A nonsmooth hybrid invariance principle applied to robust event-triggered design

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***Abstract*—We first propose a nonsmooth hybrid invariance principle with relaxed conditions stemming from the fact that flowing solutions evolve only in the tangent cone, and complete jumping solutions cannot jump ouside the jump and flow sets. We then show an application consisting in the design of event-triggered rules to stabilize a class of uncertain linear control systems. The event-triggering rule depends only on local information, that is it uses only the output signals available to the controller. The approach proposed combines a hybrid framework to describe the closed-loop system with looped functionals based techniques. The proposed design conditions are formulated in terms of linear matrix inequalities (LMIs) ensuring global robust asymptotic stability of the closed-loop system. A tunable parameter is also available to guarantee an adjustable dwell-time property of the solutions. The effectiveness of the approach is evaluated through an example borrowed from the literature.**

***Index Terms*—Event-triggered control, hybrid dynamical systems, non-smooth Lyapunov function, LaSalle invariance principle, uncertain systems**

I. Introduction

Due to its useful ability to control systems by using only seldom control updates, the literature on event-triggered designs is increasing very fast, giving more and more flexibility, see for example, [4], [10], [2], [22], [27], [16]. In the context of event-triggered control, two objectives can be pursued: 1) the controller is a priori predesigned and only the event-triggered rules have to be designed, or 2) the joint design of the control law and the event-triggering conditions has to be performed. The first case refers to the emulation approach, whereas the second one corresponds to the co-design problem. A large part of the existing works is dedicated to the design of efficient event-triggering rules, that is the design is done by emulation: see, for example, [16], [23], and references therein. Moreover, most of the results on event-triggered control consider that the full state is available, which can be unrealistic from an applicative point of view. Hence, it is interesting to address the design of event-triggered controllers by using only measured signals. Some works have addressed this challenge as, for example, [24] in which the dynamic controller is an observer-based one, [1] in which the co-design of the output feedback law and the event-triggering conditions is addressed by using the hybrid framework.

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The paper focuses on the emulation problem, when the predesigned controller is issued from a hybrid dynamic output feedback controller, with the aim of using only the available signals. Furthermore the plant data is affected by polytopic uncertainty. Hence the problem is to ensure the robustness of the event-triggered strategy. The proof technique that we adopt for our event-triggered design combines a hybrid representation of the closed-loop system with the introduction of a non-strict and non-smooth Lyapunov function. Due to this fact, we propose an extension of La Salle's invariance principle that has independent interest and can be useful in other applications involving hybrid dynamics. The extension is based on the invariance principle in [19] and [8, Ch. 8] and some observations (already made in [7]) that there is no need to check the flow and jump condition in the attractor, that the flow condition only needs to be checked in the directions of the tangent cone to the flow set (as already established in [19, Thm. 4.7]), and that possible jumps mapping outside the jump and flow sets lead to a dead end, so that one can disregard them in the Lyapunov conditions (this fact was already exploited in [8, Thm. 3.37]). A final useful feature that we inherit from [19] is that we allow for nonsmooth Lyapunov functions V that only need to be locally Lipschitz in the flow set and continuous in the jump set, and then rely on Clarke's generalized gradient [5] for dealing with flowing solutions.

The proposed event-triggered construction is obtained by adapting the recent developments arising from the stability analysis of persistent sampled-data systems (see the recent survey paper [11]), and using the new non-smooth LaSalle result discussed above. Constructive conditions are presented, in the sense that linear matrix inequality (LMI) conditions associated to a convex optimization scheme, are proposed to design the event-triggered rule ensuring robust asymptotic stability of the closed-loop system. Furthermore, differently from [1], our conditions provide a guaranteed dwell time T which can be optimized, similarly to [12]. This paper is a revised and improved version of [21] where no plant uncertainty was addressed and the proofs were not reported, nor the nonsmooth hybrid invariance principle commented above.

The paper is organized as follows. Section II provides our first contribution regarding the nonsmooth invariance principle for hybrid dynamical systems, which will be illustrated in the sequel on an output feedback event-triggered control design. Section III first formulates the problem of output feedback event-triggered control of uncertain linear systems and then develops a dedicated stability analysis employing the previous invariance principle for the nominal and uncertain cases. Section V illustrates the results on numerical examples

and compares them with some existing approach. Finally, Section VI draws some concluding remarks and perspectives.

Notation. The sets $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the sets of positive integers, positive scalars, n-dimensional vectors, $n\times n$ matrices and symmetric matrices in $\mathbb{R}^{n\times n}$. If a matrix P in $\mathbb{S}^n_+$, it means that P is symmetric positive definite. The superscript "$\top$" stands for matrix transposition, and the notation $\mathrm{He}(P)$ stands for $P+P^\top$. The Euclidean norm is denoted $|\cdot|$. Given a compact set $\mathscr{A}$, the notation $|x|_{\mathscr{A}} := \min\{|x-y|,\ y\in\mathscr{A}]\}$ indicates the distance of the vector x from the set $\mathscr{A}$.The symbols I and 0 represent the identity and the zero matrices of appropriate dimensions.

II. A nonsmooth La Salle result

Consider the hybrid dynamics:

$$\mathscr{H}\begin{cases}\dot\xi\in F(\xi), & \xi\in\mathscr{C}\\ \xi^+\in G(\xi), & \xi\in\mathscr{D}.\end{cases}\tag{1}$$

We propose here an invariance principle with nonsmooth functions stemming from the results in [7, Thm 23], [19], which rely on the nonsmooth tools of [5] (see also their use in [13]). To this end, denote *nonsmooth Lyapunov function candidate* any function $V:\mathrm{dom}\,V\to\mathbb{R}$ such that

1) $(\mathscr{C}\cup\mathscr{D})\subset\mathrm{dom}\,V$ and V is continuous in $\mathscr{C}\cup\mathscr{D}$ and locally Lipschitz [1] near each point in $\mathscr{C}\setminus\mathscr{A}$.
2) V is positive definite with respect to $\mathscr{A}$ in $\mathscr{C}\cup\mathscr{D}$ (namely, $V(\xi)=0,\ \forall\xi\in\mathscr{A}$ and $V(\xi)>0,\ \forall\xi\in(\mathscr{C}\cup\mathscr{D})\setminus\mathscr{A}$) and radially unbounded (namely for any infinite sequence of points $\xi_i\in\mathscr{C}\cup\mathscr{D}$, $\lim_{i\to\infty}|\xi_i|=\infty$ implies $\lim_{i\to\infty}V(\xi_i)=\infty$).

