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GLOBAL STABILIZATION OF A KORTEWEG–DE VRIES EQUATION WITH SATURATING DISTRIBUTED CONTROL*

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Abstract. This article deals with the design of saturated controls in the context of partial differential equations. It focuses on a Korteweg–de Vries equation, which is a nonlinear mathematical model of waves on shallow water surfaces. Two different types of saturated controls are considered. The well-posedness is proven applying a Banach fixed-point theorem, using some estimates of this equation and some properties of the saturation function. The proof of the asymptotic stability of the closed-loop system is separated in two cases: (i) when the control acts on all the domain, a Lyapunov function together with a sector condition describing the saturating input is used to conclude on the stability; (ii) when the control is localized, we argue by contradiction. Some numerical simulations illustrate the stability of the closed-loop nonlinear partial differential equation.

Key words. Korteweg–de Vries equation, stabilization, distributed control, saturating control, nonlinear system

AMS subject classifications. 93C20, 93D15, 35Q53

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1. Introduction.

In recent decades, a great effort has been made to take into account input saturations in control designs (see, e.g., [39], [15], or more recently [17]). In most applications, actuators are limited due to some physical constraints and the control input has to be bounded. Neglecting the amplitude actuator limitation can be source of undesirable and catastrophic behaviors for the closed-loop system. The standard method to analyze the stability with such nonlinear controls follows a two-step design. First the design is carried out without taking into account the saturation. In the second step, a nonlinear analysis of the closed-loop system is made when adding the saturation. In this way, we often get local stabilization results. Tackling this particular nonlinearity in the case of finite dimensional systems is already a difficult problem. However, nowadays, numerous techniques are available (see, e.g., [39], [41], [37]) and such systems can be analyzed with an appropriate Lyapunov function and a sector condition of the saturation map, as introduced in [39].

In the literature, there are few papers studying this topic in the infinite dimensional case. Among them, we can cite [18], [29], where a wave equation equipped with a saturated distributed actuator is studied, and [12], where a coupled PDE/ODE system modeling a switched power converter with a transmission line is considered. Due to some restrictions on the system, a saturated feedback has to be designed in the

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latter paper. There exist also some papers using the nonlinear semigroup theory and focusing on abstract systems [20], [34], [36].

Let us note that in [36], [34], and [20], the study of the a priori bounded controller is tackled using abstract nonlinear theory. To be more specific, for bounded [36], [34] and unbounded [34] control operators, some conditions are derived to deduce, from the asymptotic stability of an infinite dimensional linear system in abstract form, the asymptotic stability when closing the loop with saturating controller. These articles use the nonlinear semigroup theory (see, e.g., [24] or [1]).

The Korteweg–de Vries (KdV) equation

$$y_t + y_x + y_{xxx} + yy_x = 0 \tag{1.1}$$

is a mathematical model of waves on shallow water surfaces. Its controllability and stabilizability properties have been deeply studied with no constraints on the control, as reviewed in [3, 9, 32]. In this article, we focus on the following controlled KdV equation:

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + f = 0, & (t,x) \in [0,+\infty) \times [0,L], \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in [0,+\infty), \\ y(0,x) = y_0(x), & x \in [0,L], \end{cases} \tag{1.2}$$

where y stands for the state and f for the control. As studied in [30], if $f = 0$ and

$$L \in \left\{ 2\pi \sqrt{\frac{k^2+kl+l^2}{3}} \Big/ \; k,l \in \mathbb{N}^* \right\}, \tag{1.3}$$

then there exist solutions of the linearized version of (1.2), written as follows,

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x), \end{cases} \tag{1.4}$$

for which the $L^2(0,L)$-energy does not decay to zero. For instance, if $L = 2\pi$ and $y_0 = 1 - \cos(x)$ for all $x \in [0,L]$, then $y(t,x) = 1 - \cos(x)$ is a stationary solution of (1.4) conserving the energy for any time t. Note, however, that if $L = 2\pi$ and $f = 0$, the origin of (1.2) is locally asymptotically stable as stated in [8]. It is worth mentioning that there is no hope to obtain global stability, as established in [13], where an equilibrium with arbitrary large amplitude is built.

In the literature there are some methods stabilizing the KdV equation (1.2) with boundary [5], [4], [21] or distributed controls [25], [26]. Here we focus on the distributed control case. In fact, as proven in [25], [26], the feedback control $f(t,x) = a(x)y(t,x)$, where a is a positive function whose support is a nonempty open subset of $(0,L)$, makes the origin an exponentially stable solution.

In [22], in which a linear KdV equation with a saturated distributed control is considered, we use a nonlinear semigroup theory. In the case of the present paper, since the term yy_x is not globally Lipschitz, such a theory is harder to use. Thus, we aim here at studying a particular nonlinear partial differential equation without seeing it as an abstract control system and without using the nonlinear semigroup theory. In this paper, we introduce two different types of saturation borrowed from [29], [22] and [36]. In finite dimension, a way to describe this constraint is to use the classical saturation function (see [39] for a good introduction on saturated control problems) defined by

$$\mathtt{sat}(s) = \begin{cases} -u_0 & \text{if } s \leq -u_0, \\ s & \text{if } -u_0 \leq s \leq u_0, \\ u_0 & \text{if } s \geq u_0 \end{cases} \tag{1.5}$$

for some $u_0 > 0$. As in [29] and [22] we use its extension to infinite dimension for the feedback law

$$f(t,x) = \mathfrak{sat}_{\mathtt{loc}}(ay)(t,x), \tag{1.6}$$

where, for all sufficiently smooth function s and for all $x \in [0,L]$, $\mathfrak{sat}_{\mathtt{loc}}$ is defined as follows:

$$\mathfrak{sat}_{\mathtt{loc}}(s)(x) = \mathtt{sat}(s(x)). \tag{1.7}$$

Such a saturation is called localized since its image depends only on the value of s at x.

In this work, we also use a saturation operator in $L^2(0,L)$, denoted by $\mathfrak{sat}_2$, and defined by

$$\mathfrak{sat}_2(s)(x) = \begin{cases} s(x) & \text{if } \|s\|_{L^2(0,L)} \leq u_0, \\ \frac{s(x)u_0}{\|s\|_{L^2(0,L)}} & \text{if } \|s\|_{L^2(0,L)} \geq u_0. \end{cases} \tag{1.8}$$

Note that this definition is borrowed from [36] (see also [34] or [18]), where the saturation is obtained from the norm of the Hilbert space of the control operator. This saturation seems more natural when studying the stability with respect to an energy, but it is less relevant than $\mathfrak{sat}_{\mathtt{loc}}$ for applications. Figure 1 illustrates how different these saturations are.

Our first main result states that using either the localized saturation (1.7) or using the L^2 saturation map (1.8) the KdV equation (1.2) in closed loop with a saturated control is well-posed (see Theorem 2.1 below for a precise statement). Our second main

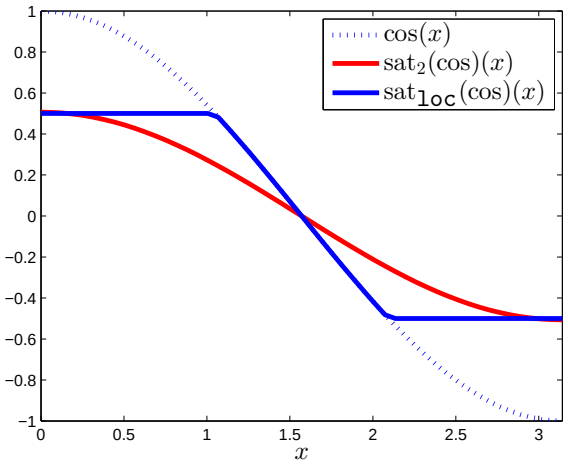

FIG. 1. $x \in [0,\pi]$. *Red:* $\mathfrak{sat}_2(\cos)(x)$ *and* $u_0 = 0.5$. *Blue:* $\mathfrak{sat}_{loc}(\cos)(x)$ *and* $u_0 = 0.5$. *Dotted lines:* $\cos(x)$.

result states that the origin of the KdV equation (1.2) in closed loop with a saturated control is globally asymptotically stable. Moreover, in the case where the control acts on all the domain and where the control is saturated with (1.8), if the initial conditions are bounded in L^2 norm, then the solution converges exponentially with a decay rate that can be estimated (see Theorem 2.2 below for a precise statement).

This article is organized as follows. In section 2, we present our main results about the well-posedness and the stability of (1.2) in presence of saturating control. Sections 3 and 4 are devoted to proving these results by using the Banach fixed-point theorem, Lyapunov techniques, and a contradiction argument. In section 5, we provide a numerical scheme for the nonlinear equation and give some simulations of the equation looped by a saturated feedback. Section 6 collects some concluding remarks and possible further research lines.

Notation. A function α is said to be a class $\mathcal{K}_\infty$ function if α is nonnegative, increasing, vanishing at 0, and such that $\lim_{s\to+\infty}\alpha(s)=+\infty$.

2. Main results. We first give an analysis of our system (1.2) when there is no constraint on the control f. To do that, letting $f(t,x):=ay(t,x)$ in (1.2), where a is a nonnegative function satisfying

$$\begin{cases} 0<a_0\le a(x)\le a_1 \quad \forall x\in\omega, \\ \text{where } \omega \text{ is a nonempty open subset of } (0,L), \end{cases} \tag{2.1}$$

then, following [31], we get that the origin of (1.2) is globally asymptotically stabilized. If $\omega=[0,L]$, then any solution to (1.2) satisfies

$$\frac{1}{2}\frac{d}{dt}\int_0^L |y(t,x)|^2dx=-\frac{1}{2}|y_x(t,0)|^2-\int_0^L a(x)|y(t,x)|^2dx\le -a_0\int_0^L |y(t,x)|^2dx, \tag{2.2}$$

which ensures an exponential stability with respect to the $L^2(0,L)$-norm. Note that the decay rate can be selected as large as we want by tuning the parameter a_0. Such a result is referred to as a rapid stabilization result.

Let us consider the KdV equation controlled by a saturated distributed control as follows:

$$\begin{cases} y_t+y_x+y_{xxx}+yy_x+\mathfrak{sat}(ay)=0, \\ y(t,0)=y(t,L)=y_x(t,L)=0, \\ y(0,x)=y_0(x), \end{cases} \tag{2.3}$$

where $\mathfrak{sat}=\mathfrak{sat}_2$ or $\mathfrak{sat}_{\mathtt{loc}}$. Since these two operators have properties in common, we will use the notation $\mathfrak{sat}$ throughout the paper. However, in some cases, we get different results. Therefore, the use of a particular saturation is specified when it is necessary.

Let us state the main results of this paper.

THEOREM 2.1 (well-posedness). *For any initial condition $y_0\in L^2(0,L)$, there exists a unique mild solution $y\in C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$ to* (2.3).

