Exponential Input-to-State Stabilization of a Class of Diagonal Boundary Control Systems with Delay Boundary Control*

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ABSTRACT

This paper deals with the exponential input-to-state stabilization with respect to boundary disturbances of a class of diagonal infinite-dimensional systems via delay boundary control. The considered input delays are uncertain and time-varying. The proposed control strategy consists in a constant-delay predictor feedback controller designed on a truncated finite-dimensional model capturing the unstable modes of the original infinite-dimensional system. We show that the resulting closed-loop system is exponential input-to-state stable with fading memory of both additive boundary input perturbations and disturbances in the computation of the predictor feedback.

1. Introduction

Feedback stabilization of finite-dimensional systems in the presence of input delays has been a very active research topic during the past decades [1, 27]. Motivated by the delay boundary control of Partial Differential Equations (PDEs), the opportunity of extending this topic to infinite-dimensional systems has recently attracted much attention [9, 29]. One of the early contributions on input delayed unstable PDEs, reported in [17], deals with a reaction-diffusion equation with a controller designed by resorting to the backstepping technique. More recently, the opportunity to use a predictor feedback for the stabilization of a reaction-diffusion equation was reported in [26]. The proposed control strategy, inspired by the early works [6, 7, 28] dealing with delay-free boundary feedback control, goes as follows. First, a finite-dimensional truncated model capturing the unstable modes of the infinite-dimensional system is obtained via spectral reduction. Then, using the Artstein transformation for handling the input delay, a predictor feedback is designed to stabilize the truncated model. Finally, the stability of the closed-loop infinite-dimensional system is assessed via a Lyapunov-based argument. This strategy was reused in [10] for the delay boundary feedback stabilization of a linear Kuramoto-Sivashinsky equation. This was then generalized to the boundary feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control for either a constant [18, 22] or a time-varying [19] input delay.

In this paper, we investigate the exponential input-to-state stabilization with respect to boundary disturbances of a class of diagonal infinite-dimensional systems via delay boundary control. In this setting, the considered input delay is uncertain and time-varying. The main motivation in achieving an input-to-state stabilization of the closed-loop system relies in the fact that the Input-to-State Stability (ISS) property, originally introduced by Sontag in [31], is one of the main tools for assessing the robustness of a system with respect to boundary disturbances. This property also plays a key role in the establishment of small gain conditions for the stability of interconnected systems [16]. Although the study of ISS properties of finite-dimensional systems has been intensively studied during the last three decades, its extension to infinite-dimensional systems, and in particular with respect to boundary disturbances, is more recent [4, 11, 12, 14, 15, 16, 20, 21, 23, 24, 25, 32, 33]. Moreover, most of these results deal with the establishment of ISS properties for open-loop stable distributed parameter systems. The literature regarding the input-to-state stabilization of open-loop unstable infinite-dimensional systems is less developed.

The present paper extends the results reported in [18, 19] regarding the use of a constant-delay predictor feedback for the delayed boundary stabilization of a class of diagonal infinite-dimensional systems. The validity of such an approach was first assessed in [18] for a constant and known input delay and then in [19] for an unknown and time-varying input delay via Lyapunov-based arguments. While such an approach allows the derivation of an ISS estimate with respect to distributed disturbances [18], it fails in the establishement of an ISS estimate, in strict form[1], with respect to boundary disturbances. In this paper, under the assumption of a sectorial condition on the eigenvalues corresponding to the modes which are not captured by the truncated model used for the design of the predictor feedback, we show that the resulting infinite-dimensional closed-loop system is exponential ISS with fading memory [16] of the boundary

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[1]More precisely, this approach only allows the derivation of an ISS estimate with respect to both the boundary perturbation and its time derivative, but not an ISS estimate in strict form, i.e., with respect to the only magnitude of the boundary perturbation.

disturbances for small variations of the time-varying delay around its nominal value. The adopted approach relies first on the extension of a small gain argument reported in [13] in order to establish the ISS property of the closed-loop truncated model, and then on the method reported in [21] for the establishment of ISS estimates with respect to boundary disturbances for diagonal infinite-dimensional systems.

This paper is organized as follows. The investigated control problem, the proposed control strategy, and the main result of this paper are introduced in Section 2. In Section 3 is reported the stability analysis of the finite-dimensional truncated model. Then, the proof of the main result of this paper, namely the ISS property of the resulting closed-loop infinite-dimensional system, is presented in Section 4. The relaxation of the assumed regularity assumptions for the boundary disturbances is discussed in Section 5. Finally, concluding remarks are formulated in Section 6.

2. Problem setting and main result

The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. All the finite-dimensional spaces $\mathbb{R}^p$ are endowed with the usual euclidean inner product $\langle x, y\rangle = x^\top y$ and the associated 2-norm $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^\top x}$. For any matrix $M \in \mathbb{R}^{p\times q}$, $\|M\|$ stands for the induced norm of M associated with the above 2-norms. For any $t_0 > 0$, we say that $\varphi \in C^0(\mathbb{R};\mathbb{R})$ is a *transition signal over* $[0, t_0]$ if $0 \le \varphi \le 1$, $\varphi|_{(-\infty,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$.

2.1. Preliminary definitions

Throughout the paper, $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ denotes a separable Hilbert space over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1 (Boundary control system [8]). Consider the abstract system taking the form:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t), & t \ge 0 \\ \mathcal{B}X(t) = v(t), & t \ge 0 \\ X(0) = X_0 \end{cases} \tag{1}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ an (unbounded) operator, $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathbb{K}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ the boundary operator, and $v : \mathbb{R}_+ \to \mathbb{K}^m$ a boundary input. We say that $(\mathcal{A}, \mathcal{B})$ is a boundary control system if

1. the disturbance-free operator $\mathcal{A}_0$, defined on the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$, is the generator of a C_0-semigroup S on $\mathcal{H}$;
2. there exists a bounded operator $B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, called a lifting operator, such that $\mathrm{R}(B) \subset D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, and $\mathcal{B}B = I_{\mathbb{K}^m}$.

Definition 2 (Riesz spectral operator [8]). Let a linear and closed operator $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{H} \to \mathcal{H}$ with simple eigenvalues λ_n and corresponding eigenvectors $\phi_n \in D(\mathcal{A}_0)$, $n \in \mathbb{N}^*$. $\mathcal{A}_0$ is a Riesz-spectral operator if

1. $\{\phi_n,\ n \in \mathbb{N}^*\}$ is a Riesz basis [5]:
   (a) $\{\phi_n,\ n \in \mathbb{N}^*\}$ is maximal, i.e., $\overline{\operatorname{span}_{n\in\mathbb{N}^*}\ \phi_n} = \mathcal{H}$;
   (b) there exist constants $m_R, M_R \in \mathbb{R}_+^*$ such that, for all $N \in \mathbb{N}^*$ and all $\alpha_1, \dots, \alpha_N \in \mathbb{K}$,
   $$m_R \sum_{n=1}^{N} |\alpha_n|^2 \le \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|_{\mathcal{H}}^2 \le M_R \sum_{n=1}^{N} |\alpha_n|^2; \tag{2}$$
2. the closure of $\{\lambda_n,\ n \in \mathbb{N}^*\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$, $[a, b] \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$.

Remark 1. Let $\{\psi_n,\ n \in \mathbb{N}^*\}$ be the biorthogonal sequence associated with $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e., $\langle \phi_n, \psi_m\rangle_{\mathcal{H}} = \delta_{n,m}$. Then ψ_n is an eigenvector of the adjoint operator $\mathcal{A}_0^*$ associated with $\overline{\lambda_n}$. Moreover, the following series expansion holds:

$$\forall z \in \mathcal{H}, \quad z = \sum_{n\ge1} \langle z, \psi_n\rangle_{\mathcal{H}}\, \phi_n. \tag{3}$$

2.2. Problem and proposed control strategy

Let $D_0 > 0$ and $\delta \in (0, D_0)$ be given. We consider the abstract boundary control system (1) for which the boundary input v takes the form:

$$v(t) = u(t - D(t)) + d_1(t) \tag{4}$$

for all $t \ge 0$ with $d_1 : \mathbb{R}_+ \to \mathbb{K}^m$ a boundary disturbance, $u : [-D_0 - \delta, +\infty) \to \mathbb{K}^m$ the boundary control with $u|_{[-D_0-\delta,0]} = 0$, and $D : \mathbb{R}_+ \to [D_0 - \delta, D_0 + \delta]$ a time-varying delay.

Assumption 1. *The disturbance-free operator $\mathcal{A}_0$ is a Riesz spectral operator.*

Then, the C_0-semigroup generated by $\mathcal{A}_0$ is given by

$$\forall z \in \mathcal{H}, \quad \forall t \ge 0, \quad S(t)z = \sum_{n\ge1} e^{\lambda_n t}\, \langle z, \psi_n\rangle_{\mathcal{H}}\, \phi_n. \tag{5}$$

Assumption 2. *There exist $N_0 \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}_+^*$ such that*

1. $\operatorname{Re}\lambda_n \le -\alpha$ *for all* $n \ge N_0 + 1$ *;*
2. $\xi \triangleq \sup\limits_{n\ge N_0+1} \left|\dfrac{\lambda_n}{\operatorname{Re}\lambda_n}\right| < \infty.$

Remark 2. If the first point of Assumption 2 holds, the second point $\xi < \infty$ is equivalent to the existence of a constant $\beta > 0$ such that $|\operatorname{Im}\lambda_n| \le \beta |\operatorname{Re}\lambda_n|$ for all $n \ge N_0 + 1$.

