Stability analysis of dissipative systems subject to nonlinear damping via Lyapunov techniques

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***Abstract*—In this article, we provide a general strategy based on Lyapunov functionals to analyse global asymptotic stability of linear infinite-dimensional systems subject to nonlinear dampings under the assumption that the origin of the system is globally asymptotically stable with a *linear damping*. To do so, we use the fact that, for any linear infinite-dimensional system that is globally exponentially stable, there exists a Lyapunov functional. Then, we derive a Lyapunov functional for the nonlinear system, which is the sum of a Lyapunov functional coming from the linear system and another term which compensates the nonlinearity. Our results are then applied to the linearized Korteweg-de Vries equation and some wave equations.**

I. Introduction

This paper is concerned with the asymptotic behavior analysis of infinite-dimensional systems subject to a nonlinear damping. These systems are composed by abstract operators generating a strongly continuous semigroup of contractions and a bounded operator representing the control operator (see e.g., [31] or [24] for the introduction of linear and nonlinear operators generating semigroups, respectively). These systems might be for instance a hyperbolic PDE, or a parabolic one or even the linearized Korteweg-de Vries equations. Assuming that a linear damping renders the origin of these systems globally asymptotically stable, we propose a general strategy to analyze the asymptotic behavior of these systems when modifying the linear damping with a nonlinearity. In contrast with the existing litterature, which uses either integral inequalities(see e.g. [2], [19]) or a frequential approach (cf. [14], [6]) or even a compactness uniqueness strategy ([33], [26], [21]), we propose here to design Lyapunov functionals to characterize our results, extending to the infinite-dimensional setting a strategy first deviced in [18] for finite-dimensional systems.

Nonlinear damping for infinite-dimensional systems. There exist many works dealing with nonlinear damping for infinite-dimensional systems. Some of them tackle specific PDEs as for instance hyperbolic ones (see e.g., [13], [19] or [2]) and others propose a general framework using abstract operators (see [29], [27], [17] and [5] for a specific case of nonlinear damping, namely the *saturation*). These papers, which deal with abstract operators, usually assume that the space where the damping takes value, namely S, is the same as the control space, namely U. However, in practice, this is not the case.

In constrast with existing works for abstract control systems, we aim here at giving a general definition of nonlinear dampings when the nonlinear damping space S is not necessarily equal to the control space U. With such a formalism, we are able to make a link between the litterature on abstract operators and the one on hyperbolic systems. At the best of our knowledge, this formalism has been introduced first in [23] in the case where the nonlinear damping is a saturation.

In many works, specific PDEs subject to a nonlinear damping have been studied. In [17], the origin of a wave equation subject to a nonlinear damping, either distributed or located at the boundary, has been proved to be globally asymptotically stable, in the case $S = U$. In [25], a similar result has been stated, but in the case where $S \neq U$. In [7], the global asymptotic stability of a PDE coupled to an ODE with a saturated feedback law at the boundary has been tackled. There exist also some papers dealing with local asymptotic stability (see [16] or [15]). Note that both situations ($S = U$ and $S \neq U$) have been tackled for the specific nonlinear partial differential equation Korteweg-de Vries equation in [21], in the case where the damping is a saturation.

Contribution. In this paper, we derive a Lyapunov functional which allows us to prove and even estimate the decay rate of the trajectories. We derive a strict and *global* Lyapunov functional if $S = U$. By global, we mean that the Lyapunov function does not depend on the initial condition, neither the decay rate. This allows to give an explicit formula for the convergence rate of the trajectories. If $S \neq U$, we have the same results except for the global character of the Lyapunov function that we are not able to obtain. We only prove that the origin of the system is semi-globally exponentially stable, meaning in particular that the decay rate of the trajectories depends on the initial condition.

Outline. Section II recalls some necessary and sufficient conditions in terms of Lyapunov functionals for infinite-dimensional systems. In Section III, nonlinear dampings for infinite-dimensional systems are introduced and our main results are stated. Their proofs are then given in Section IV. These results are illustrated in Section V on some examples,

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This research was partially supported by the iCODE Institute, research project of the IDEX Paris-Saclay, and by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the "Programme des Investissements d'Avenir", and by the European Research Council (ERC) through an ERC-Advanced Grant for the TAMING project (grant agreement 66698).

namely the linearized Korteweg-de Vries equation and some wave equations. Section VI collects some concluding remarks and further research lines to be investigated.

Acknowledgements. The authors want to warmly thank Fatiha Alabau-Boussouira and Enrique Zuazua for all the encouraging and interesting discussions and for having pointed out a large number of crucial references.

II. Lyapunov criteria for linear infinite-dimensional systems

Let H be a real Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle_H$. Let $A : D(A) \subset H \to H$ be a (possibly unbounded) linear operator whose domain $D(A)$ is dense in H. We suppose that A generates a strongly continuous semigroup of contractions denoted by $(e^{tA})_{t\geq 0}$. We use $A^\star$ to denote the adjoint operator of A.