THEOREM 2.2 (global asymptotic stability). *Given a nonempty open subset ω and the positive values a_0 and u_0, there exist a positive value $\mu^\star$ and a class $\mathcal{K}_\infty$ function $\alpha_0:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ such that for any $y_0\in L^2(0,L)$, the mild solution y of* (2.3) *satisfies*

$$\|y(t,.)\|_{L^2(0,L)}\le\alpha_0(\|y_0\|_{L^2(0,L)})e^{-\mu^\star t}\quad\forall t\ge0. \tag{2.4}$$

Moreover, in the case where $\omega=[0,L]$ and $\mathfrak{sat}=\mathfrak{sat}_2$ we can estimate locally *the decay rate of the solution. In other words, for all $r>0$, for any initial condition $y_0\in L^2(0,L)$ such that $\|y_0\|_{L^2(0,L)}\le r$, the mild solution y to* (2.3) *satisfies*

$$\|y(t,.)\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)} e^{-\mu t} \quad \forall t \geq 0, \tag{2.5}$$

where μ is defined as follows:

$$\mu := \min\left\{a_0, \frac{u_0 a_0}{r a_1}\right\}. \tag{2.6}$$

The remaining part of this paper is devoted to the proof of these results (see sections 3 and 4, respectively) and to numerical simulations to illustrate Theorem 2.2 (see section 5).

3. Well-posedness.

3.1. Linear system. Before proving the well-posedness of (2.3), let us recall some useful results on the linear system (1.4). To do that, consider the operator defined by

$$\begin{aligned} D(A) &= \{w \in H^3(0,L),\ w(0) = w(L) = w'(L) = 0\}, \\ A : w \in D(A) \subset L^2(0,L) &\longmapsto (-w' - w''') \in L^2(0,L). \end{aligned}$$

It can be proved that this operator and its adjoint operator defined by

$$\begin{aligned} D(A^\star) &= \{w \in H^3(0,L),\ w(0) = w(L) = w'(0) = 0\}, \\ A^\star : w \in D(A^\star) &\subset L^2(0,L) \longmapsto w' + w''' \end{aligned}$$

are both dissipative, which means that, for all $w \in D(A)$, $\int_0^L wA(w)dx \leq 0$, and for all $w \in D(A^\star)$, $\int_0^L wA^\star(w)dx \leq 0$.

Therefore, from [28], the operator A generates a strongly continuous semigroup of contractions which we denote by $W(t)$. We have the following theorem proven in [30] and [3].

THEOREM 3.1 (well-posedness of (1.4), [30], [3]).

- *For any initial condition $y_0 \in D(A)$, there exists a unique strong solution $y \in C(0,T;D(A)) \cap C^1(0,T;L^2(0,L))$ to* (1.4).
- *For any initial condition $y_0 \in L^2(0,L)$, there exists a unique mild solution $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ to* (1.4). *Moreover, there exists $C_0 > 0$ such that the solution to* (1.4) *satisfies*

$$\|y\|_{C(0,T;L^2(0,L))} + \|y\|_{L^2(0,T;H^1(0,L))} \leq C_0 \|y_0\|_{L^2(0,L)} \tag{3.1}$$

and the extra trace regularity

$$\|y_x(.,0)\|_{L^2(0,T)} \leq \|y_0\|_{L^2(0,L)}. \tag{3.2}$$

To ease the reading, let us denote the following Banach space, for all $T > 0$,

$$\mathcal{B}(T) := C(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

endowed with the norm

$$\|y\|_{\mathcal{B}(T)} = \sup_{t\in[0,T]} \|y(t,.)\|_{L^2(0,L)} + \left(\int_0^T \|y(t,.)\|^2_{H^1(0,L)} dt\right)^{\frac{1}{2}}. \tag{3.3}$$

Before studying the well-posedness of (2.3), we need a well-posedness result with a right-hand side. Given $g \in L^1(0,T;L^2(0,L))$, let us consider y the unique solution[1] to the following nonhomogeneous problem:

$$\begin{cases} y_t + y_x + y_{xxx} = g, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,.) = y_0. \end{cases} \tag{3.4}$$

Note that we need the following property on the saturation function, which will allow us to state that this type of nonlinearity belongs to the space $L^1(0,T;L^2(0,L))$.

LEMMA 3.2. *For all $(s,\tilde{s}) \in L^2(0,L)^2$, we have*

$$\|\mathfrak{sat}(s) - \mathfrak{sat}(\tilde{s})\|_{L^2(0,L)} \leq 3\|s-\tilde{s}\|_{L^2(0,L)}. \tag{3.5}$$

Proof. For $\mathfrak{sat} = \mathfrak{sat}_2$, please refer to [36, Theorem 5.1] for a proof. For $\mathfrak{sat} = \mathfrak{sat}_{\mathtt{loc}}$, we know from [16, p. 73] that for all $(s,\tilde{s}) \in L^2(0,L)^2$ and for all $x \in [0,L]$,

$$|\mathfrak{sat}_{\mathtt{loc}}(s(x)) - \mathfrak{sat}_{\mathtt{loc}}(\tilde{s}(x))| \leq |s(x) - \tilde{s}(x)|.$$

Thus, we get

$$\|\mathfrak{sat}_{\mathtt{loc}}(s) - \mathfrak{sat}_{\mathtt{loc}}(\tilde{s})\|_{L^2(0,L)} \leq \|s-\tilde{s}\|_{L^2(0,L)},$$

which concludes the proof of Lemma 3.2. ∎

We have the following proposition borrowed from [30, Proposition 4.1].

PROPOSITION 3.3 (see [30]). *If $y \in L^2(0,T;H^1(0,L))$, then $yy_x \in L^1(0,T;L^2(0,L))$ and the map $\psi_1 : y \in L^2(0,T;H^1(0,L)) \mapsto yy_x \in L^1(0,T;L^2(0,L))$ is continuous.*

We have also the following proposition.

PROPOSITION 3.4. *Assume $a : [0,L] \to \mathbb{R}$ satisfies (2.1). If $y \in L^2(0,T;H^1(0,L))$, then $\mathfrak{sat}(ay) \in L^1(0,T;L^2(0,L))$ and the map $\psi_2 : y \in L^2(0,T;H^1(0,L)) \mapsto \mathfrak{sat}(ay) \in L^1(0,T;L^2(0,L))$ is continuous.*

Proof. Let $y, z \in L^2(0,T;H^1(0,L))$. We have, using Lemma 3.2 and the Hölder inequality,

$$\begin{aligned} \|\mathfrak{sat}(ay) - \mathfrak{sat}(az)\|_{L^1(0,T;L^2(0,L))} &\leq 3\int_0^T \|a(y-z)\|_{L^2(0,L)} \\ &\leq 3\sqrt{L}a_1\sqrt{T}\|(y-z)\|_{L^2(0,T;H^1(0,L))}. \end{aligned} \tag{3.6}$$

∎

Plugging $z = 0$ in (3.6) yields $\mathfrak{sat}(ay) \in L^1(0,T;L^2(0,L))$ and (3.6) implies the continuity of the map ψ_2. It concludes the proof of Proposition 3.4.

[1] With $f = 0$, the existence and the unicity of y are ensured since A generates a C_0-semigroup of contractions. It follows from the semigroup theory the existence and the unicity of y when $g \in L^1(0,T;L^2(0,L))$ (see [28]).

Let us study the nonhomogenenous linear KdV equation with $y_0(x) := 0$. For any $g \in L^1(0,T;L^2(0,L))$, it is described with the following equation:

$$\begin{cases} y_t + y_x + y_{xxx} + g = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = 0. \end{cases} \tag{3.7}$$

It can be rewritten as follows:

$$\begin{cases} \dot{y} = Ay + g, \\ y(0) = 0. \end{cases} \tag{3.8}$$

By standard semigroup theory (see [28]), for any positive value t and any function $g \in L^1(\mathbb{R}_{\geq 0};L^2(0,L))$, the solution to (3.7) can be expressed as follows:

$$y(t) = \int_0^t W(t-\tau)g(\tau,x)d\tau. \tag{3.9}$$

Finally, we have the following result borrowed from [31, Lemma 2.2].

PROPOSITION 3.5 (see [31]). *There exists a positive value C_1 such that for any positive value T and any function $g \in L^1(0,T;L^2(0,L))$ the solution to* (3.7) *satisfies the following inequality:*

$$\left\| \int_0^t W(t-\tau)g(\tau,x)d\tau \right\|_{\mathcal{B}(T)} \leq C_1 \int_0^T \|g(\tau,.)\|_{L^2(0,L)}d\tau. \tag{3.10}$$

3.2. Proof of Theorem 2.1. Let us begin this section with a technical lemma.

LEMMA 3.6 (see [42]). *For any $T > 0$ and $y, z \in \mathcal{B}(T)$,*

$$\int_0^T \|(y(t,.)z(t,.))_x\|_{L^2(0,L)}dt \leq 2\sqrt{T}\|y\|_{\mathcal{B}(T)}\|z\|_{\mathcal{B}(T)}. \tag{3.11}$$

The following is a local well-posedness result.

LEMMA 3.7 (local well-posedness). *Let $T > 0$ be given. For any $y_0 \in L^2(0,L)$, there exists $T' \in [0,T]$ depending on $\|y_0\|_{L^2(0,L)}$ such that* (2.3) *admits a unique mild solution $y \in \mathcal{B}(T')$.*

Proof. We follow the strategy of [7] and [31]. We know from Proposition 3.4 that, for all $z \in L^1(0,T;L^2(0,L))$, there exists a unique mild solution to the following system:

$$\begin{cases} y_t + y_x + y_{xxx} = -zz_x - \mathfrak{sat}(az), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x). \end{cases} \tag{3.12}$$

Solution to (3.12) can be written in its integral form,

$$y(t) = W(t)y_0 - \int_0^t W(t-\tau)(zz_x)(\tau)d\tau - \int_0^t W(t-\tau)\mathfrak{sat}(az(\tau,.))d\tau. \tag{3.13}$$

For given $y_0 \in L^2(0,L)$, let r and T' be positive constants to be chosen later. We define

$$S_{T',r} = \{z \in \mathcal{B}(T'),\ \|z\|_{\mathcal{B}(T')} \leq r\}, \tag{3.14}$$

which is a closed, convex, and bounded subset of $\mathcal{B}(T')$. Consequently, $S_{T',r}$ is a complete metric space in the topology induced from $\mathcal{B}(T')$. We define a map Γ on $S_{T',r}$ by, for all $t \in [0, T']$,

$$\Gamma(z) := W(t)y_0 - \int_0^t W(t-\tau)(zz_x)(\tau)d\tau - \int_0^t W(t-\tau)\mathfrak{sat}(az(\tau,.))d\tau \ \ \forall z \in S_{T',r}. \tag{3.15}$$

We aim at proving that there exists a unique fixed point to this operator. It follows from Proposition 3.5, Lemma 3.6, and the linear estimates given in Theorem 3.1 that for every $z \in S_{T',r}$, there exists a positive value $C_2 := C_2(a_1, T, L, C_1)$ such that it holds that

$$\begin{aligned}\|\Gamma(z)\|_{\mathcal{B}(T')} &\leq C_0\|y_0\|_{L^2(0,L)} + C_1\int_0^T (\|zz_x(\tau,.)\|_{L^2(0,L)} + \|\mathfrak{sat}(az(\tau,.)\|_{L^2(0,L)})d\tau \\ &\leq C_0\|y_0\|_{L^2(0,L)} + 2C_1\sqrt{T'}\|z\|^2_{\mathcal{B}(T')} + C_2\sqrt{T'}\|z\|_{\mathcal{B}(T')},\end{aligned} \tag{3.16}$$

where the first line has been obtained with the linear estimates given in Theorem 3.1 and the estimate given in Proposition 3.5 and the second line with Lemma 3.6 and Proposition 3.4. We choose $r > 0$ and $T' > 0$ such that

$$\begin{cases} r = 2C_0\|y_0\|_{L^2(0,L)}, \\ 2C_1\sqrt{T'}r + C_2\sqrt{T'} \leq \dfrac{1}{2} \end{cases} \tag{3.17}$$

in order to obtain

$$\|\Gamma(z)\|_{\mathcal{B}(T')} \leq r \quad \forall z \in S_{T',r}. \tag{3.18}$$

Thus, with such r and T', Γ maps $S_{T',r}$ to $S_{T',r}$. Moreover, one can prove with Proposition 3.5, Lemma 3.6, and the linear estimates given in Theorem 3.1 that

$$\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{B}(T')} \leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{B}(T')} \ \ \forall z_1, z_2 \in S_{T',r}. \tag{3.19}$$

The existence of mild solutions to the Cauchy problem (2.3) follows by using the Banach fixed-point theorem [1, Theorem 5.7]. ∎

Before proving the global well-posedness, we need the following lemma inspired by [10] and [7], which implies that if there exists a solution for some $T > 0$, then the solution is unique.