The boundary feedback stabilization problem of the considered system was solved in [19] in the disturbance-free case by designing a constant-delay predictor feedback on a finite dimensional truncated model capturing the unstable modes of the infinite-dimensional system. In this paper, we go beyond the result reported in [19] by considering the impact of boundary disturbances while relaxing the assumed regularity properties and compatibility conditions. Specifically, assuming that the control input[2] u, the time-varying

[2] The construction of the control law must ensure this property.

delay D, and the boundary disturbance d_1 are of class $\mathcal{C}^1$, then, for any given initial condition $X_0 \in \mathcal{H}$, we can introduce $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ defined for all $t \geq 0$ by

$$X(t) = S(t)\{X_0 - Bv(0)\} + Bv(t) + \int_0^t S(t-s)\{\mathcal{A}Bv(s) - B\dot{v}(s)\}\,\mathrm{d}s \tag{6}$$

as the unique mild solution of (1), with control input v given by (4), associated with (D, X_0, d_1). We introduce the series expansion $X(t) = \sum_{n\geq 1} c_n(t)\phi_n$ with $c_n(t) \triangleq \langle X(t), \psi_n\rangle_{\mathcal{H}}$ the coefficient of projection of the system trajectory $X(t)$ into the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$. The use of (6), combined with (3) and (5), and an integration by parts, show that c_n satisfies

$$c_n(t) = e^{\lambda_n t}c_n(0) + \int_0^t e^{\lambda_n(t-\tau)}\left\{\langle \mathcal{A}Bv(\tau),\psi_n\rangle_{\mathcal{H}} - \lambda_n\langle Bv(\tau),\psi_n\rangle_{\mathcal{H}}\right\}\mathrm{d}\tau \tag{7}$$

for all $t \geq 0$. Thus $c_n \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{K})$ and satisfies for all $t \geq 0$ the following ODE (see also [21]):

$$\dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n\langle Bv(t),\psi_n\rangle_{\mathcal{H}} + \langle \mathcal{A}Bv(t),\psi_n\rangle_{\mathcal{H}}\,. \tag{8}$$

Let $\mathcal{E} = (e_1, e_2, \dots, e_m)$ be the canonical basis of $\mathbb{K}^m$. Then, introducing[3] $b_{n,k} \triangleq -\lambda_n\langle Be_k,\psi_n\rangle_{\mathcal{H}} + \langle \mathcal{A}Be_k,\psi_n\rangle_{\mathcal{H}}$, we obtain that

$$\begin{aligned}\dot{Y}(t) &= A_{N_0}Y(t) + B_{N_0}v(t)\\ &= A_{N_0}Y(t) + B_{N_0}\{u(t-D(t)) + d_1(t)\}\end{aligned} \tag{9}$$

with

$$Y(t) = \begin{bmatrix} c_1(t) & \dots & c_{N_0}(t)\end{bmatrix}^\top \in \mathbb{K}^{N_0}, \tag{10}$$

and the matrices $A_{N_0} = \mathrm{diag}(\lambda_1,\dots,\lambda_{N_0}) \in \mathbb{K}^{N_0\times N_0}$ and $B_{N_0} = (b_{n,k})_{1\leq n\leq N_0, 1\leq k\leq m} \in \mathbb{K}^{N_0\times m}$.

Assumption 3. *(A_{N_0}, B_{N_0}) is stabilizable.*

Under Assumption 3, one can design a predictor feedback achieving the stabilization of the truncated model (9). Then, following [19], such a predictor feedback can be successfully applied to the original infinite-dimensional system. Specifically, let $t_0, D_0 > 0$ and $\delta \in (0, D_0)$ be given. We consider a given transition signal[4] $\varphi \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ over $[0, t_0]$. We assume that $D \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$. The closed-loop system dynamics takes the following form:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t), \tag{11a}$$

$$\mathcal{B}X(t) = v(t) = u(t-D(t)) + d_1(t), \tag{11b}$$

$$u(t) = \varphi(t)\left\{ KY(t) + d_2(t) + K\int_{\max(t-D_0,0)}^t e^{(t-s-D_0)A_{N_0}}B_{N_0}u(s)\,\mathrm{d}s\right\}, \tag{11c}$$

$$X(0) = X_0 \tag{11d}$$

for any $t \geq 0$. Function Y is defined by (10). The feedback gain $K \in \mathbb{K}^{m\times N_0}$ is selected such that $A_{\mathrm{cl}} \triangleq A_{N_0} + e^{-D_0A_{N_0}}B_{N_0}K$ is Hurwitz. Functions $d_1, d_2 : \mathbb{R}_+ \to \mathbb{K}^m$ represent boundary disturbances.

Remark 3. While disturbance d_1 represents an additive disturbance in the application of the delayed boundary control u, disturbance d_2 gathers uncertainties of either/both the output measurement Y or/and the computation of the control law u that is solution of a "fixed point implicit equality" involving an integral term [3].

2.3. Well-posedness in terms of mild solutions

In the first part of this paper, we consider the following concept of mild solutions for the closed-loop system dynamics.

Definition 3. Let $(\mathcal{A},\mathcal{B})$ be an abstract boundary control system such that Assumption 1 holds. Let $t_0, D_0 > 0$, $\delta \in (0, D_0)$, a transition signal $\varphi \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ over $[0, t_0]$, and $K \in \mathbb{K}^{m\times N_0}$ be arbitrary. For a time-varying delay $D \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, an initial condition $X_0 \in \mathcal{H}$, and boundary perturbations $d_1, d_2 \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{K}^m)$, we say that $(X,u) \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})\times\mathcal{C}^1([-D_0-\delta,+\infty);\mathbb{K}^m)$ is a mild solution of (11a-11d) associated with (D, X_0, d_1, d_2) if 1) (6) holds for all $t \geq 0$ with v given by (4); 2) u satisfies (11c) for all $t \geq -D_0 - \delta$ with Y defined by (10).

The following lemma, whose proof is placed in appendix, assesses the well-posedness of the closed-loop system (11a-11d) in terms of mild solutions.

Lemma 1. *For any $D \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{K}^m)$, the closed-loop system (11a-11d) admits a unique mild solution $(X,u) \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})\times\mathcal{C}^1([-D_0-\delta,+\infty);\mathbb{K}^m)$ associated with (D, X_0, d_1, d_2).*

Remark 4. If we assume the stronger regularity assumptions $\varphi \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$, $D \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R})$, $d_1, d_2 \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{K}^m)$, and $X_0 \in D(\mathcal{A})$ such that $\mathcal{B}X_0 = d_1(0)$, it can be shown that the mild solution is actually a classical solution.

2.4. Main stability result

The stability of the closed-loop system (11a-11d) in the disturbance free case (i.e., for $d_1 = d_2 = 0$) was assessed in [18, 22] for a constant delay $D(t) = D_0$ and in [19] for an uncertain and time-varying delay $D(t)$. The objective of this paper is to study the impact of the boundary disturbances d_1 and d_2 on the system trajectories. More precisely, we derive the following result.

Theorem 2. *Let $(\mathcal{A},\mathcal{B})$ be an abstract boundary control system such that Assumptions 1, 2, and 3 hold. Let $\varphi \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ be a transition signal over $[0, t_0]$ for some $t_0 > 0$. Let $D_0 > 0$*

[3] Note that the quantity $b_{n,k}$ is independent of the specifically selected lifting operator B associated with $(\mathcal{A},\mathcal{B})$, see [18].

[4] See the notation section at the beginning of Section 2.

and $K \in \mathbb{K}^{m\times N_0}$ be such that $A_{\rm cl} = A_{N_0} + e^{-D_0 A_{N_0}} B_{N_0} K$ is Hurwitz. Let $\delta \in (0, D_0)$ be such that[5]

$$M_\lambda \|B_{N_0} K\| \left[e^{\|A_{\rm cl}\|\delta} - e^{-\lambda\delta}\right] < \lambda, \tag{12}$$

where $\lambda > 0$ and $M_\lambda \geq 1$ are such that $\|e^{A_{\rm cl}t}\| \leq M_\lambda e^{-\lambda t}$ for all $t \geq 0$. Then, there exist $\kappa \in (0, \alpha)$ and $\overline{C}_i > 0$, $1 \leq i \leq 6$, such that, for any $D \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K}^m)$, the mild solution $(X, u) \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H}) \times \mathcal{C}^1([-D_0 - \delta, +\infty); \mathbb{K}^m)$ of the closed-loop system (11a-11d) associated with (D, X_0, d_1, d_2) satisfies

$$\begin{aligned}\|X(t)\|_{\mathcal{H}} \leq \overline{C}_1 e^{-\kappa t}\|X_0\|_{\mathcal{H}} + \overline{C}_2 \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_1(\tau)\| \\ + \overline{C}_3 \sup_{\tau\in[0,\max(t-(D_0-\delta),0)]} e^{-\kappa(t-\tau)}\|d_2(\tau)\|\end{aligned} \tag{13}$$

and

$$\begin{aligned}\|u(t)\| \leq \overline{C}_4 e^{-\kappa t}\|X_0\|_{\mathcal{H}} + \overline{C}_5 \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_1(\tau)\| \\ + \overline{C}_6 \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_2(\tau)\|.\end{aligned} \tag{14}$$

for all $t \geq 0$.

The next two sections are devoted to the proof of Theorem 2. The extension of this result to continuous boundary disturbances d_1, d_2 is discussed in Section 5.