In this section, we consider the linear system given by

$$\begin{cases} \dfrac{d}{dt}z = Az, \\ z(0) = z_0. \end{cases} \tag{1}$$

Since A generates a strongly continuous semigroup of contractions, there exist both strong and weak solutions to (1). Moreover, the origin of (1) is Lyapunov stable[1] in H. Indeed, the property of contraction satisfied by $(e^{tA})_{t\geq 0}$ implies that

$$\|e^{tA}z_0\|_H \leq \|z_0\|_H. \tag{2}$$

The origin is attractive in H if, for every $z_0 \in H$, one has

$$\lim_{t\to+\infty} \|e^{tA}z_0\|_H = 0, \tag{3}$$

and this property is also referred as *strong stability* (see e.g., [2, Section 1.3]). This section aims at characterizing the decay rate of the trajectory when assuming that the origin is attractive. We first consider *global exponential stability*:

Definition 1. *[Global exponential stability] The origin of* (1) *is said to be* globally exponentially stable *if there exist two positive constants C and α such that, for any $z_0 \in H$,*

$$\|e^{tA}z_0\|_H \leq Ce^{-\alpha t}\|z_0\|_H, \qquad \forall t \geq 0. \tag{4}$$

Remark 1. *If the origin of* (1) *is globally exponentially stable in H, then, provided that the initial condition z_0 is in $D(A)$, the origin is also globally exponentially stable in $D(A)$. Indeed, since A generates a strongly continuous semigroup of contractions, then, for any initial condition $z_0 \in D(A)$, $Ae^{tA}z_0 \in H$, for all $t \geq 0$. Since* (4) *holds, this means in particular that*

$$\|e^{tA}Az_0\|_H \leq Ce^{-\alpha t}\|Az_0\|_H, \quad \forall t \geq 0.$$

[1]The origin of (1) is said to be Lyapunov stable in H if, for any positive δ, there exists a positive constant $\varepsilon = \varepsilon(\delta)$ such that

$$\|z_0\|_H \leq \varepsilon \Rightarrow \|e^{tA}z_0\|_H \leq \delta.$$

Note moreover that $e^{tA}A = Ae^{tA}$ (see e.g., [31, Proposition 2.1.5]) and $\|\cdot\|_{D(A)} := \|\cdot\|_H + \|A\cdot\|_H$. Therefore,

$$\|e^{tA}z_0\|_{D(A)} \leq Ce^{-\alpha t}\|z_0\|_{D(A)}, \quad \forall t \geq 0.$$

∘

Lyapunov functionals have been instrumental to characterize stability for dynamical systems. The main result of [9] is recalled in the following proposition.

Proposition 1. *[Exponential stability [9]] The origin of* (1) *is globally exponentially stable if and only if there exist a self-adjoint, positive definite and coercive operator $P \in \mathcal{L}(H)$ and a positive constant C such that*

$$\langle Az, Pz\rangle_H + \langle Pz, Az\rangle_H \leq -C\|z\|_H^2, \quad \forall z \in D(A). \tag{5}$$

One can choose P in the latter equation in the form

$$P = \int_0^\infty e^{sA^\star}e^{sA}ds + \alpha I_H, \tag{6}$$

with $\alpha > 0$.

Note that an operator P satisfying (5) has also been considered in the context of the asymptotic stability analysis of linear switched systems in [12]. We underline that this operator is not *explicit*, which is in contrast with the results provided in [32] and [11].

Let A be the operator defined as before. Let U be a real Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_U$. Let B be a bounded operator from U to H (i.e., $B \in \mathcal{L}(U, H)$ and let us denote by $B^\star$ its adjoint.

We consider the infinite-dimensional linear control system given by

$$\begin{cases} \dfrac{d}{dt}z = Az + Bu, \\ z(0) = z_0, \end{cases} \tag{7}$$

where u denotes the control. In addition, we will choose the following collocated feedback law

$$u = -kB^\star z, \tag{8}$$

where k is a positive constant.

The corresponding closed-loop system is then written as follows

$$\begin{cases} \dfrac{d}{dt}z = (A - kBB^\star)z := \tilde{A}z, \\ z(0) = z_0. \end{cases} \tag{9}$$

Since B is a bounded operator, the domain of $\tilde{A}$ coincides with $D(A)$. Moreover, it is easy to see that $\tilde{A}$ generates a strongly continuous semigroup of contractions. We assume that the origin of (9) is globally exponentially stable.

Example 1. Let us consider the following linear wave equation

$$\begin{cases} z_{tt} = \triangle z - a(x)z_t, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ z(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \Gamma, \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), \end{cases} \tag{10}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded connected domain with a smooth boundary $\Gamma := \partial\Omega$. The damping localization function $a(\cdot)$ is smooth, nonnegative and there exists a positive constant a_0 such that $a(x) \geq a_0$ on a non empty open subset ω of Ω. In other words, the open subset ω is actually the set where the control acts. The feedback control is said to be *globally* distributed if $\omega = \Omega$ and locally distributed if $\Omega \setminus \omega$ has a positive Lebesgue measure.

Equation (10) can be rewritten as an abstract control system (9) setting $H := H_0^1(\Omega) \times L^2(\Omega)$, $U = L^2(\Omega)$ and

$$\begin{aligned} A : D(A) \subset H &\to H \\ \begin{bmatrix} z_1 & z_2 \end{bmatrix}^\top &\mapsto \begin{bmatrix} z_2 & \triangle z_1 \end{bmatrix}, \end{aligned} \tag{11}$$

$$\begin{aligned} B : U &\to H, \\ u &\mapsto \begin{bmatrix} 0 & \sqrt{a(x)}u \end{bmatrix}^\top, \end{aligned} \tag{12}$$

where

$$D(A) := (H^2(\Omega) \cap H_0^1(0,1)) \times H_0^1(\Omega).$$

The adjoint operators of A and B are, respectively

$$\begin{aligned} A^\star : D(A) \subset H &\to H \\ \begin{bmatrix} z_1 & z_2 \end{bmatrix}^\top &\mapsto -A \begin{bmatrix} z_1 & z_2 \end{bmatrix}^\top, \end{aligned} \tag{13}$$

and

$$\begin{aligned} B^\star : H &\to U, \\ \begin{bmatrix} z_1 & z_2 \end{bmatrix}^\top &\mapsto \sqrt{a(x)} z_2. \end{aligned} \tag{14}$$

A straightforward computation, combined with some integrations by parts, shows that

$$\langle Az, z\rangle_H + \langle z, Az\rangle_H \leq 0, \quad \forall z \in D(A).$$

Hence, applying Lumer-Phillips's theorem, it follows that A generates a strongly continuous semigroup of contractions. Moreover, using [33, Theorem 2.1.], (10) is globally exponentially stable provided that ω is a neighbourhood of Γ. In particular, using Proposition 1, there exists a Lyapunov operator $P \in \mathcal{L}(H)$ such that a Lyapunov inequality holds.