In addition to using nonsmooth weak Lyapunov functions, a few desirable features of the result also come from the fact that 1) following the results in [19, Thm. 4.7 and eqn. (11)], we restrict the directions where the flow condition must be verified to those belonging to the tangent cone to the flow set $\mathscr{C}$ (see also similar observations in [16]); 2) we restrict the points where the jump condition must be verified by relying on the fact that solutions jumping outside $\mathscr{C}\cup\mathscr{D}$ reach a dead end and are not complete (see also the ideas in [8, Thm 3.37]). This fact also allows us to only require that $\mathscr{C}\cup\mathscr{D}\subset\mathrm{dom}\,V$, rather than the typical (stronger) requirement $\mathscr{C}\cup\mathscr{D}\cup G(\mathscr{D})\subset\mathrm{dom}\,V$.

Theorem 1: *Consider a compact set $\mathscr{A}$ and the hybrid system* (1) *satisfying the hybrid basic assumptions in [7, page 43], and satisfying $G(\mathscr{A}\cap\mathscr{D})\subset\mathscr{A}$.*

Assume that there exists a non-smooth Lyapunov function candidate V, such that

$$\dot V(\xi):=\max_{v\in\partial V(\xi),f\in F(\xi)\cap T_{\mathscr{C}}(\xi)}\langle v,f\rangle\le 0,\quad\forall\xi\in\mathscr{C}\setminus\mathscr{A},\tag{2}$$

$$\Delta V(\xi):=\max_{g\in G(\xi)\cap(\mathscr{C}\cup\mathscr{D})}V(g)-V(\xi)\le 0,\quad\forall\xi\in\mathscr{D}\setminus\mathscr{A},\tag{3}$$

where $\partial V(\xi)$ is the Clarke generalized gradient of V at ξ (see [5] for its definition) and $T_{\mathscr{C}}(\xi)$ denotes the tangent cone to set $\mathscr{C}$ at point ξ (see, e.g., [8, Def 5.12]).

Assume also that no complete solution keeps V constant and nonzero, namely no complete solution ξ exists satisfying $V(\xi(t,j))=V(\xi(0,0))\neq 0$, for all $(t,j)\in\mathrm{dom}\,\xi$.

Then set $\mathscr{A}$ is uniformly globally asymptotically stable (UGAS) for $\mathscr{H}$.

Remark 1: (Connection with existing results) We emphasize that most of the intuitions behind Theorem 1 are already present in the literature, but they have not been previously stated in such a compact form, which turns out to be a useful tool for stability analysis of hybrid systems. Restricting the attention to the tangent cone was already done in [19, Thm 4.7 and eqn. (11)], even though the proof provided here is somewhat more explicit. Ignoring bad jumps was done in [8, Thm 3.37], even though that theorem addressed a more general situation with unbounded attractors. Imposing the Lyapunov conditions outside the attractor $\mathscr{A}$ has also been done in [7, Thm 23], even though that result did not contain a proof. The desirable feature of our Theorem 1 is that all these relaxations are compactly and rigorously stated and proven now in a combined way. ∘

Proof. We first prove the result for the reduced dynamics $\tilde{\mathscr{H}}:=(\mathscr{C},F,\tilde{\mathscr{D}},\tilde G)$, where $\tilde G:=G\cap(\mathscr{C}\cup\mathscr{D})$ and $\tilde{\mathscr{D}}$ is the set of points in $\xi\in\mathscr{D}$ such that $\tilde G(\xi)$ is nonempty. It is immediate to see that closedness of $\mathscr{C}$ and $\mathscr{D}$, together with outer semicontinuity of G implies that $\tilde{\mathscr{D}}$ is closed and $\tilde G$ is outer semicontinuous too. Therefore $\tilde{\mathscr{H}}$ satisfies the hybrid basic assumptions in [7, page 43] in addition to $\tilde G(\tilde{\mathscr{D}})\subset(\mathscr{C}\cup\mathscr{D})$.

The reduced system $\tilde{\mathscr{H}}$ is constructed following similar steps to [8, Thm 3.37] to remove "bad jumps" mapping the state outside $\mathscr{C}\cup\mathscr{D}$ from the analysis. After proving UGAS of $\mathscr{A}$ for $\tilde{\mathscr{H}}$, we will extend the result to the original dynamics $\mathscr{H}$, which includes the bad jumps.

Proof of UGAS of $\mathscr{A}$ for $\tilde{\mathscr{H}}$. The proof follows four steps.

Step 1. *Auxiliary hybrid system $\mathscr{H}_T$.* Consider any solution x of $\tilde{\mathscr{H}}$ with $\mathrm{dom}\,x=\bigcup_{j=0}^J I^j\times\{j\}$, where possibly $J=\infty$. By definition of solution, given any $(t,j)\in\mathrm{dom}\,x$, function $x(\cdot,j):I^j\to\mathbb{R}^n$ is absolutely continuous, therefore differentiable almost everywhere in I^j. As a consequence, for almost all $\tau\in I^j$, we have that $\dot x(\tau,j)$ is well defined and belongs to $F(x(\tau,j))$. If I^j has nonempty interior, then for almost all $\tau\in I^j$, we may select a sequence h_i, $i\in\mathbb{N}$ converging to zero and such that $\tau+h_i\in I^j$, so that $x(\tau+h_i,j)\in\mathscr{C}$ (by definition of solution). As a consequence we have

$$\begin{aligned}\dot x(\tau,j) &= \lim_{h\to0^+}\tfrac{x(\tau+h,j)-x(\tau)}{h}=\lim_{i\to\infty}\tfrac{x(\tau+h_i,j)-x(\tau)}{h_i}\\ &\in T_{\mathscr{C}}(x(\tau,j)),\end{aligned}$$

where the last equality follows from the definition of tangent cone. The above derivations show that $\dot x(\tau,j)\in F(x(\tau,j))\cap T_{\mathscr{C}}(x(\tau,j)$ for almost all $\tau\in I^j$. [2] Therefore all solutions to $\tilde{\mathscr{H}}$ are also solutions to the following restricted dynamics:

$$\mathscr{H}_T\begin{cases}\dot\xi\in F(\xi)\cap T_{\mathscr{C}}(\xi), & \xi\in\mathscr{C}\\ \xi^+\in\tilde G(\xi), & \xi\in\tilde{\mathscr{D}}.\end{cases}\tag{4}$$

[1] According to [5] we say that a function is locally Lipschitz near a point if there exists a neighborhood of that point where the function is Lipschitz.

[2] Note that this observation about flowing solutions only selecting directions in the tangent cone was already made in [19, p. 2289, left column].

The converse also holds trivially because the data of $\mathscr{H}_T$ is a subset of the data of $\tilde{\mathscr{H}}$. Then solutions to $\mathscr{H}_T$ and $\tilde{\mathscr{H}}$ coincide.

