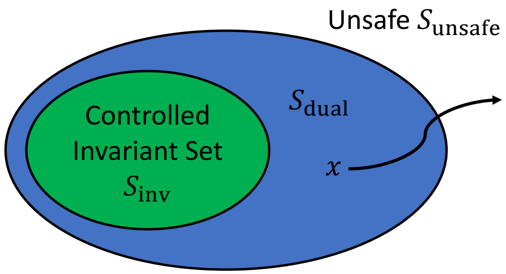


w disturbance (external input)
x_0: initial condition
u: control input
y: output
φ: specification

Plant model:

$$x_{t+1} = f(x_t, u_t, w_t)$$

Ideal result of the synthesis problems:



Approach:

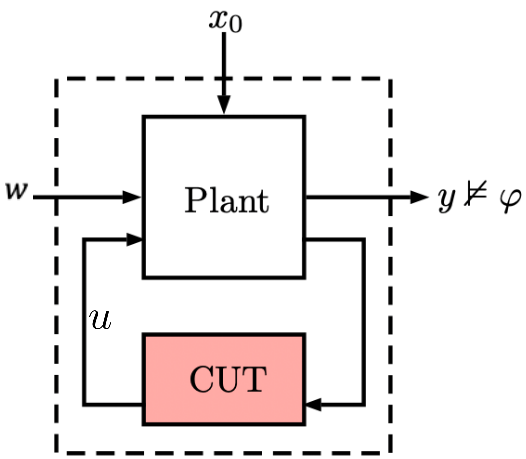
- Ignore the controller, focus on safety-critical part of the spec.
- Given the plant model and safety (invariance) part X_{safe} of the spec, consider the safety and dual reachability synthesis problems:

Invariance in X_{safe}

Reachability to

$X_{unsafe} = \neg X_{safe}$

Synthesis-guided falsification



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Invariance in X_{safe}

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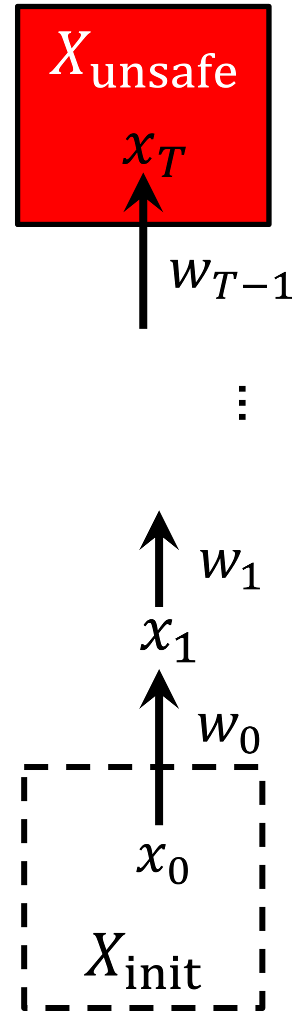
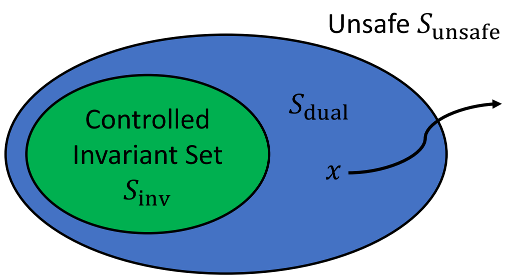


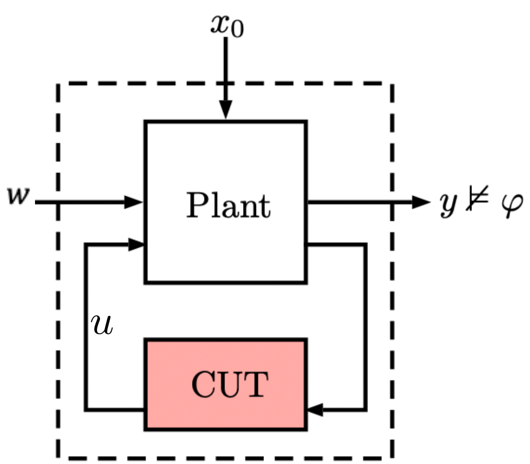
Plant model:

$$x_{t+1} = f(x_t, u_t, w_t)$$

Ideal result of the synthesis problems:



Synthesis-guided falsification



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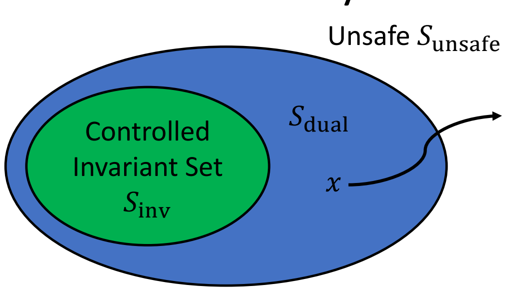
y: output
φ: specification

Plant model:

$$x_{t+1} = f(x_t, u_t, w_t)$$

Ideal result of the synthesis problems:



Approach:

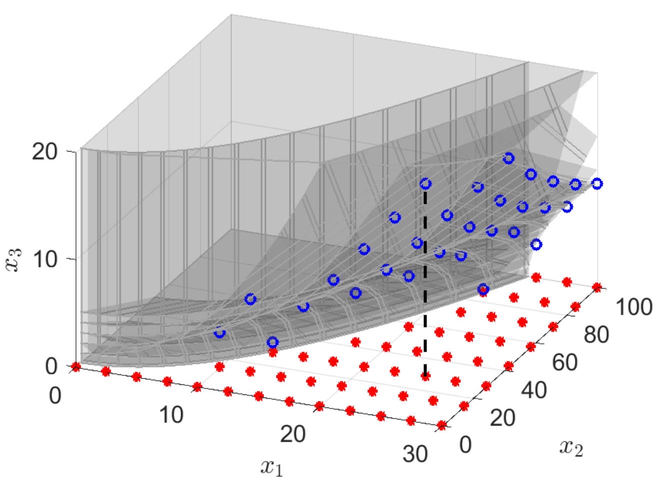
- Ignore the controller, focus on safety-critical part of the spec.
- Given the plant model and safety (invariance) part X_{safe} of the spec, consider the safety and dual reachability synthesis problems.

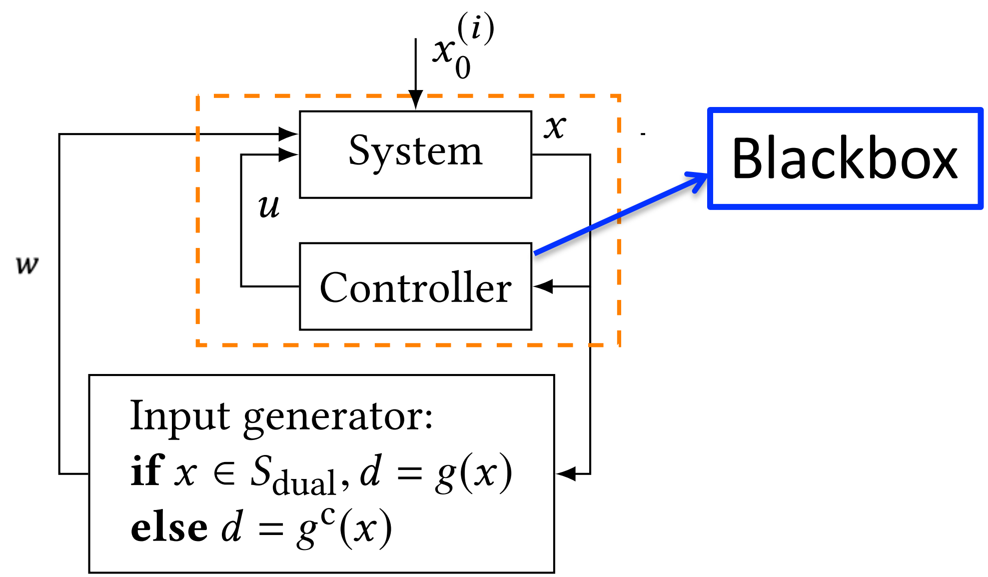
Comments:

- Synthesized X_{inv} is the validity domain W of the best safety controller (i.e., **the maximal** invariant set)
- Synthesized X_{dual} is the validity domain of "best" disturbance policy
- OK to use approximate computations/models
- Can use any synthesis approach: *iterative polytopic computations*, Hamilton Jacobi Bellman, control barrier functions, abstractions

Synthesis-guided falsification

- Hypotheses:
  - **Hard initial conditions**: boundary of the control invariant set (small # of safe control inputs)
  - **Hard external inputs**: solutions of a dual reachability problem (when out of S_{dual}, pick best effort input to get close to S_{dual})



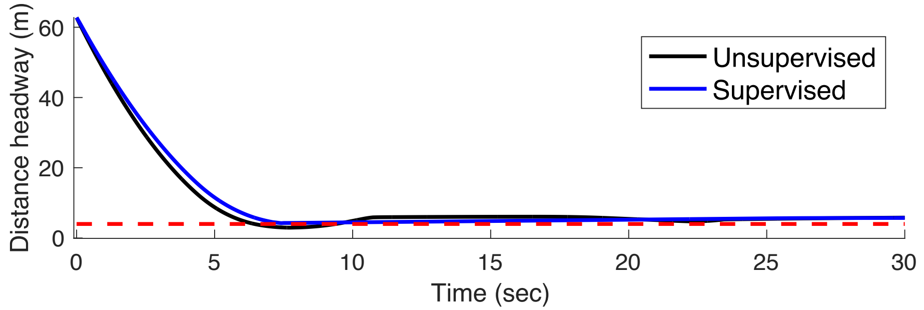


Synthesis-guided falsification

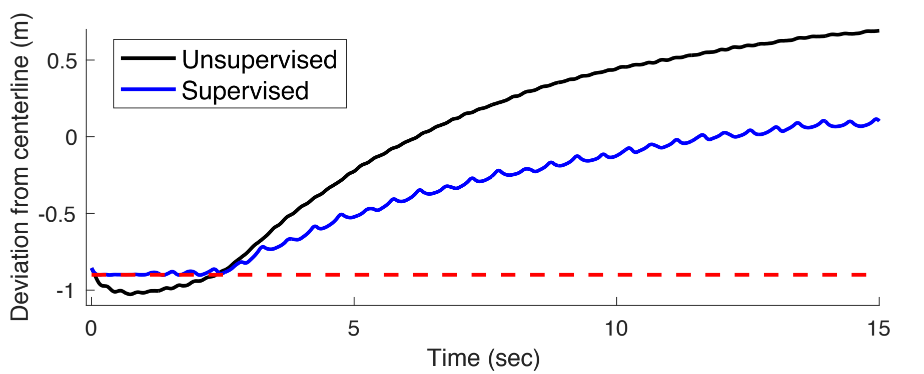
- Some example “bugs” found in the open-source autonomous driving software CommaAI (using its python source code directly!)
- Similar results with Stanford DARPA Grand Challenge code (C++ code)
- Can use the synthesis artifacts for “sandboxing” (supervising) complex controllers
- Our software is available online: integration of several driving software with car dynamics models

Can handle any (learning-based) state-feedback controller (e.g., NNs, RL, etc.)
→ so far no perception modules

Adaptive cruise control



Lane keeping

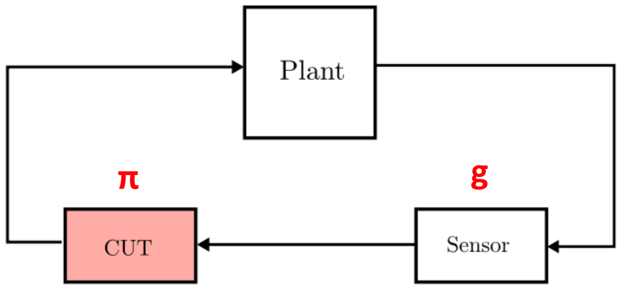


Synthesis-guided falsification with partial information

known & simple
(can be uncertain)

π

g



unknown &
complicated

known & simple
(can be noisy)

an abstraction of a
complex perception
module

Plant model including sensor:

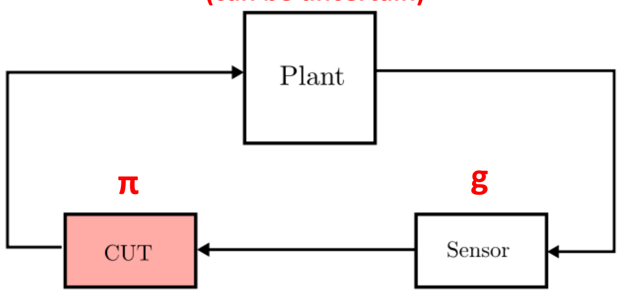
$$x_{t+1} = f(x_t, u_t, w_t)$$

$$y_t = g(x_t, v_t)$$

- Given:
  - A gray-box system
  - An unsafe set X_{unsafe}, and an initial set X_{init}
- Find one adversarial example:
  - A trajectory $x_0, x_1, \dots, x_T$
  - External inputs $w_0, w_1, \dots, w_{T-1}, v_0, v_1, \dots, v_{T-1}$
  - $x_0 \in X_{\text{init}}$ and $x_T \in X_{\text{unsafe}}$

Synthesis-guided falsification with partial information

known & simple
(can be uncertain)



unknown & complicated

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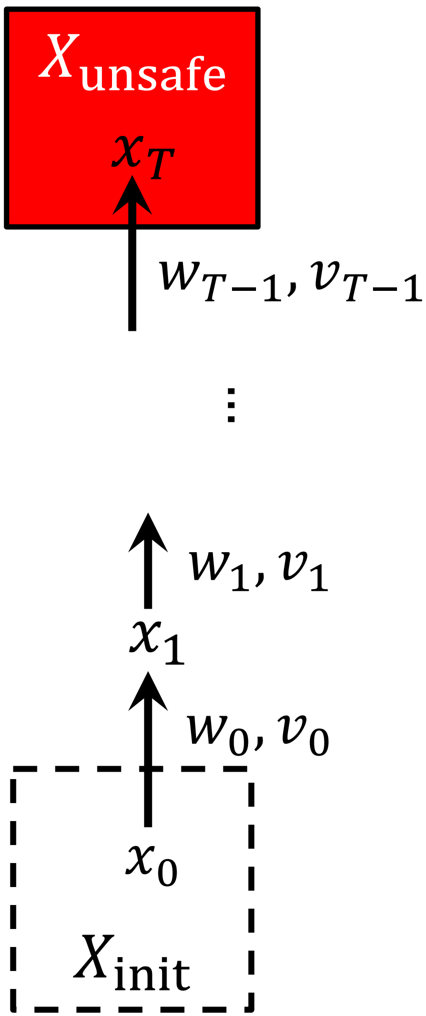
Plant model including sensor:

$$x_{t+1} = f(x_t, u_t, w_t)$$

$$y_t = g(x_t, v_t)$$

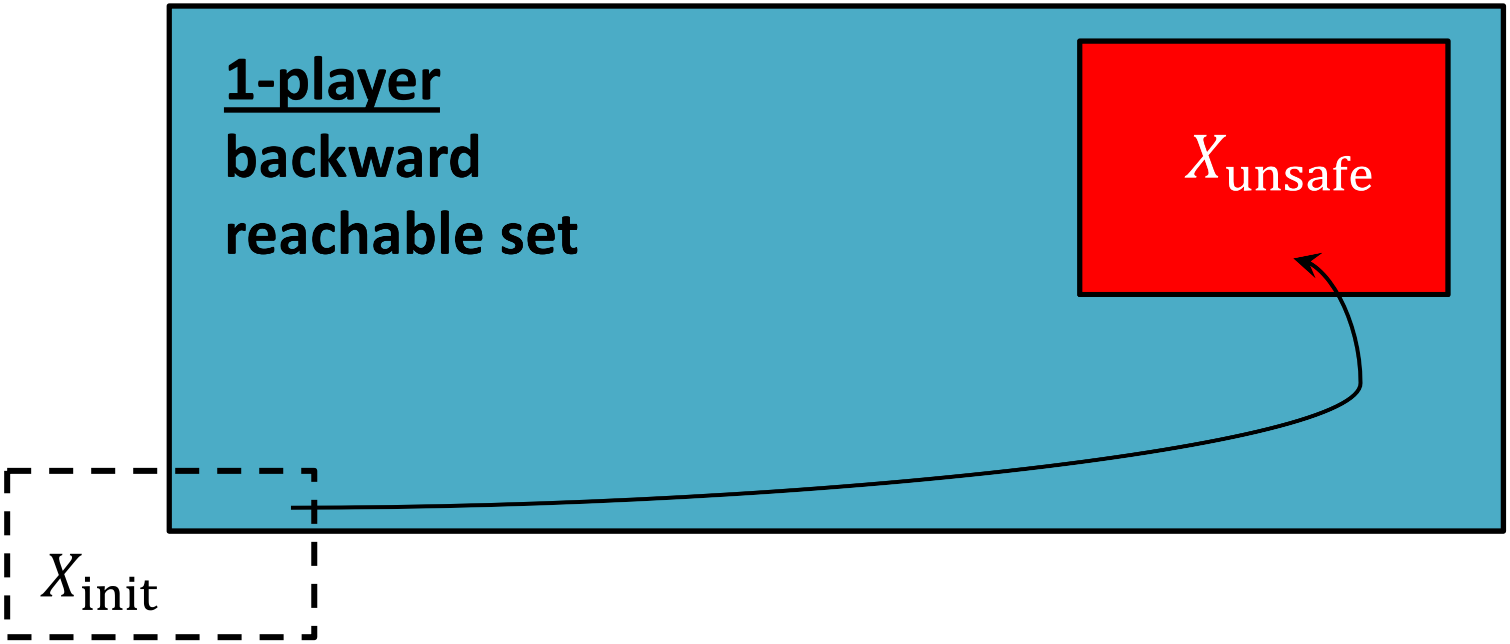


Key Idea

Verification: compute 1-player backward reachable set $= \bigcup X_k$

$$X_0 = X_{\text{unsafe}}$$
$$X_{k+1} = \mathbf{Pre}(X_k) := X_k \cup \{x \mid \exists w, v: \underbrace{f(x, \pi(x, v), w)} \in X_k\}$$

Closed-loop dynamics: complex due to π, even if f itself is simple (precisely why verification is hard)



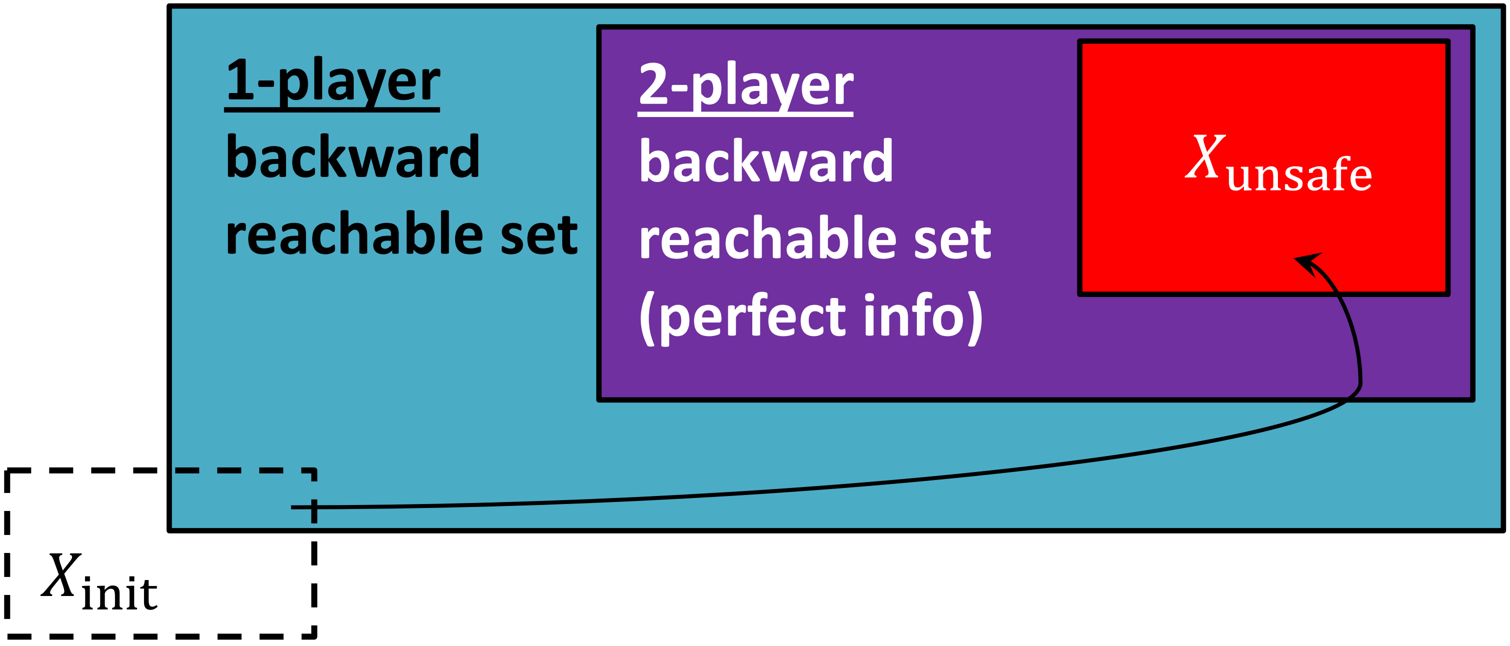
Key Idea

Synthesis: compute 2-player backward reachable set $= \bigcup \hat{X}_k$

$$\hat{X}_0 = X_{\text{unsafe}}$$
$$\hat{X}_{k+1} = \mathbf{CPre}(\hat{X}_k) := \hat{X}_k \cup \{x \mid \forall u : \underbrace{\exists w : f(x,u,w) \in \hat{X}_k}\}$$

<u>Open-loop</u> dynamics: simple, independent of π

- The adversarial examples are trivial (generic)
- Noise v is not essential for violation