Remark 5. It is interesting to note that Theorem 2, involving the assumption $\xi < +\infty$, does not introduce any constraint on the amplitude of variation of the time derivative $\dot{D}$ of the input delay D. This is in contrast with the result reported in [19] for the disturbance-free case (i.e., $d_1 = d_2 = 0$), which allows $\xi = +\infty$ but where the constant of the exponential stability property is a strictly increasing function, going to $+\infty$ at $+\infty$, of the supremum of $|\dot{D}|$. The occurence of a $\dot{D}$ term in the proof of the result reported in [19] is due to the use of a Lyapunov-based argument. As discussed in the sequel of this paper, the assumption $\xi < +\infty$ allows a proof of Theorem 2 that does not rely on such a Lyapunov-based argument.

3. Exponential ISS of the truncated model

In this section, we study the ISS property of the finite dimensional truncated model. We refer the reader to [30] for classical results about the establishment of ISS properties.

3.1. Preliminary lemma

We need the following preliminary lemma which is a disturbed version of the disturbance-free version ($p = 0$) reported in [13, Th. 2.5].

[5] Such a $\delta > 0$ always exists by a continuity argument in $\delta = 0$.

Lemma 3. *Let $A \in \mathbb{K}^{n\times n}$ be Hurwitz, $C \in \mathbb{K}^{n\times n}$, and $r > 0$. Let $\epsilon \in (0, r)$ be such that*

$$M_\lambda \|C\| \left[e^{\|A\|\epsilon} - e^{-\lambda\epsilon}\right] < \lambda, \tag{15}$$

where $\lambda > 0$ and $M_\lambda \geq 1$ are such that $\|e^{At}\| \leq M_\lambda e^{-\lambda t}$ for all $t \geq 0$. Then, there exist $\sigma, N > 0$ and $M \geq 1$ such that, for any $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ with $|d| \leq 1$, any $p, q \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K})$ with $|q| \leq 1$, and any $x_0 \in \mathcal{C}^0([-r-\epsilon, 0]; \mathbb{K}^n)$, the trajectory of

$$\dot{x}(t) = Ax(t) + q(t)C\left[x(t - r - \epsilon d(t)) - x(t - r)\right] + p(t) \tag{16a}$$

$$x(\tau) = x_0(\tau), \quad -r - \epsilon \leq \tau \leq 0 \tag{16b}$$

for $t \geq 0$ satisfies

$$\|x(t)\| \leq M e^{-\sigma t} \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|p(\tau)\| \tag{17}$$

for all $t \geq 0$.

Proof. As the case $C = 0$ is straightforward, we assume in the sequel that $C \neq 0$. The first part of the proof follows the one in [13] while considering the impact of the disturbing term p. We define, for all $t \geq 0$, $v(t) = x(t-r-\epsilon d(t)) - x(t-r)$. Let $\sigma \in (0, \lambda)$ be arbitrary, and which will be specified in the sequel. As in [13], we consider the cases $d(t) \leq 0$ and $d(t) \geq 0$ separately. In the case $d(t) \leq 0$, we have by direct integration of (16a) that, for all $t \geq r$,

$$\begin{aligned}v(t) = \left[e^{-\epsilon A d(t)} - I_n\right] x(t-r) \\ + \int_{t-r}^{t-r-\epsilon d(t)} e^{A(t-r-\epsilon d(t)-\tau)}\left[q(\tau)Cv(\tau) + p(\tau)\right] \mathrm{d}\tau\end{aligned}$$

because $t-r-\epsilon d(t) \geq t-r \geq 0$. Noting that $\|e^{-\epsilon A d(t)} - I_n\| \leq e^{\|A\|\epsilon} - 1$,

$$\begin{aligned}&\left\|\int_{t-r}^{t-r-\epsilon d(t)} e^{A(t-r-\epsilon d(t)-\tau)} q(\tau) C v(\tau)\,\mathrm{d}\tau\right\| \\ &\leq M_\lambda \|C\| \int_{t-r}^{t-r-\epsilon d(t)} e^{-\lambda(t-r-\epsilon d(t)-\tau)}\,\|v(\tau)\|\,\mathrm{d}\tau \\ &\leq M_\lambda \|C\| e^{-\lambda(t-r-\epsilon d(t))} \int_{t-r}^{t-r-\epsilon d(t)} e^{(\lambda-\sigma)\tau} \times e^{\sigma\tau}\,\|v(\tau)\|\,\mathrm{d}\tau \\ &\leq M_\lambda \|C\| e^{-\sigma(t-r)} e^{\sigma\epsilon d(t)} \frac{1 - e^{(\lambda-\sigma)\epsilon d(t)}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ &\leq M_\lambda \|C\| e^{-\sigma(t-r)} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\|,\end{aligned}$$

where it has been used that $-1 \leq d(t) \leq 0$, and, similarly,

$$\begin{aligned}&\left\|\int_{t-r}^{t-r-\epsilon d(t)} e^{A(t-r-\epsilon d(t)-\tau)} p(\tau)\,\mathrm{d}\tau\right\| \\ &\leq M_\lambda e^{-\sigma(t-r)} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|,\end{aligned}$$

we obtain that, for all $t \geq r$ such that $d(t) \leq 0$,

$$\begin{aligned} e^{\sigma t}\|v(t)\| \leq \left[e^{\|A\|\epsilon} - 1\right] e^{\sigma r} \times e^{\sigma(t-r)}\|x(t-r)\| \\ + M_\lambda \|C\| e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ + M_\lambda e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned} \tag{18}$$

Now, in the case $d(t) \geq 0$, we have by direct integration of (16a) that, for all $t \geq r + \epsilon$,

$$\begin{aligned} v(t) = -\left[e^{\epsilon A d(t)} - I_n\right] x(t - r - \epsilon d(t)) \\ - \int_{t-r-\epsilon d(t)}^{t-r} e^{A(t-r-\tau)} \left[q(\tau) C v(\tau) + p(\tau)\right] \mathrm{d}\tau \end{aligned}$$

because $t - r \geq t - r - \epsilon d(t) \geq t - r - \epsilon \geq 0$. Then, we deduce that, for all $t \geq r + \epsilon$ such that $d(t) \geq 0$,

$$\begin{aligned} e^{\sigma t}\|v(t)\| \\ \leq \left[e^{\|A\|\epsilon} - 1\right] e^{\sigma(r+\epsilon)} \times e^{\sigma(t-r-\epsilon d(t))}\|x(t - r - \epsilon d(t))\| \\ + M_\lambda \|C\| e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r-\epsilon,t-r]} e^{\sigma\tau}\|v(\tau)\| \\ + M_\lambda e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r-\epsilon,t-r]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned} \tag{19}$$

Combining (18-19), we obtain that, for all $t \geq r + \epsilon$,

$$\begin{aligned} \sup_{\tau\in[r+\epsilon,t]} e^{\sigma\tau}\|v(\tau)\| \\ \leq \left[e^{\|A\|\epsilon} - 1\right] e^{\sigma(r+\epsilon)} \sup_{\tau\in[0,t-r]} e^{\sigma\tau}\|x(\tau)\| \\ + M_\lambda \|C\| e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ + M_\lambda e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned} \tag{20}$$

Now, integrating (16a) over $[0, t]$, we obtain for all $t \geq 0$,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} \left[q(\tau) C v(\tau) + p(\tau)\right] \mathrm{d}\tau.$$

As $x(0) = x_0(0)$, straightforward estimations show that, for all $t \geq 0$,

$$\begin{aligned} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\| \leq M_\lambda \|x_0(0)\| + \frac{M_\lambda \|C\|}{\lambda - \sigma} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \\ + \frac{M_\lambda}{\lambda - \sigma} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned} \tag{21}$$

From (20-21), we deduce that, for all $t \geq r + \epsilon$,

$$\begin{aligned} \sup_{\tau\in[r+\epsilon,t]} e^{\sigma\tau}\|v(\tau)\| \leq M_\lambda e^{\sigma(r+\epsilon)} \left[e^{\|A\|\epsilon} - 1\right] \|x_0(0)\| \\ + \delta \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ + \frac{\delta}{\|C\|} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|, \end{aligned}$$

where

$$\delta \triangleq \frac{M_\lambda \|C\|}{\lambda - \sigma} e^{\sigma r} \left[e^{\sigma\epsilon}(e^{\|A\|\epsilon} - 1) + 1 - e^{-(\lambda-\sigma)\epsilon}\right].$$

From the small gain assumption (15) and a continuity argument in $\sigma = 0$, we select $\sigma \in (0, \lambda)$ such that $\delta < 1$. We deduce that, for all $t \geq 0$,

$$\begin{aligned} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \leq \frac{M_\lambda e^{\sigma(r+\epsilon)} \left[e^{\|A\|\epsilon} - 1\right]}{1 - \delta} \|x_0(0)\| \\ + \sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ + \frac{\delta}{\|C\|(1-\delta)} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned}$$

Using (21), we obtain that, for all $t \geq 0$,

$$\begin{aligned} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\| \\ \leq M_\lambda \left\{1 + \frac{M_\lambda \|C\| e^{\sigma(r+\epsilon)} \left[e^{\|A\|\epsilon} - 1\right]}{(1-\delta)(\lambda-\sigma)}\right\} \|x_0(0)\| \\ + \frac{M_\lambda \|C\|}{\lambda - \sigma} \sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ + \frac{M_\lambda}{(1-\delta)(\lambda-\sigma)} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned} \tag{22}$$

