Time-delays in Physics:

From advection to time-delay systems

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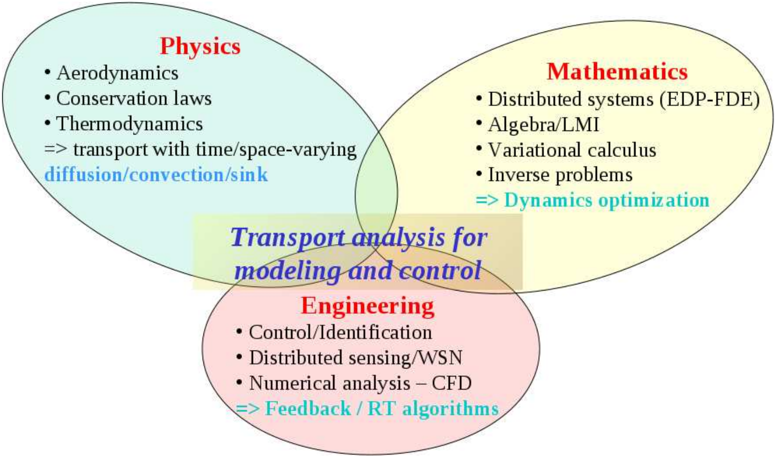
34th Int. Sum. School on Auto. Control, Grenoble, July 5, 2013.

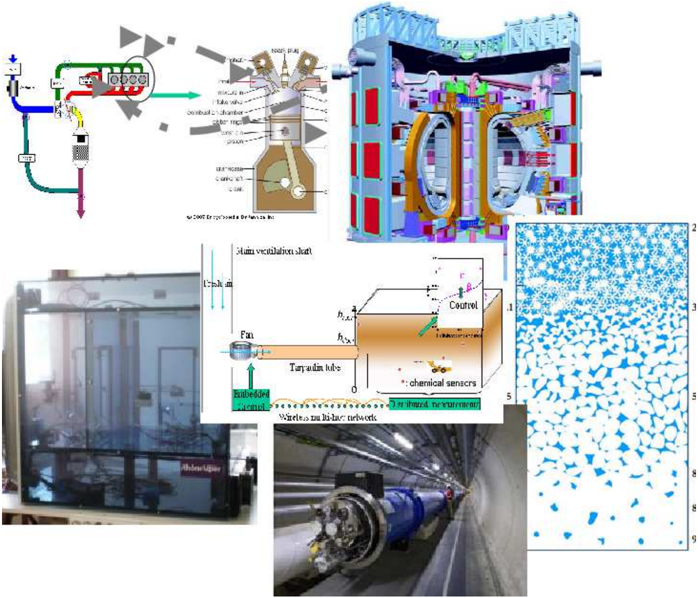
Analytical Framework



$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \mathcal{A}(\zeta, \mathbf{x}, t) + \nabla \mathcal{D}(\nabla \cdot \zeta, \zeta, \mathbf{x}, t) = \mathcal{S}_o(u, \mathbf{x}, t) - \mathcal{S}_i(\zeta, \mathbf{x}, t)$$

$$y = g(\zeta, \mathbf{x}, t)$$

Applications



Conservation laws

- Model from physics: subatomic, atomics or molecular, microscopic, macroscopic, astronomical scale
- I.e. fluid dynamics = study of the interactive motion and behavior of a large number of elements
- System of interacting elements as a continuum
- Consider an elementary volume that contains a sufficiently large number of molecules with well defined mean velocity and mean kinetic energy
- At each point we can thus infer, e.g. velocity, temperature, pressure, entropy etc.