LEMMA 3.8. *Let $T > 0$ and $a : [0, L] \to \mathbb{R}$ satisfying (2.1). There exists $C_{11} := C_{11}(T, L) > 0$ such that for every $y_0, z_0 \in L^2(0, L)$ for which there exist mild solutions y and z of*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + \mathfrak{sat}(ay) = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x), \end{cases} \tag{3.20}$$

and

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + \mathfrak{sat}(az) = 0, \\ z(t,0) = z(t,L) = z_x(t,L) = 0, \\ z(0,x) = z_0(x), \end{cases} \tag{3.21}$$

these solutions satisfy

$$\int_0^T \int_0^L (z_x(t,x) - y_x(t,x))^2 dxdt \leq e^{C_{11}(1+\|y\|_{L^2(0,T;H^1(0,L))} + \|z\|_{L^2(0,T;H^1(0,L))})} \int_0^L (z_0(x) - y_0(x))^2 dx, \tag{3.22}$$

$$\int_0^T \int_0^L (z(t,x) - y(t,x))^2 dxdt \leq e^{C_{11}(1+\|y\|_{L^2(0,T;H^1(0,L))} + \|z\|_{L^2(0,T;H^1(0,L))})} \int_0^L (z_0(x) - y_0(x))^2 dx. \tag{3.23}$$

Proof. We follow the strategy of [10] and [7]. Let us assume that for given $y_0 \in L^2(0,L)$, there exist $T > 0$ and two different solutions y and z to (3.20) and (3.21), respectively, defined on $[0,T] \times [0,L]$. Then $\Delta := z - y$ defined on $[0,T] \times [0,L]$ is a mild solution of

$$\begin{cases} \Delta_t + \Delta_x + \Delta_{xxx} = -y\Delta_x - z_x\Delta - (\mathfrak{sat}(az) - \mathfrak{sat}(ay)), \\ \Delta(t,0) = \Delta(t,L) = \Delta_x(t,L) = 0, \\ \Delta(0,x) = z_0(x) - y_0(x). \end{cases} \tag{3.24}$$

Integrating by parts in

$$\int_0^L 2x\Delta(\Delta_t + \Delta_x + \Delta_{xxx} + y\Delta_x + z_x\Delta + \mathfrak{sat}(az) - \mathfrak{sat}(ay))dx = 0, \tag{3.25}$$

and using the boundary conditions of (3.24), we readily get

$$\frac{d}{dt}\int_0^L x\Delta^2 dx + 3\int_0^L \Delta_x^2 dx = \int_0^L \Delta^2 dx - 2\int_0^L xy\Delta\Delta_x dx + \int_0^L z\Delta^2 dx + 4\int_0^L xz\Delta\Delta_x dx - \int_0^L x\Delta(\mathfrak{sat}(az) - \mathfrak{sat}(ay))dx. \tag{3.26}$$

By the boundary conditions and the continuous Sobolev embedding $H_0^1(0,L) \subset C([0,T])$, there exists $C_3 = C_3(L) > 0$ such that

$$2\left|\int_0^L xy\Delta\Delta_x dx\right| \leq C_3\|y_x\|_{L^2(0,L)} \int_0^L |x\Delta\Delta_x| dx. \tag{3.27}$$

Thus,

$$2\left|\int_0^L xy\Delta\Delta_x dx\right| \leq \frac{1}{2}\int_0^L \Delta_x^2 dx + \frac{C_3^2}{2}\|y_x\|^2_{L^2(0,L)} L \int_0^L x\Delta^2 dx. \tag{3.28}$$

Similarly,

$$4\left|\int_0^L xz\Delta\Delta_x dx\right| \leq \frac{1}{2}\int_0^L \Delta_x^2 dx + 2C_3^2\|z_x\|^2_{L^2(0,L)} \int_0^L x\Delta^2 dx. \tag{3.29}$$

Moreover, since $\mathfrak{sat}$ is globally Lipschitz with constant 3 (as stated in Lemma 3.2) and for all $x \in [0, L]$, $a(x) \leq a_1$, we use a Hölder inequality to get

$$\begin{aligned}(3.30)\qquad \left| \int_0^L x\Delta(\mathfrak{sat}(az) - \mathfrak{sat}(ay))dx \right| &\leq \|x\Delta\|_{L^2(0,L)} \|\mathfrak{sat}(az) - \mathfrak{sat}(ay)\|_{L^2(0,L)} \\ &\leq 3\|a(x)\Delta\|_{L^2(0,L)} \|x\Delta\|_{L^2(0,L)} \\ &\leq 3a_1 \int_0^L x\Delta^2 dx.\end{aligned}$$

Note that, from [10, Lemma 16], for every $\phi \in H^1(0, L)$ with $\phi(0) = 0$, and every $d \in [0, L]$,

$$(3.31)\qquad \int_0^L \phi^2 dx \leq \frac{d^2}{2} \int_0^L \phi_x^2 dx + \frac{1}{d} \int_0^L x\phi^2 dx.$$

Thus, from (3.31) there exists $C_4 > 0$ such that

$$\int_0^L \Delta^2 dx \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + C_4 \int_0^L x\Delta^2 dx.$$

Moreover, with the boundary conditions of z and the Sobolev embedding $H_0^1(0, L) \subset C([0, T])$, there exists $C_5 = C_5(L) > 0$ such that

$$2\left| \int_0^L z\Delta^2 dx \right| \leq C_5 \|z_x\|_{L^2(0,L)} \int_0^L \Delta^2 dx.$$

Hence, using the boundary conditions of Δ and (3.31) with $d := \min\{C_5^{-1/2} \|z_x\|_{L^2(0,L)}^{-1/2}, L\}$, there exists $C_6 = C_6(L) > 0$ such that

$$(3.32)\qquad 2\int_0^L z\Delta^2 dx \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + C_6\Big(1 + \|z_x\|_{L^2(0,L)}^{3/2}\Big) \int_0^L x\Delta^2 dx.$$

Finally, there exists $C_7 = C_7(L) > 0$ such that

$$(3.33)\qquad \frac{d}{dt} \int_0^L x\Delta^2 dx + \int_0^L \Delta_x^2 dx \leq C_7\Big(1 + \|y_x\|_{L^2(0,L)}^2 + \|z_x\|_{L^2(0,L)}^2\Big) \int_0^L x\Delta^2 dx.$$

In particular,

$$(3.34)\qquad \frac{d}{dt} \int_0^L x\Delta^2 dx \leq C_7 \left(1 + \|y_x\|_{L^2(0,L)}^2 + \|z_x\|_{L^2(0,L)}^2\right) \int_0^L x\Delta^2 dx.$$

Using the Grönwall lemma, the last inequality, and the initial conditions of Δ, we get, for every $t \in [0, T]$,

$$(3.35)\qquad \int_0^L x\Delta^2(t, x)dx \leq e^{C_7\left(T + \|y\|_{L^2(0,T;H^1(0,L))}^2 + \|z\|_{L^2(0,T;H^1(0,L))}^2\right)} \int_0^L x(z_0(x) - y_0(x))^2 dx,$$

and thus we obtain the existence of $C_8 = C_8(T, L)$ such that

$$\begin{aligned}(3.36)\qquad &\int_0^T \int_0^L (z_x(t, x) - y_x(t, x))^2 dxdt \\ &\quad \leq e^{C_8\left(\|y\|_{L^2(0,T;H^1(0,L))}^2 + \|z\|_{L^2(0,T;H^1(0,L))}^2\right)} \int_0^L (z_0(x) - y_0(x))^2 dx.\end{aligned}$$

Similarly, integrating by parts in

$$\int_0^L \Delta(\Delta_t + \Delta_x + \Delta_{xxx} + y\Delta_x + z_x\Delta + \mathfrak{sat}(az) - \mathfrak{sat}(ay))dx = 0 \tag{3.37}$$

we get, using the boundary conditions of Δ,

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Delta^2 dx + \frac{1}{2}\Delta_x^2(t,0) = -\int_0^L (y\Delta_x - 2z\Delta_x)\Delta dx - \int_0^L \Delta(\mathfrak{sat}(az) - \mathfrak{sat}(ay))dx. \tag{3.38}$$

Moreover,

$$-\int_0^L (y\Delta_x - 2z\Delta_x)\Delta \le \int_0^L \Delta_x^2 dx + \int_0^L \left(\frac{1}{2}y^2 + 2z^2\right)\Delta^2 dx \tag{3.39}$$

and

$$\left|\int_0^L \Delta(\mathfrak{sat}(az) - \mathfrak{sat}(ay))dx\right| \le 3a_1 \int_0^L \Delta^2 dx. \tag{3.40}$$

Thanks to the continuous Sobolev embedding $H_0^1(0,L) \subset C([0,L])$, (3.40), and (3.39), there exists $C_9 = C_9(L) > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Delta^2 dx \le \int_0^L \Delta_x^2 dx + C_9\left(\|y_x\|^2_{L^2(0,L)} + \|z_x\|^2_{L^2(0,L)} + 1\right)\int_0^L \Delta^2 dx. \tag{3.41}$$

Thus applying the Grönwall lemma, we get the existence of $C_{10} = C_{10}(L) > 0$ such that

$$\begin{aligned}\int_0^L (z(t,x) - y(t,x))^2 dx \\ \le e^{C_{10}\left(1+\|y\|^2_{L^2(0,T;H^1(0,L))}+\|z\|^2_{L^2(0,T;H^1(0,L))}\right)}\int_0^L (z_0(x) - y_0(x))^2 dx.\end{aligned} \tag{3.42}$$

With the use (3.36) and (3.42), it concludes the proof of Lemma 3.8. ∎

We aim at removing the smallness condition given by T' in Lemma 3.7, following [7]. Since we have the local well-posedness, we only need to prove the following a priori estimate for any mild solution to (2.3).