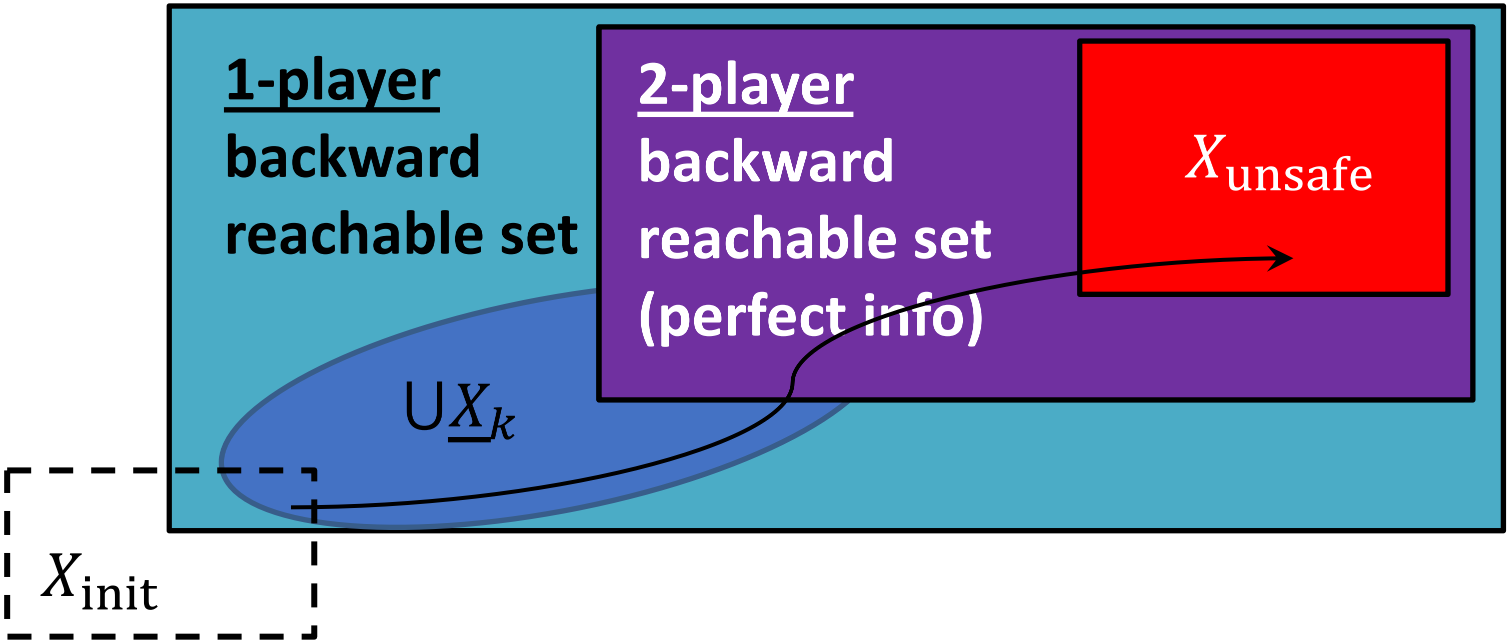


Key Idea

Synthesis guided Falsification: compute $\cup \underline{X}_k$, where

1. $\underline{X}_0 \subseteq$ 2−player backward reachable set
2. $Y_{k+1} = \mathbf{CPre}_y(\underline{X}_k) := \{y \mid \forall u: \exists x, v: y = g(x,v), f(x,u,w) \in \underline{X}_k\}$
3. $y_{k+1} \in Y_{k+1}$, $u_{k+1} = \pi(y_{k+1})$,
4. $\underline{X}_{k+1} = \mathbf{Pre}(\underline{X}_k \mid y_{k+1}) := \{x \mid \exists w, v: y_{k+1} = g(x,v), f(x, u_{k+1}, w) \in \underline{X}_k\}$

independent of π!

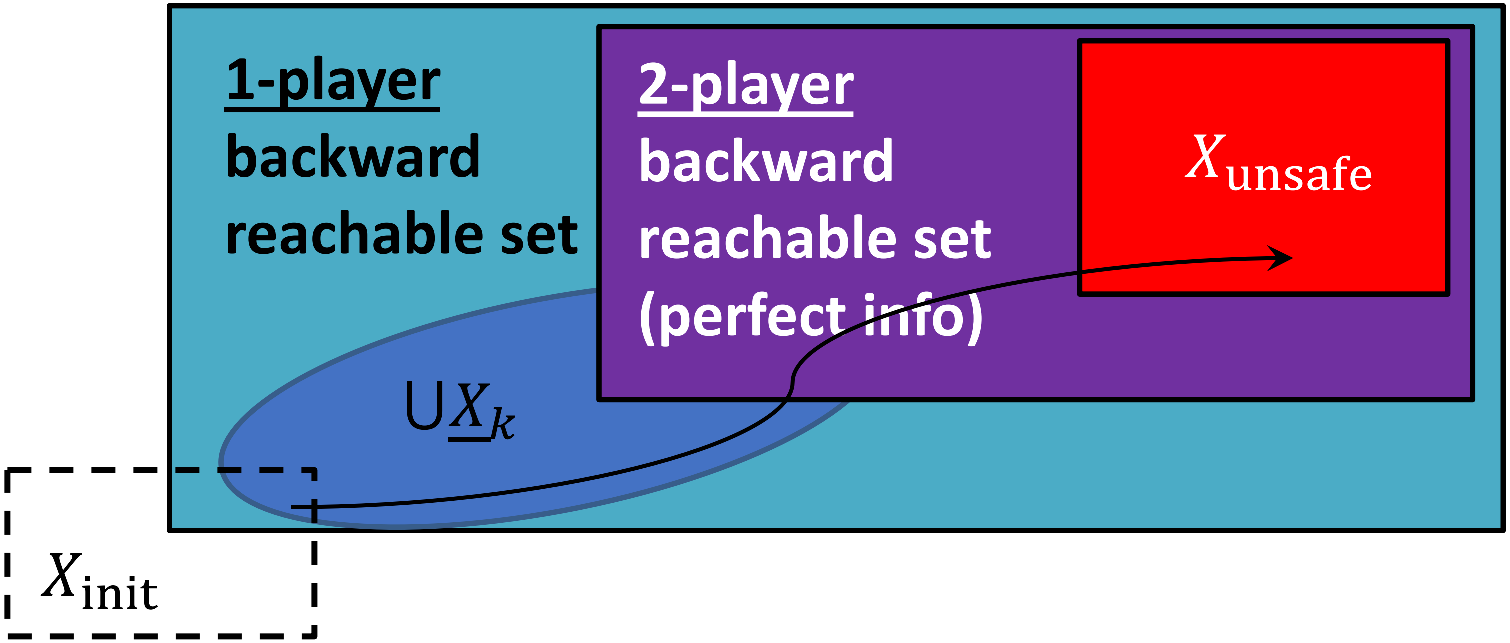


Key Idea

Synthesis guided Falsification: compute $\cup \underline{X}_k$, where

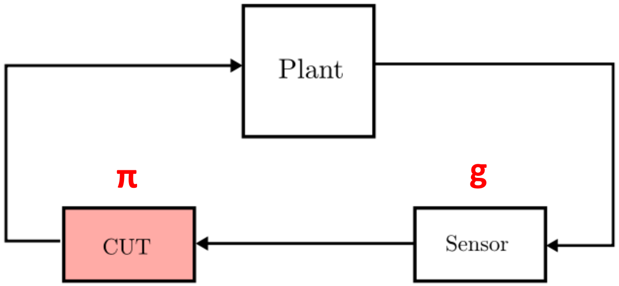
1. $\underline{X}_0 \subseteq$ 2−player backward reachable set
2. $Y_{k+1} = \mathbf{CPre}_{\nu}(X_k)$: From where it is challenging to satisfy the spec
3. query at the challenging region of the state space
4. $\underline{X}_{k+1} = \mathbf{Pre}(\underline{X}_k | y_{k+1})$ 1-player backward reachability based on the query result

independent of π!