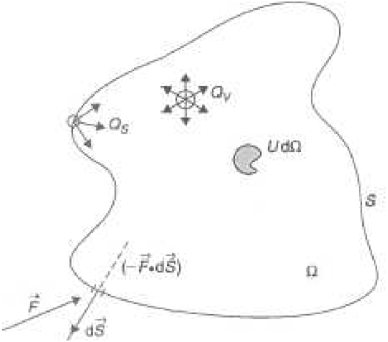
General form of a conservation law

- Conservation: the variation of a conserved (intensive) flow quantity U in a given volume results from internal sources and a quantity, the *flux*, crossing the boundary
- Fluxes and sources depend on space-time coordinates, + on the fluid motion
- Not all flow quantities obey conservation laws. Fluid flows fully described by the conservation of
  1. mass
  2. momentum (3-D vector)
  3. energy

  $\Rightarrow$ 5 equations
- Other quantities can be used but will not take the form of a conservation law

Scalar conservation law [Hirsch 2007]

Consider:
- a scalar quantity per unit volume U,
- an arbitrary volume Ω fixed in space (control volume) bounded by
- a closed surface S (control surface) crossed by the fluid flow



- Total amount of U inside Ω: $\int_\Omega Ud\Omega$ with variation per unit time $\frac{\partial}{\partial t}\int_\Omega Ud\Omega$
- Flux = amount of U crossing S per unit time: $F_n dS = \vec{F}\cdot d\vec{S}$ with $d\vec{S}$ outward normal, and net total contribution $-\oint_S \vec{F}\cdot d\vec{S}$ ($\vec{F} > 0$ when entering the domain)
- Contribution of volume and surface sources: $\int_\Omega Q_V d\Omega + \oint_S \vec{Q}_S \cdot d\vec{S}$

Scalar conservation law (2)

Provides the integral conservation form for quantity U:

$$\frac{\partial}{\partial t}\int_\Omega U\, d\Omega + \oint_S \vec{F}\cdot d\vec{S} = \int_\Omega Q_V\, d\Omega + \oint_S \vec{Q}_S \cdot d\vec{S}$$

- valid $\forall$ fixed S and Ω, at any point in the flow domain
- internal variation of U depends only of fluxes through S, not inside
- no derivative/gradient of F: may be discontinuous and admit shock waves

$\Rightarrow$ relate to conservative numerical scheme at the discrete level (e.g. conserve mass)

Differential form of a conservation law

Obtained using Gauss' theorem $\oint_S \vec{F} \cdot d\vec{S} = \int_\Omega \vec{\nabla} \cdot \vec{F}\, d\Omega$ as:

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{F} = Q_V + \vec{\nabla} \cdot \vec{Q}_S \Leftrightarrow \frac{\partial U}{\partial t} + \vec{\nabla} \cdot (\vec{F} - \vec{Q}_S) = Q_V$$

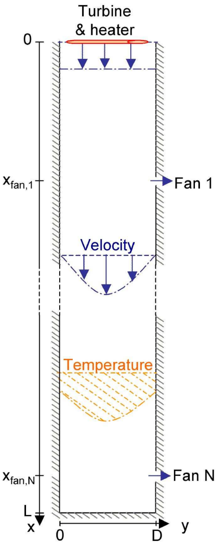
- the effective flux $(\vec{F} - \vec{Q}_S)$ appears exclusively in the gradient operator $\Rightarrow$ way to recognize conservation laws
- more restrictive than the integral form as the flux has to be differentiable (excludes shocks)
- fluxes and source definition provided by the quantity U considered

Convection-diffusion form of a convection law

Flux = convective transport + molecular agitation (even at rest)

- Convective flux:
  - amount of U carried away or transported by the flow (velocity $\vec{v}$): $\vec{F}_C = \vec{v}U$
  - for fluid density $U = \rho$, local flux through $d\vec{S}$ is the local mass flow rate: $\vec{F}_C \cdot d\vec{S} = \rho\vec{v} \cdot d\vec{S} = d\vec{m}$ (e.g. kg/s)
  - for $U = \rho u$ (u the quantity per unit mass, e.g. concentration), $\vec{F}_C \cdot d\vec{S} = \rho u\vec{v} \cdot d\vec{S} = u\, d\vec{m}$
- Diffusive flux:
  - macroscopic effect of molecular thermal agitation
  - from high to low concentration, in all directions, proportional to the concentration difference
  - Fick's law: $\vec{F}_D = -\kappa\rho\vec{\nabla}u$, where κ is the diffusion coefficient (m^2/s)
- Provides the transport equation:
  $$\frac{\partial \rho u}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}u) = \vec{\nabla} \cdot (\kappa\rho\vec{\nabla}u) + Q_V + \vec{\nabla} \cdot \vec{Q}_S$$
  $\Rightarrow$ Backbone of all mathematical modeling of fluid flow phenomena