Step 2. *Proof of [19, eqn (2)].* Consider now any solution x to $\mathscr{H}_T$ never reaching $\mathscr{A}$ and notice that due to the stated Lipschitz assumption, the generalized gradient $\partial V(x)$ is well defined for all $x \in \mathscr{C} \setminus \mathscr{A}$. Then following similar reasonings to those in [25, page 99], given any $(t,j) \in \mathrm{dom}\, x$, we obtain that (2) implies $\frac{d}{dt}V(x(t,j)) \leq \dot{V}(x(t,j)) \leq 0$ for almost all $t \in I^j$, as long as I^j has nonempty interior. Moreover, for any $(t,j) \in \mathrm{dom}\, x$ such that $(t,j+1) \in \mathrm{dom}\, x$, we have from (3) that $V(x(t,j+1)) \leq V(x(t,j))$. Integrating over all I^j and taking sums, we obtain that

$$V(x(t',j')) \leq V(x(t,j)), \forall (t',j') \succeq (t,j), \tag{5}$$

where $(t',j') \in \mathrm{dom}\, x$ and $(t,j) \in \mathrm{dom}\, x$.

Let us now extend the proof of (5) also for solutions reaching $\mathscr{A}$. To this end, we first prove (strong) forward invariance of $\mathscr{A}$. In particular, since $G(\mathscr{A} \cap \mathscr{D}) \subset \mathscr{A}$, then $\tilde{G}(\mathscr{A} \cap \tilde{\mathscr{D}}) \subset G(\mathscr{A} \cap \mathscr{D}) \subset \mathscr{A}$ and no solution can leave $\mathscr{A}$ across a jump. Thus any solution ξ_{bad} possibly leaving $\mathscr{A}$, must do this while flowing. Let us show that this is impossible, by contradiction. From continuity of flowing solutions and closedness of $\mathscr{A}$, if ξ_{bad} leaves $\mathscr{A}$ during flow, then there exist $(t_1,j) \in \mathrm{dom}\ \xi_{\mathrm{bad}}$ and $(t_2,j) \in \mathrm{dom}\ \xi_{\mathrm{bad}}$ with $t_2 > t_1$, such that $\xi(t_1,j) \in \mathscr{A}$ and $\xi(t_2,j) \notin \mathscr{A}$, for all $t \in (t_1,t_2]$, implying $0 = V(\xi(t_1,j)) < V(\xi(t_2,j))$, due to positive definiteness of V. Since the solution is flowing, $\xi(t,j) \in \mathscr{C} \setminus \mathscr{A}$, for all $t \in (t_1,t_2]$, and thus integrating (2) as done before (5), exploiting $\frac{d}{dt}V(x(t,j)) \leq \dot{V}(x(t,j)) \leq 0$, we obtain $V(\xi(t_2,j)) \leq V(\xi(t_1,j))$, thereby establishing a contradiction. As a consequence, $\mathscr{A}$ is (strongly) forward invariant.

Let us now use forward invariance of $\mathscr{A}$ to prove (5) also for solutions passing through $\mathscr{A}$. Consider any solution x reaching $\mathscr{A}$ for some $(t^*,j^*) \in \mathrm{dom}\, x$. Then the reasoning before (5) proves (5) for any $(t',j') < (t^*,j^*)$. Since the solution remains in $\mathscr{A}$ for all times larger than (t^*,j^*), then (5) is trivially satisfied for all $(t',j') \geq (t^*,j^*)$ from positive definiteness of V. We conclude this step by emphasizing that (5) holds for all solutions to $\mathscr{H}_T$ and that this property coincides with [19, eqn (2)] with selection $u_c \equiv 0$ and $u_d \equiv 0$.

Step 3. *Proof of global convergence to $\mathscr{A}$.* To prove convergence we use [19, Thm 4.3] as detailed below. Consider the abstract hybrid system defined in [19, Def. 2.3] and notice that by [19, Thm. 2.6], solutions to $\tilde{\mathscr{H}}$ satisfy that definition. Since we proved in Step 1 that solutions to $\tilde{\mathscr{H}}$ and $\mathscr{H}_T$ coincide, then also solutions to $\mathscr{H}_T$ form an abstract hybrid system as in [19, Def. 2.3] and join the desirable properties established in [19, Thm 4.3]. In particular, since [19, eqn (2)] holds for all solutions with function V (this has been proven in Step 2 above), then each complete solution approaches the largest weakly invariant subset of some level set of V (nothing needs to be checked about convergence of compact solutions, therefore we only focus on complete solutions). Since by assumption no complete solution keeps V constant and nonzero, then all complete solutions approach the zero level set of V, thereby completing the proof of global convergence.

Step 4. *Proof of UGAS of $\mathscr{A}$.* Stability of $\mathscr{A}$ follows from equation (5), continuity, positive definiteness and radial unboundedness of V, implying the existence of class $\mathscr{K}_\infty$ upper and lower bounds for V, depending on the distance to $\mathscr{A}$. Then stability plus global convergence imply UGAS from [8, Thm 7.12].

Proof of UGAS of $\mathscr{A}$ for $\mathscr{H}$. Extending the UGAS property of $\mathscr{A}$ from $\tilde{\mathscr{H}}$ to $\mathscr{H}$ amounts to accounting for the solutions of $\mathscr{H}$ that are not solutions of $\tilde{\mathscr{H}}$. Such solutions correspond to the non-complete solutions jumping from $\mathscr{D}$ to $G(\mathscr{D}) \setminus (\mathscr{C} \cup \mathscr{D})$ and are not complete by definition because the corresponding final jump is a dead end. As a consequence, global convergence to $\mathscr{A}$ for $\tilde{\mathscr{H}}$ (which is a property of complete solutions) implies global convergence to $\mathscr{A}$ for $\mathscr{H}$. To prove stability of $\mathscr{A}$ for $\mathscr{H}$, we may follow similar steps to those in [8, Thm 3.37], which rely on the property that there exists $\gamma \in \mathscr{K}_\infty$ such that

$$|g|_{\mathscr{A}} \leq \gamma(|\xi|_{\mathscr{A}}), \quad \forall \xi \in \mathscr{D}, \forall g \in G(\xi). \tag{6}$$