LEMMA 3.9. *For given $T > 0$, there exists $G := G(T) > 0$ such that for any $y_0 \in L^2(0,L)$, for any $0 < T' \le T$, and for any mild solution $y \in \mathcal{B}(T')$ to (2.3), it holds that*

$$\|y\|_{\mathcal{B}(T')} \le G\|y_0\|_{L^2(0,L)} \tag{3.43}$$

and

$$\|y\|_{L^2(0,L)} \le \|y_0\|_{L^2(0,L)}. \tag{3.44}$$

Proof. Let us fix $0 < T' \le T$. We multiply the first equation in (2.3) by y and integrate on $(0,L)$. Using the boundary conditions in (2.3), we get the following estimates:

$$\int_0^L yy_x dx = 0, \quad \int_0^L yy_{xxx} dx = \frac{1}{2}|y_x(t,0)|^2, \quad \int_0^L y^2 y_x dx = 0.$$

Using the fact that $\mathfrak{sat}$ is odd, we get that

$$\frac{1}{2}\frac{d}{dt}\|y(t,.)\|^2_{L^2(0,L)} \leq -\frac{1}{2}|y_x(t,0)|^2 - \int_0^L y\mathfrak{sat}(ay)dx \leq 0, \tag{3.45}$$

which implies (3.44). Moreover, using again (3.45), there exists $C_{12} = C_{12}(L) > 0$ such that

$$\|y\|_{L^\infty(0,T';L^2(0,L)} \leq C_{12}\|y_0\|_{L^2(0,L)}. \tag{3.46}$$

It remains to prove a similar inequality for $\|y_x\|_{L^2(0,T';L^2(0,L))}$ to achieve the proof. We multiply (2.3) by xy, integrate on $(0,L)$, and use

$$\int_0^L xyy_x dx = -\frac{1}{2}\|y\|^2_{L^2(0,L)}, \quad \int_0^L yy_{xxx} dx = \frac{3}{2}\|y_x\|^2_{L^2(0,L)},$$

and

$$\begin{aligned} -\int_0^L xy^2 y_x dx = \frac{1}{3}\int_0^L y^3(t,x)dx &\leq \frac{1}{3}\sup_{x\in[0,L]}|y(t,x)|\|y\|^2_{L^\infty(0,T;L^2(0,L))} \\ &\leq \frac{\sqrt{L}}{3}\|y_x\|_{L^2(0,L)}\|y\|^2_{L^\infty(0,T;L^2(0,L))} \\ &\leq \frac{\sqrt{L}\delta}{6}\|y_x\|_{L^2(0,L)} + \frac{\sqrt{L}}{6\delta}\|y\|^4_{L^\infty(0,T;L^2(0,L))}, \end{aligned} \tag{3.47}$$

where δ is chosen as $\delta := \frac{3}{\sqrt{L}}$. In this way, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L |x^{1/2}y(t,.)|^2 dx - \frac{1}{2}\int_0^L y^2 dx \\ &\quad + \frac{3}{2}\int_0^L |y_x|^2 dx - \frac{1}{3}\int_0^L |y|^3 dx = -\int_0^L x\mathfrak{sat}(ay)ydx. \end{aligned} \tag{3.48}$$

We get, using (3.47) and the fact that $\mathfrak{sat}$ is odd, that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L |x^{1/2}y(t,.)|^2 dx + \int_0^L |y_x|^2 dx \\ &\quad \leq \frac{1}{2}\|y\|^2_{L^\infty(0,T;L^2(0,L))} + \frac{L}{18}\|y\|^4_{L^\infty(0,T;L^2(0,L))}. \end{aligned} \tag{3.49}$$

Using (3.46) and the Grönwall inequality, we get the existence of a positive value $C_{13} = C_{13}(L) > 0$ such that

$$\|y_x\|_{L^2(0,T';L^2(0,L))} \leq C_{13}\|y_0\|_{L^2(0,L)}, \tag{3.50}$$

which concludes the proof of Lemma 3.9. ∎

Using a classical extension argument, Lemmas 3.7, 3.9, and 3.8, for any $T > 0$, we can conclude that there exists a unique mild solution in $\mathcal{B}(T)$ to (2.3). Indeed, with Lemma 3.7, we know that there exists $T' \in (0,T)$ such that there exists a unique solution to (2.3) in $\mathcal{B}(T')$. Moreover, Lemma 3.9 allows us to state the existence of a mild solution to (2.3) for every $T > 0$: since the solution y to (2.3) is bounded by its initial condition for every $T' > 0$ belonging to $[0,T]$ as stated in (3.44), we know that there exists a solution to (2.3) in $\mathcal{B}(T)$. Finally, Lemma 3.8 implies that there exists a unique mild solution to (2.3) in $\mathcal{B}(T)$. This concludes the proof of Theorem 2.1.

Remark 3.10. In [31], the generalized KdV equation

$$\begin{cases} y_t + y_x + y_{xxx} + b(y)y_x + ay = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x) \end{cases} \tag{3.51}$$

is considered, where the function $a : [0,L] \to \mathbb{R}$ satisfies (2.1) and where $b : \mathbb{R} \to \mathbb{R}$ satisfies the following growth condition:

$$b(0) = 0,\ |b^{(j)}(\mu)| \leq C\left(1 + |\mu|^{p-j}\right) \quad \forall \mu \in \mathbb{R} \tag{3.52}$$

for $j = 0$ if $1 \leq p < 2$ and for $j = 0, 1, 2$ if $p \geq 2$.

The saturated version of (3.51) is

$$\begin{cases} y_t + y_x + y_{xxx} + b(y)y_x + \mathfrak{sat}(ay) = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x). \end{cases} \tag{3.53}$$

The strategy followed in [31] can be followed easily to prove the same result as Theorem 2.1 for (3.53). Note that in [31], provided that the initial condition satisfies some compatibility conditions, the well-posedness is proved for a solution in $C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$, where $s \in [0,3]$. The authors proved this result by looking at $v = y_t$ which solves an equation equivalent to (3.51). In our case, it seems harder to prove such a result. Since the saturation operator introduces some nonsmoothness, $v = y_t$ does not solve an equation equivalent to (3.53).

4. Global asymptotic stability. Let us begin by introducing the following definition.

DEFINITION 4.1. *System* (2.3) *is said to be* semiglobally exponentially stable *in* $L^2(0,L)$ *if for any* $r > 0$ *there exist two constants* $K := K(r) > 0$ *and* $\mu := \mu(r) > 0$ *such that for any* $y_0 \in L^2(0,L)$ *such that* $\|y_0\|_{L^2(0,L)} \leq r$*, the mild solution* $y = y(t,x)$ *to* (2.3) *satisfies*

$$\|y(t,.)\|_{L^2(0,L)} \leq K\|y_0\|_{L^2(0,L)}e^{-\mu t} \qquad \forall t \geq 0. \tag{4.1}$$

Following [31], we first show that (2.3) is semiglobally exponentially stable in $L^2(0,L)$. From this result, we will be able to prove the global uniform exponential stability of (2.3). To do that, we state and prove a technical lemma that allows us to bound the saturation function with a linear function as long as the initial condition is bounded. Then we separate our proof into two cases. The first one deals with the case $\omega = [0,L]$ and $\mathfrak{sat} = \mathfrak{sat}_2$, while the second one deals with the case $\omega \subseteq [0,L]$ whatever the saturation is. The tools to tackle these two cases are different. The goal of the next three sections is to prove the following result

PROPOSITION 4.2 (semiglobal exponential stability). *For all* $y_0 \in L^2(0,L)$ *with* $\|y_0\|_{L^2(0,L)} \leq r$*, the system* (2.3) *is semiglobally exponentially stable in* $L^2(0,L)$.

Moreover, if $\omega = [0,L]$ *and* $\mathfrak{sat} = \mathfrak{sat}_2$*, inequality* (4.1) *holds with* $K = 1$ *and* μ *can be estimated as done in Theorem* 2.2.

4.1. Technical lemma. Before starting the proof of the Proposition 4.2, let us state and prove the following lemma.

LEMMA 4.3 (sector condition). *Let r be a positive value, a function $a : [0, L] \to \mathbb{R}$ satisfying* (2.1) *and $k(r)$ defined by*

$$k(r) = \min\left\{\frac{u_0}{a_1 r}, 1\right\}. \tag{4.2}$$

(i) *Given $\mathfrak{sat} = \mathfrak{sat}_2$ and $s \in L^2(0, L)$ such that $\|s\|_{L^2(0,L)} \le r$, we have*

$$(\mathfrak{sat}_2(a(x)s(x)) - k(r)a(x)s(x))\, s(x) \ge 0 \quad \forall x \in [0, L]. \tag{4.3}$$

(ii) *Given $\mathfrak{sat} = \mathfrak{sat}_{loc}$ and $s \in L^\infty(0, L)$ such that, for all $x \in [0, L]$, $|s(x)| \le r$, we have*

$$(\mathfrak{sat}_{loc}(a(x)s(x)) - k(r)a(x)s(x))\, s(x) \ge 0 \quad \forall x \in [0, L]. \tag{4.4}$$

Proof.

(i) We first prove item (i) of Lemma 4.3. Two cases may occur:
1. $\|as\|_{L^2(0,L)} \ge u_0$;
2. $\|as\|_{L^2(0,L)} \le u_0$.

The first case implies that, for all $x \in [0, L]$,

$$\mathfrak{sat}_2(a(x)s(x)) = \frac{a(x)s(x)}{\|as\|_{L^2(0,L)}} u_0.$$

Thus, for all $x \in [0, L]$,

$$(\mathfrak{sat}_2(a(x)s(x)) - k(r)a(x)s(x))\, s(x) = a(x)s(x)^2 \left(\frac{u_0}{\|as\|_{L^2(0,L)}} - k(r)\right).$$

Since

$$\frac{u_0}{\|as\|_{L^2(0,L)}} \ge \frac{u_0}{a_1\|s\|_{L^2(0,L)}} \ge \frac{u_0}{a_1 r} \ge k(r),$$

we obtain

$$(\mathfrak{sat}_2(a(x)s(x)) - k(r)a(x)s(x))\, s(x) \ge 0.$$

Now, let us consider the case $\|as\|_{L^2(0,L)} \le u_0$. We have, for all $x \in [0, L]$,

$$\mathfrak{sat}_2(a(x)s(x)) = a(x)s(x),$$

and then, for all $x \in [0, L]$,

$$(\mathfrak{sat}_2(a(x)s(x)) - k(r)a(x)s(x))s(x) = a(x)s(x)^2(1 - k(r)) \ge 0.$$

(ii) We now deal with item (ii) of Lemma 4.3. Let us pick $x \in [0, L]$ and consider the two following cases:
1. $|a(x)s(x)| \ge u_0$;
2. $|a(x)s(x)| \le u_0$.

The first case implies either $a(x)s(x) \ge u_0$ or $a(x)s(x) \le -u_0$.
Since these two possibilities are symmetric, we just deal with the case $a(x) \ge u_0$. We have

$$\mathfrak{sat}_{\mathtt{loc}}(a(x)s(x)) = u_0,$$

and then

$$(\mathfrak{sat}_{\mathtt{loc}}(a(x)s(x)) - k(r)a(x)s(x))\, s(x) = u_0 s(x) - k(r)a(x)s^2(x)$$
$$\geq (u_0 - k(r)a(x)r)\, s(x) \geq \left(u_0 - \frac{u_0}{a_1 r}a(x)r\right) s(x) \geq 0.$$

The second case implies that $\mathfrak{sat}_{\mathtt{loc}}(a(x)s(x)) = a(x)s(x)$, and then $(\mathfrak{sat}_{\mathtt{loc}}(a(x)s(x) - k(r)a(x)s(x))s(x) = (1-k(r))a(x)^2 s(x)^2 \geq 0$. This concludes the proof of the second item of Lemma 4.3. ∎

4.2. Proof of Proposition 4.2 when $\omega = [0, L]$ and $\mathfrak{sat} = \mathfrak{sat}_2$. Now we are able to prove Proposition 4.2 when $\omega = [0, L]$ and $\mathfrak{sat} = \mathfrak{sat}_2$. Let $r > 0$ and $y_0 \in L^2(0, L)$ be such that $\|y_0\|_{L^2(0,L)} \leq r$.