III. Main results

A. Nonlinear damping functions

As it has been noticed at the beginning of the section, we want to study the asymptotic behavior of the origin of (7) with (8) modified by a nonlinearity, namely the *nonlinear damping function*. We provide next the definition of the nonlinear damping function.

Definition 2. *[Nonlinear damping functions on S] Let S be a real Banach space equipped with the norm $\|\cdot\|_S$. Assume moreover that (U, S) is a rigged Hilbert space[2], i.e., S is a dense subspace of U and that the following inclusions hold*

$$S \subseteq U \subseteq S'. \tag{15}$$

[2]We refer the interested reader to [10] for more details on rigged Hilbert spaces

In particular, the duality pairing between S and S' is compatible with the inner product on U, in the sense that

$$(u, v)_{S' \times S} = \langle u, v\rangle_U, \quad \forall u \in U = U' \subset S', \quad \forall v \in S \subset U. \tag{16}$$

A function $\sigma : U \to S$ is said to be a nonlinear damping function on U *if there exists positive constants C_1 and C_2 such that the following properties hold true.*

1. *The function σ is locally Lipschitz.*
2. *The function σ is maximal monotone, that is: for all $s_1, s_2 \in U$, σ satisfies*
$$\langle \sigma(s_1) - \sigma(s_2), s_1 - s_2\rangle_U \geq 0. \tag{17}$$
3. *For any $s \in U$, one has*
$$\|\sigma(s) - C_1 s\|_{S'} \leq C_2 h(\|s\|_S)\langle \sigma(s), s\rangle_U, \tag{18}$$
where $h : \mathbb{R} \to \mathbb{R}_+$ is a continuous and non decreasing function satisfying $h(0) > 0$.

Example 2. [Some examples of nonlinear damping functions] We provide two sets of examples depending on the fact that $S = U$ or not.

1. Suppose that $S := U$. The saturation studied in [29], [17] and [21] is defined as follows, for all $s \in U$,
$$\mathfrak{sat}_U(s) := \begin{cases} \dfrac{s}{\|s\|_U} s_0 & \text{if } \|s\| \geq s_0, \\ s & \text{if } \|s\| \leq s_0 \end{cases} \tag{19}$$
where the positive constant s_0 is called the saturation level. This operator clearly satisfies Item 1. of Definition 2. The fact that this operator is globally Lipschitz is proven in [29]. Moreover, one verifies easily that this operator satisfies Item 3. of Definition 2. In [27], this operator is proved to be m-dissipative, which implies that it is maximal monotone.
2. Let $L^\infty(0,1)$ and $L^2(0,1)$ be the spaces of $\mathbb{R}^n$-valued functions defined on $(0,1)$ which are essentially bounded or square integrable. Suppose that $S := L^\infty(0,1)$ and $U = L^2(0,1)$. In this case, S' is the space of finitely additive measures[3]. It contains the space L^1, and L^1 is continuously embedded in S' via the operator $u \mapsto \int_0^1 u dx$ (see [3, Remark 7, Page 102]). Therefore, it is clear that (U, S) is a rigged Hilbert space. Moreover, one can write via the latter embedding that
$$\langle u, v\rangle_{L^2(0,1)} \leq \|v\|_{L^1(0,1)} \|u\|_{L^\infty(0,1)}. \tag{20}$$
For this case, we give two examples: the first one is a saturation, while the second one is borrowed from [19].
   (i) *(L^∞-saturation) in dimension one* Standard saturation functions can be defined as follows:
$$\begin{aligned} L^2(0,1) &\to L^\infty(0,1), \\ s &\mapsto \mathfrak{sat}_{L^\infty(0,1)}(s), \end{aligned} \tag{21}$$

[3]Let Σ be an algebra of sets of a given set Ω. A function $\lambda : \Sigma \to \bar{\mathbb{R}}$ is said to be a finitely additive signed measure if: (i) $\lambda(\emptyset) = 0$; (ii) given $K_1, K_2 \in \Sigma$, disjoint subsets, $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$. The corresponding space, which is a Banach space, is endowed with the norm of total variation.

where $\mathfrak{sat}_{L^\infty(0,1)}(s)(\cdot) = \sigma(s(\cdot))$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is a non decreasing, locally Lipschitz function verifying, for some positive constants C_1, C_2, that $|\sigma(s) - C_1 s| \leq C_2 \sigma(s)s$ for every $s \in \mathbb{R}$. For instance, arctan, tanh and the standard saturation functions $\sigma_0(s) = \frac{s}{\max(1,|s|)}$ are saturation functions. In all these cases, the function h appearing in (18) can be taken equal to one. Note moreover that saturation functions are uniformly bounded.

(ii) We have also the following nonlinear damping function, also called *weak damping*, and borrowed from [19, Theorem 2.]

$$\sigma(s) \leq c|s|^q, \forall s \in \mathbb{R}$$

with $c > 0$ and $q \geq 1$ and such that $\sigma(0) = 0$ and $\sigma'(0) > 0$.

Note that examples of nonlinear dampings that do not satisfy Properties 1 and 2 are provided in [19].

