Boundary Output Feedback Stabilization of Nonlinear Diffusion Equations

Hugo Lhachemi, Ionuţ Munteanu, and Christophe Prieur

***Abstract*—This paper studies the problem of output feedback stabilization of a nonlinear diffusion equation, generally referred to as the generalized porous media equation. This equation occurs in a number of physical phenomena involving fluid flow, heat transfer or nonlinear diffusion. The generalized porous media equation differs from classical reaction-diffusion equations due to the presence of a nonlinear version of the Laplacian operator. This nonlinearity induces numerous challenges in terms of stability assessment and control design. In this context, the present paper reports one of the very first contributions regarding the output feedback control of the porous media equation. The reported approach takes advantage of the monotonicity property of the nonlinearity and adopts a perturbation-based approach by leveraging spectral reduction methods.**

***Index Terms*—Generalized porous media equation, output feedback, monotonicity**

I. Introduction

A. Nonlinear diffusion: description and relevance of the model

This paper addresses the control design problem of the following so-called porous media equation [18], described by

$$z_t = (\beta(z))_{xx} + g(z), \tag{1a}$$
$$z(0,t) = 0,\ \beta(z(1,t)) = u(t), \tag{1b}$$
$$z(x,0) = z_0(x), \tag{1c}$$

for $t > 0$ and $x \in (0,1)$. Here, the unknown function is $z = z(x,t)$. We used the notation $(\cdot)_t = \partial(\cdot)/\partial t$ and $(\cdot)_x = \partial(\cdot)/\partial x$ for the partial derivatives with respect to time t and space x, respectively. Function β is nonlinear, piecewise differentiable and strictly monotone such that $\beta(0) = 0$. Function g is assumed to be Lipschitz continuous such that $g(0) = 0$. Additional regularity assumptions will be introduced next for ensuring the well-posedness and for deriving an adequate control strategy. The input u stands for the boundary control.

When β is linear, (1) reduces to a traditional semilinear reaction-diffusion equation which has been largely studied in the literature. The porous media equation embraces the case of nonlinear diffusion terms captured by a nonlinear function β. The introduction of this model was originally motivated by the study of fluid flows through porous media, giving its name to the equation, modeled by $z_t = (z^m)_{xx}$ for $m \in \mathbb{N}^*$. One of its most well-known applications is the description of the flow of an isotropic gas through a porous medium, as modeled independently by Leibenzon [12] and Muskat [16] around 1930. An earlier application of the porous media equation can be found in the study of groundwater infiltration by Boussisnesq in 1903 [1]. Later, this class of equations also arised in the modeling of a number of other physical phenomena such as non-homogeneous media, nonlinear heat transfers, Stefan problem for phase separation, see, e.g., [6]. See also the heat radiation in plasma model developed by Zel'dovich and coworkers around 1950 in [19] and the monograph [18, Section 2.1.1]. In the case of heat transfers, the process is described in the presence of a heat source g as

$$T_t = (kT_x)_x + g(T) \tag{2}$$

where T is the temperature and k is the thermal conductivity. It appears that the thermal conductivity may depend itself on the temperature, i.e., $k = k(T)$. In this case, one can recast (2) into

$$T_t = (\beta(T))_{xx} + g(T) \tag{3}$$

where the function β is given by $\beta(T) = \int_0^T k(s)\,\mathrm{d}s$. Under this form, Equation (3) is a particular case of the porous media Equation (1). In the case of the Stefan problem that models the evolution of the boundary layer between two phases of a material undergoing a phase change, the modeling consists of solving heat equations in both regions, which is modeled by (1) with $\beta(r) = c_1 r$ in a region and $\beta(r) = c_2 r$ in the other region, for two different constants $c_1, c_2 > 0$. For further details, see [6]. These equations that model nonlinear diffusion phenomena are sometimes referred to as generalized porous media equations.

B. State-of-the-art on control design and contributions

Feedback boundary control strategies for the porous media equation have been reported in the literature for few particular cases of the β function and in the sole case of a state-feedback. The case $\beta(r) = r^m$ with no reaction term ($g = 0$) was studied in [8], [9]. The particular case $\beta(r) = r^2$ corresponds to the so-called Boussinesq equation, related to groundwater flow. Internal control have been reported in [11] while a boundary control was studied in [17]. It is also worth mentionning the case of the Burgers' equation, often seen as a toy model related to the Navier-Stokes equation [10]. The case of a more general β function was studied in [15]. For other PDE equations related to the porous media equation see, e.g., [18]. Besides these references, quasi-linear heat equations have also been studied [6].

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One of the main challenges that arises in the feedback control of the nonlinear diffusion equation lies in the regularity of the system trajectories. Indeed, while for linear functions β the open-loop equation (1) is well-posed in $L^2(0,1)$, the case of a nonlinear β functions requires to embed the study into $H^{-1}(0,1)$, see e.g. [2], [18]. This regularity makes challenging the control design problem of the nonlinear diffusion equation. In this context, the contribution of this paper is threefold:

1) we report an observer-based output feedback control strategy for the stabilization of a class of nonlinear diffusion equation;
2) we assess the well-posedness of the resulting closed-loop system;
3) we guarantee the global exponential stability of the closed-loop dynamics in suitable norm.

As far as we know, there is no available result on the output feedback control for the nonlinear diffusion equation (1). This work presents a first result in that direction.

The approach developed in this paper relies on spectral reduction methods. Indeed, one can observe that with $\beta(r) = \alpha r$ for some $\alpha > 0$, equation (1) reduces to the semilinear heat equation described by $z_t = \alpha z_{xx} + g(z)$. The literature is plentiful of results for the either state or output feedback stabilization of such a system [13]. One of the main approaches lies in the use of the fact that the Laplace operator $-\partial_{xx}$ has a countable spectrum with underlying eigenvectors that form an orthonormal basis. This allows to carry on spectral reduction methods to reduce the PDE into a countable infinite number of ordinary differential equations whose only a finite number of them have to be actively controlled. Such an approach does not solely hold for the heat equation but actually applies to PDEs for which the associated unbounded operator presents (generalized) eigenvectors that form a Riesz basis; see e.g. the case of the wave equation in [4]. The key idea of this paper is to leverage these methods by writing

$$(\beta(z))_{xx} = \alpha z_{xx} + (\beta(z) - \alpha z)_{xx} \tag{4}$$

for some $\alpha > 0$. We interpret here $\alpha\partial_{xx}$ as the leading operator for spectral reduction while $(\beta - \alpha\cdot)_{xx}$ is seen as a perturbation. Proceeding this way, the spectral decomposition approach becomes a suitable approach for the well-posedness and the stabilization by a suitable output feedback law.

C. Outline of the paper

The outline of the paper is the following: in Section II we introduce the main notations of this work, we discuss about the well-posedness of the plant and the control objective. Then, in Section III we employ the spectral decomposition method and design the corresponding finite-dimensional observer-based controller. In Section IV we state and prove the main stabilization result. To illustrate the derived results, we provide a numerical example in Section V. The paper ends with an appendix assessing the well-posedness of the closed-loop system.