Multiplying (2.3) by y, integrating with respect to x on $(0, L)$ yields

$$\frac{1}{2}\frac{d}{dt}\int_0^L |y(t,x)|^2 dx \leq -\int_0^L \mathfrak{sat}_2(ay(t,x))y(t,x)dx. \tag{4.5}$$

Note that from (3.44), we get

$$\|y\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)} \leq r. \tag{4.6}$$

Thus, using Lemma 4.3 and (4.5), it implies that

$$\frac{1}{2}\frac{d}{dt}\int_0^L |y(t,x)|^2 dx \leq -\int_0^L k(r)a_0|y(t,x)|^2 dx. \tag{4.7}$$

Applying the Grönwall lemma leads to

$$\|y(t,.)\|_{L^2(0,L)} \leq e^{-\mu t}\|y_0\|_{L^2(0,L)}, \tag{4.8}$$

where μ is defined in the statement of Theorem 2.2. It concludes the proof of Proposition 4.2 when $\omega = [0, L]$ and when $\mathfrak{sat} = \mathfrak{sat}_2$.

Remark 4.4. The constant μ depends on u_0, r, and a_0. Thus, although we have proven an exponential stability, the rapid stabilization is still an open question. Moreover, in the case $a(x) = a_0 = a_1$ for all $x \in [0, L]$, which is the case where the gain is constant, we obtain that

$$\mu = \min\left\{a_0, \frac{u_0}{r}\right\}.$$

4.3. Proof of Proposition 4.2 when $\omega \subsetneq [0, L]$. In this section, we have $\mathfrak{sat} = \mathfrak{sat}_2$ or $\mathfrak{sat} = \mathfrak{sat}_{\mathtt{loc}}$. We follow the strategy of [31] and [3]. We use a contradiction argument. It is based on the following unique continuation result.

THEOREM 4.5 (see [33]). *Let $u \in L^2(0, T; H^3(0, L))$ be a solution of*

$$u_t + u_x + u_{xxx} + uu_x = 0$$

such that

$$u(t, x) = 0 \ \ \forall t \in (t_1, t_2),\ \forall x \in \omega,$$

with ω an open nonempty subset of $(0, L)$. Then

$$u(t, x) = 0 \ \ \forall t \in (t_1, t_2),\ \forall x \in (0, L).$$

Moreover, the following lemma will be used.

LEMMA 4.6 (Aubin–Lions lemma, [35, Corollary 4]). *Let $X_0 \subset X \subset X_1$ be three Banach spaces with X_0, X_1 reflexive spaces. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. Then $\{h \in L^p(0,T;X_0)/\, \dot{h} \in L^q(0,T;X_1)\}$ embeds compactly in $L^p(0,T;X)$ for any $1 < p,\, q < \infty$.*

Let us now start the proof of Proposition 4.2. Let $r > 0$ and $y_0 \in L^2(0,L)$ be such that

$$\|y_0\|_{L^2(0,L)} \leq r. \tag{4.9}$$

As in the proof of Lemma 3.9, with multiplier techniques applied to (2.3), we obtain

(4.10)
$$\|y(t,.)\|^2_{L^2(0,L)} = \|y_0\|^2_{L^2(0,L)} - \int_0^t |y_x(\sigma,0)|^2 d\sigma - 2\int_0^t \int_0^L \mathfrak{sat}(ay) y dx d\sigma \quad \forall t \in [0,T]$$

and

$$\|y\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{8T+2L}{3}\|y_0\|^2_{L^2(0,L)} + \frac{TC}{27}\|y_0\|^4_{L^2(0,L)}. \tag{4.11}$$

Moreover, multiplying (2.3) by $(T-t)y$, we obtain after performing some integrations by parts

$$\begin{aligned} T\|y_0\|^2_{L^2(0,L)} \leq \int_0^T \int_0^L |y(t,x)|^2 dx dt + \int_0^T (T-t)|y_x(t,0)|^2 dt \\ + 2\int_0^T (T-t)\int_0^L \mathfrak{sat}(ay) y dx dt. \end{aligned} \tag{4.12}$$

Note that, since $\mathfrak{sat}$ is an odd function, (4.10) implies that, for all $t \in [0,T]$,

$$\|y(t,.)\|^2_{L^2(0,L)} \leq \|y_0\|^2_{L^2(0,L)}. \tag{4.13}$$

From now on, we will separate the proof into two cases: $\mathfrak{sat} = \mathfrak{sat}_2$ and $\mathfrak{sat} = \mathfrak{sat}_{\mathtt{loc}}$.

Case 1. $\mathfrak{sat} = \mathfrak{sat}_2$.

Using (3.44), we have

$$\|y(T,.)\|_{L^2(0,L)} \leq r,$$

and we can apply the first item of Lemma 4.3. The inequality (4.10) becomes

$$\|y(T,.)\|^2_{L^2(0,L)} \leq \|y_0\|^2_{L^2(0,L)} - \int_0^T |y_x(t,0)|^2 dt - 2\int_0^T \int_0^L ak(r) y^2 dx dt. \tag{4.14}$$

Let us state a claim that will be useful in the following.

Claim 1. *For any $T > 0$ and any $r > 0$ there exists a positive constant $C_{14} = C_{14}(T,r)$ such that for any solution y to (2.3) with an initial condition $y_0 \in L^2(0,L)$ such that $\|y_0\|_{L^2(0,L)} \leq r$, it holds that*

$$\|y_0\|^2_{L^2(0,L)} \leq C_{14}\left(\int_0^T |y_x(t,0)|^2 dt + 2\int_0^T \int_0^L k(r) a |y(t,x)|^2 dx dt\right). \tag{4.15}$$

Let us assume Claim 1 for the time being. Then (4.10) implies

$$\|y(kT,.)\|^2_{L^2(0,L)} \leq \gamma^k \|y_0\|^2_{L^2(0,L)} \qquad \forall k \geq 0,\ \forall t \geq 0, \tag{4.16}$$

where $\gamma \in (0,1)$. From (4.13), we have $\|y(t,.)\|_{L^2(0,L)} \leq \|y(kT,.)\|_{L^2(0,L)}$ for $kT \leq t \leq (k+1)T$. Thus we obtain, for all $t \geq 0$,

$$\|y(t,.)\|^2_{L^2(0,L)} \leq \frac{1}{\gamma}\|y_0\|_{L^2(0,L)} e^{\frac{\log\gamma}{T}t}. \tag{4.17}$$

In order to prove Claim 1, since the solution to (2.3) satisfies (4.12), it is sufficient to prove that there exists some constant $C_{15} := C_{15}(T,L) > 0$ such that

$$\int_0^T \int_0^L |y|^2 dxdt \leq C_{15}\left(\int_0^T |y_x(t,0)|^2 dt + 2\int_0^T \int_0^L k(r)ay^2 dxdt\right) \tag{4.18}$$

provided that $\|y_0\|_{L^2(0,L)} \leq r$. We argue by contradiction to prove the existence of such a constant C_{15}.

Suppose (4.18) fails to be true. Then, there exists a sequence of mild solutions $\{y^n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(T)$ of (2.3) with

$$\|y^n(0,.)\|_{L^2(0,L)} \leq r \tag{4.19}$$

and such that

$$\lim_{n\to+\infty} \frac{\|y^n\|^2_{L^2(0,T;L^2(0,L))}}{\int_0^T |y^n_x(t,0)|^2 dt + 2\int_0^T \int_0^L k(r)a(y^n)^2 dxdt} = +\infty. \tag{4.20}$$

Note that (4.19) implies with (4.13) that

$$\|y^n(t,.)\|_{L^2(0,L)} \leq r \quad \forall t \in [0,T]. \tag{4.21}$$

Let $\lambda^n := \|y^n\|_{L^2(0,T;L^2(0,L))}$ and $v^n(t,x) = \frac{y^n(t,x)}{\lambda^n}$. Notice that $\{\lambda^n\}_{n\in\mathbb{N}}$ is bounded, according to (4.21). Hence, there exists a subsequence that we continue to denote by $\{\lambda^n\}_{n\in\mathbb{N}}$ such that

$$\lambda^n \to \lambda \geq 0 \quad \text{as } n \to +\infty.$$

Then v^n fullfills

$$\begin{cases} v^n_t + v^n_x + v^n_{xxx} + \lambda^n v^n v^n_x + \frac{\mathfrak{sat}_2(a\lambda^n v^n)}{\lambda^n} = 0, \\ v^n(t,0) = v^n(t,L) = v^n_x(t,L) = 0, \\ \|v^n\|_{L^2(0,T;L^2(0,L))} = 1, \end{cases} \tag{4.22}$$

and, due to (4.20), we obtain

$$\int_0^T |v^n_x(t,0)|^2 dt + 2\int_0^T \int_0^L ak(r)(v^n)^2 dxdt \to 0 \text{ as } n \to +\infty. \tag{4.23}$$

It follows from (4.12) that $\{v^n(0,.)\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,L)$. Note also that from (4.11) $\{v^n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1(0,L))$. Thus we see that $\{v^n v^n_x\}_{n\in\mathbb{N}}$ is a subset of $L^2(0,T;L^1(0,L))$. In fact,

$$\|v^n v_x^n\|_{L^2(0,T;L^1(0,L))} \leq \|v^n\|_{C(0,T;L^2(0,L))}\|v^n\|_{L^2(0,T;H^1(0,L))}. \tag{4.24}$$

Moreover, we have that $\{\frac{\mathfrak{sat}_2(a\lambda^n v^n)}{\lambda^n}\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;L^2(0,L))$. Indeed, from Lemma 3.2

$$\begin{aligned}\left\|\frac{\mathfrak{sat}_2(a\lambda^n v^n)}{\lambda^n}\right\|_{L^2(0,T;L^2(0,L))} &\leq 3\|av^n\|_{L^2(0,T;L^2(0,L))}\\ &\leq 3a_1\sqrt{L}\|v^n\|_{L^2(0,T;H^1(0,L))}.\end{aligned} \tag{4.25}$$

Thus $\{v^n v_x^n\}_{n\in\mathbb{N}}$ and $\{\frac{\mathfrak{sat}_2(a\lambda^n v^n)}{\lambda^n}\}_{n\in\mathbb{N}}$ are also subsets of $L^2(0,T;H^{-2}(0,L))$ since $L^2(0,L)\subset L^1(0,L)\subset H^{-1}(0,L)\subset H^{-2}(0,L)$. Combined with (4.22) it implies that $\{v_t^n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;H^{-2}(0,L))$. Since $\{v^n\}_{n\in\mathbb{N}}$ is a bounded sequence of $L^2(0,T;H^1(0,L))$, we get with Lemma 4.6 that a subsequence of $\{v^n\}_{n\in\mathbb{N}}$ also denoted by $\{v^n\}_{n\in\mathbb{N}}$ converges strongly in $L^2(0,T;L^2(0,L))$ to a limit v. Moreover, with the last line of (4.22), it holds that $\|v\|_{L^2(0,T;L^2(0,L))}=1$.