It remains now to evaluate $\sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\|$. To do so, we note from the definition of v that

$$\begin{aligned} \sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\ \leq 2e^{\sigma(r+\epsilon)} \left(\sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + \sup_{\tau\in[0,2\epsilon]} \|x(\tau)\|\right). \end{aligned} \tag{23}$$

Introducing $V(t) = \|x(t)\|^2/2$, the evaluation of $\dot{V}$ via (16a), the use of Young's inequality, and recursive estimations over time intervals of finite length $r - \epsilon > 0$ show the existence of constants $\gamma_0, \gamma_1 > 0$, which only depend on A, C, r, ϵ, such that

$$\sup_{\tau\in[0,2\epsilon]} \|x(\tau)\| \leq \gamma_0 \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + \gamma_1 \sup_{\tau\in[0,2\epsilon]} e^{\sigma\tau}\|p(\tau)\|. \tag{24}$$

Combining (22-24), we deduce the existence of constants $M \geq 1$ and $N > 0$ such that, for all $t \geq 0$,

$$\begin{aligned} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\| \\ \leq M \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,\max(t,2\epsilon)]} e^{\sigma\tau}\|p(\tau)\|. \end{aligned} \tag{25}$$

To conclude, it remains to show that $\sup_{\tau\in[0,\max(t,2\epsilon)]} e^{\sigma\tau}\|p(\tau)\|$ can be replaced by $\sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|$ in (25). This is obviously true for $t \geq 2\epsilon$, as well as $t = 0$ because $M \geq 1$. Thus, we focus on the case $0 < t < 2\epsilon$. Let $T \in (0, 2\epsilon)$ be arbitrary. Let $\zeta_n \in C^0(\mathbb{R}_+; \mathbb{R})$ with $0 \leq \zeta_n \leq 1$, $\zeta_n|_{[0,T]} = 1$

and $\zeta_n\big|_{[T+(2\epsilon-T)/n,+\infty)} = 0$ for $n \geq 1$. We define $p_n = \zeta_n p \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{K})$ and we denote by x_n the solution of (16a-16b) associated with the initial condition x_0 and the disturbance p_n. As $p_n(t) = p(t)$ for all $0 \leq t \leq T$, we obtain that $x_n(t) = x(t)$ for all $0 \leq t \leq T$. Therefore, we obtain by applying (25) to x_n at time $t = T$ that, for all $n \geq 1$,

$$\begin{aligned}
&\sup_{\tau\in[0,T]} e^{\sigma\tau}\|x(\tau)\| \\
&\leq M \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,2\epsilon]} e^{\sigma\tau}\|p_n(\tau)\| \\
&\underset{n\to+\infty}{\longrightarrow} M \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,T]} e^{\sigma\tau}\|p(\tau)\|,
\end{aligned}$$

where the limit holds by a continuity argument. Thus, the claimed estimate (17) holds.

3.2. Study of the truncated model

We apply the result of Lemma 3 to the study of the finite-dimensional truncated model composed of (9) and (11c).

Lemma 4. *Under the assumptions of Theorem 2, there exist $\sigma, C_1, C_2, C_3, \overline{C}_4, \overline{C}_5, \overline{C}_6 > 0$ such that, for any $D \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{K}^m)$, Y defined by (10), where $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ is the mild solution of the closed-loop system (11a-11d) associated with (D, X_0, d_1, d_2), satisfies*

$$\begin{aligned}
\|Y(t)\| \leq C_1 e^{-\sigma t}\|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \\
+ C_3 \sup_{\tau\in[0,\max(t-(D_0-\delta),0)]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|
\end{aligned} \tag{26}$$

for all $t \geq 0$. Furthermore, the control law u satisfies (14) with $\kappa = \sigma$ for all $t \geq 0$.

Proof. Let $\delta \in (0, D_0)$ satisfying the small gain condition (12) be given. Let $\sigma, N > 0$ and $M \geq 1$ be the constants provided by Lemma 3 for $A = A_{\mathrm{cl}}$, $C = B_{N_0}K$, $q = 1$, $r = D_0$, and $\epsilon = \delta$. We introduce the Artstein transformation [1, 27] by defining, for all $t \geq 0$,

$$Z(t) = Y(t) + \int_{t-D_0}^{t} e^{(t-D_0-s)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s. \tag{27}$$

As $u(\tau) = 0$ for $\tau \leq 0$, we obtain that $u = \varphi K Z + \varphi d_2$. Taking the time derivative, (9) yields for all $t \geq 0$

$$\begin{aligned}
\dot{Z}(t) = &\left\{ A_{N_0} + \varphi(t)e^{-D_0 A_{N_0}} B_{N_0} K \right\} Z(t) \\
&+ B_{N_0} K \left\{ [\varphi Z](t - D(t)) - [\varphi Z](t - D_0) \right\} \\
&+ B_{N_0} d_1(t) + \varphi(t) e^{-D_0 A_{N_0}} B_{N_0} d_2(t) \\
&+ B_{N_0} \left\{ [\varphi d_2](t - D(t)) - [\varphi d_2](t - D_0) \right\}.
\end{aligned} \tag{28}$$

We first study the case $t \geq t_1 \triangleq t_0 + D_0 + \delta$. We have for all $t \geq t_1$ that $\varphi(t) = \varphi(t - D_0) = \varphi(t - D(t)) = 1$ and thus

$$\begin{aligned}
\dot{Z}(t) = &A_{\mathrm{cl}} Z(t) + B_{N_0} K \left\{ Z(t - D(t)) - Z(t - D_0) \right\} \\
&+ B_{N_0} d_1(t) + e^{-DA_{N_0}} B_{N_0} d_2(t) \\
&+ B_{N_0} \left\{ d_2(t - D(t)) - d_2(t - D_0) \right\}.
\end{aligned}$$

Consequently, it follows from (17) that, for all $t \geq t_1$,

$$\begin{aligned}
&\|Z(t)\| \\
&\leq M e^{-\sigma(t-t_1)} \sup_{\tau\in[t_0,t_1]} \|Z(\tau)\| + \tilde{N}_1 \sup_{\tau\in[t_1,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \\
&\quad + \tilde{N}_2 \sup_{\tau\in[t_0,t]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|,
\end{aligned} \tag{29}$$

with $\tilde{N}_1 = N\|B_{N_0}\|$ and

$$\tilde{N}_2 = N\left\{ \|e^{-D_0 A_{N_0}} B_{N_0}\| + \|B_{N_0}\| e^{\sigma D_0}(e^{\sigma\delta} + 1) \right\}.$$

We now study the case $0 \leq t \leq t_1$. Introducing $W(t) = \|Z(t)\|^2/2$, the evaluation of $\dot{W}$ via (28), the use of Young's inequality, and recursive estimations over time intervals of finite length $D_0 - \delta > 0$ show the existence of $\gamma_0, \gamma_1, \gamma_2 > 0$, which only depend of $A_{N_0}, B_{N_0}, K, D_0, \delta, t_0$, such that

$$\begin{aligned}
\|Z(t)\| \leq \gamma_0 e^{-\sigma t}\|Y(0)\| + \gamma_1 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \\
+ \gamma_2 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|,
\end{aligned} \tag{30}$$

for all $0 \leq t \leq t_1$, where it has been used $Z(0) = Y(0)$. Thus, combining (29-30), we obtain that, for all $t \geq 0$,

$$\begin{aligned}
\|Z(t)\| \leq \gamma_3 e^{-\sigma t}\|Y(0)\| + \gamma_4 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \\
+ \gamma_5 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|
\end{aligned} \tag{31}$$

with $\gamma_3 = M e^{\sigma t_1}\gamma_0$, $\gamma_4 = M e^{\sigma t_1}\gamma_1 + \tilde{N}_1$, and $\gamma_5 = M e^{\sigma t_1}\gamma_2 + \tilde{N}_2$. From $u = \varphi K Z + \varphi d_2$ and (2), we deduce that (14) holds for $\kappa = \sigma$ with $\overline{C}_4 = \|K\|\gamma_3/\sqrt{m_R}$, $\overline{C}_5 = \|K\|\gamma_4$, and $\overline{C}_6 = \|K\|\gamma_5 + 1$. Now, using estimates (2), (14) with $\kappa = \sigma$, and (31) into (27) and the fact that, for $i \in \{1, 2\}$,

$$\begin{aligned}
\int_{\max(t-D_0,0)}^{t} \sup_{\tau\in[0,s]} e^{-\sigma(s-\tau)}\|d_i(\tau)\|\,\mathrm{d}s \\
\leq D_0 e^{\sigma D_0} \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_i(\tau)\|,
\end{aligned}$$

we obtain that

$$\begin{aligned}
\|Y(t)\| \leq C_1 e^{-\sigma t}\|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \\
+ C_3 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|
\end{aligned} \tag{32}$$

for all $t \geq 0$, with

$$\begin{aligned}
C_1 &= \gamma_3/\sqrt{m_R} + e^{D_0(\sigma+\|A_{N_0}\|)}\|B_{N_0}\|\overline{C}_4/\sigma, \\
C_2 &= \gamma_4 + D_0 e^{D_0(\sigma+\|A_{N_0}\|)}\|B_{N_0}\|\overline{C}_5, \\
C_3 &= \gamma_5 + D_0 e^{D_0(\sigma+\|A_{N_0}\|)}\|B_{N_0}\|\overline{C}_6.
\end{aligned}$$