II. Preliminaries and control objective

A. Notation and properties

We denote by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ the set of all positive integers. Spaces $\mathbb{R}^n$ are endowed with the Euclidean scalar product $\langle \cdot, \cdot \rangle_n$ and norm $\|\cdot\|_n$. $L^2(0,1)$ stands for the space of square Lebesgue integrable functions on $(0,1)$ and is endowed with the inner product $\langle z, w \rangle = \int_0^1 z(x)w(x)dx$ with associated norm denoted by $\|\cdot\|$. For any integer m, the $m-$order Sobolev space is denoted by $H^m(0,1)$ and is endowed with its usual norm denoted by $\|\cdot\|_{H^m}$. We set $H_0^1(0,1)$ for the completion of the space of infinitely differentiable functions, which are nonzero only on a compact subset of $(0,1)$, with respect to the Sobolev norm $\|\cdot\|_{H^1}$. We denote by $H^{-1}(0,1)$ the dual space of $H_0^1(0,1)$ (for details, see e.g. [7, Section 5.9.1]). For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that P is positive semi-definite (resp. positive definite). We set I for the identity matrix. We shall frequently use the Young's inequality:

$$ab \leq \epsilon a^2 + C_\epsilon b^2 \text{ where } C_\epsilon = \frac{1}{4\epsilon} \tag{5}$$

for all $a, b \in \mathbb{R}$ and all $\epsilon > 0$.

Let us next introduce the Dirichlet-Laplace operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^2(0,1)$, $\mathcal{A}z = -z_{xx}$, $\forall z \in \mathcal{D}(\mathcal{A}) = \left\{z \in H^2(0,1) : z(0) = z(1) = 0\right\}$. It is well known that the Dirichlet-Laplace operator has a countable increasing set of positive eigenvalues, $\left\{\lambda_j = \pi^2 j^2\right\}_{j\in\mathbb{N}^*}$, which accumulate to infinity, and with the corresponding eigenfunctions set $\left\{\phi_j = \sqrt{2}\sin(j\pi x)\right\}_{j\in\mathbb{N}^*}$ that forms an orthonormal basis in $L^2(0,1)$. For details, see e.g. [7, Section 6.5]). Then, the Parseval's identity reads as $\|z\|^2 = \sum_{j=1}^{\infty} \langle z, \phi_j \rangle^2$, $\forall z \in L^2(0,1)$. Moreover, $\mathcal{A}$ induces an isomorphism from $H_0^1(0,1)$ onto $H^{-1}(0,1)$. Therefore, we can define an inner product in $H^{-1}(0,1)$ as

$$\langle z, w \rangle_{H^{-1}} = \left\langle \mathcal{A}^{-1/2}z, \mathcal{A}^{-1/2}w \right\rangle, \ \forall z, w \in H^{-1}(0,1).$$

We denote by $\|\cdot\|_{H^{-1}}$ the corresponding norm. We have

$$\|z\|_{H^{-1}}^2 = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle z, \phi_j \rangle^2, \quad \forall z \in H^{-1}(0,1).$$

B. Main assumptions and examples from the literature

We make the following standing assumptions for the functions β and g.

Assumption 1: The function $\beta : \mathbb{R} \to \mathbb{R}$ is bijective, piecewise differentiable, with nonzero derivative, satisfies $\beta(0) = 0$, and is strictly monotone, i.e., there exists $\alpha > 0$ such that

$$(\beta(r) - \beta(s))(r - s) \geq \alpha(r - s)^2, \quad \forall r, s \in \mathbb{R}. \tag{6}$$

⌟

Assumption 2: Function $g : H^{-1}(0,1) \to H^{-1}(0,1)$ is Lipschitz continuous and satisfies $g(0) = 0$. ⌟

Assumption 3: There exist $\rho > 0$ and $L_g \in \mathbb{R}$ such that

$$|\beta(r) - \alpha r| \leq \rho|r|, \quad \forall r \in \mathbb{R}, \tag{7a}$$

$$\|g(z) - L_g z\|_{H^{-1}} \leq \rho\|z\|_{H^{-1}}, \quad \forall z \in H^{-1}(0,1). \tag{7b}$$

⌟

The last assumption motivates the definition $f = g - L_g\mathbf{1}$, hence we have

$$g = L_g\mathbf{1} + f \tag{8}$$

where f is seen as a perturbation around the linear operator $L_g\mathbf{1}$, $\mathbf{1}$ standing for the identity operator.

Remark 1: The introduced assumptions are compatible with the nonlinear heat transfert model (3) *with* $\beta(T) = \int_0^T k(s)\,\mathrm{d}s$ *provided that* k *is positive, continuous, and* $|k(s)-\alpha| \leq \rho$ *for all* $s \in \mathbb{R}$. *This framework is also compatible with the Stefan problem for which* β *is piecewise linear, see [6]. Note however that in the case of the porous media equation for which* $\beta(r) = r^m,\ m > 1$, *Assumptions 1 and 3 fail to hold.* ⌟

Remark 2: The monotonicity assumption (6) *for* β, *as well as its piecewise differentiability, are needed for well-posedness assessment in virtue of the results in [2, Corrolary 31] as decribed in the next subsection. However, for stability analysis, we merely need the following significantly weaker property:* $\beta(r)r \geq 0$ *for all* $r \in \mathbb{R}$. ⌟

C. Discussion on the regularity of the system trajectories

The well-posedness of the open-loop plant (1), that is for $u = 0$, has been intensively studied in the literature, see [2, Corollary 31] and the monograph on porous media equations [18]. From the current state of the art, the existence of solutions can be proved either in the space $L^1(0,1)$ or in the space $H^{-1}(0,1)$. While the well-posedness in the space $L^1(0,1)$ seems more attractive at first sight, we focus in this paper on solutions in the space $H^{-1}(0,1)$. The main reason is that even if $L^1(0,1)$ is a Banach space, it is not a Hilbert space, i.e., its norm is not associated with an inner product. However, the possibility to dispose of an inner product gives powerful tools for control design in order to carry on spectral reduction methods. This is the key reason why we focus our study on solutions in the space $H^{-1}(0,1)$.

D. Control objective

Based on the above well-posedness considerations, our objective is to derive an output feedback control strategy that achieves the exponential stabilization of (1) in H^{-1}-norm using the distributed measurement

$$y(t) = {}_{H^{-1}}\langle z(t), \sigma\rangle_{H_0^1} = \int_0^1 \sigma(x)z(x,t)\mathrm{d}x,\ t > 0, \tag{9}$$

where $\sigma \in H_0^1(0,1)$.