Therefore, having in mind (4.23), we get

$$\begin{aligned}&\|v_x(.,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L ak(r)v^2dxdt\\ &\quad\leq \liminf_{n\to+\infty}\left\{\|v_x^n(.,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L ak(r)(v^n)^2dxdt\right\} = 0.\end{aligned} \tag{4.26}$$

Thus,

$$ak(r)v^2(t,x)=0 \ \ \forall x\in[0,L],\forall t\in(0,T), \ \text{ and } \ v_x(t,0)=0 \ \ \forall t\in(0,T). \tag{4.27}$$

and therefore

$$v(t,x)=0 \ \ \forall x\in\omega,\forall t\in(0,T), \ \text{ and } \ v_x(t,0)=0 \ \ \forall t\in(0,T). \tag{4.28}$$

We obtain that the limit function v satisfies

$$\begin{cases} v_t + v_x + v_{xxx} + \lambda v v_x = 0,\\ v(t,0)=v(t,L)=v_x(t,L)=0,\\ \|v\|_{L^2(0,T;L^2(0,L))}=1\end{cases} \tag{4.29}$$

with $\lambda\geq 0$. Let us consider $u:=v_t$ which satisfies

$$\begin{cases} u_t + u_x + u_{xxx} + \lambda v_x u + \lambda v u_x = 0,\\ u(t,0)=u(t,L)=u_x(t,L)=0,\end{cases} \tag{4.30}$$

with $u(0,.)=-v'(0,.)-v'''(0,.)-\lambda v(0,.)v'(0,.)\in H^{-3}(0,L)$ and

$$u(t,x)=0 \ \ \forall x\in\omega,\forall t\in(0,T), \ \text{ and } \ u_x(t,0)=0 \ \ \forall t\in(0,T).$$

Let us recall the following result.

LEMMA 4.7 (see [26, Lemma 3.2]). *There exists a positive value $C_{16}(T,r)>0$ such that for any solution u to (4.30) where v is solution to (4.29), it holds that*

$$\|u_x(.,0)\|^2_{L^2(0,T)} + \|u(0,.)\|^2_{H^{-3}(0,L)} \geq C_{16}\|u(0,.)\|^2_{L^2(0,L)}. \tag{4.31}$$

Applying the result of this lemma, we get $u(0,.) \in L^2(0,L)$ and therefore $u = v_t \in \mathcal{B}(T)$. Since $v, v_t \in L^2(0,T;H^1(0,L))$ and $v \in C([0,T];H^1(0,L))$, we can conclude that $vv_x \in L^2(0,T;L^2(0,L))$. In this way, $v_{xxx} = -v_t - v_x - \lambda v v_x \in L^2(0,T;L^2(0,L))$ and therefore $v \in L^2(0,T;H^3(0,L))$. Finally, using Theorem 4.5, we obtain

$$v(t,x) = 0 \quad \forall x \in [0,L],\ t \in [0,T].$$

Thus we get a contradiction with $\|v\|_{L^2(0,T;L^2(0,L))} = 1$. It concludes the proof of Claim 1 and thus Lemma 4.2 in the case where $\mathfrak{sat} = \mathfrak{sat}_2$.

Case 2. $\mathfrak{sat} = \mathfrak{sat}_{\mathtt{loc}}$.

Following the same strategy as before, we write the following claim.

Claim 2. *For any $T > 0$ and any $r > 0$, there exists a positive constant $C_{17} = C_{17}(T,r)$ such that for any mild solution y to* (2.3) *with an initial condition $y_0 \in L^2(0,L)$ such that $\|y_0\|_{L^2(0,L)} \le r$, it holds that*

$$\|y_0\|^2_{L^2(0,L)} \le C_{17}\left(\int_0^T |y_x(t,0)|^2 dt + 2\int_0^T\int_0^L \mathfrak{sat}_{loc}(ay(t,x))y(t,x)dtdx\right). \tag{4.32}$$

If Claim 2 holds, we obtain also (4.17) for a suitable choice of γ and we end the proof of Lemma 4.2 when $\mathfrak{sat} = \mathfrak{sat}_{\mathtt{loc}}$. Due to (4.12), we see that in order to prove Claim 2, it is sufficient to obtain the existence of $C_{18} > 0$ such that

$$\begin{aligned}\int_0^T\int_0^L |y(t,x)|^2 dtdx \\ \le C_{18}\left(\int_0^T |y_x(t,0)|^2 dt + 2\int_0^T\int_0^L \mathfrak{sat}_{\mathtt{loc}}(ay(t,x))y(t,x)dtdx\right).\end{aligned} \tag{4.33}$$

We argue by contradiction to prove (4.33). Then, we assume that there exists a sequence of mild solutions $\{y^n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(T)$ to (2.3) with

$$\|y^n(0,.)\|_{L^2(0,L)} \le r \tag{4.34}$$

and such that

$$\lim_{n\to+\infty} \frac{\|y^n\|^2_{L^2(0,T;L^2(0,L))}}{\int_0^T |y^n_x(t,0)|^2 dt + 2\int_0^T\int_0^L \mathfrak{sat}_{\mathtt{loc}}(ay^n(t,x))y^n(t,x)dtdx} = +\infty. \tag{4.35}$$

Note that (4.34) implies with (4.13) that

$$\|y^n(t,.)\|_{L^2(0,L)} \le r \quad \forall t \in [0,T]. \tag{4.36}$$

Note that we have, from (4.9) and (4.11),

$$\|y^n\|^2_{L^2(0,T;H^1(0,L))} \le \beta,$$

where

$$\beta := \frac{8T+2L}{3}r^2 + \frac{TC}{27}r^4.$$

Moreover, due to the Poincaré inequality and the left Dirichlet boundary condition of (2.3), we obtain

$$\sup_{x\in[0,L]}|y^n(t,x)|\le\sqrt{L}\|y^n(t,.)\|_{H^1(0,L)}\quad\forall t\in[0,T]. \tag{4.37}$$

Thus, we see that

$$\int_0^T|y^n(t,x)|^2dt\le L\|y^n\|^2_{L^2(0,T;H^1(0,L))}\le L\beta. \tag{4.38}$$

Now let us consider $\Omega_i\subset[0,T]$ defined as follows:

$$\Omega_i=\left\{t\in[0,T],\ \sup_{x\in[0,L]}|y(t,x)|>i\right\}. \tag{4.39}$$

In the following, we will denote by Ω_i^c its complement. It is defined by

$$\Omega_i^c=\left\{t\in[0,T],\ \sup_{x\in[0,L]}|y(t,x)|\le i\right\}. \tag{4.40}$$

Since the function $t\mapsto\sup_{x\in[0,L]}|y^n(t,x)|^2$ is a nonnegative function, we have

$$\int_0^T\sup_{x\in[0,L]}|y^n(t,x)|^2dt\ge\int_{\Omega_i}\sup_{x\in[0,L]}|y^n(t,x)|^2dt\ge i^2\nu(\Omega_i), \tag{4.41}$$

where $\nu(\Omega_i)$ denotes the Lebesgue measure of Ω_i. Therefore, with (4.38), we obtain

$$\nu(\Omega_i)\le\frac{L\beta}{i^2}. \tag{4.42}$$

We deduce from the previous equation that

$$\max\left(T-\frac{L\beta}{i^2},0\right)\le\nu(\Omega_i^c)\le T. \tag{4.43}$$

Moreover, with the second item of Lemma 4.3, we have, for all $i\in\mathbb{N}$,

$$\begin{aligned}\int_0^T\int_0^L\mathfrak{sat}_{\mathtt{loc}}(ay^n)y^ndtdx&=\int_{\Omega_i}\int_0^L\mathfrak{sat}_{\mathtt{loc}}(ay^n)y^ndtdx+\int_{\Omega_i^c}\int_0^L\mathfrak{sat}_{\mathtt{loc}}(ay^n)y^ndtdx\\&\ge\int_{\Omega_i^c}\int_0^L\mathfrak{sat}_{\mathtt{loc}}(ay^n)y^ndtdx\\&\ge\int_{\Omega_i^c}\int_0^Lak(i)(y^n)^2dtdx.\end{aligned} \tag{4.44}$$

Let $\lambda^n:=\|y^n\|_{L^2(0,T;L^2(0,L))}$ and $v^n(t,x)=\frac{y^n(t,x)}{\lambda^n}$. Notice that $\{\lambda^n\}_{n\in\mathbb{N}}$ is bounded, according to (4.36). Hence, there exists a subsequence, which we continue to denote by $\{\lambda^n\}_{n\in\mathbb{N}}$ such that

$$\lambda^n\to\lambda\ge0,\ \text{as } n\to+\infty.$$

Then, v^n fullfills

$$\begin{cases}v^n_t+v^n_x+v^n_{xxx}+\lambda^nv^nv^n_x+\frac{\mathfrak{sat}_{\mathtt{loc}}(a\lambda^nv^n)}{\lambda^n}=0,\\v^n(t,0)=v^n(t,L)=v^n_x(t,L)=0,\\\|v^n\|_{L^2(0,T;L^2(0,L))}=1,\end{cases} \tag{4.45}$$

and, due to (4.35),

$$\int_0^T |v_x^n(t,0)|^2 dt + 2\int_0^T \int_0^L \frac{\mathfrak{sat}_{\mathtt{loc}}(a\lambda^n v^n)}{\lambda^n} v^n dt dx \to 0 \text{ as } n \to +\infty.$$

Moreover, due to (4.44), we have, for all $i \in \mathbb{N}$,

$$\int_0^T |v_x^n(t,0)|^2 dt + 2\int_{\Omega_i^c} \int_0^L ak(i)(v^n)^2 dt dx \to 0 \text{ as } n \to +\infty. \tag{4.46}$$

Note that from Lemma 3.2,

$$\left\| \frac{\mathfrak{sat}_{\mathtt{loc}}(a\lambda^n v^n)}{\lambda^n} \right\|_{L^2(0,T;L^2(0,L))} \leq 3a_1\sqrt{L}\|v^n\|_{L^2(0,T;H^1(0,L)}, \tag{4.47}$$

and therefore the sequence $\{\frac{\mathfrak{sat}_{\mathtt{loc}}(a\lambda^n v^n)}{\lambda^n}\}_{n\in\mathbb{N}}$ is a subset of $L^2(0,T;L^2(0,L))$. In addition, $\{v^n v_x^n\}_{n\in\mathbb{N}}$ is a bounded sequence of $L^2(0,T;L^1(0,L))$. Note that $L^1(0,L) \subset L^2(0,L) \subset H^{-2}(0,L)$; thus $\{\frac{\mathfrak{sat}_{\mathtt{loc}}(a\lambda^n v^n)}{\lambda^n}\}_{n\in\mathbb{N}}$ and $\{v^n v_x^n\}_{n\in\mathbb{N}}$ are bounded sequences of $L^2(0,T;H^{-2}(0,L))$. Since $v_t^n = -v_x^n - v_{xxx}^n - \lambda^n v^n v_x^n - \frac{\mathfrak{sat}_{\mathtt{loc}}(a\lambda^n v^n)}{\lambda^n}$, we know that $\{v_t^n\}_{n\in\mathbb{N}}$ is a subset of $L^2(0,T;H^{-2}(0,L))$. Since $\{v^n\}_{n\in\mathbb{N}}$ is a subset of $L^2(0,T;H^1(0,L))$, we obtain from Lemma 4.6 that $\{v^n\}_{n\in\mathbb{N}}$ converges strongly to a function v in $L^2(0,T;L^2(0,L))$. Futhermore, with (4.46) and due to the nonnegativity of k, we have, for all $i \in \mathbb{N}$,

$$ak(i)v(t,x) = 0 \ \ \forall x \in [0,L], \forall t \in \Omega_i^c, \ \text{ and } \ v_x(t,0) = 0 \ \ \forall t \in (0,T). \tag{4.48}$$