A toy example

known & simple
(can be uncertain)



π

g

unknown & complicated

known & simple
(can be noisy)

an abstraction of a complex perception module

Plant model including sensor:

$$x_{t+1} = f(x_t, u_t, w_t)$$

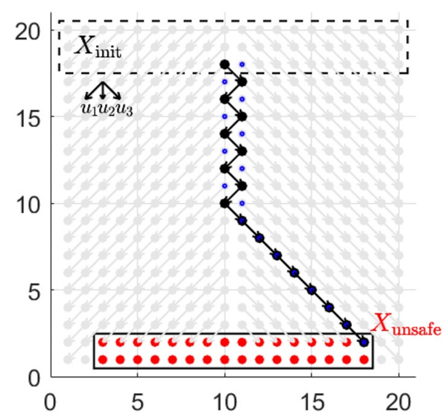
$$y_t = g(x_t, v_t)$$

Obstacle avoidance:

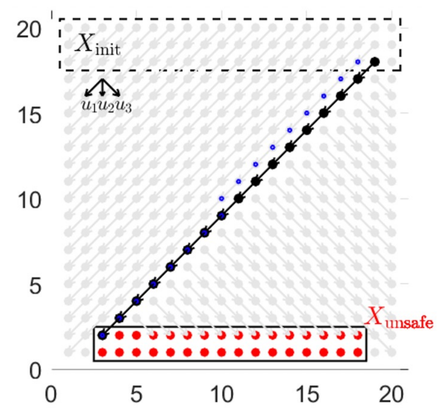
Moving down stream
Needs to avoid obstacle
Can go left or right (controller decisions)

Obstacle

Two different controllers for this task (unknown to the algorithm):



Controller 1



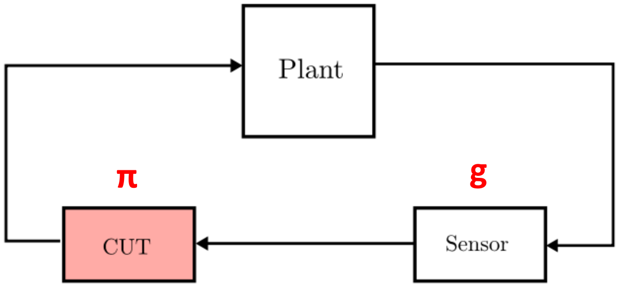
Controller 2

A toy example

known & simple
(can be uncertain)

π

g



unknown &
complicated

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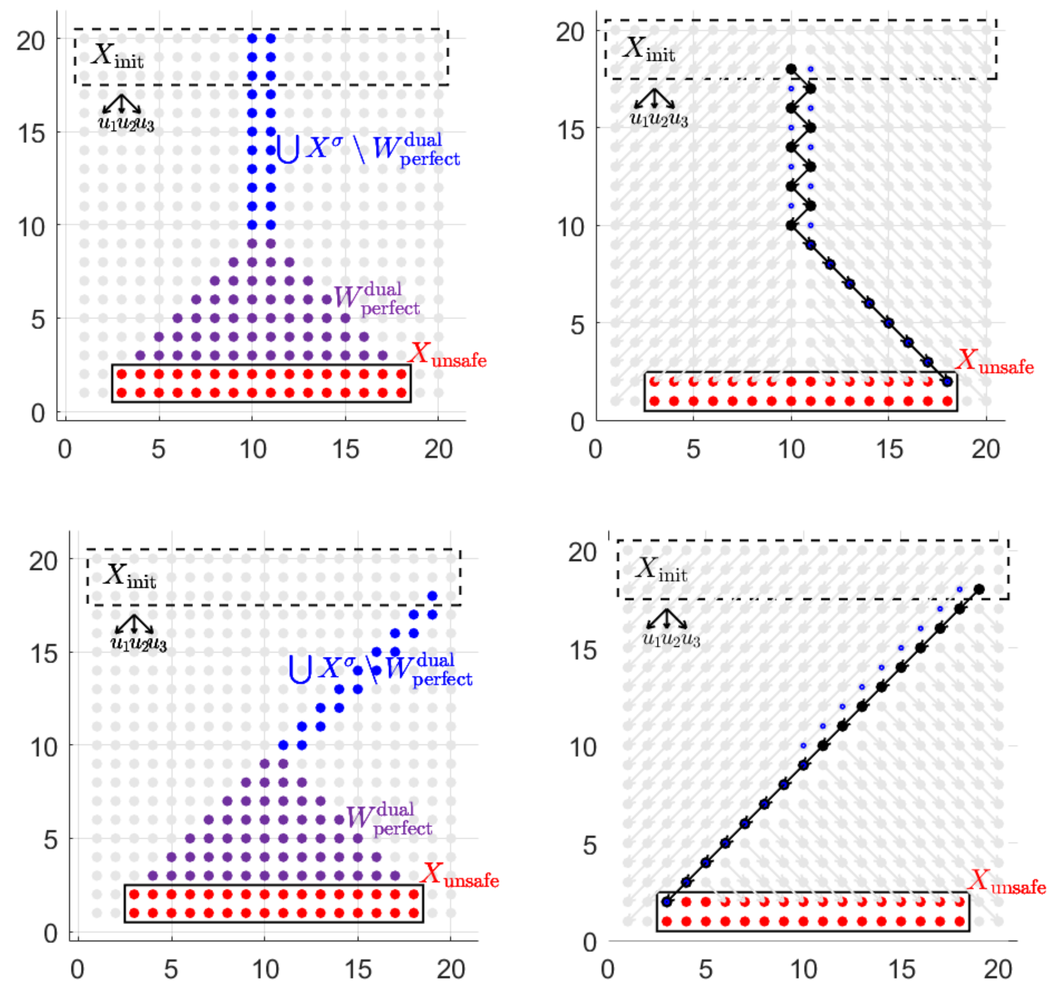
an abstraction of a
complex perception
module

Plant model including sensor:

$$x_{t+1} = f(x_t, u_t, w_t)$$

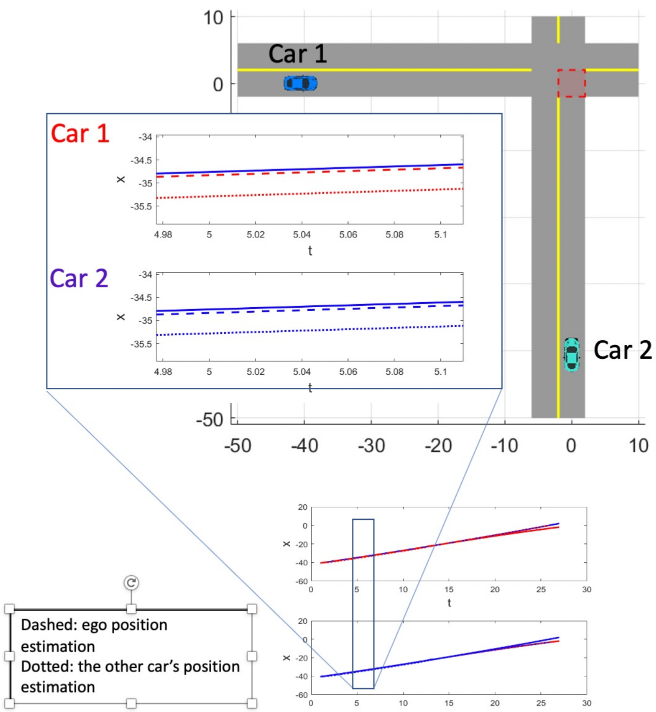
$$y_t = g(x_t, v_t)$$

Interesting insight: falsifying "sensor noise" trajectories at the discontinuities of the controller

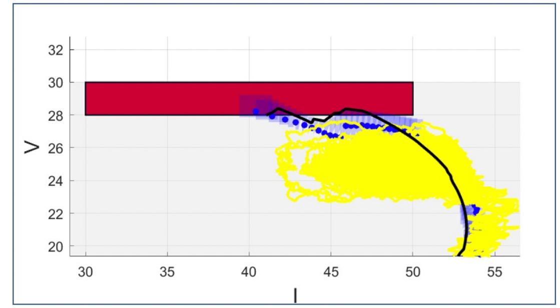


More examples with complex models/ controllers

Two cars at an intersection, 8-D dynamics, complex hybrid MPC controllers, each car has partial information of the other