Remark 3: Since the nonlinear diffusion equation is well-posed in $H^{-1}(0,1)$, *that is* $z(\cdot,t) \in H^{-1}(0,1)$, *it is not possible to consider pointwise measurement of the form* $z(\xi_m,t)$ *for some* $\xi_m \in [0,1]$ *because it is not well-defined. However, in practice, an in-domain pointwise measurement, such as temperature measurement, is generally an ideal version (obtained as the limit when* $\delta \to 0$*) of a local measurement obtained as the average value of the measured quantity over a small spatial domain:* $y(t) = \int_0^1 \sigma(x)z(x,t)\,\mathrm{d}x$ *with* $\operatorname{supp}\sigma \subset [\xi_m - \delta, \xi_m + \delta]$ *for some small* $\delta > 0$ *and with* $\int_0^1 \sigma(x)\,\mathrm{d}x = 1$ *(for further details, see [5, Example 1.1.3]). This case fits within our framework provided* $\sigma \in H_0^1(0,1)$. ⌟

III. Spectral reduction and control design

A. Spectral reduction

Let us denote $z_n(t) = \langle z(\cdot,t), \phi_n\rangle$, $n \geq 1$. We proceed with the spectral reduction of the problem: multiplying equation (1) by ϕ_n, integrating over $(0,1)$, and integrating by parts, the dynamics of the nth mode is expressed by

$$\frac{d}{dt}z_n(t) = -\lambda_n\left\langle\beta(z(\cdot,t)),\phi_n\right\rangle - u(t)\phi_n'(1) + \left\langle g(z(\cdot,t)),\phi_n\right\rangle.$$

Remark 4: Since the solutions to (1) *are in* $H^{-1}(0,1)$ *[2], [18], they are expressed under a variational form. Thus, the above relation is rigorously derived by lifting the non-homogeneous boundary condition (as done in appendix) and applying the variational definition of the solutions with the particular test function* $\phi_n,\ n \geq 1$. ⌟

Following the idea (4) and in view of (8), we have

$$\frac{d}{dt}z_n(t) = (-\alpha\lambda_n + L_g)z_n - u(t)\phi_n'(1) + R_n(t) + f_n(t) \tag{10}$$

for all $t > 0$ and $n \geq 1$ where

$$f_n(t) = \left\langle f(z(\cdot,t)),\phi_n\right\rangle, \tag{11a}$$

$$R_n(t) = -\lambda_n\left\langle\beta(z(\cdot,t)) - \alpha z(\cdot,t),\phi_n\right\rangle. \tag{11b}$$

The idea is now to carry on the control design by considering the last two terms R_n and f_n of (10) as perturbations. To do so, let $\delta > 0$ be the targeted exponential decay rate for the closed-loop system solutions. We fix the integer $N_0 \geq 0$ so that $-\alpha\lambda_{N_0+1} + L_g < -\delta$. Introducing now

$$\begin{aligned}
&Z_1 = (z_1\ z_2\ \dots\ z_{N_0})^\top,\\
&A_1 = \operatorname{diag}(-\alpha\lambda_1 + L_g, -\alpha\lambda_2 + L_g, \dots, -\alpha\lambda_{N_0} + L_g),\\
&B_1 = (\phi_1'(1)\ \phi_2'(1)\ \dots\ \phi_{N_0}'(1))^\top,\\
&\mathcal{R}_1 = (R_1\ R_2\ \dots\ R_{N_0})^\top,\\
&\mathcal{F}_1 = (f_1\ f_2\ \dots\ f_{N_0})^\top,
\end{aligned}$$

we infer that

$$\frac{d}{dt}Z_1 = A_1Z_1 - B_1u + \mathcal{R}_1 + \mathcal{F}_1.$$

The pair (A_1, B_1) is controllable; see [13] for details. Hence, there exists a spectrum shifting matrix $K \in \mathbb{R}^{1\times N_0}$ such that $A_1 - B_1K$ is Hurwitz with spectral abscissa strictly less than $-\delta$.

Let an integer $N \geq N_0$ to be selected latter on. In order to define the control strategy, let us introduce the following additional definitions:

$$\begin{aligned}
&Z_2 = (z_{N_0+1}\ z_{N_0+2}\ \dots\ z_N)^\top,\\
&\mathcal{R}_2 = (R_{N_0+1}\ R_{N_0+2}\ \dots\ R_N)^\top,\\
&\mathcal{F}_2 = (f_{N_0+1}\ f_{N_0+2}\ \dots\ f_N)^\top,\\
&A_2 = \operatorname{diag}(-\alpha\lambda_{N_0+1} + L_g, \dots, -\alpha\lambda_N + L_g),\\
&B_2 = (\phi_{N_0+1}'(1)\ \phi_{N_0+2}'(1)\ \dots\ \phi_N'(1))^\top.
\end{aligned}$$

We take u of the form $u(t) = KU(t)$, where the vector-valued function $U(\cdot)$ will be precised latter on. Then, we can rewrite (10) as

$$\frac{d}{dt}Z_1 = A_1Z_1 - B_1KU + \mathcal{R}_1 + \mathcal{F}_1 \tag{12a}$$

$$\frac{d}{dt}Z_2 = A_2 Z_2 - B_2 K U + \mathcal{R}_2 + \mathcal{F}_2, \tag{12b}$$

$$\frac{d}{dt}z_n = (-\alpha\lambda_n + L_g) z_n - \phi_n'(1) K U + R_n + f_n, \quad n \geq N+1. \tag{12c}$$

B. Control strategy

The adopted control strategy takes the form

$$\frac{d}{dt}\hat{Z}_1 = A_1 \hat{Z}_1 - B_1 K U - L\left\{ \int_0^1 \sigma(x) \left(\sum_{k=1}^{N} \phi_k(x) \hat{z}_k \right) \mathrm{d}x - y \right\}, \tag{13a}$$

$$\frac{d}{dt}\hat{Z}_2 = A_2 \hat{Z}_2 - B_2 K U, \tag{13b}$$

$$U = \hat{Z}_1, \tag{13c}$$

where $\hat{Z}_1 = (\hat{z}_1, \ldots, \hat{z}_{N_0})^\top$ and $\hat{Z}_2 = (\hat{z}_{N_0+1}, \ldots, \hat{z}_N)^\top$. Here, $L \in \mathbb{R}^{N_0 \times 1}$ is the vector consisting of the observer gains.

C. Representation of the closed-loop system dynamics

We now write the closed-loop system dynamics in suitable form for stability analysis. To do so, we first denote the estimation errors $E_1 = Z_1 - \hat{Z}_1$ and $E_2 = Z_2 - \hat{Z}_2$. Using the definition (9) of the output y and the series expansion $z = \sum_{n\geq 1} z_n \phi_n$, we note that

$$\begin{aligned}
&\int_0^1 \sigma(x) \left(\sum_{k=1}^{N} \phi_k(x) \hat{z}_k \right) dx - y \\
&= \int_0^1 \sigma(x) \left(\sum_{k=1}^{N} \phi_k(x) \hat{z}_k - \sum_{n\geq 1} z_n \phi_n(x) \right) dx \\
&= \sum_{k=1}^{N_0} (\hat{z}_k - z_k)\sigma_k + \sum_{k=N_0+1}^{N} (\hat{z}_k - z_k)\sigma_k - \zeta \\
&= C_1 E_1 + C_2 E_2 + \zeta
\end{aligned}$$