Consider now the following nonlinear dynamics

$$\begin{cases} \frac{d}{dt} z = A_\sigma(z), \\ z(0) = z_0, \end{cases} \tag{22}$$

where the nonlinear operator A_σ is defined as follows

$$\begin{aligned} A_\sigma : D(A_\sigma) \subset H &\to H \\ z &\mapsto Az - \sqrt{k} B \sigma(\sqrt{k} B^\star z). \end{aligned} \tag{23}$$

with $D(A_\sigma)$ the domain of A_σ. Since B is bounded, one clearly has that $D(A_\sigma) = D(A)$.

For the latter system, there exist many results related to its well-posedness and the asymptotic stability of its origin. The following theorem collects some of them.

Theorem 1. *[Well-posedness and global asymptotic stability]*

(i) *Suppose that σ is a nonlinear damping. Therefore, A_σ generates a strongly continuous semigroup of contractions $(W_\sigma(t))_{t\geq 0}$. In particular, there exist a unique strong solution to* (22) *for every initial condition $z_0 \in D(A)$. Moreover, the following functions*

$$t \mapsto \|W_\sigma(t) z_0\|_H, \; t \mapsto \|A_\sigma W_\sigma(t) z_0\|_H, \tag{24}$$

are nonincreasing.

(ii) *Supposing that all the assumptions of the latter item hold and assuming moreover that $D(A)$ is compactly embedded in H, then the origin of* (22) *is globally asymptotically stable, i.e., for every $z_0 \in D(A)$,*

$$\lim_{t \to +\infty} \|W_\sigma(t) z_0\|_H^2 = 0. \tag{25}$$

The proof of the first item is based on the result given in [28, Lemma 2.1, Part IV, page 165], which proves that any linear m-dissipative operator with a bounded Lipschitz operator is a m-dissipative operator. Thanks to [24, Corollary 3.7, page 53], the well-posedness result follows readily. The second item has been proved in the specific case of hyperbolic systems in [8] (differentiable nonlinear damping) and [13] (non differentiable nonlinear damping). The proof of this item relies on the use of the LaSalle's Invariance Principle. For the well-posedness and the global asymptotic stability of the closed-loop system (22) in the case where σ is a saturation, we refer the interested reader to [27], [29] or more recently to [20].

Remark 2. *In some cases, it is not immediate to check whether $D(A)$ is compactly embedded in H. For instance, we know that this holds for hyperbolic systems [13] or the linearized Korteweg-de Vries equation [20]. Note that the global asymptotic stability does not give any information on the decay rate of the trajectory of the systems. Here, we do not aim at just proving that the origin of* (22) *is globally asymptotically stable, but rather at characterizing the decay rate of trajectories.* ∘

B. Global asymptotic stability results

The remaining parts of the paper aim at characterizing precisely the stability properties of the origin of (22). Before stating our main results, let us provide some stability definitions.

Definition 3. *[Semi-global exponential stability] The origin of* (22) *is said to be* semi-globally exponentially stable in $D(A)$ *if, for any positive r and any z_0 in $D(A)$ satisfying $\|z_0\|_{D(A)} \leq r$, there exist two positive constants $\mu := \mu(r)$ and $K := K(r)$ such that*

$$\|W_\sigma(t) z_0\|_H \leq K e^{-\mu t} \|z_0\|_H, \quad \forall t \geq 0. \tag{26}$$

This definition is inspired by [21], which focuses on a particular nonlinear damping function, namely the *saturation*. It is well known that a linear finite-dimensional system subject to a saturated controller cannot be globally exponentially stabilized (see [30]).

We are now in position to state the main result of our paper.

Theorem 2. *[Semi-global exponential stability] Consider that σ in* (22) *is a nonlinear damping function satisfying Item 1. and 3. of Definition 2. Then, we have the following results*

1. *If $S = U$, there exists a $\mathcal{KL}$ function β[4] such that*

$$\|W_\sigma(t) z_0\|_H \leq \beta(\|z_0\|_H, t), \quad \forall t \geq 0. \tag{27}$$

2. *If $S \neq U$, assume that σ is also maximal monotone (i.e., σ satisfies Item 2. of Def. 2) and that the following inequality holds*

$$\|B^\star s\|_S \leq c_S \|s\|_{D(A)}, \quad \forall s \in D(A). \tag{28}$$

Hence, the origin of (22) *is semi-globally exponentially stable in $D(A)$.*

Note that, in Item 2. of Theorem 2, the stability is not uniform with respect to the initial condition. This seems to

[4] A continuous function $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \beta(s,t) \in \mathbb{R}_+$ is said to be $\mathcal{KL}$ if $t \mapsto \beta(s,\cdot)$ is strictly increasing, vanishes at 0 and such that $\lim_{s\to+\infty} \beta(t,s) = +\infty$, and if $t \mapsto \beta(\cdot,t)$ is strictly decreasing and is such that $\lim_{t\to+\infty} \beta(s,t) = 0$, for each fixed s.

be unsastifactory, but it is common in the PDE litterature (see for instance [19, Theorem 2.], where a similar result is stated).

IV. Proof of Theorem 2

We split the proof of Theorem 2 into two cases. Firstly, we tackle Item 1. of Theorem 2 and then Item 2. Indeed, the Lyapunov functionals considered in the two cases are different. In both cases, each argument is itself divided into two steps. First, we find a strict Lyapunov functional and then we prove the asymptotic stability of the origin of (22).