To conclude the proof, it remains to show that we can substitute in estimate (32) the term $\sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|$ by the

term $\sup_{\tau\in[0,\max(t-(D_0-\delta),0)]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|$. To do so, let $T \geq 0$ be arbitrary. Let $\zeta_n \in C^1(\mathbb{R};\mathbb{R})$ with $0 \leq \zeta_n \leq 1$ be such that $\zeta_n|_{(-\infty,T-(D_0-\delta)]} = 1$ and $\zeta_n|_{[T-(D_0-\delta)+(D_0-\delta)/n,+\infty]} = 0$ for $n \geq 1$. Then we define $d_{2,n} = \zeta_n d_2 \in C^1(\mathbb{R}_+;\mathbb{K}^m)$. Thus we can introduce (X_n, u_n) the mild solution of (11a-11d) associated with $(D, X_0, d_1, d_{2,n})$. From [18, Lem. 5.3] (see also [3] in the case $\varphi = 1$), it can be seen that $X|_{[0,T]} = X_n|_{[0,T]}$ and $u|_{[-D_0-\delta,T-(D_0-\delta)]} = u_n|_{[-D_0-\delta,T-(D_0-\delta)]}$. Applying the ISS estimate (32) at time $t = T$ to X_n for any $n \geq 1$, we obtain that

$$\begin{aligned}
&\|Y(T)\| \\
&\leq C_1 e^{-\sigma T}\|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau\in[0,T]} e^{-\sigma(T-\tau)}\|d_1(\tau)\| \\
&\quad + C_3 \sup_{\tau\in[0,T]} e^{-\sigma(T-\tau)}\|\zeta_n(\tau)d_2(\tau)\|.
\end{aligned}$$

By letting $n \to +\infty$, a continuity argument shows that (26) holds at time $t = T$. As $T \geq 0$ is arbitrary, this concludes the proof.

4. Exponential ISS of the infinite-dimensional system

This section is devoted to the proof of the main result of this paper: namely, Theorem 2. Let $\sigma > 0$ be provided by Lemma 4. Let $0 < \kappa < \min(\alpha, \sigma)$ be given and define $\epsilon = \kappa/\alpha \in (0,1)$. First, we infer from $\xi = \sup_{n\geq N_0+1} \left|\frac{\lambda_n}{\mathrm{Re}\,\lambda_n}\right| < \infty$ that the following estimate holds true for all $n \geq N_0 + 1$

$$\begin{aligned}
\left|\frac{\lambda_n}{\mathrm{Re}\,\lambda_n + \kappa}\right| &= \left|\frac{\lambda_n}{\mathrm{Re}\,\lambda_n}\right| \times \left|\frac{\mathrm{Re}\,\lambda_n}{\mathrm{Re}\,\lambda_n + \kappa}\right| \\
&\leq \xi \left|\frac{-\alpha}{-\alpha+\kappa}\right| \leq \frac{\alpha\xi}{\alpha-\kappa},
\end{aligned} \tag{33}$$

where we have used the facts that $\mathrm{Re}\,\lambda_n \leq -\alpha < -\kappa < 0$ and the function $x \to x/(x+\kappa)$ is positive and strictly increasing for $x \in (-\infty, -\kappa)$. Now, from the integral expression (7) of the coefficient of projection c_n and using the definition of the boundary input (4), we have for all $t \geq 0$

$$\begin{aligned}
&|c_n(t)| \\
&\leq e^{\mathrm{Re}\,\lambda_n t}|c_n(0)| \\
&\quad + |\lambda_n| \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\,\langle Bu(\tau - D(\tau)), \psi_n\rangle_{\mathcal{H}}\,| \,\mathrm{d}\tau \\
&\quad + \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\,\langle \mathcal{A}Bu(\tau - D(\tau)), \psi_n\rangle_{\mathcal{H}}\,| \,\mathrm{d}\tau \\
&\quad + |\lambda_n| \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\,\langle Bd_1(\tau), \psi_n\rangle_{\mathcal{H}}\,| \,\mathrm{d}\tau \\
&\quad + \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\,\langle \mathcal{A}Bd_1(\tau), \psi_n\rangle_{\mathcal{H}}\,| \,\mathrm{d}\tau.
\end{aligned} \tag{34}$$

We now evaluate the different terms on the right hand side of (34). Denoting by $e_1, \ldots, e_m$ the canonical basis of $\mathbb{K}^m$, we have $d_1(t) = \sum_{k=1}^m d_{1,k}(t)e_k$ with $d_{1,k}(t) \in \mathbb{K}$. Then, noting that $|d_{1,k}(t)| \leq \|d_1(t)\|$, we have for all $n \geq N_0 + 1$ and $t \geq 0$

$$\begin{aligned}
&|\lambda_n| \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\,\langle Bd_1(\tau), \psi_n\rangle_{\mathcal{H}}\,| \,\mathrm{d}\tau \\
&\leq |\lambda_n| \sum_{k=1}^m |\,\langle Be_k, \psi_n\rangle_{\mathcal{H}}\,| \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)}\|d_1(\tau)\| \,\mathrm{d}\tau \\
&\leq |\lambda_n| \sum_{k=1}^m |\,\langle Be_k, \psi_n\rangle_{\mathcal{H}}\,| \int_0^t e^{(1-\epsilon)\,\mathrm{Re}\,\lambda_n(t-\tau)} \,\mathrm{d}\tau \\
&\quad \times \sup_{\tau\in[0,t]} e^{\epsilon\,\mathrm{Re}\,\lambda_n(t-\tau)}\|d_1(\tau)\| \\
&\leq \frac{1}{1-\epsilon}\left|\frac{\lambda_n}{\mathrm{Re}\,\lambda_n}\right| \sum_{k=1}^m |\,\langle Be_k, \psi_n\rangle_{\mathcal{H}}\,| \sup_{\tau\in[0,t]} e^{-\epsilon\alpha(t-\tau)}\|d_1(\tau)\| \\
&\leq \frac{\alpha\xi}{\alpha-\kappa} \sum_{k=1}^m |\,\langle Be_k, \psi_n\rangle_{\mathcal{H}}\,| \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_1(\tau)\|.
\end{aligned} \tag{35}$$

Similarly, we have for all $n \geq N_0 + 1$ and $t \geq 0$

$$\begin{aligned}
&\int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\,\langle \mathcal{A}Bd_1(\tau), \psi_n\rangle_{\mathcal{H}}\,| \,\mathrm{d}\tau \\
&\leq \frac{1}{\alpha-\kappa} \sum_{k=1}^m |\,\langle \mathcal{A}Be_k, \psi_n\rangle_{\mathcal{H}}\,| \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_1(\tau)\|.
\end{aligned} \tag{36}$$

We now estimate the two remaining integral terms involving the control input u on the right-hand side of (34). We note that these two integrals are null for $t \leq D_0 - \delta$ because $u(\tau) = 0$ for $\tau \leq 0$. Thus we focus on the case $t \geq D_0 - \delta$. First, we evaluate the following integral:

$$\begin{aligned}
\mathcal{I}_n(t) &= \int_0^t e^{-\mathrm{Re}\,\lambda_n \tau}\|u(\tau - D(\tau))\| \,\mathrm{d}\tau \\
&= \int_0^t e^{-\mathrm{Re}\,\lambda_n \tau} \chi_{\{s-D(s)\geq 0\}}(\tau)\|u(\tau - D(\tau))\| \,\mathrm{d}\tau
\end{aligned}$$

for $t \geq D_0 - \delta$. To do so, we note that

$$\begin{aligned}
&\int_0^t e^{-\mathrm{Re}\,\lambda_n \tau} \chi_{\{s-D(s)\geq 0\}}(\tau) \sup_{s\in[0,\tau-D(\tau)]} e^{-\kappa((\tau-D(\tau))-s)}\|d_i(s)\| \,\mathrm{d}\tau \\
&\leq \frac{e^{\kappa(D_0+\delta)}}{|\,\mathrm{Re}\,\lambda_n + \kappa|} e^{-(\mathrm{Re}\,\lambda_n+\kappa)t} \sup_{s\in[0,t-(D_0-\delta)]} e^{\kappa s}\|d_i(s)\|.
\end{aligned}$$

Therefore, recalling that $0 < \kappa < \sigma$, the use of (14) provided by Lemma 4 yields

$$\mathcal{I}_n(t) \leq \frac{e^{\kappa(D_0+\delta)}}{|\,\mathrm{Re}\,\lambda_n + \kappa|} e^{-\mathrm{Re}\,\lambda_n t}\Delta(t)$$

with

$$\begin{aligned}
\Delta(t) &= \overline{C}_4 e^{-\kappa t}\|X_0\|_{\mathcal{H}} + \overline{C}_5 \sup_{s\in[0,t-(D_0-\delta)]} e^{-\kappa(t-s)}\|d_1(s)\| \\
&\quad + \overline{C}_6 \sup_{s\in[0,t-(D_0-\delta)]} e^{-\kappa(t-s)}\|d_2(s)\|.
\end{aligned}$$