Condition (6) is guaranteed under our assumptions that $\mathscr{A}$ is compact, $G(\mathscr{A} \cap \mathscr{D}) \subset \mathscr{A}$, and G is outer semi-continuous and locally bounded (which is guaranteed from the hybrid basic assumptions in [7, page 43]). Indeed, to establish (6), it is enough to have that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$G((\mathscr{A} + \delta\mathscr{B}) \cap \mathscr{D}) \subset \mathscr{A} + \varepsilon\mathscr{B}, \tag{7}$$

where $\mathscr{B}$ denotes the closed unit ball. Let us prove (7) by contradiction. Namely suppose that there exists $\varepsilon > 0$ such that, for each $i \in \mathbb{N}$ there exists $\xi_i \in (\mathscr{A} + i^{-1}\mathscr{B}) \cap \mathscr{D}$ and $g_i \in G(\xi_i)$ such that $|g_i|_{\mathscr{A}} > \varepsilon$. Without loss of generality (due to local boundedness of G and compactness of $\mathscr{A}$), let $(\xi_i, g_i)_{i \in \mathbb{N}}$ be a converging sequence converging to (ξ, g). Since $\mathscr{A}$ and $\mathscr{D}$ are closed, then $\xi \in \mathscr{A} \cap \mathscr{D}$. Moreover $|g|_{\mathscr{A}} \geq \varepsilon$ and, by outer semi-continuity, $g \in G(\xi)$. This is a contradiction to $G(\mathscr{A} \cap \mathscr{D}) \subset \mathscr{A}$. ■

III. Application to event-triggered control

A. Hybrid representation of sampled-data systems

Consider a linear system fed by an output feedback sampled-data control given by the hybrid dynamical system

$$\begin{cases} \dot{x} &= Ax + Bu, \\ \dot{u} &= 0, \qquad\qquad (x,u,\sigma) \in \mathscr{C}, \\ \dot{\sigma} &\in g_T(\sigma), \end{cases}$$
$$\begin{cases} x^+ &= x, \\ u^+ &= KCx, \qquad (x,u,\sigma) \in \mathscr{D}, \\ \sigma^+ &= 0, \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n$ represents the state of the system and $u \in \mathbb{R}^m$ represents the zero order holder of the system input since the last sampling time. The output of the system y is given by

$$y = Cx \in \mathbb{R}^p. \tag{9}$$

Such a system (8)-(9) can appear when connecting, for instance, a linear continuous plant with a dynamic output feedback controller (see the example Section V). Then, to study this kind of systems, the hybrid formalism of [7], [17], [18] can be used. Matrices A,B,C characterize the system dynamics and matrix K corresponds to the controller gain. While C is assumed to be constant and known, let us assume that matrices A and B are constant but uncertain, such that

$$[A\ B] \quad \in \quad Co\{\ [A_i\ B_i]\ \}_{i\in\mathscr{I}}, \tag{10}$$

for some constant and known matrices A_i and B_i, for $i \in \mathscr{I}$ where $\mathscr{I}$ is a bounded subspace of $\mathbb{N}$. Timer $\sigma \in [0,2T]$ flows by keeping track of the elapsed time since the last sample (where it was reset to zero) according to the following set-valued dynamics:

$$g_T(\sigma) \quad := \quad \begin{cases} 1 & \sigma \leq 2T \\ [0,1] & \sigma = 2T, \end{cases} \tag{11}$$

whose rationale is that whenever $\sigma < 2T$, its value exactly represents the elapsed time since the last sample, moreover $\sigma \in [T,2T]$ implies that at least T seconds have elapsed since the last sample.[3] In (8), the so-called flow and jump sets $\mathscr{C}$ and $\mathscr{D}$ must be suitably selected to induce a desirable behavior on the sampled-data system and are the available degrees of freedom in the design of the event-triggered algorithm.

The problem we intend to solve in this section is reported below and corresponds to an emulation problem (see, for example [9], [26], [15], [23] and the references therein) since we assume that the controller is given.

Problem 1: Given an uncertain linear plant and a hybrid controller defined by matrices A_i,B_i for $i \in \mathscr{I}$ and K,C, design an event-triggering rule, with a prescribed dwell-time T that makes the closed-loop system (8)-(11) globally asymptotically stable to a compact set wherein $x = 0$ and $u = 0$.

B. Event-triggered design

In order to address Problem 1, we focus on hybrid dynamics (8) for suitably selecting the flow and jump sets $\mathscr{C}$ and $\mathscr{D}$ whose role is precisely to rule when a sampling should happen, based on the available signals to the controller, namely output $y = Cx$, the last sampled input u and timer σ. Then, we select the following sets $\mathscr{C}$ and $\mathscr{D}$:

$$\mathscr{C} \quad := \quad \mathscr{F} \cup \{\sigma \in [0,T]\} \tag{12a}$$

$$\mathscr{D} \quad := \quad \mathscr{J} \cap \{\sigma \in [T,2T]\}, \tag{12b}$$

where sets $\mathscr{F}$ and $\mathscr{J}$ are selected as

$$\mathscr{F} := \left\{(x,u) : \begin{bmatrix} y \\ s-Ky \end{bmatrix}^\top M \begin{bmatrix} y \\ s-Ky \end{bmatrix} \leq 0 \right\}, \tag{12c}$$

$$\mathscr{J} := \left\{(x,u) : \begin{bmatrix} y \\ s-Ky \end{bmatrix}^\top M \begin{bmatrix} y \\ s-Ky \end{bmatrix} \geq 0 \right\}, \tag{12d}$$

[3]Note that the use of a set-valued map for the right hand side g_T of the flow equation for σ enables us to confine the timer σ to a compact set $[0,2T]$, while at the same time using dynamics whose right hand sides are outer semicountinuous set-valued mappings, thereby satisfying the regularity conditions in [8, As. 6.5] and enjoying the desirable robustness properties of stability of compact attractors established in [8, Ch. 7].

where matrix $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{R}^{(p+m)\times(p+m)}$ has to be designed, and y is defined in (9). Solution (12) to the considered event-triggered problem is parametrized by M and T.

Note that the jump set selection in (12b) ensures that all solutions satisfy a dwell-time constraint corresponding to T. Indeed, jumps are inhibited unless timer $\sigma \geq T$, which, according to (11), implies that at least T ordinary time elapses between each pair of consecutive sampling times. In the following developments, the dwell-time T will be a parameter for the design of the matrix M that defines the flow and jump sets. The contribution of the next sections is to provide LMI-based design rules for M in the nominal and uncertain cases.

Remark 2: The definition of the flow and jump sets provided in (12) meets the one provided in the recent paper [14]. The novelty of this definition relies on the consideration of a general matrix M. For example, selecting $M_2 = 0$ leads to the definition of the flow and jump sets usually employed in the literature, issued from an Input-to-State (or Input-to-Output) analysis. See [14] for more details. ◦

C. LMI-based design of M: nominal case

In this section, we will assume that matrices A,B are constant and known (e.g. $\mathscr{I} = \{0\}$). The next theorem is the corresponding design result.