where $\sigma_n = \langle \sigma, \phi_n \rangle$, $n \geq 1$. Here $C_1 = (\sigma_1 \ \sigma_2 \ \ldots \ \sigma_{N_0})$, $C_2 = (\sigma_{N_0+1} \ \sigma_{N_0+2} \ \ldots \ \sigma_N)$, and $\zeta = \sum_{n\geq N+1} \sigma_n z_n$. Combining the above identity with (12) and (13), we infer that

$$\frac{d}{dt}\hat{Z}_1 = [A_1 - B_1 K]\hat{Z}_1 + LC_1 E_1 + LC_2 E_2 + L\zeta, \tag{14a}$$

$$\frac{d}{dt}E_1 = (A_1 - LC_1)E_1 - LC_2 E_2 - L\zeta + \mathcal{R}_1 + \mathcal{F}_1, \tag{14b}$$

$$\frac{d}{dt}\hat{Z}_2 = A_2 \hat{Z}_2 - B_2 K \hat{Z}_1, \tag{14c}$$

$$\frac{d}{dt}E_2 = A_2 E_2 + \mathcal{R}_2 + \mathcal{F}_2. \tag{14d}$$

For latter purpose we need to rescale some of the variables. More precisely, we set

$$\tilde{Z}_2 = \Lambda_2^{-1} \hat{Z}_2, \quad \tilde{B}_2 = \Lambda_2^{-1} B_2$$

where $\Lambda_2 := \mathrm{diag}(\lambda_{N_0+1}, \ldots, \lambda_N)$. Introducing now the finite-dimensional state vector

$$X = \mathrm{col}(\hat{Z}_1,\ E_1,\ \tilde{Z}_2,\ E_2)$$

and the matrices

$$F = \begin{bmatrix} A_1 - B_1 K & LC_1 & 0 & LC_2 \\ 0 & A_1 - LC_1 & 0 & -LC_2 \\ -\tilde{B}_2 K & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix},$$

we rewrite (14) as

$$\frac{d}{dt}X = FX + \mathbb{L}\zeta + G\ (R\ \mathbb{F}) \tag{15}$$

where $\mathbb{L} = \mathrm{col}(L, -L, 0, 0)$, $R = \mathrm{col}(\mathcal{R}_1, \mathcal{R}_2)$,$\mathbb{F} = \mathrm{col}(\mathcal{F}_1, \mathcal{F}_2)$.

The finite-dimensional system (15) is complemented with the rest of the modes:

$$\frac{d}{dt}z_n = (-\alpha\lambda_n + L_g) z_n - \phi_n'(1) K \hat{Z}_1 + R_n + f_n,\ n \geq N+1. \tag{16}$$

In view of the matrix F, the observer matrix L must be computed in order to place the eigenvalues of $A_1 - LC_1$. This is possible if the pair (A_1, C_1) is observable. Since the matrix A_1 is diagonal with simple eigenvalues while the measurment is scalar, the application of the Hautus test shows that the pair (A_1, C_1) is observable if and only if the following assumption holds.

Assumption 4: The function $\sigma \in H_0^1(0,1)$ is such that

$$\sigma_n = \sqrt{2} \int_0^1 \sigma(x) \sin(n\pi x)\, \mathrm{d}x \neq 0, \quad 1 \leq n \leq N_0.$$

⌟

This allows to select the observer gain matrix $L \in \mathbb{R}^{N_0}$ such that $A_1 - LC_1$ is Hurwitz with spectral abscissa strictly less than $-\delta$. In this case, since the three matrix $A_1 - B_1 K$, $A_1 - LC_1$ and A_2 are Hurwitz, a block matrix reasoning shows that the matrix F is Hurwitz as-well.

IV. Stability analysis

Our main result is stated by the following theorem.

Theorem 1: Let $L_g \in \mathbb{R}$ and $\alpha, \delta > 0$ be given. Let $N_0 \geq 0$ be such that $-\alpha\lambda_{N_0+1} + L_g < -\delta$. Under Assumption 4, let $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_1 - B_1 K$ and $A_1 - LC_1$ have a spectral abscissa strictly less than $-\delta$. Then, there exist N large enough and ρ small enough, such that for any β and g obeying Assumptions 1-3, the closed-loop system formed by the nonlinear diffusion equation (1), the output (9), and the controller (13) admits a unique strong solution in $C^1(0,\infty;\ H^{-1}(0,1))$. Moreover, this closed-loop system is exponentially stable: there exists $M > 0$ so that, for all $z_0 \in H^{-1}(0,1)$, $\hat{Z}_1(0) \in \mathbb{R}^{N_0}$, $\hat{Z}_2(0) \in \mathbb{R}^{N-N_0}$ and for all $t \geq 0$, it holds

$$\begin{aligned}
&\|z(\cdot,t)\|_{H^{-1}}^2 + \|\hat{Z}_1(t)\|_{N_0}^2 + \|\hat{Z}_2(t)\|_{N-N_0}^2 \\
&\leq M e^{-2\delta t} \left(\|z_0\|_{H^{-1}}^2 + \|\hat{Z}_1(0)\|_{N_0}^2 + \|\hat{Z}_2(0)\|_{N-N_0}^2 \right).
\end{aligned}$$

Proof : The well-posedness of the coupling between the generalized porous media equation (1), the output (9), and the

controller (13) is shown in the appendix. The remainder of this section is devoted to the proof of the stability part of Theorem 1. To do so, we define the Lyapunov function candidate as

$$V(X,z) = X^\top P X + \sum_{n\geq N+1} \frac{1}{\lambda_n} z_n^2$$

for $X \in \mathbb{R}^{2N}$, $z = \sum_{n\geq N+1} z_n \phi_n \in H^{-1}(0,1)$, and for some $P \succ 0$ to be defined. The derivation of V, along the solutions to (15)-(16) gives

$$\begin{aligned}\dot V + 2\delta V &= X^\top \left\{F^\top P + PF + 2\delta P\right\} X + 2X^\top P \mathbb{L}\zeta \\ &+ 2X^\top PG(R+\mathbb{F}) + 2\sum_{n\geq N+1} \frac{1}{\lambda_n}(-\alpha\lambda_n + L_g + \delta) z_n^2 \\ &+ 2\sum_{n\geq N+1} \frac{1}{\lambda_n}(-\phi_n'(1) K\hat Z_1 + R_n + f_n) z_n. \end{aligned} \tag{17}$$