With $u(t) = \sum_{k=1}^{m} u_k(t)e_k$ where $u_k(t) \in \mathbb{K}$, we have that $|u_k(t)| \leq \|u(t)\|$ for all $1 \leq k \leq m$. Recalling that $\mathrm{Re}\,\lambda_n \leq -\alpha < -\kappa < 0$ for any $n \geq N_0 + 1$, we obtain that, for all $n \geq N_0 + 1$ and $t \geq D_0 - \delta$,

$$
\begin{aligned}
&|\lambda_n| \int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\langle Bu(\tau - D(\tau)), \psi_n\rangle_{\mathcal{H}}| \,\mathrm{d}\tau \\
&\leq |\lambda_n| \sum_{k=1}^{m} |\langle Be_k, \psi_n\rangle_{\mathcal{H}}| e^{\mathrm{Re}\,\lambda_n t} \mathcal{I}_n(t) \\
&\leq \frac{\alpha\xi e^{\kappa(D_0+\delta)}}{\alpha-\kappa} \sum_{k=1}^{m} |\langle Be_k, \psi_n\rangle_{\mathcal{H}}| \Delta(t) \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t e^{\mathrm{Re}\,\lambda_n(t-\tau)} |\langle \mathcal{A}Bu(\tau - D(\tau)), \psi_n\rangle_{\mathcal{H}}| \,\mathrm{d}\tau \\
&\leq \sum_{k=1}^{m} |\langle \mathcal{A}Be_k, \psi_n\rangle_{\mathcal{H}}| e^{\mathrm{Re}\,\lambda_n t} \mathcal{I}_n(t) \\
&\leq \frac{e^{\kappa(D_0+\delta)}}{\alpha-\kappa} \sum_{k=1}^{m} |\langle \mathcal{A}Be_k, \psi_n\rangle_{\mathcal{H}}| \Delta(t). \qquad (38)
\end{aligned}
$$

Based on (35-38), we deduce from (34), Young's inequality, and the fact $\|Y(0)\|^2 = \sum_{n=1}^{N_0} |c_n(0)|^2$ that, for all $t \geq 0$,

$$
\begin{aligned}
\sum_{n\geq N_0+1} |c_n(t)|^2 \leq \tilde{C}_1 e^{-2\kappa t}\|X_0\|_{\mathcal{H}}^2 + \tilde{C}_2 \sup_{\tau\in[0,t]} e^{-2\kappa(t-\tau)}\|d_1(\tau)\|^2 \\
+ \tilde{C}_3 \sup_{\tau\in[0,\max(t-(D_0-\delta),0)]} e^{-2\kappa(t-\tau)}\|d_2(\tau)\|^2, \qquad (39)
\end{aligned}
$$

where constants $\tilde{C}_i$ are given by

$$
\begin{aligned}
\tilde{C}_0 &= \alpha^2\xi^2 \sum_{k=1}^{m} \|Be_k\|_{\mathcal{H}}^2 + \sum_{k=1}^{m} \|\mathcal{A}Be_k\|_{\mathcal{H}}^2, \\
\tilde{C}_1 &= \frac{4}{m_R}\left(1 + \frac{2m\overline{C}_4^2 e^{2\kappa(D_0+\delta)}}{(\alpha-\kappa)^2}\tilde{C}_0\right), \\
\tilde{C}_2 &= \frac{8m\left(1+\overline{C}_5 e^{\kappa(D_0+\delta)}\right)^2}{m_R(\alpha-\kappa)^2}\tilde{C}_0, \\
\tilde{C}_3 &= \frac{8m\overline{C}_6^2 e^{2\kappa(D_0+\delta)}}{m_R(\alpha-\kappa)^2}\tilde{C}_0.
\end{aligned}
$$

Consequently, as

$$
\begin{aligned}
\|X(t)\|_{\mathcal{H}} &\leq \sqrt{M_R \sum_{n\geq 1} |c_n(t)|^2} \\
&\leq \sqrt{M_R}\left(\|Y(t)\| + \sqrt{\sum_{n\geq N_0+1} |c_n(t)|^2}\right),
\end{aligned}
$$

we obtain from (26) and (39) that the ISS estimate (13) holds with $\overline{C}_i = \sqrt{M_R}\left(C_i + \sqrt{\tilde{C}_i}\right)$, $1 \leq i \leq 3$, which concludes the proof.

5. Extension of the main result to continuous boundary perturbations

The result stated in Theorem 2 deals with mild solutions associated with continuously differentiable boundary disturbances. However, as shown in [21], the satisfaction of an ISS estimate, combined with the introduction of a proper concept of weak solutions, can be employed to easily extend the obtained ISS estimate to boundary disturbances exhibiting relaxed regularity assumptions. Such a concept of weak solutions extends to abstract boundary control systems the concept of weak solutions originally introduced for infinite-dimensional nonhomogeneous Cauchy problems in [2] and further investigated in [8, Def. 3.1.6, Thm. 3.1.7, A.5.29] under a variational from. In this context and adopting the approach reported in [21], we introduce the following concept of weak solution for the closed-loop dynamics (11a-11d).

Definition 4. Let $(\mathcal{A}, \mathcal{B})$ be an abstract boundary control system such that Assumption 1 holds. Let $t_0, D_0 > 0$, $\delta \in (0, D_0)$, a transition signal $\varphi \in C^1(\mathbb{R};\mathbb{R})$ over $[0, t_0]$, and $K \in \mathbb{K}^{m\times N_0}$ be arbitrary. For a time-varying delay $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, an initial condition $X_0 \in \mathcal{H}$, and boundary perturbations $d_1, d_2 \in C^0(\mathbb{R}_+;\mathbb{K}^m)$, we say that $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ is a weak solution of the abstract boundary control system (11a-11d) associated with (D, X_0, d_1, d_2) if for any $T > 0$ and any $z \in C^0([0,T]; D(\mathcal{A}_0^*)) \cap C^1([0,T];\mathcal{H})$ with[6] $\mathcal{A}_0^* z \in C^0([0,T];\mathcal{H})$ and $z(T) = 0$, we have:

$$
\begin{aligned}
&\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t)\right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= -\langle X_0, z(0)\rangle_{\mathcal{H}} \qquad (41) \\
&\quad + \int_0^T \left\langle B(u(t - D(t)) + d_1(t)), \mathcal{A}_0^* z(t)\right\rangle_{\mathcal{H}} \mathrm{d}t \\
&\quad - \int_0^T \langle \mathcal{A}B(u(t - D(t)) + d_1(t)), z(t)\rangle_{\mathcal{H}} \mathrm{d}t,
\end{aligned}
$$

with B an arbitrarily given lifting operator associated with $(\mathcal{A}, \mathcal{B})$ and where the control input u satisfies $u|_{[-D_0-\delta,0]} = 0$ and, for all $t \geq 0$,

$$
\begin{aligned}
u(t) = \varphi(t)\Bigg\{ &KY(t) + d_2(t) \qquad (42) \\
&+ K\int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s \Bigg\}
\end{aligned}
$$

with Y defined by (10).

[6] Such a function z is called a test function over $[0, T]$.

In particular, using the definition of the mild solutions (6) into the left hand side of (41), it is easy to show based on an integration by parts, the basic properties of the C_0-semigroups, and the fundamental theorem of calculus that any mild solution is also a weak solution.

Remark 6. Following [21, Sec. 4], we have the following facts.

- Definition 4 is independent of a specifically selected lifting operator in the sense that the right hand side of (41) is unchanged when switching between different lifting operators B associated with $(\mathcal{A}, \mathcal{B})$.
- X_0 is the initial condition of the weak solution in the sense that if $(X,u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ is a weak solution associated with $(D, X_0, d_1, d_2) \in \mathcal{H} \times C^0(\mathbb{R}_+;\mathbb{K}^m) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ of the abstract boundary control system (11a-11d), then $X(0) = X_0$.

We first state a preliminary result about the uniqueness of the weak solutions for the studied problem.

Lemma 5. *For any $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in C^0(\mathbb{R}_+;\mathbb{K}^m)$, there exists at most one weak solution $(X,u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ associated with (D, X_0, d_1, d_2) of the closed-loop system (11a-11d).*

Proof. By linearity, it is sufficient to show that if $(X,u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ satisfies

$$\begin{aligned}&\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\ &= \int_0^T \left\langle Bu(t - D(t)), \mathcal{A}_0^* z(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\ &\quad - \int_0^T \langle \mathcal{A}Bu(t - D(t)), z(t) \rangle_{\mathcal{H}} \, \mathrm{d}t\end{aligned}$$

for all $T > 0$ and for all test function z over $[0,T]$ with $u|_{[-D_0-\delta,0]} = 0$ and

$$u(t) = \varphi(t)K \left\{ Y(t) + \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s) \, \mathrm{d}s \right\}$$

for all $t \geq 0$, then $X = 0$ and $u = 0$. We proceed by induction by showing that $X|_{[0,n(D_0-\delta)]=0}$ and $u|_{[-D_0-\delta,(n-1)(D_0-\delta)]=0}$ for all $n \geq 1$.

Initialization: From $u|_{[-D_0-\delta,0]} = 0$, we obtain for $T = D_0 - \delta$ that:

$$\int_0^{D_0-\delta} \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t = 0$$

for any test function z over $[0, D_0 - \delta]$ because $t - D(t) \leq 0$ and thus $u(t - D(t)) = 0$ for all $t \leq D_0 - \delta$. Using the test function $z(t) = e^{-\overline{\lambda}_k t} \int_{D_0-\delta}^{t} \langle X(\tau), \psi_k \rangle_{\mathcal{H}} \, e^{\overline{\lambda}_k \tau} \, \mathrm{d}\tau \, \psi_k$, we obtain that $\mathcal{A}_0^* z(t) + \dfrac{\mathrm{d}z}{\mathrm{d}t}(t) = \langle X(t), \psi_k \rangle_{\mathcal{H}} \, \psi_k$ whence $\int_0^{D_0-\delta} |\langle X(t), \psi_k \rangle_{\mathcal{H}}|^2 \, \mathrm{d}t = 0$. By continuity of X, we infer that $\langle X(t), \psi_k \rangle_{\mathcal{H}} = 0$ for all $t \in [0, D_0 - \delta]$ and all $k \geq 1$. As $\{\psi_k, \ k \in \mathbb{N}^*\}$ forms a Riesz basis, this yields $X(t) = 0$ for all $t \in [0, D_0 - \delta]$.