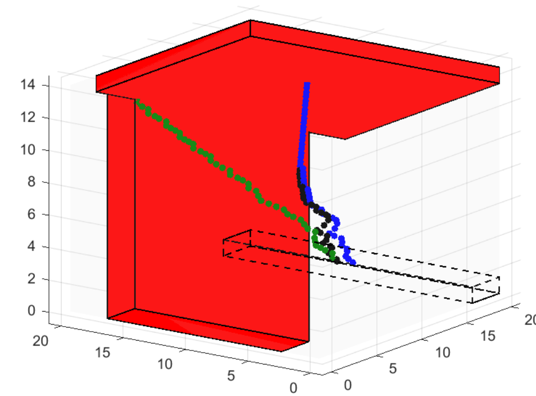
Buck converter with rule-based switching controller → forced to overvoltage



- Black: falsifying trajectory
- Yellow: simulations with random noise & disturbance (no violation)

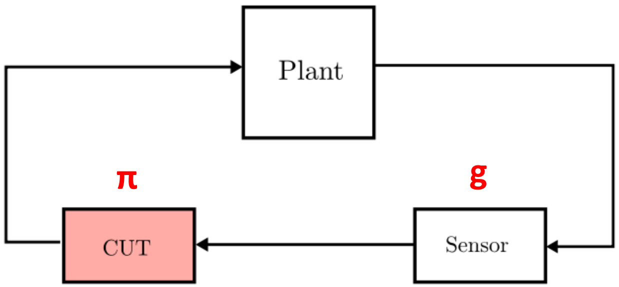
More complex specification including a deadline

$$\square(x \in X_{\text{safe}}) \wedge \lozenge_{[0,T]}\square(x \in X_{\text{target}})$$

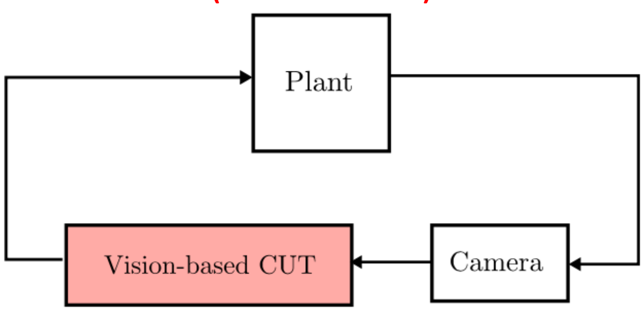


Falsification with perception in-the-loop

So far, we have

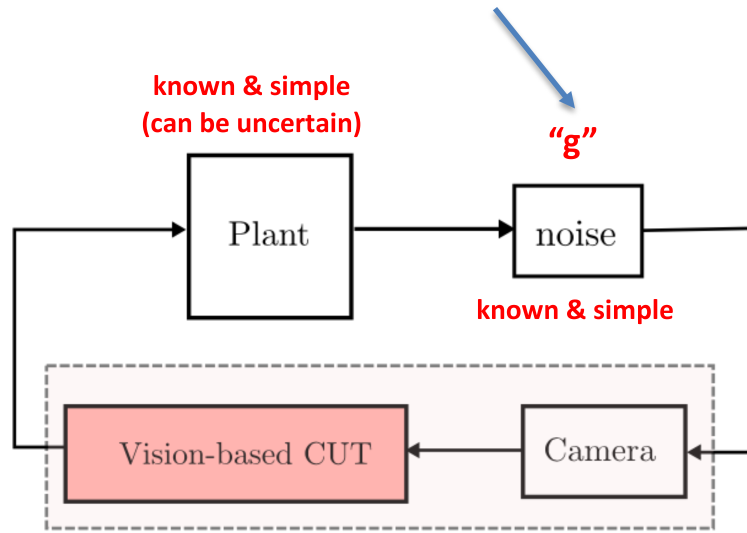


Yet, we want



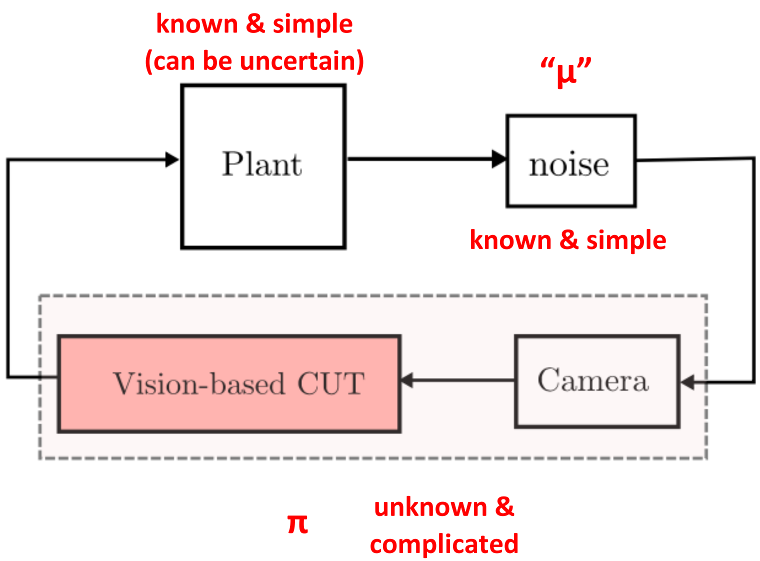
$\approx$

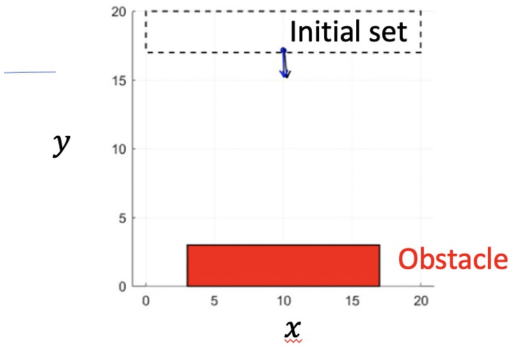
Searching for falsifying noise (adversary) in the semantic (state) space instead of image space



Vision-based CUT can be a modular design or an end-to-end controller..

"Adversarial inputs" for dynamic decision making with vision in-the-loop





Learn an end-to-end vision-based neural network controller from demonstrations generated by a state-feedback MPC

- Input: low resolution image
- Output: control input u (i.e., acceleration along x-direction)
- Camera model implemented in Matlab (or CARLA)

- a vehicle moving on the 2d plane
- dynamics along x-direction: double integrator

$$x(t+1) = x(t) + \tau * v_x(t)$$
$$v_x(t+1) = v_x(t) + \tau * u\,(t)$$

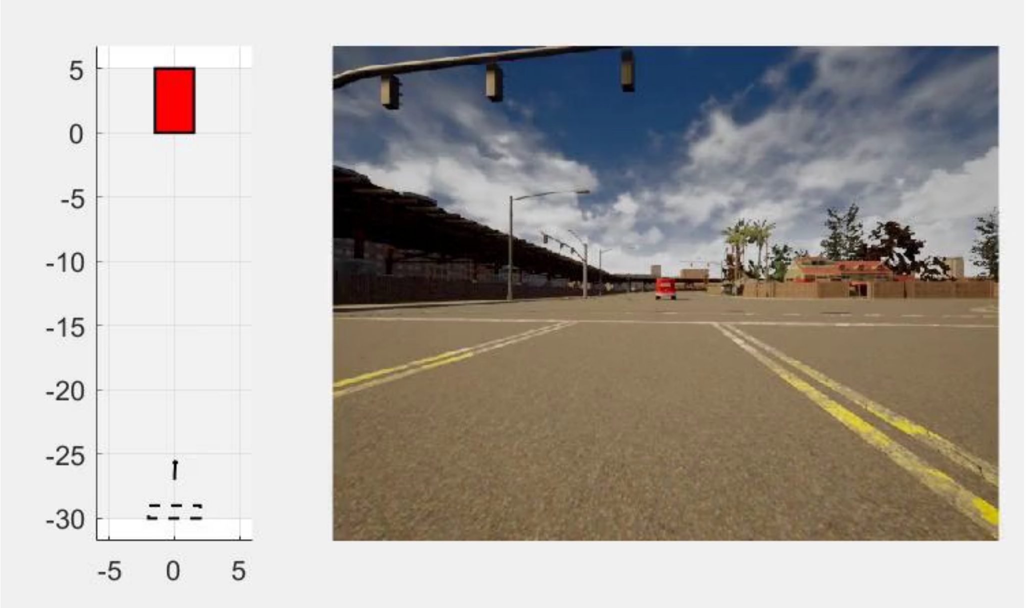
acceleration c is the control input

- dynamics along y-direction:

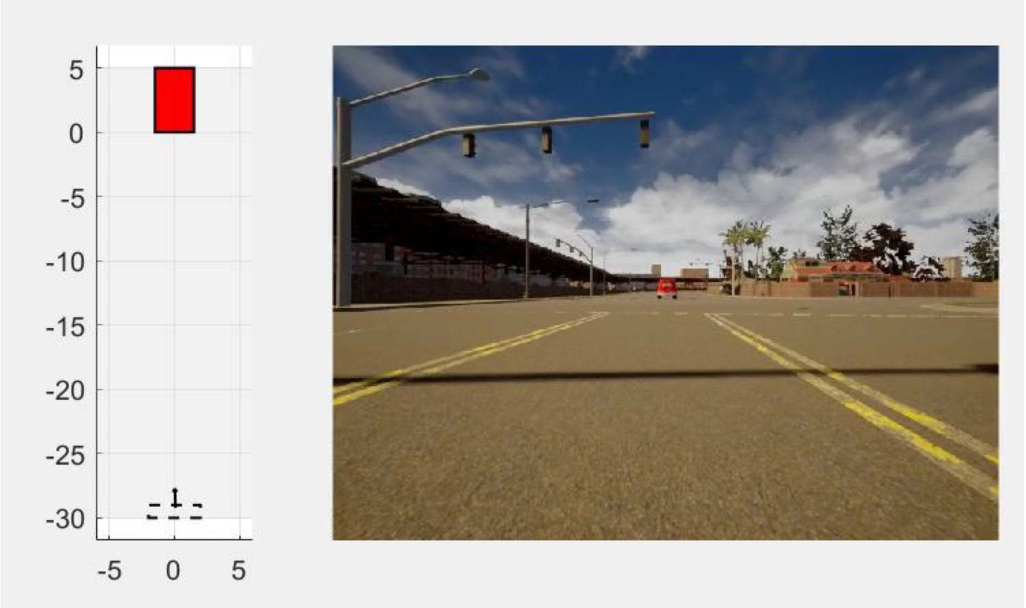
constant velocity + bounded disturbance w

$$y(t+1) = y(t) + \tau * \left(v_y(t) + w(t)\right)$$

Randomly perturbed images



Adversarially perturbed images with our falsification algorithm (similar perturbation magnitude)



Koopman-Inspired Safety Control for Unknown Nonlinear Systems

$z = \psi(x)$

Nonlinear system
$x^+ = f(x, u)$

$x = Cz$

Koopman over-approximation
$z^+ \in Az + Bu + W$

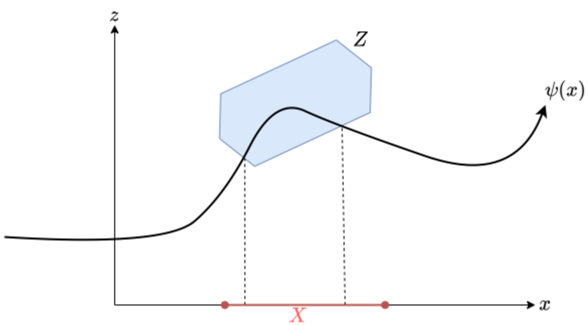
Hard

Easy

Inner approximates

Backward Reachable Set
X_t

Backward Reachable Set
Z_t



Two key ingredients:

- **Koopman over-approximation (KoA):** a simulation-like relation between the original system and Koopman-inspired abstraction

- **Implicit inner-approximation** Z of target set X where $\{x \mid \psi(x) \in Z\} \subseteq X$.

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Koopman over-approximation
$z^+ \in Az + Bu + W$

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Backward Reachable Set
X_t

Backward Reachable Set
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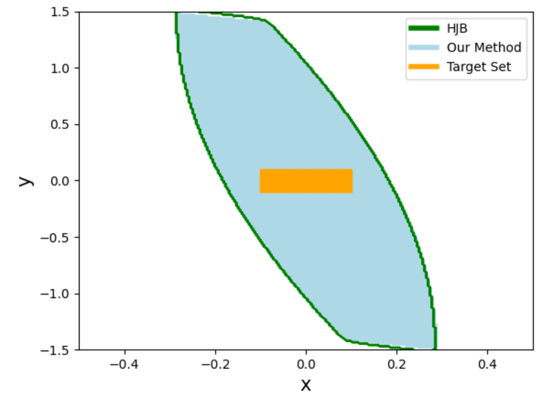
Forced Duffing Oscillator



Some properties:

- Any lifting function ψ including x in its coordinates can be used;
- The Koopman over-approximation is learned from data;
- If we can estimate local Lipschitz constants, can improve computation further by updating the linear representation locally

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Koopman-Inspired Safety Control for Unknown Nonlinear Systems

Nonlinear system
$x^+ = f(x,u)$

$z = \psi(x)$ ⇄ $x = Cz$

Koopman over-approximation
$z^+ \in A_i z + B_i u + W_i$ if $x \in D_i$

Hard ↓

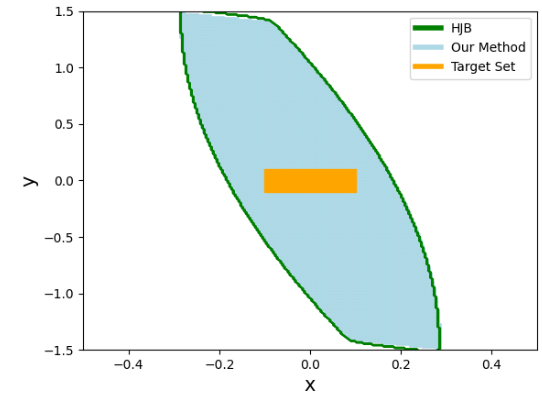
Easy ↓

Backward Reachable Set
X_t

Inner approximates ←

Backward Reachable Set
Z_t

Forced Duffing Oscillator

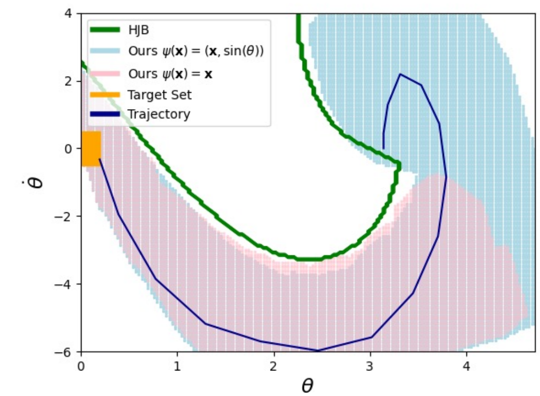


Single linearization is not enough:

- Different over-approximations are learned over local subdomains (leading to a PWA system) for better accuracy:
  - Experiments show that to obtain BRSs with similar sizes, the Koopman over-approximation requires less pieces than direct linearization (hybridization).