Case 1: $S = U$**.**

Set $\tilde{A} = A - C_1 BB^\star$, where the positive constant C_1 is given in (18) and $P \in \mathcal{L}(H)$ is defined in (5). Consider the following candidate Lyapunov functional

$$\tilde{V}(z) := \langle Pz, z\rangle_H + M\int_0^{\|z\|^2} \sqrt{v}h(\|B\|_{\mathcal{L}(U,H)}\sqrt{v})dv, \quad (29)$$

where M is a sufficiently large positive constant to be chosen later and h is the function defined in Item 3 of Definition 2. This function, inspired by [18], is positive definite and coercive. Indeed, since $h(0) > 0$,

$$\begin{aligned}\int_0^{\|z\|_H^2} \sqrt{v}h(\|B\|_{\mathcal{L}(U,H)}\sqrt{v})dv \geq & h(0)\int_0^{\|z\|_H^2}\sqrt{v}dv \\ \geq & \frac{2h(0)}{3}\|z\|_H^3.\end{aligned} \quad (30)$$

Noticing that there exists $\alpha > 0$ such that $\alpha\|z\|^2 \leq \langle Pz, z\rangle_H \leq \|P\|_{\mathcal{L}(H)}\|z\|^2$, one has

$$\begin{aligned}&\alpha\|z\|^2 + Mh(0)\frac{2}{3}\|z\|_H^3 \leq \tilde{V} \leq \\ &\|P\|_{\mathcal{L}(H)}\|z\|^2 + M\|z\|_H^2 h(\|B\|_{\mathcal{L}(U,H)}\|z\|_H).\end{aligned} \quad (31)$$

Applying Cauchy-Schwarz's inequality, one has

$$\begin{aligned}\frac{d}{dt}\langle Pz, z\rangle_H = & \langle Pz, \tilde{A}z\rangle_H + \langle P\tilde{A}z, z\rangle_H \quad (32)\\ & + \langle Pz, B(C_1B^\star z - \sigma(B^\star z)\rangle_H \\ & + \langle PB(C_1B^\star z - \sigma(B^\star z), z\rangle_H \\ \leq & -C\|z\|_H^2 \\ & + 2\langle B^\star Pz, C_1B^\star z - \sigma(B^\star z)\rangle_U \quad (33)\\ \leq & -C\|z\|_H^2 \quad (34)\\ & + 2\|B^\star Pz\|_U\|C_1B^\star z - \sigma(B^\star z)\|_U,\end{aligned}$$

Using Item 3. of Definition 2, the fact that $B^\star$ is bounded in U and the fact that the trajectory z is in $D(A)$, it yields

$$\begin{aligned}\frac{d}{dt}\langle Pz, z\rangle_H \leq & -C\|z\|_H^2 + 2C_2\|B^\star\|_{\mathcal{L}(H,U)}\|P\|_{\mathcal{L}(H)}\|z\|_H \\ & \cdot h(\|B^\star z\|_U)\langle B^\star z, \sigma(B^\star z)\rangle_U \\ \leq & -C\|z\|_H^2 + 2C_2\|B^\star\|_{\mathcal{L}(H,U)}\|P\|_{\mathcal{L}(H)}\|z\|_H \\ & \cdot h(\|B\|_{\mathcal{L}(U,H)}\|z\|_U)\langle B^\star z, \sigma(B^\star z)\rangle_U.\end{aligned}$$

Secondly, using the dissipativity of the operator A, one has, for all $z \in D(A)$

$$\begin{aligned}M\frac{d}{dt}\int_0^{\|z\|^2}&\sqrt{v}h(\|B\|_{\mathcal{L}(U,H)}\sqrt{v})dv = \\ & M\|z\|_H h(\|B\|_{\mathcal{L}(U,H)}\|z\|_H)(\langle Az, z\rangle_H \\ & + \langle z, Az\rangle - 2\langle B^\star z, \sigma(B^\star z)\rangle_U) \\ \leq & -2M\|z\|_H h(\|B\|_{\mathcal{L}(U,H)}\|z\|_H)\langle B^\star z, \sigma(B^\star z)\rangle_U\end{aligned}$$

Hence, if one chooses M as

$$M = C_2\|B^\star\|_{\mathcal{L}(H,U)}\|P\|_{\mathcal{L}(H)}, \quad (35)$$

one obtains, after adding the above two equations, that

$$\frac{d}{dt}\tilde{V}(z) \leq -C\|z\|_H^2, \ \forall z \in D(A) \quad (36)$$

We claim that, once a trajectory enters the unit ball in the H-topology, then it converges exponentially to the origin. Indeed, let $t^\star$ the first time such that $\|W_\sigma(t)z_0\|_H = 1$, with $t^\star = 0$ if $\|z_0\|_H \leq 1$. Since $(W_\sigma(t))_{t\geq 0}$ is a strongly continuous semigroup of contractions one has $\|W_\sigma(t)z_0\|_H \leq \|W_\sigma(t^\star)z_0\|_H \leq 1$, for all $t \geq t^\star$. This implies that, for all $t \geq t^\star$, one has $\|W\sigma(t)z_0\|_H^3 \leq \|W_\sigma(t)z_0\|_H^2$. Therefore, since h is non decreasing, (31) reduces in the unit ball to

$$\alpha\|z\|_H^2 \leq \tilde{V}(z) \leq (\|P\|_{\mathcal{L}(H)} + Mh(\|B\|_{\mathcal{L}(U,H)}))\|z\|_H^2. \quad (37)$$

and, therefore, (36) to

$$\frac{d}{dt}\tilde{V}(z(t)) \leq -CC_V\tilde{V}(z(t)), \ \forall t \geq t^\star, \quad (38)$$

where $C_V := \frac{1}{\|P\|_{\mathcal{L}(H)} + Mh(\|B\|_{\mathcal{L}(U,H)})}$. Then, it is clear that the claim holds true.