Euler and Navier-Stokes equations



- From the conservation of mass, momentum and energy:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\vec{v}\\ \rho E\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\vec{v}\\ \rho\vec{v}^T\otimes\vec{v}+p\mathbf{I}-\tau\\ \rho\vec{v}H-\tau\cdot\vec{v}-k\nabla T\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \dot{q}\end{bmatrix},$$

with shear stress (Navier-Stokes only)

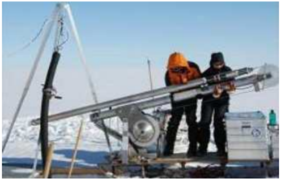
$$\begin{bmatrix}\tau_{xx}\\ \tau_{xy}\\ \tau_{yy}\end{bmatrix}=\begin{bmatrix}\lambda\\ \mu\\ \lambda\end{bmatrix}(\nabla\cdot\vec{v})+2\mu\begin{bmatrix}u_x\\ 0\\ v_y\end{bmatrix}$$

and viscosity [*Stokes & Sutherland*]

$$\lambda=-\frac{2}{3}\mu \quad and \quad \frac{\mu}{\mu_{sl}}=\left(\frac{T}{T_{sl}}\right)^{3/2}\frac{T_{sl}+110}{T+110}.$$

- Discrete boundary conditions (potential numerical instabilities).

Motivating example: Firn inverse modeling and polutant emissions tracking

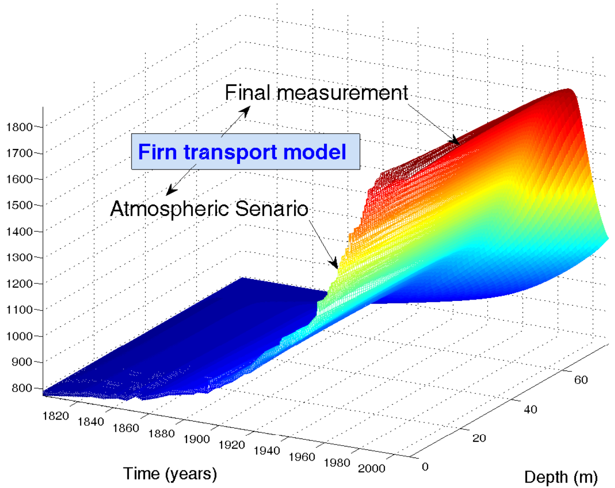


Trace gas measurements in interstitial air from polar firn

- allow to reconstruct their atmospheric concentration time trends over the last 50 to 100 years
- provides a unique way to reconstruct the recent anthropogenic impact on atmospheric composition
- extends to hundreds of thousands of years in ice (e.g. Vostok $\approx$ 800 000 y)

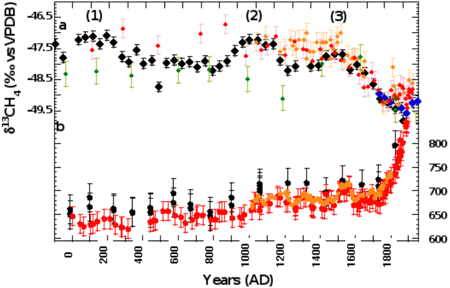
Converting depth-concentration profiles in firn into atmospheric concentration histories requires models of trace gas transport in firn

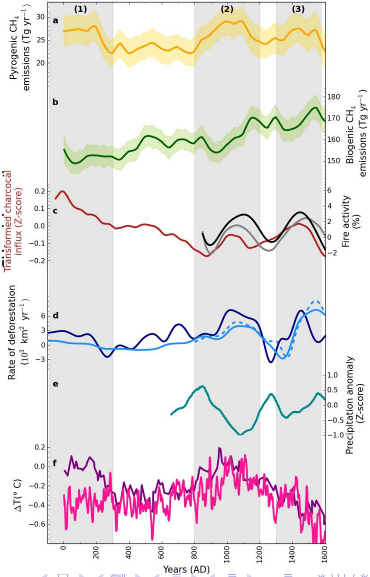
I.e. CH_4 transport at NEEM (Greenland) in firn



I.e. CH_4 from NEEM and EUROCORE (Greenland) in ice

Natural and anthropogenic variations in methane sources during the past two millennia [Sapart et al., Nature 2012]





Reflect changes in both pyrogenic and biogenic sources.