  Why do we need hybridization in the lifted space? See Liu, Ozay, Sontag IFAC WC'23 paper on non-existence of linear immersions for systems with multiple omega limit sets

Inverted Pendulum



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Koopman-Inspired Safety Control for Unknown Nonlinear Systems

$z = \psi(x)$

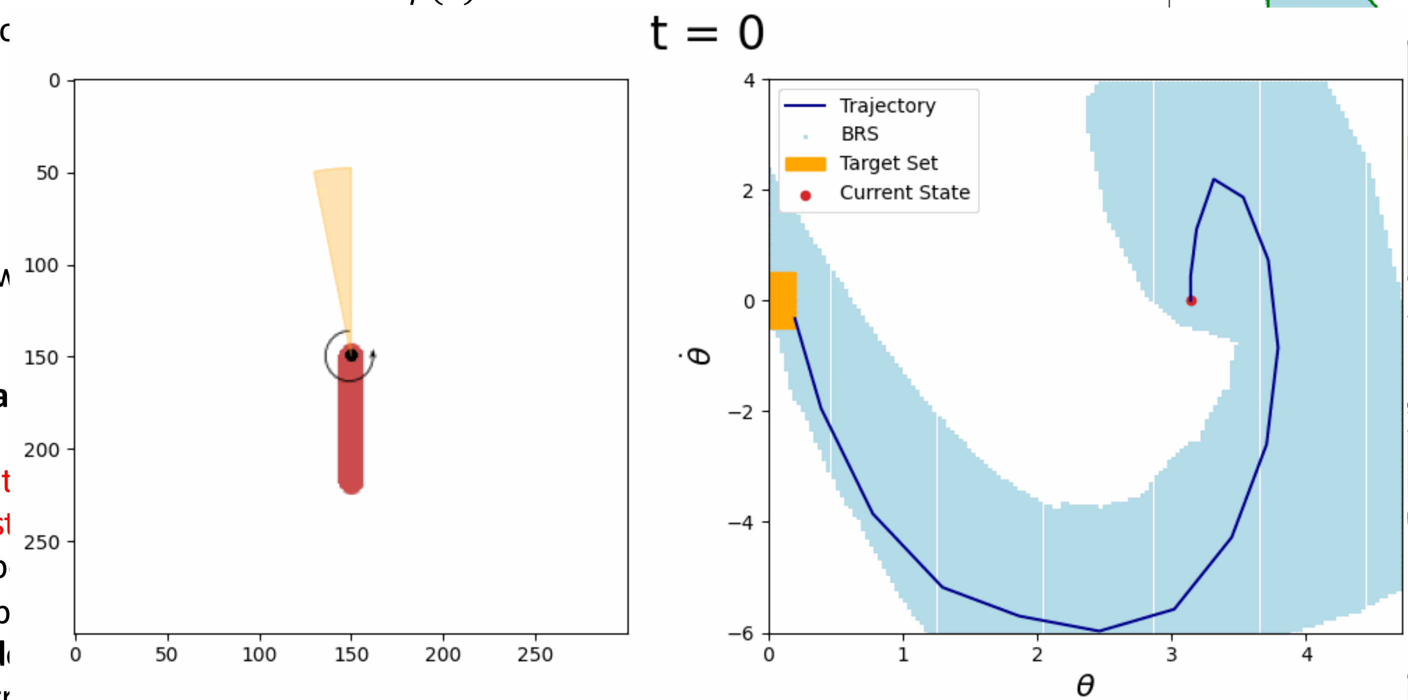
Nc

Backw

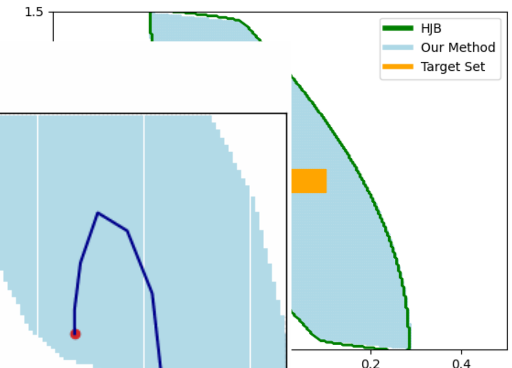
Single linea

- Different
  PWA syst
  - Exp
    app

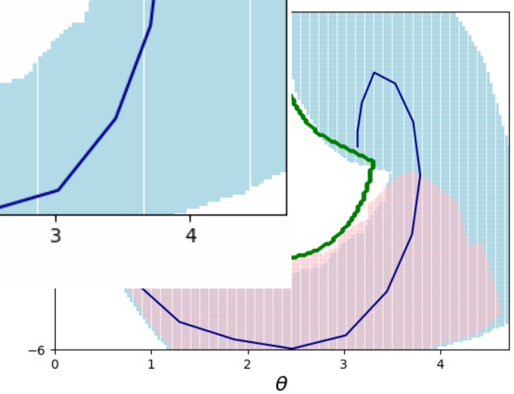
Why d
tomorr
with multiple omega limit sets



Forced Duffing Oscillator



ndulum



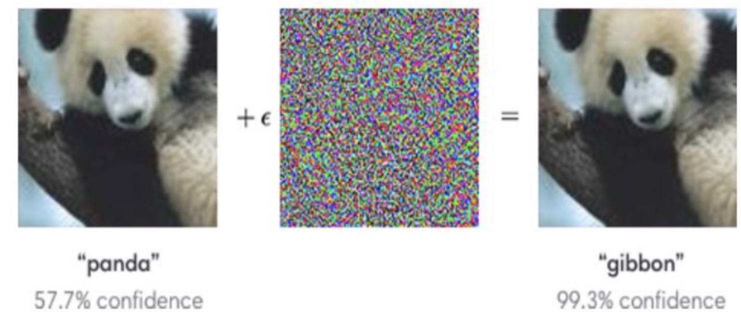
Balim, Aspeel, Liu, Ozay, L-CSS'23

Summary and conclusions

Key takeaways:

- Zonotopes and constrained zonotopes for backward reachability → applications in synthesis and falsification
- A new framework: synthesis-guided falsification:
  - Leads to explainable counterexamples
  - Works with blackbox controllers (code)
  - Extends to vision/perception-based control or end-to-end learning controllers
- An interesting connection between adversarial examples in machine learning and those in decision-making
- We can also do backward reachability for nonlinear systems using liftings

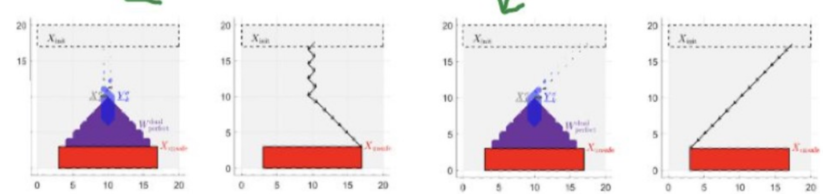
Machine learning:



Adversarial examples occur at decision boundaries in classification

Decision making (obstacle avoidance):

Controller 1: Controller 2:



Adversarial examples occur at decision boundaries

i.e., discontinuities of the controller

Linear programming (LP)

- An optimization problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \ldots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \ldots, p \end{array}$$

is an LP problem if f_i for $i=0,\ldots,m$ and h_i for $i=1,\ldots,p$ are affine functions. An LP is typically written in the following form:

$$\begin{array}{ll} \text{minimize}_x & c_0^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

In other words, an LP problem is an optimization problem whose objective function is linear and feasible set is defined by a polyhedron.