Now, it remains to estimate the behavior of trajectories of (22) before they enter the unit ball. Since the function $X \in \mathbb{R}_+ \mapsto \int_0^X \sqrt{v}h(\|B\|_{\mathcal{L}(U,H)}\sqrt{v})dv + \alpha X \in \mathbb{R}_+$ is increasing, it befines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. It has therefore an increasing inverse function, that we denote by g. Then, along any trajectory of (22),

$$\frac{d}{dt}\tilde{V}(z) \leq -g(\tilde{V}(z)). \quad (39)$$

From here, one can characterize the asymptotic behavior of (22). Indeed, there exist three positive constants C_3, C_4 and C_5 such that, for $\|z_0\|_H$ large enough and $t \in \left[0, \frac{C_4}{C_5}\|z_0\|_H - 1\right]$,

$$\|W_\sigma(t)z_0\|_H \leq C_3\sqrt{g \circ G}(C_4\|z_0\|_H - C_5 t), \quad (40)$$

where G is given by $G(\|z\|_H) := \int_1^{\|z\|_H}\frac{dv}{g(v)}$. It is then clear that there exists a function β of class $\mathcal{KL}$ as given in Item 1. of Theorem 2, composed by an exponential function and $g \circ G$. This concludes the proof of Theorem 2 in the case where $S = U$.

Case 2: $S \neq U$**.**

In this case, we are not able to control the term $\|C_1B^\star z - \sigma(B^\star z)\|_U$ in (33) with Item 3. of Definition 2. To tackle this term, the inequality (28) together with Item 1. of Theorem

1 will be used in order to prove that the origin of (22) is semi-globally exponentially stable.

Let $\tilde{V}(z)$ be the Lyapunov function candidate defined by

$$z \in D(A) \mapsto \tilde{V}(z) := \langle Pz, z\rangle_H + M\|z\|_H^2, \tag{41}$$

where $M > 0$ will be selected later.

First, using the dissipativity of the operator A, one has

$$\frac{d}{dt}M\|z\|_H^2 \leq -2M\langle B^\star z, \sigma(B^\star z)\rangle_U. \tag{42}$$

Second, perfoming similar computations than in the case $S = U$, one obtains

$$\frac{d}{dt}\langle Pz, z\rangle_H \leq -C\|z\|_H^2 + 2\langle B^\star Pz, C_1 B^\star z - \sigma(B^\star z)\rangle_U. \tag{43}$$

It remains now to control the term

$$2\langle B^\star Pz, C_1 B^\star z - \sigma(B^\star z)\rangle_U.$$

We now assume that we have a strong solution for (22), whose initial condition $z_0 \in D(A)$ is such that

$$\|z_0\|_{D(A)} \leq r,\ \|z_0\|_H \leq r, \tag{44}$$

for some positive r. Since (U, S) is a rigged Hilbert space, hence the following holds

$$\langle B^\star Pz, C_1 B^\star z - \sigma(B^\star z)\rangle_U = (C_1 B^\star z - \sigma(B^\star z), B^\star Pz)_{S'\times S}.$$

Hence, applying Cauchy-Schwarz's inequality, one obtains

$$\begin{aligned}\frac{d}{dt}\langle Pz, z\rangle_H \leq& -C\|z\|_H^2 \\ &+ 2\|B^\star Pz\|_S\|C_1 B^\star z - \sigma(B^\star z)\|_{S'}.\end{aligned} \tag{45}$$

Moreover, thanks to (28), one has

$$\|B^\star Pz\|_S \leq c_S\|Pz\|_{D(A)} \tag{46}$$

and

$$\begin{aligned}\|C_1 B^\star z - \sigma(B^\star z)\|_{S'} &\leq C_2 h(\|B^\star z\|_S)\langle B^\star z, \sigma(B^\star z)\rangle_U \\ &\leq C_2 h(c_S\|z\|_{D(A)})\langle B^\star z, \sigma(B^\star z)\rangle_U,\end{aligned} \tag{47}$$

where we have used the fact that h is non decreasing and Item 3. of Definition 2 in the second one.

Now, using (24), the fact that $P \in \mathcal{L}(D(A))$ and the dissipativity of the strong solution, which comes from Item 2. of Theorem 1 and which can be written as follows:

$$\begin{aligned}\|PW_\sigma(t)z_0\|_{D(A)} &\leq \|P\|_{\mathcal{L}(D(A))}\|z_0\|_{D(A)}, \\ \|W_\sigma(t)z_0\|_{D(A)} &\leq \|z_0\|_{D(A)},\end{aligned} \tag{48}$$

one has

$$\begin{aligned}\frac{d}{dt}\langle Pz, z\rangle_H \leq& -C\|z\|_H^2 \\ &+ 2c_S h(\|B^\star\|r)rC_2\|P\|_{\mathcal{L}(D(A)}\langle B^\star z, \sigma(B^\star z)\rangle_U.\end{aligned} \tag{49}$$

Therefore, if one selects M such that

$$M = c_S C_2 h(\|B^\star\|r)r\|P\|_{\mathcal{L}(D(A))}, \tag{50}$$

it follows

$$\frac{d}{dt}\tilde{V}(z) \leq -C\|z\|_H^2. \tag{51}$$

Note that we have, for all $z \in H$

$$\langle Pz, z\rangle_H \leq \|P\|_{\mathcal{L}(H)}\|z\|_H. \tag{52}$$

Hence, it yields

$$\begin{aligned}\frac{d}{dt}\tilde{V}(z) \leq& -\frac{C}{2\|P\|_{\mathcal{L}(H)}}\langle Pz, z\rangle_H - \frac{C}{2}\|z\|_H^2 \\ \leq& -\mu\tilde{V}(z),\end{aligned} \tag{53}$$

where

$$\mu := \min\left(\frac{C}{2\|P\|_{\mathcal{L}(H)}}, \frac{C}{2M}\right). \tag{54}$$