Thus, since for all $i \in \mathbb{N}$, $k(i)$ is strictly positive, we have

$$v(t,x) = 0 \ \ \forall x \in \omega, \forall t \in \Omega_i^c, \ \text{ and } \ v_x(t,0) = 0 \ \ \forall t \in (0,T). \tag{4.49}$$

We obtain

$$v(t,x) = 0 \ \ \forall x \in \omega, \forall t \in \bigcup_{i\in\mathbb{N}} \Omega_i^c, \ \text{ and } \ v_x(t,0) = 0 \ \ \forall t \in (0,T). \tag{4.50}$$

Since, with (4.43), we know that $\nu\left(\bigcup_{i\in\mathbb{N}} \Omega_i^c\right) = T$, we get that, for almost every $t \in [0,T]$,

$$v(t,x) = 0 \ \ \forall x \in \omega \quad \text{and} \ \ v_x(t,0) = 0. \tag{4.51}$$

We obtain that v fullfills

$$\begin{cases} v_t + v_x + v_{xxx} + \lambda v v_x = 0, \\ v(t,0) = v(t,L) = v_x(t,L) = 0, \ , \ \|v\|_{L^2(0,T;L^2(0,L))} = 1. \end{cases} \tag{4.52}$$

Thus v is a solution to a KdV equation. In particular, it belongs to $\mathcal{B}(T)$ and is consequently in $C(0,T;L^2(0,L))$. Therefore, (4.51) becomes

$$v(t,x) = 0 \ \ \forall x \in \omega, \forall t \in [0,T], \ \text{ and } \ v_x(t,0) = 0 \ \ \forall t \in (0,T). \tag{4.53}$$

We are in the same situation as (4.29). Therefore we obtain once again a contradiction. We can conclude that Claim 2 is true. It concludes the proof of Lemma 4.2 when $\mathfrak{sat} = \mathfrak{sat}_{\mathtt{loc}}$ and completes the proof of Proposition 4.2.

Remark 4.8. Since the strategy followed in the last section is to argue by contradiction, we cannot estimate the exponential rate μ. However, such a proof allows us to prove the local exponential stability of the solution whatever the saturation $\mathfrak{sat}$ is.

4.4. Proof of Theorem 2.2. We are now in position to prove Theorem 2.2, following [31]. By Proposition 4.2, there exists $\mu^\star > 0$ such that if

$$\|\tilde{y}_0\|_{L^2(0,L)} \leq 1, \tag{4.54}$$

then the corresponding solution $\tilde{y}$ to (2.3) satisfies

$$\|\tilde{y}(t,.)\|_{L^2(0,L)} \leq K_1 \|\tilde{y}_0\|_{L^2(0,L)} e^{-\mu^\star t} \qquad \forall t \geq 0 \tag{4.55}$$

for some constants $K_1 \geq 1$ which depends only on $\|\tilde{y}_0\|_{L^2(0,L)}$. In addition, for a given $r > 0$, there exist two constants $K_r > 0$ and $\mu_r > 0$ such that if $\|y_0\|_{L^2(0,L)} \leq r$, then any mild solution y to (2.3) satisfies

$$\|y(t,.)\|_{L^2(0,L)} \leq K_r \|y_0\|_{L^2(0,L)} e^{-\mu_r t} \qquad \forall t \geq 0. \tag{4.56}$$

Consequently, setting $T_r := \mu_r^{-1} \ln(rK_r)$, we have

$$\|y_0\|_{L^2(0,L)} \leq r \Rightarrow \|y(T_r,.)\|_{L^2(0,L)} \leq 1.$$

Therefore, using (4.55), we obtain

$$\begin{array}{rcll} \|y(t,.)\|_{L^2(0,L)} & \leq & K_1 \|y(T_r,.)\|_{L^2(0,L)} e^{-\mu^\star (t-T_r)} & \forall t \geq T_r, \\ & \leq & K_1 K_r \|y_0\|_{L^2(0,L)} e^{\mu^\star T_r} e^{-\mu^\star t} & \forall t \geq 0. \end{array} \tag{4.57}$$

Thus it concludes the proof of Theorem 2.2.

Remark 4.9. As was noticed in Remark 3.10, the same result as Theorem 2.2 can be obtained for (3.51) following the strategy of [31]. Note that in [31], a stabilization in $H^3(0,L)$ is obtained. The authors used a strategy similar to the one described in Remark 3.10. Hence, it seems harder to obtain such a result for (3.53), since the saturation introduces some nonsmoothness.

5. Simulations. In this section we provide some numerical simulations showing the effectiveness of our control design. In order to discretize our KdV equation, we use a finite difference scheme inspired by [27]. The final time is denoted T_{final}. We choose $(N_x + 1)$ points to build a uniform spatial discretization of the interval $[0, L]$ and $(N_t + 1)$ points to build a uniform time discretization of the interval $[0, T_{final}]$. We pick a space step defined by $dx = L/N_x$ and a time step defined by $dt = T_{final}/N_t$. We approximate the solution with the following notation $y(t,x) \approx Y_j^i$, where i denotes the time and j the space discrete variables.

Some used approximations of the derivative are given by

$$\mathcal{D}_- y = \frac{Y_j^i - Y_{j-1}^i}{dx} \tag{5.1}$$

and

$$\mathcal{D}_+ y = \frac{Y_{j+1}^i - Y_j^i}{dx}. \tag{5.2}$$

As in [27], we choose the numerical scheme $y_x(t,x) \approx \frac{1}{2}(\mathcal{D}_+ + \mathcal{D}_-)(Y_j^i) := \mathcal{D}(Y_j^i)$ and $y_t(t,x) \approx \frac{Y_j^{i+1} - Y_j^i}{dt}$. For the other differentiation operator, we use $y_{xxx}(t,x) \approx \mathcal{D}_+ \mathcal{D}_+ \mathcal{D}_-(Y_j^i)$.

Let us introduce a matrix notation. Let us consider the matrices $D_-, D_+, D \in \mathbb{R}^{N_x \times N_x}$ given by

$$D_- = \frac{1}{dx}\begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \; D_+ = \frac{1}{dx}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 1 \\ 0 & \dots & \dots & 0 & -1 \end{bmatrix}, \tag{5.3}$$

$$D := \frac{1}{2}(D_+ + D_-) \tag{5.4}$$

and let us define $\mathcal{A} = D_+D_+D_- + D$ and $\mathcal{C} = \mathcal{A} + dtI$, where I is the identity matrix in $\mathcal{M}_{N_x \times N_x}(\mathbb{R})$. Note that we choose this forward difference approximation in order to obtain a positive definite matrix $\mathcal{C}$.

Moreover, for each discrete time i, we denote $Y^i := \begin{bmatrix} Y_1^i & Y_2^i & \dots & Y_{N_x+1}^i \end{bmatrix}^\top$.

Thus, inspired by [27], we consider a completely implicit numerical scheme for the approximation of the nonlinear problem (2.3) which reads as follows:

$$\begin{cases} \dfrac{Y_j^{i+1} - Y_j^i}{dt} + (\mathcal{A}Y^{i+1})_j + \frac{1}{2}\left(D[(Y^{i+1})^2]\right)_j + \mathfrak{sat}(a_\delta Y_j^{i+1}) = 0, \quad j = 1, \dots N_x, \\ Y_1^i = Y_{N_x+1}^i = Y_{N_x}^i = 0, \\ Y^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} y_0(x)dx, \end{cases} \tag{5.5}$$

where $x_{j+\frac{1}{2}} = \left(j + \frac{1}{2}dx\right)$, $x_j = jdx$, and Y^1 denotes the discretized version of the initial condition $y_0(x)$. Note that a_δ is the approximation of the damping function $a = a(x)$ and is given by $a_\delta = (a_j)_{J=1}^{N_x} \in \mathbb{R}^{N_x}$, where each component a_j is defined by $a_j := \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} a(x)dx$.

Since we have the nonlinearities yy_x and $\mathfrak{sat}(ay)$, we use an iterative Newton fixed-point method to solve the nonlinear system

$$\mathcal{C}Y^{i+1} = Y^i - dt\frac{1}{2}D(Y^{i+1})^2 - dt\mathfrak{sat}\left(a_\delta Y^{i+1}\right).$$

With $N_{iter} = 5$, which denotes the number of iterations of the fixed-point method, we get good approximations of the solutions. Note that for sufficiently large N_{iter} the solutions can be approximated with this fixed-point method.

Given Y^1 satisfying (5.5), the following is the structure of the algorithm used in our simulations.

For $i = 1 : N_t$

- $Y_1^i = Y_{N_x}^i = Y_{N_x+1}^i = 0$;
- Setting $J(1) = Y^i$, for all $k \in \{1, \dots, N_{iter}\}$, solve

$$J(k+1) = C^{-1}\left(Y^i - dt\frac{1}{2}D(J(k))^2 - dt\mathfrak{sat}(a_\delta J(k))\right)$$

Set $Y^{i+1} = J(N_{iter})$

end

In order to illustrate our theoretical results, we perform some simulations with $L = 2\pi$, for which we know that the linearized KdV equation is not asymptotically stable. To be more specific, letting $y_0(x) = 1 - \cos(x)$ and $f = 0$, it holds that the energy $\|y\|^2_{L^2(0,L)}$ of the linearized equation (1.4) remains constant for all $t \geq 0$. Let us perform a simulation of (2.3) with these parameters.