Due to Young's inequality (5), we have

$$\begin{aligned}&\sum_{n\geq N+1} \frac{1}{\lambda_n}\phi_n'(1) K\hat Z_1 z_n \\ &\leq \frac{\alpha}{3}\sum_{n\geq N+1} z_n^2 + 3C_\alpha \sum_{n\geq N+1} \frac{[\phi_n'(1)]^2}{\lambda_n^2}(K\hat Z_1)^2 \\ &= \frac{\alpha}{3}\sum_{n\geq N+1} z_n^2 + \frac{6C_\alpha}{\pi^2}\sum_{n\geq N+1}\frac{1}{n^2}(K\hat Z_1)^2 \\ &\leq \frac{\alpha}{3}\sum_{n\geq N+1} z_n^2 + \frac{6C_\alpha}{N\pi^2}\hat Z_1^\top K^\top K \hat Z_1 \\ &= \frac{\alpha}{3}\sum_{n\geq N+1} z_n^2 + \frac{6C_\alpha}{N\pi^2} X^\top \tilde K^\top \tilde K X \end{aligned} \tag{18}$$

where $\tilde K := (K,0,0,0)$. Above, we have used the fact that $\sum_{n\geq N+1}\frac{1}{n^2} \leq \frac{1}{N}$. Next, due to the monotonicity property (6) of β, we have

$$\begin{aligned}&\sum_{n\geq N+1}\frac{1}{\lambda_n} R_n z_n = \sum_{n\geq 1}\frac{1}{\lambda_n} R_n z_n - \sum_{n=1}^N \frac{1}{\lambda_n} R_n z_n \\ &\overset{(11b)}{=} -\sum_{n\geq 1}\langle \beta(z) - \alpha z, \phi_n\rangle \langle z, \phi_n\rangle - \sum_{n=1}^N \frac{1}{\lambda_n} R_n z_n \\ &= -\int_0^1 [\beta(z) - \alpha z] z dx - \sum_{n=1}^N \frac{1}{\lambda_n} R_n z_n \\ &\overset{(6)}{\leq} -\sum_{n=1}^N \frac{1}{\lambda_n} R_n z_n \overset{(11b)}{=} \sum_{n=1}^N \langle \beta(z) - \alpha z, \phi_n\rangle z_n \\ &\overset{(5)}{\leq} \rho \sum_{n=1}^N z_n^2 + C_\rho \sum_{n=1}^N \langle \beta(z) - \alpha z, \phi_n\rangle^2 \\ &\leq \rho \sum_{n=1}^N z_n^2 + C_\rho \|\beta(z) - \alpha z\|^2 \\ &\overset{(7a)}{\leq} \rho \sum_{n=1}^N z_n^2 + C_\rho \rho^2 \|z\|^2 = \rho \sum_{n=1}^N z_n^2 + \frac{\rho}{4}\|z\|^2 \\ &= \frac{5\rho}{4}\sum_{n=1}^N z_n^2 + \frac{\rho}{4}\sum_{n\geq N+1} z_n^2. \end{aligned} \tag{19}$$

We denote by $\hat z_n$ and e_n, for $1 \leq n \leq N_0$, the components of $\hat Z_1$ and E_1, respectively. Hence we have $e_n = z_n - \hat z_n$, giving $\sum_{n=1}^{N_0} z_n^2 = \sum_{n=1}^{N_0}(\hat z_n^2 + 2e_n\hat z_n + e_n^2)$. Similarly, we denote by $\hat z_n$, $\tilde z_n$ and e_n, for $N_0{+}1 \leq n \leq N$, the components of $\hat Z_2$, $\tilde Z_2$, and E_2, respectively. Thus we have $e_n = z_n - \hat z_n = z_n - \lambda_n \tilde z_n$, giving $\sum_{n=N_0+1}^N z_n^2 = \sum_{n=N_0+1}^N (\lambda_n^2 \tilde z_n^2 + 2\lambda_n \tilde z_n e_n + e_n^2)$,. Overall, we infer that

$$\sum_{n=1}^N z_n^2 = X^\top \Omega X \tag{20}$$

where

$$\Omega = \begin{bmatrix} I & I & O & O \\ I & I & O & O \\ O & O & \Lambda_2^2 & \Lambda_2 \\ O & O & \Lambda_2 & I \end{bmatrix}. \tag{21}$$

It thus follows by (19)-(20) that

$$\sum_{n\geq N+1}\frac{1}{\lambda_n} R_n z_n \leq \frac{5\rho}{4} X^\top \Omega X + \frac{\rho}{4}\sum_{n=N+1}^\infty z_n^2. \tag{22}$$

Finally, again using Young's inequality, we get

$$\begin{aligned}\sum_{n\geq N+1}\frac{1}{\lambda_n} f_n z_n &\leq \frac{1}{2}\sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 + \frac{1}{2}\sum_{n\geq N+1}\frac{1}{\lambda_n} f_n^2 \\ &\leq \frac{1}{2}\sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 + \frac{1}{2}\|f(z)\|_{H^{-1}}^2 \\ &\overset{(7b)}{\leq} \frac{1}{2}\sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 + \frac{1}{2}\rho^2\|z\|_{H^{-1}}^2 \\ &= \frac{1}{2}\sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 + \frac{1}{2}\rho^2\sum_{n=1}^\infty \frac{1}{\lambda_n} z_n^2 \\ &\leq \frac{1}{2}\rho^2\sum_{n=1}^N z_n^2 + \frac{1}{2}(1+\rho^2)\sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 \\ &\overset{(20)}{=} \frac{\rho^2}{2} X^\top \Omega X + \frac{1}{2}(1+\rho^2)\sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2. \end{aligned} \tag{23}$$

Setting $\tilde X = \mathrm{col}(X, \zeta, R+\mathbb{F})$ and inserting into the relation (17) the estimates (18)-(23), we deduce that

$$\begin{aligned}\dot V + 2\delta V &\leq \tilde X^\top \begin{bmatrix} \Phi_0 & P\mathbb{L} & PG \\ \mathbb{L}^\top P & 0 & 0 \\ G^\top P & 0 & 0 \end{bmatrix} \tilde X \\ &+ 2\sum_{n\geq N+1}\frac{1}{\lambda_n}\left[-\left(\frac{2\alpha}{3} - \frac{\rho}{4}\right)\lambda_n + L_g + \delta + \frac{1}{2}(1+\rho^2)\right] z_n^2 \end{aligned} \tag{24}$$

where $\Phi_0 = F^\top P + PF + 2\delta P + \frac{12C_\alpha}{N\pi^2}\tilde K^\top \tilde K + \left(\frac{5\rho}{2} + \rho^2\right)\Omega$. Note that the application of Cauchy-Schwarz inequality gives:

$$\begin{aligned}\zeta^2 &= \left(\sum_{n\geq N+1}\sigma_n z_n\right)^2 \\ &\leq \sum_{n\geq N+1}\lambda_n\sigma_n^2 \sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 \\ &\leq \|\sigma\|_{H_0^1}^2 \sum_{n\geq N+1}\frac{1}{\lambda_n} z_n^2 \end{aligned} \tag{25}$$

while using Parseval formula we get that

$$\begin{aligned}
&\eta(R+\mathbb{F})^\top(R+\mathbb{F}) = \eta\sum_{n=1}^{N}(R_n+f_n)^2\\
&\overset{(11)}{\leq} 2\eta\sum_{n=1}^{N}\lambda_n^2\left\langle\beta(z)-\alpha z,\phi_n\right\rangle^2 + 2\eta\sum_{n=1}^{N}\left\langle f(z),\phi_n\right\rangle^2\\
&\leq 2\eta\lambda_N^2\sum_{n=1}^{N}\left\langle\beta(z)-\alpha z,\phi_n\right\rangle^2 + 2\eta\lambda_N\sum_{n=1}^{N}\frac{1}{\lambda_n}\left\langle f(z),\phi_n\right\rangle^2\\
&\leq 2\eta\lambda_N^2\sum_{n=1}^{\infty}\left\langle\beta(z)-\alpha z,\phi_n\right\rangle^2 + 2\eta\lambda_N\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\langle f(z),\phi_n\right\rangle^2\\
&= 2\eta\lambda_N^2\int_0^1(\beta(z)-\alpha z)^2dx + 2\eta\lambda_N\|f(z)\|_{H^{-1}}^2\\
&\overset{(7)}{\leq} 2\eta\lambda_N(\lambda_N+1)\rho^2\|z\|^2\\
&= 2\eta\lambda_N(\lambda_N+1)\rho^2\left(X^\top\Omega X+\sum_{n\geq N+1}z_n^2\right). \qquad (26)
\end{aligned}$$