Theorem 2: *Assume that there exist matrices $P \in \mathbb{S}^n$, $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{S}^{p+m}$ satisfying*

$$\begin{aligned} &\Psi_M(A,B) := \begin{bmatrix} He(PA_{cl}) - C^\top M_1 C & PB - C^\top M_2 \\ B^\top P - M_2^\top C & -M_3 \end{bmatrix} < 0, \\ &\Phi(A,B,T) := \\ &\quad \left(\Lambda(A,B,T)\begin{bmatrix} I \\ KC \end{bmatrix}\right)^\top P \Lambda(A,B,T)\begin{bmatrix} I \\ KC \end{bmatrix} - P < 0, \end{aligned} \tag{13}$$

with $A_{cl} := A + BKC$ and

$$\Lambda(A,B,T) := \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T} \in \mathbb{R}^{n\times 2n}. \tag{14}$$

Then the compact attractor

$$\mathscr{A} := \{(x,u,\sigma) :\ x = 0, u = 0, \sigma \in [0,2T]\}, \tag{15}$$

is globally asymptotically stable for the nominal closed-loop dynamics (8), (12).

The proof of our theorems is based on the use of a non-smooth Lyapunov function and the La Salle result of Section II. In particular, we use the following function:[4]

$$V(x,u,\sigma) := \underbrace{e^{-\rho \min\{\sigma,T\}} \left| \Lambda(T - \min\{\sigma,T\}) \begin{bmatrix} x \\ u \end{bmatrix} \right|_P^2}_{=:V_0(x,u,\sigma)} + \underbrace{\eta |u|^2}_{=:V_u(u)}, \tag{16}$$

with Λ given in (14), and where ρ and η are sufficiently small positive scalars selected later.

Before exposing the proof of Theorem 2, which is reported in Section IV, several remarks are stated.

Remark 3: The LMI conditions can be interpreted as follows

[4]Here we use the standard notation $|z|_P^2 := z^\top P z$.

- The condition $\Psi_M(A,B) < 0$ imposes that the Lyapunov function V in (16) is decreasing while flowing with $\sigma \geq T$ (which requires $(x,u) \in \mathscr{F}$).
- The condition $\Phi(A,B,T) < 0$ can be interpreted as an asymptotic stability criterion for system (8) when the control updates are performed periodically with a period T, which motivates the union and intersection in (12a) and (12b). This condition also guarantees that the Lyapunov function V in (16) is non-increasing while flowing and when $\sigma < T$.

$\circ$

Remark 4: The dwell-time T appears as a tuning parameter of the event-triggered control system (8)-(12). The interest of the proposed approach with respect to the literature, where the dwell-time is computed a posteriori, resides in the fact that Theorem 2 includes a guaranteed dwell-time T as a tuning parameter. In particular, if one can find a solution to the LMI conditions (13) for a given parameter T, then this same T can be employed in the definition of the flow and jumps sets (12) and becomes a guaranteed dwell time for all solutions. This method can be compared to [23] or [1] where a similar triggering rule includes a dwell time constraint. Nevertheless, compared to these contributions, the dwell time T appears as a parameter for the design of event trigger algorithm. $\circ$

Remark 5: Theorem 2 presents a similar result as compared to the preliminary contribution presented in [21], where an additional reset control component is included in the jump dynamics of system (8)-(12). The advances with respect to [21] will be presented in the next section to assess robust stability of uncertain system (8)-(10). $\circ$

D. LMI-based design of M: uncertain case

In the case where matrices A, B and parameter T are known and constant, inequality $\Phi(A,B,T) < 0$ can be easily implemented and verified. However, when matrices A and B are not assumed to be know anymore, but become uncertain, verifying inequality $\Phi(A,B,T) < 0$ for any pair (A,B) in (10) becomes a difficult nonlinear problem. In this section, we propose a method to guarantee this inequality even for uncertain matrices A and B. This method is taken from [20, Thm 1] and is based on the recent developments arising from stability analysis of persistent sampled-data systems (see the recent survey paper [11]). In particular, the following lemma follows in a straightforward way from the looped-functional approach developed in [20], [3]. To avoid repetition, the proof is omitted.

Lemma 1: *For a given positive scalar T and matrices A_i, B_i, K, C as defined in* (10)*, if there exist $P, Z \in \mathbb{S}^n_+$, $Q, U \in \mathbb{S}^n$, $R \in \mathbb{R}^{n\times n}$ and $Y_i \in \mathbb{R}^{2n\times n}$, $i = 1,\dots,m$ such that the inequalities*

$$\begin{array}{rcl} \Theta_1(A_i,B_i,T) := & F_0(A_i,B_i,T) + TF_1(A_i,B_i) & < \ 0, \\ \Theta_2(A_i,B_i,T) := & \begin{bmatrix} F_0(A_i,B_i,T) & TY_i \\ \star & -TZ \end{bmatrix} & < \ 0, \end{array} \tag{17}$$

hold for all $i = 1,\dots,m$ with

$$\begin{aligned} F_0(A_i,B_i,T) &:= T\mathrm{He}\{e_{0i}^\top P e_1 - Y_i e_{12} - e_{12}^\top R e_2\} \\ &\quad - e_{12}^\top Q e_{12} - e_2^\top T X e_2, \\ F_1(A_i,B_i) &:= \mathrm{He}[e_{0i}^\top Q e_{12} + e_{0i}^\top R e_2] + e_{0i}^\top Z e_{0i} + 2e_2^\top X e_2, \end{aligned}$$

and $e_{0i} := \begin{bmatrix} A_i & B_i KC \end{bmatrix}$, $e_1 := \begin{bmatrix} I_n & 0 \end{bmatrix}$, $e_2 := \begin{bmatrix} 0 & I_n \end{bmatrix}$ and $e_{12} := \begin{bmatrix} I_n & -I_n \end{bmatrix}$, then inequality $\Phi(A,B,T) < 0$ in (13) *holds for each pair (A,B) satisfying* (10)*.*

The theorem below reports our main robust result solving Problem 1. It is based on the nonsmooth hybrid Lyapunov function introduced in (16), which is weak in the sense that it does not increase both during flow and across jumps (samplings) of the proposed event-triggered sampled-data system. The proof then relies on the nonsmooth invariance principle of Section II. The details are given in Section IV.