We first simulate our system in the case where the damping is not localized. We use the saturation function $\mathfrak{sat}_2$. Given $a_0 = 1$, $T_{final} = 6$, and $L = 2\pi$, Figure 2 shows the solution to (1.2), denoted by y_w, with the unsaturated control $f = a_0 y_w$ and starting from y_0. Figure 3 illustrates the simulated solution with the same initial condition and a saturated control $f = \mathfrak{sat}_2(a_0 y)$, where $u_0 = 0.5$. Figure 4 gives the evolution of the control with respect to the time and the space. We check in Figures 2

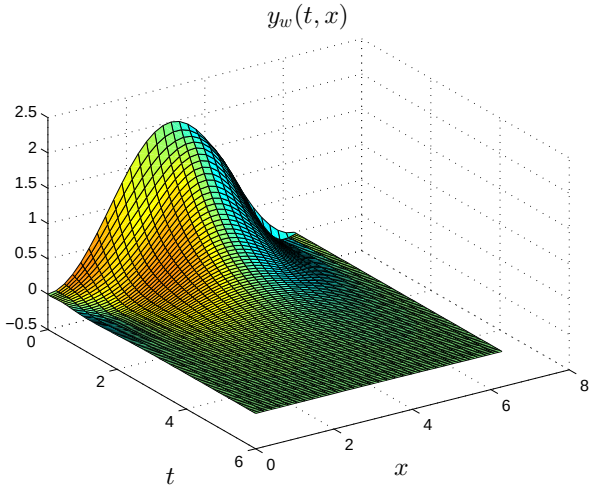

FIG. 2. *Solution $y_w(t,x)$ with the control $f = a_0 y_w$, where $\omega = [0, L]$.*

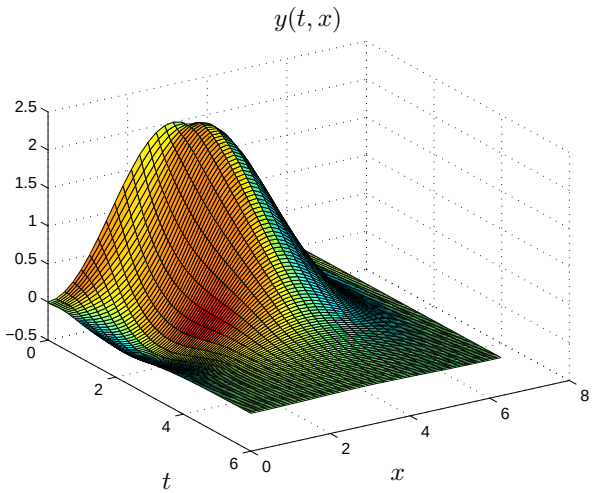

FIG. 3. *Solution $y(t,x)$ with the control $f = \mathfrak{sat}_2(a_0 y)$, where $\omega = [0, L]$, $u_0 = 0.5$.*

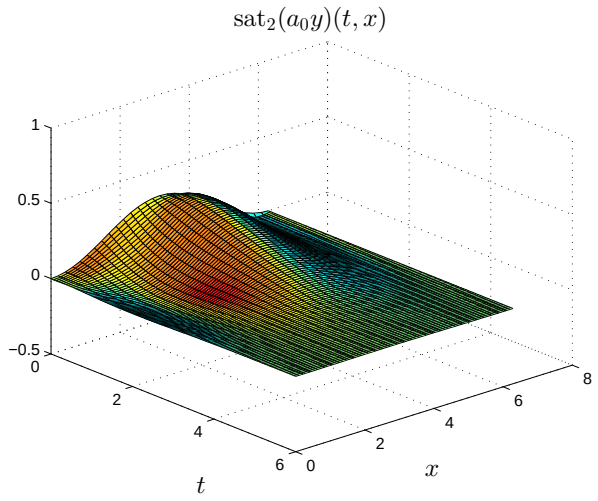

FIG. 4. *Control* $f = \mathfrak{sat}_2(a_0 y)(t,x)$, *where* $\omega = [0, L]$, $u_0 = 0.5$.

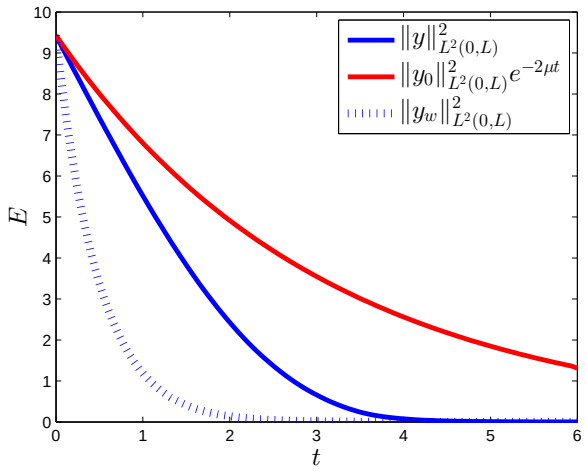

FIG. 5. *Blue: Time evolution of the energy function* $\|y\|^2_{L^2(0,L)}$ *with a saturation* $u_0 = 0.5$ *and* $a_0 = 1$. *Red: Time evolution of the theoritical energy* $\|y_0\|^2_{L^2(0,L)} e^{-2\mu t}$. *Dotted line: Time evolution of the solution without saturation* y_w *and* $a_0 = 1$.

and 3 that the solution to (2.3) converges to 0 with the unsaturated and the saturated controls as proven in Theorem 2.2.

The evolution of the L^2-energy of the solution in these two cases is given by Figure 5. With $\|y_0\|_{L^2(0,L)} := 3.07$ and the values of u_0, a_0, and a_1, the value μ is computed numerically following the formula (2.6) given in Theorem 2.2. It is equal to $\mu = 0.3257$. We deduce from the second point of Theorem 2.2 that the energy function $\|y\|^2_{L^2(0,L)}$ converges exponentially to 0 with an explicit decay rate given by μ as stated in Theorem 2.2.

We now focus on the case where the damping is localized. We close the loop with the saturated controller $f = \mathfrak{sat}_{\mathtt{loc}}(ay)$, where a is defined by $a(x) = a_0 = 1$, for all $x \in \omega := \left[\frac{1}{3}L, \frac{2}{3}L\right]$.

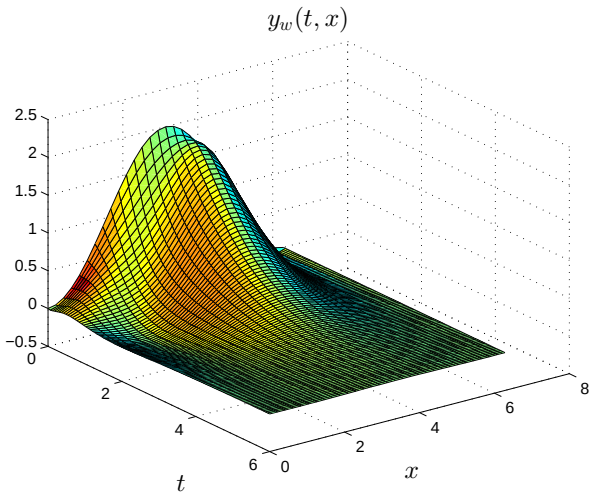

FIG. 6. *Solution $y_w(t,x)$ with a localized feedback law without saturation.*

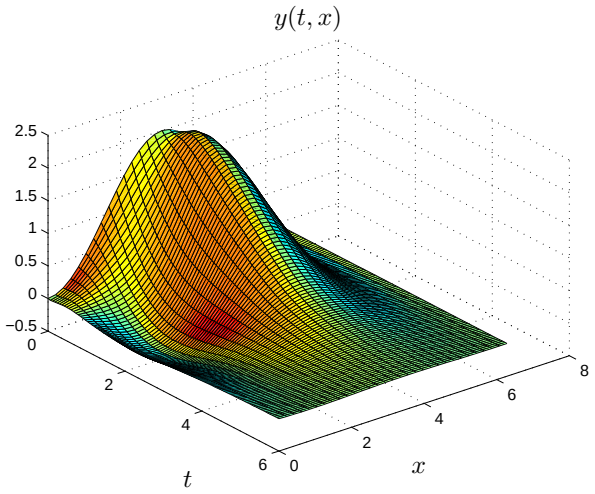

FIG. 7. *Solution $y(t,x)$ with a localized feedback law saturated; $u_0 = 0.5$.*

Given $T_{final} = 6$, Figure 6 shows the simulated solution of (1.2), denoted by y_w, with a localized control that is not saturated and starting from y_0. Figure 7 illustrates the simulated solution to (2.3) with the same initial condition, but with a localized saturated control whose saturation level is given by $u_0 = 0.5$. We check, in Figures 6 and 7, that the mild solution to (2.3) converges to 0 as stated in Theorem 2.2. Moreover, Figure 8 gives the evolution of the control with respect to the time and the space.

The evolution of the L^2-energy of the solution in these two last cases is given by Figure 9. We can see that the energy function $\|y\|^2_{L^2(0,L)}$ converges exponentially to 0 as stated in Proposition 4.2. However, contrary to the case $\mathfrak{sat} = \mathfrak{sat}_2$ and $\omega = [0, L]$, we cannot have an estimation of the decay rate since our proof is based on a contradiction argument.

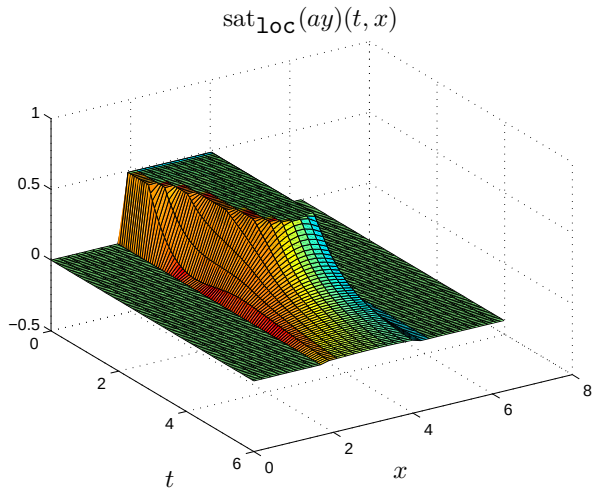

FIG. 8. *Control* $f = \mathfrak{sat}_{loc}(ay)(t,x)$, *where* $\omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$, $u_0 = 0.5$.

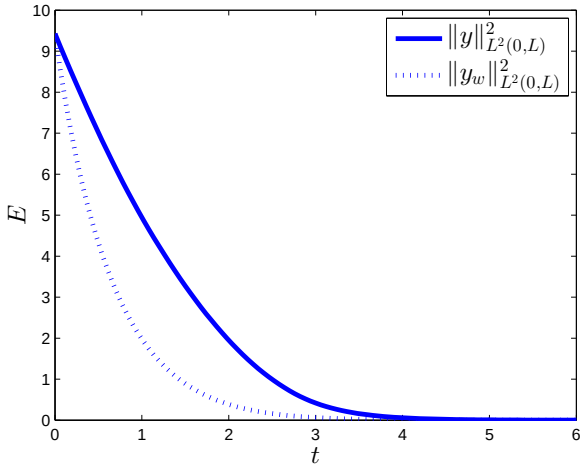

FIG. 9. *Blue: Time evolution of the energy function* $\|y\|^2_{L^2(0,L)}$ *with a saturation* $u_0 = 0.5$, $a_0 = 1$, *and* $\omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$. *Dotted line: Time evolution of the solution without saturation* y_w *with* $a_0 = 1$ *and* $\omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$.

6. Conclusion. In this paper, we have studied the well-posedness and the asymptotic stability of a KdV equation with saturated distributed controls. The well-posedness issue has been tackled by using the Banach fixed-point theorem. The stability has been studied with two different methods: in the case where the control acts on all the domain saturated with $\mathfrak{sat}_2$, we used a sector condition and Lyapunov theory for infinite dimensional systems; in the case where the control acts only on a part of the domain saturated with either $\mathfrak{sat}_2$ or $\mathfrak{sat}_{\mathtt{loc}}$, we argued by contradiction. We illustrated our results on some simulations, which showed that the smaller the saturation level, the slower the convergence to zero.

To conclude, let us state some questions arising in this context:

1. Can a saturated localized damping stabilize in $H^3(0,L)$ a generalized KdV equation, as done in the unsaturated case in [31] and [19]?
2. Is it possible to saturate other damping terms, for instance, the one suggested in [25] and used in [23] which dissipates the H^{-1}-norm in the unsaturated case?
3. Some boundary controls have been already designed in [4], [11], [38], or [5]. By saturating these controllers, are the corresponding equations still stable?
4. Can a constraint other than the saturation can be considered, for instance, the backlash studied in [40] or the quantization [14]?
5. Can we apply the same method for other nonlinear partial differential equations, for instance, the Kuramoto–Sivashinsky equation [2], [6]?

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