Using estimates (25)-(26) into (24) we infer that

$$\dot V + 2\delta V \leq \tilde X^\top\Theta\tilde X + 2\sum_{n\geq N+1}\frac{1}{\lambda_n}\Phi_{2,n}z_n^2$$

with

$$\Theta = \begin{bmatrix} \Phi_1 & P\mathbb{L} & PG\\ \mathbb{L}^\top P & -\mu & 0\\ G^\top P & 0 & -\eta I\end{bmatrix},$$

where

$$\begin{aligned}
\Phi_1 &= \Phi_0 + 2\eta\lambda_N(\lambda_N+1)\rho^2\Omega\\
&= F^\top P + PF + 2\delta P + \frac{12C_\alpha}{N\pi^2}\tilde K^\top\tilde K\\
&\quad + \left[\frac{5}{2}+(1+2\eta\lambda_N(\lambda_N+1))\,\rho\right]\rho\Omega
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{2,n} = &-\left(\frac{2\alpha}{3}-\frac{\rho}{4}-\eta\lambda_N(\lambda_N+1)\rho^2\right)\lambda_n\\
&+ L_g + \delta + \frac{\mu\|\sigma\|_{H_0^1}^2}{2} + \frac{1}{2}(1+\rho^2).
\end{aligned}$$

Hence, assuming there exist $N\geq N_0$, $\rho,\mu,\eta>0$, and $P\succ 0$ such that $\Theta\preceq 0$ and $\Phi_{2,n}\leq 0$ for all $n\geq N_1$, we infer that $\dot V+2\delta V\leq 0$, giving the claimed stability estimate.

Let us show that for N large enough, there always exists $\rho>0$ small enough, $\mu,\eta>0$ and $P\succ 0$ such that $\Theta\preceq 0$ and $\Phi_{2,n}\leq 0$ for all $n\geq N+1$. To do so, let P be such that $F^\top P+PF+2\delta P=-I$. The application of the Lemma reported in Appendix of [13] shows that $\|P\|=O(1)$ as $N\to+\infty$. Let us also fix $\mu=\eta=\sqrt{\lambda_N}$ and $\rho=1/\lambda_N^3$. Noting that $\Omega\preceq 2\lambda_N^2 I$, we infer from he Schur complement that $\Theta\preceq 0$ for N large enough. Furthermore, selecting N large so that $\frac{2\alpha}{3}-\frac{\rho}{4}-\eta\lambda_N(\lambda_N+1)\rho^2=\frac{2\alpha}{3}-\frac{1}{4\lambda_N^3}-\eta\frac{\lambda_N+1}{\lambda_N^3}\rho^2>0$, we infer that $\Phi_{2,n}\leq\Phi_{2,N+1}$ for all $n\geq N+1$ while, based on the above selection of $\rho,\eta,\mu>0$, it can be seen that $\Phi_{2,N+1}\to-\infty$ as $N\to+\infty$. This completes the proof. □

V. Numerical example

We report in this section a numerical example. Inspired by the Stefan problem, we consider the case of the nonlinear diffusion equation (1) with the below piecewise linear functions β, g. First we define $\beta(r)=c_{\beta,1}r$ for $|r|\leq r_\beta$, $\beta(r)=c_{\beta,1}r_\beta+c_{\beta,2}(r-r_\beta)$ for $r>r_\beta$, and $\beta(r)=-c_{\beta,1}r_\beta+c_{\beta,2}(r+r_\beta)$ for $r<-r_\beta$ with $c_{\beta,1},c_{\beta,2},r_\beta>0$. Function g is defined by $g(z)=\phi(\|z\|_{H^{-1}})\frac{z}{\|z\|_{H^{-1}}}$ where $\phi(r)=c_{\phi,1}r$ for $0\leq r\leq r_\phi$ and $\phi(r)=c_{\phi,1}r_\phi+c_{\phi,2}(r-r_\phi)$ for $r>r_\phi$ with $c_{\phi,1},c_{\phi,2},r_\phi>0$. It can be checked that β,g do satisfy Assumptions 1-3. For numerical simulation, we set $c_{\beta,1}=0.5$, $c_{\beta,2}=1$, $c_{\phi,1}=5$, $c_{\phi,2}=7$, and $r_\beta=r_\phi=1$.

The numerical scheme consists of a finite-element method with 100 nodes. The time-domain evolution of the solution of the open-loop system (1) for the initial condition $z_0(x)=10(1+5x^2)\sin(\pi x)$ shows that the system trajectories do not converge to the origin, confirming the instability of the plant. We thus consider the closed-loop system formed by the nonlinear diffusion equation (1), the output (9) with $\sigma(x)=x(1-x)$, and the controller (13). We set $K=-1.59$ and $L=66.13$ while the dimension of the observer is set as $N=3$. The simulation of the closed-loop system from the above initial condition z_0 is depicted in Fig. 1, in compliance with the theoretical predictions.

VI. Conclusion

This paper reports a control design solution to the problem of output feedback control of the generalized porous media equation. The adopted approach takes advantage of spectral reduction methods by adopting a perturbation-based approach. As a possible future research direction is the extension of these results to the multi-dimensional case by following the ideas from the recent paper [14].