Heredity: Let $n \geq 1$ be such that $X|_{[0,n(D_0-\delta)]} = 0$ and $u|_{[-D_0-\delta,(n-1)(D_0-\delta)]=0}$. Then, $Y|_{[0,n(D_0-\delta)]} = 0$ and thus

$$u(t) = \varphi(t)K \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s) \, \mathrm{d}s$$

for all $t \in [0, n(D_0-\delta)]$, whence (see [3] and [18, Lem. 5.3]) $u(t) = 0$ for all $t \in [0, n(D_0-\delta)]$. Thus, $u(t-D(t)) = 0$ for all $t \in [0, (n+1)(D_0-\delta)]$ and we obtain with $T = (n+1)(D_0-\delta)$ that

$$\int_0^{(n+1)(D_0-\delta)} \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t = 0$$

for all test function z over $[0, (n+1)(D_0-\delta)]$. Using the test function $z(t) = e^{-\overline{\lambda}_k t} \int_{(n+1)(D_0-\delta)}^{t} \langle X(\tau), \psi_k \rangle_{\mathcal{H}} \, e^{\overline{\lambda}_k \tau} \, \mathrm{d}\tau \, \psi_k$, we obtain that $\mathcal{A}_0^* z(t) + \dfrac{\mathrm{d}z}{\mathrm{d}t}(t) = \langle X(t), \psi_k \rangle_{\mathcal{H}} \, \psi_k$ and thus we infer that $\int_0^{(n+1)(D_0-\delta)} |\langle X(t), \psi_k \rangle_{\mathcal{H}}|^2 \, \mathrm{d}t = 0$. By continuity of X, we infer that $\langle X(t), \psi_k \rangle_{\mathcal{H}} = 0$ for all $t \in [0, (n+1)(D_0 - \delta)]$ and all $k \geq 1$. As $\{\psi_k, \ k \in \mathbb{N}^*\}$ forms a Riesz basis, this yields $X(t) = 0$ for all $t \in [0, (n+1)(D_0 - \delta)]$. This completes the proof by induction.

We can now state the main result of this section whose proof is an adapation of [21, Thm. 3].

Theorem 6. *In the context of both assumptions and conclusions of Theorem 2, for any $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in C^0(\mathbb{R}_+;\mathbb{K}^m)$, there exists a unique weak solution $(X,u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ associated with (D, X_0, d_1, d_2) of the abstract boundary control system (11a-11d). Furthermore, this weak solution satisfies the ISS estimates (13-14) for all $t \geq 0$.*

Proof. We consider $\delta \in (0, D_0)$, $\kappa \in (0, \alpha)$, and the constants $\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4, \overline{C}_5, \overline{C}_6 > 0$ as provided by Theorem 2. Let $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in C^0(\mathbb{R}_+;\mathbb{K}^m)$ be given. The uniqueness follows from Lemma 5. We prove the existence.

For a given $T > 0$, as $C^1([0,T];\mathbb{K}^m)$ is a dense subset of $C^0([0,T];\mathbb{K}^m)$, we introduce $d_{1,n}, d_{2,n} \in C^1([0,T];\mathbb{K}^m)$ such that $(d_{1,n})_n$ and $(d_{2,n})_n$ converge uniformly over $[0,T]$ to $d_1|_{[0,T]}$ and $d_2|_{[0,T]}$, respectively. We introduce $(X_n, u_n) \in C^0([0,T];\mathcal{H}) \times C^1([-D_0-\delta, T];\mathbb{K}^m)$ the unique mild solution of the abstract boundary control system (11a-11d) over $[0,T]$ associated with $(D, X_0, d_{1,n}, d_{2,n})$. By linearity of (11a-11d), $(X_n - X_m, u_n - u_m)$ is the unique mild solution of the abstract boundary control system (11a-11d) over $[0,T]$ associated with $(D, 0, d_{1,n} - d_{1,m}, d_{2,n} - d_{2,m})$. Thus, we deduce from (13-14) that $(X_n)_n$ and $(u_n)_n$ are Cauchy sequences of the Banach spaces $C^0([0,T];\mathcal{H})$ and $C^0([0,T];\mathbb{K}^m)$, respectively. Thus $X_n \underset{n\to+\infty}{\longrightarrow} X \in C^0([0,T];\mathcal{H})$ and $u_n \underset{n\to+\infty}{\longrightarrow} u \in C^0([0,T];\mathbb{K}^m)$ and (X,u) satisfies the estimates (13-14) for

any $t \in [0,T]$. It is easy to see from (13-14) that the obtained X and u are independent of the selected approximating sequences $(d_{1,n})_n$ and $(d_{2,n})_n$ but only on D, X_0, d_1, and d_2.

For any given $0 < T_1 < T_2$, let $(X_1, u_1) \in C^0([0,T_1];\mathcal{H})\times C^0([0,T_1];\mathbb{K}^m)$ and $(X_2,u_2) \in C^0([0,T_2];\mathcal{H})\times C^0([0,T_2];\mathbb{K}^m)$ be the result of the above construction over the time intervals $[0,T_1]$ and $[0,T_2]$, respectively. By restricting the approximating sequences of $d_1|_{[0,T_2]}$ and $d_2|_{[0,T_2]}$ to the interval $[0,T_1]$, we obtain approximating sequences of $d_1|_{[0,T_1]}$ and $d_2|_{[0,T_1]}$. Then, we infer that $X_2|_{[0,T_1]} = X_1$ and $u_2|_{[0,T_1]} = u_1$. Consequently, we can define $(X,u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ such that $(X|_{[0,T]}, u|_{[0,T]})$ is the result of the above construction for any $T > 0$. Then, (X,u) satisfy the estimates (13-14) for all $t \geq 0$.

It remains to show that (X,u) is actually the weak solution associated with (D, X_0, d_1, d_2) of the abstract boundary control system (11a-11d). Let $T > 0$ be arbitrarily given. Let $(d_{1,n})_n \in C^1([0,T];\mathbb{K}^m)^{\mathbb{N}}$ and $(d_{2,n})_n \in C^1([0,T];\mathbb{K}^m)^{\mathbb{N}}$ be approximating sequences, compliant with the procedure of the previous paragraphs, converging to $d_1|_{[0,T]}$,and $d_2|_{[0,T]}$, respectively. Thus, the corresponding sequence of mild solutions $(X_n,u_n)_n$ converges uniformly to $(X|_{[0,T]}, u|_{[0,T]})$. As mild solutions are weak solutions, we obtain that, for any test function z over $[0,T]$ and any $n \geq 1$,

$$\begin{aligned}
&\int_0^T \left\langle X_n(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t)\right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= -\langle X_0, z(0)\rangle_{\mathcal{H}} \\
&\quad + \int_0^T \left\langle B(u_n(t-D(t)) + d_{1,n}(t)), \mathcal{A}_0^* z(t)\right\rangle_{\mathcal{H}} \mathrm{d}t \\
&\quad - \int_0^T \left\langle \mathcal{A}B(u_n(t-D(t)) + d_{1,n}(t)), z(t)\right\rangle_{\mathcal{H}} \mathrm{d}t
\end{aligned}$$

with $u_n|_{[-D_0-\delta,0]} = 0$ and

$$u_n(t) = \varphi(t)\Bigg\{ KY_n(t) + d_{2,n}(t) + K\int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u_n(s)\,\mathrm{d}s \Bigg\}$$

for all $t \in [0,T]$, where

$$Y_n(t) = \begin{bmatrix} \langle X_n(t), \psi_1\rangle_{\mathcal{H}} & \dots & \left\langle X_n(t), \psi_{N_0}\right\rangle_{\mathcal{H}} \end{bmatrix}^\top.$$

By letting $n \to +\infty$, we obtain that (41-42) hold over $[0,T]$. As both $T > 0$ and the test function z over $[0,T]$ have been arbitrarily selected, this concludes the proof.

6. Conclusion

This paper has investigated the input-to-state stabilization with respect to boundary disturbances of a class of diagonal infinite-dimensional systems via delay boundary control. First, a preliminary lemma regarding the robustness of a constant-delay predictor feedback with respect to uncertain and time-varying input delays has been derived under the form of an ISS estimate with fading memory of the disturbance input. This result was applied to a truncated model capturing the unstable modes of the studied infinite-dimensional system. Finally, this ISS property was extended to the closed-loop infinite-dimensional system, first considering mild solutions and then for weak solutions associated with disturbances exhibiting relaxed regularity assumptions.

A. Proof of Lemma 1

As $u|_{[-D_0-\delta,0]} = 0$, (6) is equivalent over $[0, D_0-\delta]$ to

$$X(t) = S(t)\{X_0 - Bd_1(0)\} + Bd_1(t) + \int_0^t S(t-s)\{\mathcal{A}Bd_1(s) - B\dot{d_1}(s)\}\,\mathrm{d}s,$$

which is well and uniquely defined as an element of $C^0([0, D_0-\delta];\mathcal{H})$ with associated control input $u = 0 \in C^1([-D_0-\delta, 0];\mathbb{K}^m)$.