Theorem 3: *Assume that there exist matrices $P \in \mathbb{S}^n$, $M := \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{S}^{p+m}$, and matrices $Z \in \mathbb{S}^n_+$, $Q, U \in \mathbb{S}^n$, $R \in \mathbb{R}^{n\times n}$ and $\Xi_i \in \mathbb{R}^{2n\times n}$, $i = 1,\dots,m$ satisfying conditions $\Psi_M(A_i,B_i) < 0$, $\Theta_j(A_i,B_i,T) < 0$ for all $i \in \mathscr{I}$, $j = 1,2$, where Θ_i and Ψ_M are given in* (17) *and* (13)*, respectively. Then the compact attractor $\mathscr{A}$ in* (15) *is globally asymptotically stable for the uncertain closed-loop dynamics (8)-(11), (12).*

Remark 6: Theorem 3 represents an extension of Theorem 2 to the case of uncertain systems. The only difference relies on condition $\Phi(A,B,T) < 0$ in (13) that, in the uncertain case, becomes nonlinear with respect to matrices A, B and has been then replaced by the linear conditions $\Theta_j(A_i,B_i,T) < 0$, with $i \in \mathscr{I}$, $j = 1,2$. $\circ$

E. Optimization

In this section we propose an optimization process for the selection of the matrix M and the associated triggering rule. From the LMI $\Psi_M < 0$ in (13), the matrices $\mathrm{He}(PA_{cli}) - C^\top M_1 C$ and $-M_3$ are required to be negative definite. Then a natural optimization procedure consists in the minimization of the effect of the off-diagonal term $PB_i - C^\top M_2$. This optimization can be performed by minimizing the size of the positive definite matrix M_3 appearing on the diagonal. Obtaining small values of the diagonal term $-M_3$ will indeed reduce also the off-diagonal term in (13). This optimization problem can be reformulated in terms of an LMI optimization as follows

$$\begin{array}{rcl} \min\limits_{P,M} Tr(M_3), \text{ subject to:} & P > I, \ M_1 < 0, & \\ \Psi_M(A_i,B_i) & < \ 0, & \\ \Theta_j(A_i,B_i,T) & < \ 0, \ j = 1,2, & \forall i \in \mathscr{I}. \end{array} \tag{18}$$

In the optimization problem (18), the additional constraint $P > I$ has been imposed for well conditioning the LMI constraints. In addition, constraint $M_1 < 0$ has been included in order to obtain negative definiteness of $\mathrm{He}(PA_{cli})$ in (13), which avoids exponentially unstable continuous dynamics, thereby giving more graceful inter-sample transients.

Furthermore, note that minimizing the trace of M_3 aims at increasing the negativity of matrix M_3, which, in turns, leads to larger flow sets (see equation (12)). Indeed, the set of vectors x for which $x^\top M_3 x \leq 0$ becomes increasingly larger for matrices M_3 with decreasing trace. Since the jump set is the closed complement of the flow set, it is expected that solutions will flow longer and jump less in light of larger flow sets.

IV. Proof of Theorems 2 and 3

We will show below that function V in (16) is a non-strict Lyapunov function for the nominal and uncertain closed

loops. Then, we will make use of the nonsmooth invariant principle presented in Section II. Note first that $G(\xi) \in \mathscr{A}$ for all $\xi \in \mathscr{D} \cap \mathscr{A}$, because $(x^+, u^+) = (0,0)$ for $(x,u) = (0,0)$. The rest of the proof focuses on showing the assumptions of Theorem 1 with $\mathscr{A}$ in (15) and V in (16).

We first show that, for any $\rho > 0$ and $\eta > 0$, V is a positive definite function with respect to the compact attractor $\mathscr{A}$ in (15). To this end, let us denote $\xi := (x,u,\sigma)$. Then note that $\xi = (x,u,\sigma) \in \mathscr{A}$ implies $x = 0$ and $s = 0$, so that clearly $V(\xi) = 0$ for all such values of ξ. To show that $V(\xi) > 0$ for all $\xi \notin \mathscr{A}$, first note that $(x,u,\sigma) \notin \mathscr{A}$, implies either $u \neq 0$, or $x \neq 0$. If $u \neq 0$, then $V(\xi) > 0$, because the second term V_u is positive for any $\eta > 0$. Assume now that $u = 0$, which implies $x \neq 0$ if $\xi \notin \mathscr{A}$. The first term in V then becomes

$$V_0(x,0,\sigma) = \mathrm{e}^{-\rho \min\{\sigma,T\}} \left| \mathrm{e}^{A(T-\min\{\sigma,T\})} x \right|_P,$$

which is nonzero because of the global invertibility of the (matrix) exponential, coupled with $P > 0$ and $x \neq 0$. Then V is positive definite with respect to $\mathscr{A}$. Radial unboundedness of V trivially follows from the fact that $V_u(u)$ goes to infinity when u grows unbounded, and whenever u is bounded and x grows unbounded, certainly $V_0(\xi)$ in (16) goes to infinity.

Let us now prove inequality (2). To this end we only focus on the V_0 component of V because $\dot{u} = 0$ along the dynamics, therefore V_u remains constant during flow. Noticing that in (12a) the flow set is the union of two sets, let us split the analysis in three cases:

• Case 1: $\sigma \in [0,T)$. In this case we may exploit the following identities that follow from the fact that $\dot{\sigma} = 1$, that $\left[\begin{smallmatrix}\dot{x}\\ \dot{u}\end{smallmatrix}\right] = A_f \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]$, and that $\frac{d}{d\tau}\Lambda(\tau) = \Lambda(\tau) A_f$, where $A_f := \left[\begin{smallmatrix}A & B\\ 0 & 0\end{smallmatrix}\right]$:

$$\begin{aligned}
\dot{\varphi}(\sigma) &:= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-\rho \min\{\sigma,T\}} = -\rho\, \mathrm{e}^{-\rho \min\{\sigma,T\}} \\
\frac{\mathrm{d}}{\mathrm{d}t} \left(\Lambda(T - \min\{\sigma,T\}) \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\right) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\Lambda(T-\sigma) \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\right) \\
&= -\Lambda(T-\sigma) A_f \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] + \Lambda(T-\sigma) \left[\begin{smallmatrix}A_f x\\ 0\end{smallmatrix}\right] = 0.
\end{aligned}$$

Based on the above inequalities, we clearly get in this first case:

$$\begin{aligned}
\dot{V}(\xi) &= \underbrace{-\rho\, \mathrm{e}^{-\rho \min\{\sigma,T\}}}_{\dot{\varphi}} |\Lambda(T - \min\{\sigma,T\}) \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]|_P^2 + 0 + 0 \\
&= -\rho V_0(\xi) \leq 0, \qquad (19)
\end{aligned}$$

which proves negative semidefiniteness of $\dot{V}$ for this first case.

• Case 2: $\sigma = T$. In this case we may have $(x,u) \notin \mathscr{F}$ or $(x,u) \in \mathscr{F}$. If $(x,u) \notin \mathscr{F}$, then $T_{\mathscr{C}}(\xi) \cap F(\xi)$ is empty because any flowing solution would select $\dot{\tau} = 1$ and would exit set $\mathscr{C}$. Then we only need to check the case with $(x,u) \in \mathscr{F}$, which is dealt with in Case 3 treated next.