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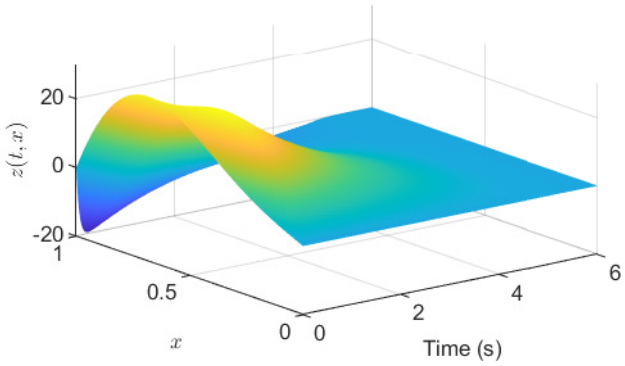
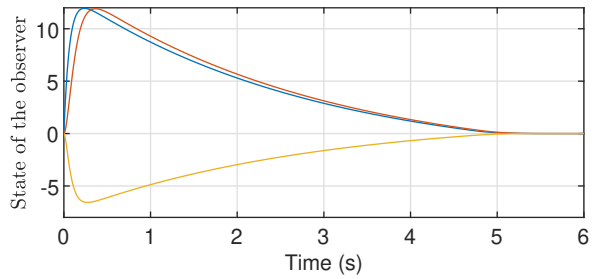
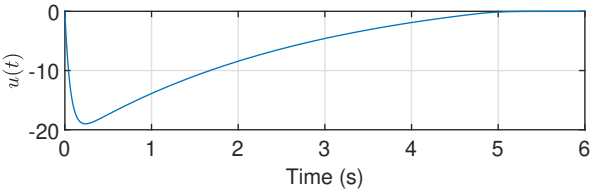


(a) State of the PDE $z(\cdot,t)$



(b) State of the observer (13a)-(13b)



(c) Input $u(t)$

Fig. 1. Time-domain behavior of the closed-loop system from the initial condition $z_0(x) = 10(1+5x^2)\sin(\pi x)$

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APPENDIX

Let $z_0 \in H^{-1}(0,1)$, $\hat{Z}_1^0 \in \mathbb{R}^{N_0}$, $\hat{Z}_2^0 \in \mathbb{R}^{N-N_0}$ be arbitrary but fixed. In this section, we prove that for these initial data there exists a unique solution to the system consisting of the generalized porous media equation (1), the output (9), and the controller (13), which reads as

$$z_t(x,t) = (\beta(z(\cdot,t)))_{xx}(x) + g(z(\cdot,t))(x), \tag{27a}$$
$$z(0,t) = 0,\ \beta(z(1,t)) = K\hat{Z}_1(t), \tag{27b}$$
$$\frac{d}{dt}\hat{Z}_1(t) = [A_1 - B_1K]\hat{Z}_1(t), \tag{27c}$$
$$- L\left\{\int_0^1 \sigma(x)\left(\sum_{k=1}^N \phi_k(x)\hat{z}_k(t)\right)dx - y(t)\right\} \tag{27d}$$
$$\frac{d}{dt}\hat{Z}_2(t) = A_2\hat{Z}_2(t) - B_2K\hat{Z}_1(t), \tag{27e}$$
$$z(x,0) = z_0(x),\ \hat{Z}_1(0) = \hat{Z}_1^0,\ \hat{Z}_2(0) = \hat{Z}_2^0. \tag{27f}$$

We restrict the study of (27) on an interval $[0,T]$, with $T > 0$ selected latter, and we split the dynamics into two parts. More precisely, letting $\hat{W}_1(\cdot) \in C^1([0,T];\ \mathbb{R}^{N_0})$, we show that there exists a unique solution to

$$z_t(x,t) = (\beta(z(\cdot,t)))_{xx}(x) + g(z(\cdot,t))(x), \tag{28a}$$
$$z(0,t) = 0,\ \beta(z(1,t)) = K\hat{W}_1(t),\ t \in [0,T]. \tag{28b}$$

To this end, we lift the boundary condition (28b) into the equation (28a), by introducing, for each $v \in \mathbb{R}$, the function $D(x,v) := \beta^{-1}(xv)$, $x \in (0,1)$. Setting $w(\cdot) := z(\cdot) - D(\cdot, K\hat{W}_1(\cdot))$, we equivalently rewrite (28) as

$$w_t = (\tilde{\beta}(w, D(K\hat{W}_1)))_{xx} + \tilde{g}(w, D(K\hat{W}_1)), \tag{29a}$$
$$w(0,t) = 0,\ w(1,t) = 0,\ t \in [0,T], \tag{29b}$$

where [1]

$$\tilde{\beta}(r,d) := \beta(r+d) - \beta(d),\ \forall r,d \in \mathbb{R}$$

and

$$\tilde{g}(w,t) := -\frac{d}{dt}D(K\hat{W}_1) + g(w + D(K\hat{W}_1)).$$

Note that

$$\frac{d}{dt}D(K\hat{W}_1)(x,t) = x\frac{1}{\beta'(\beta^{-1}(xK\hat{W}_1)(t))}K\frac{d}{dt}\hat{W}_1(t)$$

exists due to the hypotheses. It is easy to see that $\tilde{\beta}$ remains monotone, in the sense that, for each $d \in \mathbb{R}$, we have

$$(\tilde{\beta}(r_1,d) - \tilde{\beta}(r_2,d))(r_1 - r_2) \geq \alpha(r_1 - r_2)^2,\ \forall r_1, r_2 \in \mathbb{R}.$$

Moreover, the function $\tilde{g}$ is Lipschitz in $H^{-1}(0,1)$ with respect to the first variable, uniformly with respect to time. Therefore, [18, Theorem 10.9] assures the existence of a unique strong solution $w \in C([0,T];\ H^{-1}(0,1))$ to (29), which implies that there exists a unique solution $z \in C([0,T];\ H^{-1}(0,1))$ to (28).

Now, given $z \in C([0,T];\ H^{-1}(0,1))$, we recall the notation $y(t) = {}_{H^{-1}}\langle z(t), \sigma\rangle_{H_0^1}$ and the estimate

$$|y(t)| \leq \|\sigma\|_{H_0^1}\|z(t)\|_{H^{-1}},\ \forall t \in [0,T].$$

[1] We have used the fact that $(\beta(D(K\hat{W}_1)))_{xx} = 0$.

Therefore, the classical theory related to linear ordinary differential systems ensures the existence of a unique solution $(\hat{W}_1, \hat{W}_2)^\top := (\hat{w}_1, \dots, \hat{w}_{N_0}, \hat{w}_{N_0+1}, \dots, \hat{w}_N)^\top$ in $C^1([0,T];\ \mathbb{R}^N)$ to the system

$$\frac{d}{dt}\hat{W}_1 = [A_1 - B_1K]\hat{Z}_1 \tag{30a}$$

$$- L\left\{\int_0^1 \sigma(x)\left(\sum_{k=1}^N \phi_k \hat{w}_k\right)dx - y\right\}, \tag{30b}$$

$$\frac{d}{dt}\hat{W}_2 = A_2\hat{W}_2 - B_2K\hat{W}_1,\ t\in[0,T]. \tag{30c}$$

Next, we follow a classical fixed point argument. More precisely, taking advantage of the above, we can introduce the map $S : C^1([0,T];\mathbb{R}^{N_0}) \to C^1([0,T];\mathbb{R}^{N_0})$ as: for $\hat{W}_1 \in C^1([0,T];\mathbb{R}^{N_0})$ we construct the unique solution $z \in C([0,T];H^{-1}(0,1))$ to (28). Using the latter, we construct $(\overline{W}_1, \overline{W}_2) \in C^1([0,T];\mathbb{R}^N)$, the unique solution to (30). We define $S(\hat{W}_1) = \overline{W}_1$.