After integration of the above differential inequality, one obtains

$$\tilde{V}(W_\sigma(t)z_0) \leq e^{-\mu t}\tilde{V}(z_0), \quad \forall t \geq 0. \tag{55}$$

Hence,

$$\|W_\sigma(t)z_0\|_H^2 \leq \frac{\|P\|_{\mathcal{L}(H)} + M}{M}e^{-\mu t}\|z_0\|_H^2. \tag{56}$$

Since M depends on the bound of the initial condition, the origin of (22) is semi-globally exponentially stable for any strong solution to (22). It concludes the proof of Item 2. of Theorem 2.□

Remark 3 (Improved decay rate)**.** *The decay rate for the case where $S = U$ can be characterized more precisely when specifying the damping σ and the operator A. Indeed, if the nonlinear damping σ is given by any saturation function given by Item 2(i) of Example 2, then h is reduced to the identity and, thus, we have $\|W_\sigma(t)z_0\|_H \leq C_3(C_4\|z_0\|_H - C_5 t)$, for all $t \in \left[0, \frac{C_4}{C_5}\|z_0\|_H - 1\right]$. It is a linear decay of the trajectories with large initial conditions. It is moreover optimal in the case where A is a skew-symmetric operator (i.e., $A^\star = -A$). Indeed,*

$$\frac{d}{dt}\|z\|_H^2 = -2\langle B\sigma(B^\star z), z\rangle_H \geq -C_\sigma\|B\|_{\mathcal{L}(U,H)}\|z\|_H, \tag{57}$$

where C_σ is the bound of the saturation function σ. Therefore, for all $t \geq 0$, $\frac{d}{dt}\|z\|_H \geq -C_\sigma\|B\|_{\mathcal{L}(U,H)}$, which implies that

$$\|z\|_H \geq \|z_0\|_H - 2C_\sigma\|B\|_{\mathcal{L}(U,H)}t, \tag{58}$$

and the decay rate is at most linear as long as the r.h.s. of the above inequality is larger than one, i.e. $\|z_0\|_H$ is large enough. ◦

V. Illustrative examples

A. Linearized Korteweg-de Vries equation

As a first example, let us focus on the following partial differential equation, which is the Korteweg-de Vries equation,

$$\begin{cases} z_t(t,x) + z_x(t,x) + z_{xxx}(t,x) = \\ \quad -a(x)z(t,x), \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t,0) = z(t,L) = z_x(t,L) = 0, \quad t \in \mathbb{R}_{\geq 0}, \\ z(0,x) = z_0(x), \quad x \in [0, L], \end{cases} \tag{59}$$

where L is a positive constant, ω is a nonempty open subset of $(0, L)$ and $a(x)$ is a smooth bounded nonnegative function satisfying $a(x) \geq a_0$ for all $x \in \omega$ for some positive constant a_0.

This equation can be written in an abstract way as in (9) if one sets $H = U = L^2(0, L)$,

$$\begin{aligned} A : D(A) \subset L^2(0,L) &\to L^2(0,L), \\ z &\mapsto -z' - z''', \end{aligned} \tag{60}$$

where

$$D(A) := \{z \in H^3(0, L) \mid z(0) = z(L) = z'(L) = 0\}, \tag{61}$$

and

$$\begin{aligned} B : H &\to U \\ u &\mapsto \sqrt{a(x)}u. \end{aligned} \tag{62}$$

The adjoint operators of A and B are, respectively

$$\begin{aligned} A^\star : D(A^\star) \subset H &\to H, \\ z &\mapsto z' + z''', \end{aligned} \tag{63}$$

with $D(A^\star) := \{z \in H^3(0, L) \mid z(0) = z(L) = z'(0) = 0\}$, and

$$\begin{aligned} B^\star : U &\to H \\ z &\mapsto \sqrt{a(x)}z. \end{aligned} \tag{64}$$

A straightforward computation, together with some integrations by parts, shows that $\langle Az, z\rangle_H + \langle z, Az\rangle_H \leq 0$, for all $z \in D(A)$. Since A is a closed linear operator and $D(A)$ is dense in H, according to Lümer-Phillips' theorem (see e.g., [31, Theorem 3.8.4., Page 103]), it follows that A generates a strongly continuous semigroup of contractions. Note that, according to [4, Section 4], the origin of this system is globally exponentially stable. Note that, in the case where $\omega = (0, L)$, the Lyapunov functional of the linear system is simply given by $V(z) := \frac{1}{2}\|z\|^2_{L^2(0,L)}$ and let $a_1 > 0$ be an upper bound of the function a. We will now illustrate our main results given in Theorem 2 in the case where the linear damping is modified by a nonlinear damping. We aim also at giving explicit decay rates when $\omega = (0, L)$ and when specifying the nonlinearity.