We proceed by induction. Assume that, for a given $n \in \mathbb{N}^*$, there exists a unique couple $(X,u) \in C^0([0,n(D_0-\delta)];\mathcal{H})\times C^1([-D_0-\delta,(n-1)(D_0-\delta)])$ such that (6) holds over $[0,n(D_0-\delta)]$ and (11c) holds over $[-D_0-\delta,(n-1)(D_0-\delta)]$. In particular, reproducing the developments reported in Subsection 2.2, c_n is of class C^1 over $[0,n(D_0-\delta)]$, and thus $Y \in C^1([0,n(D_0-\delta)],\mathbb{K}^{N_0})$. We show that there exists a unique couple $(\tilde{X},\tilde{u}) \in C^0([0,(n+1)(D_0-\delta)];\mathcal{H})\times C^1([-D_0-\delta,n(D_0-\delta)];\mathbb{K}^m)$ such that

$$\tilde{X}(t) = S(t)\{X_0 - B\tilde{v}(0)\} + B\tilde{v}(t) + \int_0^t S(t-s)\{\mathcal{A}B\tilde{v}(s) - B\dot{\tilde{v}}(s)\}\,\mathrm{d}s \tag{43}$$

with $\tilde{v}(t) = \tilde{u}(t-D(t)) + d_1(t)$ for all $t \in [0,(n+1)(D_0-\delta)]$ and where the control law is characterized by $\tilde{u}|_{[-D_0-\delta,0]} = 0$ and, for all $t \in [0,n(D_0-\delta)]$,

$$\tilde{u}(t) = \varphi(t)\Bigg\{ K\tilde{Y}(t) + d_2(t) + K\int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0}\tilde{u}(s)\,\mathrm{d}s \Bigg\} \tag{44}$$

with

$$\tilde{Y}(t) = \begin{bmatrix} \left\langle \tilde{X}(t), \psi_1\right\rangle_{\mathcal{H}} & \dots & \left\langle \tilde{X}(t), \psi_{N_0}\right\rangle_{\mathcal{H}} \end{bmatrix}^\top.$$

In particular, the induction assumption yields $\tilde{X}|_{[0,n(D_0-\delta)]} = X$ and $\tilde{u}|_{[-D_0-\delta,(n-1)(D_0-\delta)]} = u$. Moreover, as $t - D(t) \leq n(D_0-\delta)$ for $t \leq (n+1)(D_0-\delta)$, we have $\tilde{Y}(t) = Y(t)$, i.e., the control input $\tilde{u}$ over the time interval $[0,n(D_0-\delta)]$ is only defined by X over the time interval $[0,n(D_0-\delta)]$ and does not depend on $\tilde{X}$ over $[n(D_0-\delta),(n+1)(D_0-\delta)]$. As $Y \in C^0([0,n(D_0-\delta)];\mathbb{K}^{N_0})$, we obtain from [18, Lem. 5.3] (see also [3] in the case $\varphi = 1$) that the control $\tilde{u}$ given by

the implicit equation (44) is well and uniquely defined on $[-D_0-\delta, n(D_0-\delta)]$ as an element of $C^0([-D_0-\delta, n(D_0-\delta)]; \mathbb{K}^m)$ and is such that $\tilde{u}|_{[-D_0-\delta,(n-1)(D_0-\delta)]} = u$. Introducing

$$Z(t) = Y(t) + \int_{t-D_0}^{t} e^{(t-D_0-s)A_{N_0}} B_{N_0} \tilde{u}(s)\, \mathrm{d}s,$$

which is such that $Z \in C^1([0, n(D_0-\delta)]; \mathbb{K}^{N_0})$, we can write $\tilde{u}(t) = \varphi(t)KZ(t) + \varphi(t)d_2(t)$ for all $t \in [0, n(D_0-\delta)]$. Thus $\tilde{u} \in C^1([-D_0-\delta, n(D_0-\delta)])$ and we obtain that (43) is well defined over $[-D_0-\delta, n(D_0-\delta)]$. This completes the proof by induction.

References

[1] Artstein, Z., 1982. Linear systems with delayed controls: a reduction. IEEE Transactions on Automatic Control 27, 869–879.

[2] Ball, J., 1977. Strongly continuous semigroups, weak solutions, and the variation of constants formula. Proceedings of the American Mathematical Society 63, 370–373.

[3] Bresch-Pietri, D., Prieur, C., Trélat, E., 2018. New formulation of predictors for finite-dimensional linear control systems with input delay. Systems & Control Letters 113, 9–16.

[4] Bribiesca Argomedo, F., Witrant, E., Prieur, C., 2012. D^1-Input-to-state stability of a time-varying nonhomogeneous diffusive equation subject to boundary disturbances, in: American Control Conference (ACC), 2012, IEEE. pp. 2978–2983.

[5] Christensen, O., et al., 2016. An Introduction to Frames and Riesz Bases. Springer.

[6] Coron, J.M., Trélat, E., 2004. Global steady-state controllability of one-dimensional semilinear heat equations. SIAM Journal on Control and Optimization 43, 549–569.

[7] Coron, J.M., Trélat, E., 2006. Global steady-state stabilization and controllability of 1D semilinear wave equations. Communications in Contemporary Mathematics 8, 535–567.

[8] Curtain, R.F., Zwart, H., 2012. An Introduction to Infinite-Dimensional Linear Systems Theory. volume 21. Springer Science & Business Media.

[9] Fridman, E., Orlov, Y., 2009. Exponential stability of linear distributed parameter systems with time-varying delays. Automatica 45, 194–201.

[10] Guzmán, P., Marx, S., Cerpa, E., 2019. Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control .

[11] Jacob, B., Nabiullin, R., Partington, J.R., Schwenninger, F.L., 2018. Infinite-dimensional input-to-state stability and orlicz spaces. SIAM Journal on Control and Optimization 56, 868–889.

[12] Jacob, B., Schwenninger, F.L., Zwart, H., 2019. On continuity of solutions for parabolic control systems and input-to-state stability. Journal of differential equations 266, 6284–6306.

[13] Karafyllis, I., Krstic, M., 2013. Delay-robustness of linear predictor feedback without restriction on delay rate. Automatica 49, 1761–1767.

[14] Karafyllis, I., Krstic, M., 2016. ISS with respect to boundary disturbances for 1-D parabolic PDEs. IEEE Transactions on Automatic Control 61, 3712–3724.

[15] Karafyllis, I., Krstic, M., 2017. ISS in different norms for 1-D parabolic PDEs with boundary disturbances. SIAM Journal on Control and Optimization 55, 1716–1751.

[16] Karafyllis, I., Krstic, M., 2019. Input-to-State Stability for PDEs. Springer.

[17] Krstic, M., 2009. Control of an unstable reaction-diffusion PDE with long input delay. Systems & Control Letters 58, 773–782.

[18] Lhachemi, H., Prieur, C., 2019. Feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control. arXiv preprint arXiv:1902.05086 .

[19] Lhachemi, H., Prieur, C., Shorten, R., 2019a. An LMI condition for the robustness of constant-delay linear predictor feedback with respect to uncertain time-varying input delays. Automatica 109, 108551.

[20] Lhachemi, H., Saussié, D., Zhu, G., Shorten, R., 2019b. Input-to-state stability of a clamped-free damped string in the presence of distributed and boundary disturbances. IEEE Transactions on Automatic Control, in press .

[21] Lhachemi, H., Shorten, R., 2019. ISS property with respect to boundary disturbances for a class of Riesz-spectral boundary control systems. Automatica 109, 108504.

[22] Lhachemi, H., Shorten, R., Prieur, C., 2019c. Control law realification for the feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control. IEEE Control Systems Letters 3, 930–935.

[23] Mironchenko, A., Karafyllis, I., Krstic, M., 2019. Monotonicity methods for input-to-state stability of nonlinear parabolic pdes with boundary disturbances. SIAM Journal on Control and Optimization 57, 510–532.

[24] Mironchenko, A., Wirth, F., 2016. Restatements of input-to-state stability in infinite dimensions: what goes wrong, in: Proc. of 22th International Symposium on Mathematical Theory of Systems and Networks (MTNS 2016), pp. 667–674.

[25] Mironchenko, A., Wirth, F., 2018. Characterizations of input-to-state stability for infinite-dimensional systems. IEEE Transactions on Automatic Control 63, 1692–1707.

[26] Prieur, C., Trélat, E., 2019. Feedback stabilization of a 1D linear reaction-diffusion equation with delay boundary control. IEEE Transactions on Automatic Control 64, 1415–1425.

[27] Richard, J.P., 2003. Time-delay systems: an overview of some recent advances and open problems. Automatica 39, 1667–1694.

[28] Russell, D.L., 1978. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. SIAM Review 20, 639–739.

[29] Solomon, O., Fridman, E., 2015. Stability and passivity analysis of semilinear diffusion pdes with time-delays. International Journal of Control 88, 180–192.

[30] Sontag, E.D., 2008. Input to state stability: Basic concepts and results, in: Nonlinear and optimal control theory. Springer, pp. 163–220.

[31] Sontag, E.D., Apr. 1989. Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control 34, 435–443.

[32] Zheng, J., Zhu, G., 2018a. A De Giorgi iteration-based approach for the establishment of ISS properties for Burgers' equation with boundary and in-domain disturbances. IEEE Transactions on Automatic Control, in press .

[33] Zheng, J., Zhu, G., 2018b. Input-to-state stability with respect to boundary disturbances for a class of semi-linear parabolic equations. Automatica 97, 271–277.