Now, let $\hat{W}_1^i \in C^1([0,T];\mathbb{R}^{N_0}),\ i=1,2$. We construct $z^i \in C([0,T];H^{-1}(0,1)),\ i=1,2$, solutions to (28) corresponding to $\hat{W}^i,\ i=1,2$, respectively. We immediately see that

$$(z^1-z^2)_t = \ (\beta(z^1)-\beta(z^2))_{xx} + g(z^1) - g(z^2), \tag{31a}$$

$$(z^1-z^2)(0,t) = \ 0, \tag{31b}$$

$$\beta(z^1(1,t)) - \beta(z^2(1,t)) = K(\hat{W}_1^1 - \hat{W}_1^2)(t). \tag{31c}$$

Successively multiplying (31) by $\phi_j,\ j=1,2,...,$ in $L^2(0,1)$, we get that

$$\frac{d}{dt}\left\langle z^1-z^2,\phi_j\right\rangle = -K(\hat{W}_1^1-\hat{W}_1^2)\phi_j'(1) - \lambda_j\left\langle \beta(z^1)-\beta(z^2),\phi_j\right\rangle + \left\langle g(z^1)-g(z^2),\phi_j\right\rangle,$$

for $j = 1,2,....$ Then, multiplying the above relation by $\frac{1}{\lambda_j}\left\langle z^1-z^2,\phi_j\right\rangle$, and summing over j, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|z^1-z^2\|_{H^{-1}}^2 = - K(\hat{W}_1^1-\hat{W}_1^2)\sum_{j=1}^\infty \frac{\phi_j'(1)}{\lambda_j}\left\langle z^1-z^2,\phi_j\right\rangle - \left\langle \beta(z^1)-\beta(z^2), z^1-z^2\right\rangle + \left\langle g(z^1)-g(z^2), z^1-z^2\right\rangle_{H^{-1}}.$$

Making use of the Young's inequality, the monotonicity of β and the Lipschitzianity of g, from the above we get

$$\begin{aligned}&\frac{d}{dt}\|z^1-z^2\|_{H^{-1}}^2\\ &\le 2C_\alpha|K(\hat{W}_1^1-\hat{W}_1^2)|^2\sum_{j=1}^\infty\left(\frac{\phi_j'(1)}{\lambda_j}\right)^2 + 2\alpha\|z^1-z^2\|^2\\ &- 2\alpha\|z^1-z^2\|^2 + \frac{1}{2}\|z^1-z^2\|_{H^{-1}}^2\\ &\le c_1\|\hat{W}_1^1-\hat{W}_1^2\|_{N_0}^2 + c_2\|z^1-z^2\|_{H^{-1}}^2,\end{aligned}$$

for some constants $c_1, c_2 > 0$. Hence, we obtain that

$$\|z^1(t)-z^2(t)\|_{H^{-1}}^2 \le c_1\int_0^t e^{c_2(t-s)}\|\hat{W}_1^1(s)-\hat{W}_1^2(s)\|_{N_0}^2 ds,$$

for all $t\in[0,T]$. This yields

$$\begin{aligned}&\|z^1-z^2\|_{C([0,T];H^{-1}(0,1))}\\ &\le \left(c_1\frac{e^{c_2T}-1}{c_2}\right)^{\frac{1}{2}}\|\hat{W}_1^1-\hat{W}_1^2\|_{C^1([0,T];\mathbb{R}^{N_0})}.\end{aligned} \tag{32}$$

Now, let $\overline{W}^i = (\overline{W}_1^i, \overline{W}_2^i) \in C^1([0,T];\mathbb{R}^N),\ i=1,2$, the solutions to (30) corresponding to $y^i(t) = \int_0^1 \sigma(x)z^i(x,t)dx,\ i=1,2$, respectively. Setting the matrix H as

$$H := \begin{pmatrix} A_1 - B_1K - LC_1 & -LC_2 \\ -B_2K & A_2\end{pmatrix},$$

we observe that

$$\frac{d}{dt}(\overline{W}^1-\overline{W}^2) = H(\overline{W}^1-\overline{W}^2) + \mathcal{Y}, \tag{33}$$

where

$$\mathcal{Y} = \begin{pmatrix} L(y^1-y^2)\\ 0\end{pmatrix}.$$

We note by (32) that there exists $c_3 > 0$ independent to T such that

$$\begin{aligned}\|\mathcal{Y}\|_{C([0,T])} &= \|L\int_0^1\sigma(x)(z^1(x,\cdot)-z^2(x,\cdot))dx\|_{C([0,T])}\\ &\le c_3\left(e^{c_2T}-1\right)^{\frac{1}{2}}\|\hat{W}_1^1-\hat{W}_1^2\|_{C^1([0,T];\mathbb{R}^{N_0})}.\end{aligned} \tag{34}$$

Invoking the variation of constants formula in (33), we get that

$$(\overline{W}^1-\overline{W}^2)(t) = \int_0^t e^{(t-s)H}\mathcal{Y}(s)ds,$$

for all $t\in[0,T]$. Taking into account relation (34), the above implies that there exists a constant $c_4 > 0$ such that

$$\begin{aligned}&\|\overline{W}^1-\overline{W}^2\|_{C([0,T];\mathbb{R}^N)}\\ &\le c_4(e^{c_2T}-1)^{\frac{1}{2}}(e^{\|H\|T}-1)\|\hat{W}_1^1-\hat{W}_1^2\|_{C^1([0,T];\mathbb{R}^{N_0})}.\end{aligned}$$

Inserting this into (33) and using (34), we conclude that there exists a constant $c_5 > 0$ such that

$$\begin{aligned}&\|\overline{W}^1-\overline{W}^2\|_{C^1([0,T];\mathbb{R}^N)}\\ &\le c_5(e^{c_2T}-1)^{\frac{1}{2}}e^{\|H\|T}\|\hat{W}_1^1-\hat{W}_1^2\|_{C^1([0,T];\mathbb{R}^{N_0})}.\end{aligned}$$

Recalling that, by definition, $S(\hat{W}_1^i) = \overline{W}_1^i,\ i=1,2$, we immediately deduce from the above that

$$\begin{aligned}&\|S(\hat{W}_1^1)-S(\hat{W}_1^2)\|_{C^1([0,T];\mathbb{R}^{N_0})}\\ &\le c_5(e^{c_2T}-1)^{\frac{1}{2}}e^{\|H\|T}\|\hat{W}_1^1-\hat{W}_1^2\|_{C^1([0,T];\mathbb{R}^{N_0})}\end{aligned}$$

for all $T > 0$. Thus, taking T small enough, S is a contraction in $C^1([0,T];\mathbb{R}^{N_0})$. Then, the contraction mapping principle ensures that S has a unique fixed point, denoted by $\hat{Z}_1$, which, in fact, defines a unique solution $(z, \hat{Z}_1, \hat{Z}_2)$ to the coupled system (27), on $[0,T]$, for any initial data. Since T does not depend on the initial data and the system (27) is autonomous, we can define a unique global solution by following the classical Peano procedure [3]. This is achieved by iteratively extending the solution defined on the interval $[0,nT]$ for some positive integer n to the interval $[0,(n+1)T]$ by defining the solution on $[nT,(n+1)T]$ by invoking the previously developed approach.