The PDE modified with the nonlinear damping reads

$$\begin{cases} z_t(t,x) + z_x(t,x) + z_{xxx}(t,x) = \\ \quad - \sqrt{a(x)}\sigma(\sqrt{a(x)}z(t,x)), \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \quad t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x), \quad x \in [0, L], \end{cases} \tag{65}$$

In the case where $S = U$, the result follows easily, since the operator $B^\star$ does not have to satisfy any regularity property, which is in contrast with the case where $S \neq U$. In the case where σ is defined by (19) and where $\omega = (0, L)$, the function h reduces to be the identity and the Lyapunov function is therefore given by $\tilde{V}(z) := \frac{1}{2}\|z\|^2_{L^2(0,L)} + \frac{2}{3}\sqrt{a_1}\|z\|^3_{L^2(0,L)}$. Following proof of Item 1. of Theorem 2, we obtain that

$$\beta(s,t) := \begin{cases} \left(\frac{2}{3}\sqrt{a_1}\right)^{-\frac{1}{3}} \left[V_1^{\frac{1}{3}} s - a_1 V_1^{-\frac{2}{3}} t\right] & s \geq 1. \\ \sqrt{2V_1}\, s \exp\left(-\frac{a_1 V_1}{2} t\right) & s \leq 1 \end{cases} \tag{66}$$

where $V_1 = \frac{1}{2} + \frac{2}{3}\sqrt{a_1}$.

Consider now the saturation function σ defined in (21), i.e. $\sigma = \mathfrak{sat}_{L^\infty(0,1)}$. In order to check whether (28) holds, the following result, obtained in [22], is needed:

Lemma 1. *[[22], Lemma 4.] For all $z \in D(A)$, there exists a positive constant Δ such that*

$$\|z\|_{H_0^1(0,L)} \leq \Delta \|z\|_{D(A)}.$$

Using the above mentionned result together with the fact that the space $H_0^1(0, L)$ is continuously embedded in $L^\infty(0, L)$, that is due to Rellich-Kondrachov Theorem [3, Theorem 9.16, page 285], one obtains that $\|z\|_S \leq \Delta\|z\|_{D(A)}$. Since $B^\star$ is bounded in L^∞-norm, there exists a positive constant c_B such that $\|B^\star z\|_S \leq c_B\|z\|_S$, and then

$$\|B^\star z\|_S \leq c_B\Delta\|z\|_{D(A)}.$$

Therefore, (28) holds for the linear Korteweg-de Vries equation. Consider now that $\omega = [0, L]$. In this case, considering a positive constant r and initial condition z_0 satisfying $\|z_0\|_{D(A)} \leq r$, we can provide explicitely the asymptotic behavior of the trajectory, that is:

$$\|z(t,\cdot)\|_{L^2(0,L)} \leq \exp\left(-\frac{a_1}{1 + c_B\Delta r} t\right) \|z_0\|_H, \quad \forall t \geq 0. \tag{67}$$

B. Wave equations

Consider (10) in Example 1 for the case where the damping is modified by a nonlinear damping function satisfying all the items of Definition 2. Then, the equation reads as follows

$$\begin{cases} z_{tt} = \triangle z - \sqrt{a(x)}\sigma(\sqrt{a(x)}z_t), \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ z(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \Gamma, \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), \end{cases} \tag{68}$$

As before, the case $S = U$ follows easily. Therefore, assuming that the origin of (10) holds, there exists a strict and global Lyapunov functional for (68). Note that, in contrast with the Korteweg-de Vries equation, the explicit Lyapunov function for the wave equation correponding to the operator P is difficult to obtain, even in the case where the damping acts in all the domain

Assume now that $S = L^\infty(\Omega)$. Note that the inequality given by (28) does not hold if the dimension of x is higher or equal to 2. Then, assume that $\Omega := [0, 1]$. This implies that

$$\begin{aligned} \|B^\star z\|_S =& \|a(\cdot)z_t(t,\cdot)\|_{L^\infty(\Omega)} \\ \leq& \|a(\cdot)\|_{L^\infty(\Omega)}\|z_t(t,\cdot)\|_{L^\infty(\Omega)}. \end{aligned} \tag{69}$$

Since $H_0^1(\Omega)$ embedds continuously in $L^\infty(\Omega)$ due to Rellich-Kondrachov theorem [3, Theorem 9.16, page 285], there exists a positive constant C_Ω such that

$$\|B^\star z\|_S \leq C_\Omega \|a(\cdot)\|_{L^\infty(\Omega)} \|z_t(t,\cdot)\|_{H_0^1(\Omega)}. \tag{70}$$

Noticing that

$$\|(z, z_t)\|_{D(A)} := \|z\|_{H^2(\Omega)\cap H_0^1(\Omega)} + \|z_t\|_{H_0^1(\Omega)}, \tag{71}$$

then

$$\|B^\star z\|_S \leq C_\Omega \|a(\cdot)\|_{L^\infty(\Omega)} \|(z, z_t)\|_{D(A)}. \tag{72}$$

This implies that (28) holds for (10) with $c_S := C_\Omega \|a(\cdot)\|_{L^\infty(\omega)}$. Hence, one can apply Theorem 2. In particular, the origin of (68) is semi-globally exponentially stable in $D(A)$ for any nonlinear damping function σ satisfying all the items of Definition 2.

VI. Conclusion

In this paper, we have characterized the asymptotic behavior of a family of linear infinite-dimensional systems subject to a nonlinear damping by means of Lyapunov functionals. These Lyapunov functionals are the sum of two terms: the first one is based on the Lyapunov operator coming from the linear system and the second term is added in order to compensate the nonlinearities. From this work, there exist many research lines which can be pursued further. Below, we have listed some of them.

- Unfortunetaly, our strategy in the case where $S \neq U$ (i.e., $S = L^\infty(\Omega)$) does not work for the wave equation in dimension equal to or larger than two . It might be interesting to investigate a weaker property than (28) in order to characterize precisely the asymptotic behavior of the wave equation subject to a nonlinear damping;
- It might be also interesting to investigate ISS properties of such linear infinite-dimensional systems subject to a nonlinear damping. The case where $S = U$ has been tackled in [23], but the case $S \neq U$ seems harder to obtain. We could also follow the strategy employed in [1], which proposes an explicit way for the construction of Lyapunov functionals.

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