• Case 3: $(x,u) \in \mathscr{F}$ and $\sigma \geq T$. In this case we have that $\dot{\sigma} \in [0,1]$, namely $\dot{\varphi}(\sigma) \leq 0$ and

$$V_0(\xi) = \varphi(T) |\Lambda(0) \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]|_P^2 = \varphi(T) |x|_P^2.$$

Then, along flowing solutions we obtain:

$$\begin{aligned}
\dot{V}(\xi) &= 2x^\top P(Ax + Bu) + \dot{\varphi}(\sigma) |x|_P^2 \\
&\leq 2x^\top P((A + BKC)x + Be) + \begin{bmatrix} y \\ e \end{bmatrix}^\top (M - M) \begin{bmatrix} y \\ e \end{bmatrix} \\
&= \begin{bmatrix} x \\ e \end{bmatrix}^\top \Psi_M(A,B) \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} y \\ e \end{bmatrix}^\top M \begin{bmatrix} y \\ e \end{bmatrix}, \qquad (20)
\end{aligned}$$

where we used $e = u - Ky$.

Since $\Psi_M(A,B)$ is affine with respect to the plant matrix (A,B), guaranteeing $\Psi_M(A_i,B_i) < 0$, $i = 1,\ldots,m$, ensures that for any matrix $[A,B]$ in $Co\{[A_i,B_i]\}_{i=1\ldots,m}$, the LMI $\Psi_M(A,B) < 0$ also holds. As a consequence, due to the strict inequality, there exists a sufficiently small $\varepsilon > 0$ such that $\Psi_M(A,B) < -\varepsilon I$, which implies, together with the fact that $(x,u) \in \mathscr{F}$, so that the last term of (20) is also non-positive,

$$\dot{V}(\xi) \leq -\varepsilon \left| \begin{bmatrix} x \\ u - Ky \end{bmatrix} \right|^2, \text{ if } (x,u) \in \mathscr{F}, \sigma \geq T. \qquad (21)$$

Thus $\dot{V}(\xi) \leq 0$ also in this second subset of $\mathscr{C}$, and inequality (2) holds.

Consider now inequality (3). We need to specify here the values of the positive scalars ρ and η in (16). Let us now split the analysis for Theorems 2 and 3. For Theorem 3, we apply Lemma 1 whose assumptions hold due to the stated hypotheses of the theorem. In particular, Lemma 1 implies $\Phi(A,B,T) < 0$. For the proof of Theorem 2, we also have $\Phi(A,B,T) < 0$ by assumption. Using the strict inequality in $\Phi(A,B,T) < 0$, and positive definiteness of P, we obtain that, for both Theorems 2 and 3, there exist sufficiently small $\rho > 0$ and then $\eta > 0$ guaranteeing:

$$\begin{aligned}
\left(\Lambda(T) \begin{bmatrix} I \\ KC \end{bmatrix}\right)^\top P \left(\Lambda(T) \begin{bmatrix} I \\ KC \end{bmatrix}\right) &\leq \mathrm{e}^{-4\rho T} P, \\
\eta K^\top C^\top C K &\leq \mathrm{e}^{-2\rho T}(1 - \mathrm{e}^{-2\rho T}) P. \qquad (22)
\end{aligned}$$

With this selection of ρ, noticing that from the definition of $\mathscr{D}$ in (12b) we have $\sigma \geq T$, and that from the jump map we also have $\sigma^+ = 0$ and $u^+ = KCx$, we obtain, using (22), that for all $\xi \in \mathscr{D}$,

$$\begin{aligned}
V^+(\xi) &= \left|\Lambda(T) \left[\begin{smallmatrix} I \\ KC \end{smallmatrix}\right] x\right|_P^2 + \eta |KCx|^2 \qquad (23) \\
&\leq \left(\mathrm{e}^{-4\rho T} + \mathrm{e}^{-2\rho T}(1 - \mathrm{e}^{-2\rho T})\right) x^\top P x \\
&= \mathrm{e}^{-2\rho T} x^\top P x = \mathrm{e}^{-\rho T}\, \mathrm{e}^{-\rho T} \left|\Lambda(0) \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]\right|_P^2 \\
&= \mathrm{e}^{-\rho T} V_0(\xi) \leq \mathrm{e}^{-\rho T} V(\xi)
\end{aligned}$$

which proves the strict decrease of the Lyapunov function, across any jump outside $\mathscr{A}$, thereby establishing inequality (3).

Let us now complete the proof by showing that no "bad" complete solution exists, which keeps V constant and nonzero. If any such "bad" complete solution exists, then it has to start outside $\mathscr{A}$ and it cannot jump because otherwise from (23), a decrease of V would be experienced across the jump. Since the "bad" complete solution ξ_{bad} flows forever, then it flows for more than T ordinary time, so that its σ component will eventually become larger than T. At that point, for ξ_{bad}

to be a solution (that flows) it must also be that the (x,u) components eventually belong to $\mathscr{F}$. However, if they belong to $\mathscr{F}$ and the solution flows, from inequality (21) we get that x must be identically zero (otherwise V would decrease), and consequently, since $y = Cx$ would also be identically zero, it must be that also u is identically zero (again from (21)). This contradicts the fact that the "bad" solution evolves with V constant and nonzero.

Since all the assumptions of Theorem 1 have been proven, the proof follows directly from applying that result.

V. Example

Consider a linear plant taken from [6], [1] given by

$$\left\{ \begin{array}{rcl} \dot{x}_p & = & A_p(\omega)x_p + B_p(\omega)u_p, \\ y_p & = & C_p x_p, \end{array} \right. \tag{24}$$

with matrices

$$A_p(\omega) := \begin{bmatrix} 0 & 1 \\ -2 & 3+\omega \end{bmatrix},\ B_p(\omega) := \begin{bmatrix} 0 \\ 1+0.1\omega \end{bmatrix},\ C_p^\top := \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

where $\omega \in \Omega := [-\omega_0, \omega_0]$ represents a constant uncertainty affecting the system for some positive constant ω_0. This plant is coupled to a dynamic output feedback controller of the form

$$\left\{ \begin{array}{rcl} \dot{x}_c & = & A_c x_c + B_c y_p, \\ u_p & = & C_c x_c + D_c y_p, \end{array} \right. \tag{25}$$

where the parameter of the controller has been obtained using an optimization process provided in [1]

$$\begin{array}{ll} A_c := \left[\begin{array}{cc} 1.0919 & -1.1422 \\ 4.9734 & -6.1425 \end{array} \right], & B_c := \left[\begin{array}{c} 16.7501 \\ 64.6472 \end{array} \right], \\ C_c := \left[\begin{array}{cc} 0.1157 & -0.0928 \end{array} \right], & D_c := 0. \end{array}$$

Denoting $x := \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]$, the whole dynamics described by (24) and (25) can be reformulated as system (8) with

$$\left[\begin{array}{c|c} A(\omega) & B(\omega) \\ \hline K & C \end{array} \right] \in \left\{ \left[\begin{array}{cc|cc} A_p(\omega) & 0 & B_p(\omega) & 0 \\ 0 & A_c & 0 & B_c \\ \hline D_c & C_c & C_p & 0 \\ I & 0 & 0 & I \end{array} \right], \omega \in \Omega \right\}. \tag{26}$$

A. Nominal case : $\omega_0 = 0$

This controller, designed in [1] has already shown some improvements with respect to the literature (for instance with respect to [6]). Indeed, the authors obtained a dwell-time $T = 0.0114s$. With our approach solutions to the conditions of Theorem 2 exist for values of the design parameter T up to $0.11s$, which is ten times larger than the solution provided in [1]. This demonstrates the potential of the proposed method.

Figure 1 shows simulations of system (8), (12), (26), obtained for several values of $T = 0.02, 0.05$ and $0.10s$ and where matrix M results from the optimization problem (18). The initial conditions are $x_{p0} = [10\ \ -5]^\top$, $x_{c0} = [0\ 0]^\top$ and $\sigma = 0$. In addition, below each figure, the number N_u of control updates required during the simulation time of $20s$ is indicated. As a first comment on this figure, one can see that increasing T (from left to right in the figure) leads to a notable reduction of the number of control updates. It should also be noticed that the reduction of the number of control updates has an impact on the performance of the controller. This can be interpreted in terms of the classical trade-off between the number of control updates and the performance of the closed-loop system.

ω_0	0	0.04	0.08	0.12	0.139
Th.2	0.114	–	–	–	–
Th.3	0.112	0.100	0.070	0.028	0.008

TABLE I
Maximal dwell time T_{max} leading to feasibility of the conditions in Theorems 2 and 3 for several values of ω_0.

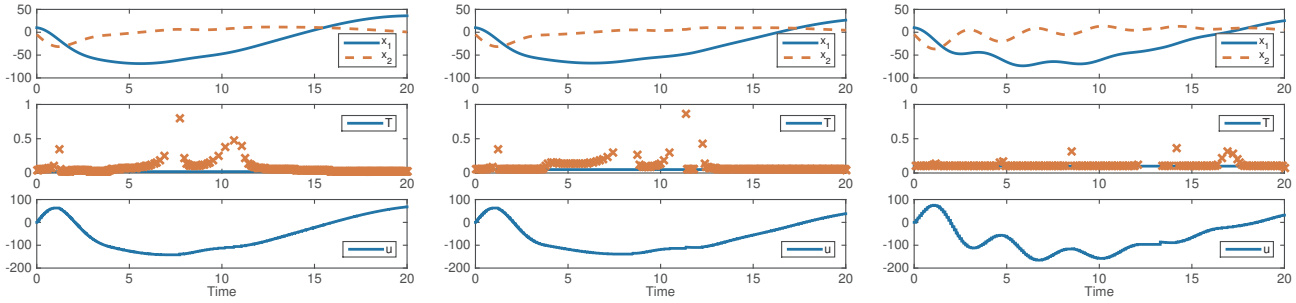
B. Uncertain case : $\omega_0 \neq 0$

We assess now the impact of model uncertainties on the event-triggered rule resulting from Theorem 3. As a first test, selecting $\omega_0 = 0$ and applying Theorem 3 allows evaluating its conservatism with respect to Theorem 2. The first column of Table I presents the maximal allowable values for the dwell time T for which the conditions of Theorems 2 and 3 are feasible. When $\omega_0 = 0$, a fair comparison can be done and shows that the two maximal values of T are quite close, which demonstrates that Theorem 3 is not too conservative with respect to Theorem 2. Note that Theorem 3 is based on the looped-functional approach, which has been refined in several papers such as [3]. It is then possible to also use alternative existing stability conditions for sampled-data systems instead of Lemma 1 (see [3] for details). Table I also shows that increasing ω_0 in Theorem 3 reduces the maximal value of the dwell time parameter T. Figure 2 also depicts the effects of the uncertainties on the event-triggered control performance. This figure presents the variations of the average number of control updates N_u, resulting from 80 simulations, with respect to the dwell time parameter T. These simulations result from the application of Theorem 2 for $\omega_0 = 0$ (black solid trace) and Theorem 3 otherwise (all the other traces). The general trend of this figure is that increasing T reduces N_u.

As confirmed by Table I, for large values of T and of ω_0, the conditions of Theorem 3 become infeasible, as shown in Figure 2 by the vertical dotted lines. Moreover, the figure also shows that increasing ω_0 increases the average number of control updates, as expected. It can be also seen that for large values of T and small values of ω_0, the average number of control updates are very similar. This can be interpreted by the fact that when T is large, the control updates are mostly periodic, as for the nominal case presented in Figure 1 when $T = 0.1$ (rightmost figure).

VI. Conclusion

In this paper, we provided a stability theorem for uncertain linear systems controlled by means of a dynamic output feedback controller. The contribution is twofold. On the first hand, a new invariance principle has been derived using nonstrict and nonsmooth Lyapunov functions for hybrid systems. Secondly we apply this invariance principle to design a new event-triggered algorithm yielding robust asymptotic stability for the



$N_u = 434$ $N_u = 255$ $N_u = 165$

Fig. 1. Figure representing the state of the plant x_p, the inter-sampling times (with the dwell time T) and the control input u issued from Theorem 2 with $T = 0.02$ (left), $T = 0.05$ (middle) and $T = 0.10$ (right).

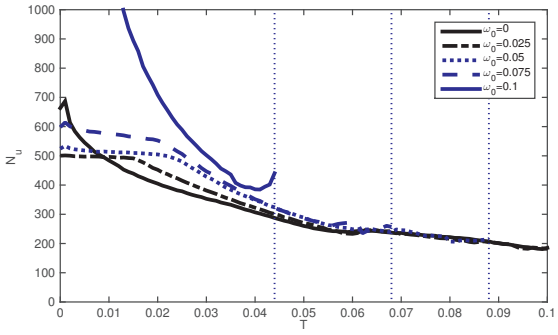


Fig. 2. Evolution of the average number of control updates N_u with respect to the dwell time parameter T for several values of ω_0. The vertical dotted lines represent the limit values of the dwell time parameter for which the conditions of Theorem 3 are feasible for $\omega_0 = 0.1$ (left), $\omega_0 = 0.075$ (middle) and $\omega_0 = 0.05$ (right).

closed-loop system. Moreover numerically tractable conditions allow to guarantee an adjustable dwell time of the solutions. Future work involves providing more advanced theoretical conditions in order to address the co-design problem. In other words, we envision an extension of the current work in order to simultaneously design the feedback stabilizer and its event-triggered sampled data implementation.

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