Correlated with both natural climate variability and changes in human population, land-use and important events in history: decline of the Roman Empire & Han dynasty, and Medieval period.

Solving the air continuity for bubbles in polar firns and ice cores [Witrant, Martinerie et al., ACP 2012, IFAC SSSC 2013]

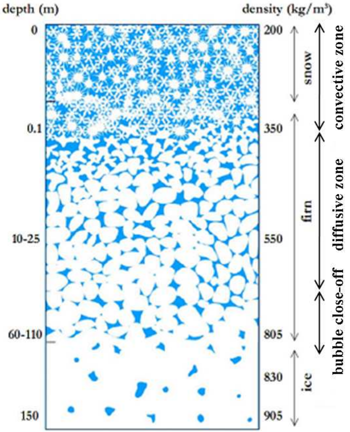
From poromechanics, firn = system composed of the ice lattice, gas connected to the surface (open pores) and gas trapped in bubbles (closed pores). Air transport is driven by:

$$\frac{\partial[\rho_{ice}(1-\epsilon)]}{\partial t} + \nabla[\rho_{ice}(1-\epsilon)\vec{v}] = 0$$

$$\frac{\partial[\rho^o_{gas} f]}{\partial t} + \nabla[\rho^o_{gas} f(\vec{v} + \vec{w}_{gas})] = -\vec{r}^{o\to c}$$

$$\frac{\partial[\rho^c_{gas}(\epsilon - f)]}{\partial t} + \nabla[\rho^c_{gas}(\epsilon - f)\vec{v}] = \vec{r}^{o\to c}$$

with appropriate boundary and initial conditions.



Scheme adapted from [Sowers et al.'92, Lourantou'08].

Firn example: from distributed to lumbed dynamics

- Defining $q = \rho^c_{gas}(\epsilon - f)$ and considering the 1-D case, we have to solve

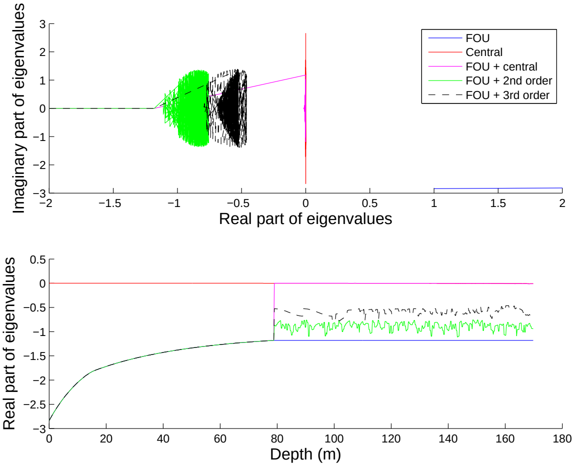
$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial z}[qv] = r^{o\to c}$$

- Approximate $\partial[qv]/\partial z$, i.e. on uniform mesh:
  - backward difference: $(u_z)_i = \frac{u_i - u_{i-1}}{\Delta z} + \frac{\Delta z}{2}(u_{zz})_i$
  - central difference: $(u_z)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta z_i} - \frac{\Delta z^2}{6}(u_{zzz})_i$
  - other second order: $(u_z)_i = \frac{u_{i+1} + 3u_i - 5u_{i-1} + u_{i-2}}{4\Delta z_i} + \frac{\Delta z^2}{12}(u_{zzz})_i - \frac{\Delta z^3}{8}(u_{zzzz})_i$
  - third order: $(u_z)_i = \frac{2u_{i+1} + 3u_i - 6u_{i-1} + u_{i-2}}{6\Delta z_i} - \frac{\Delta z^3}{12}(u_{zzzz})_i$
- Provides the computable lumped model:

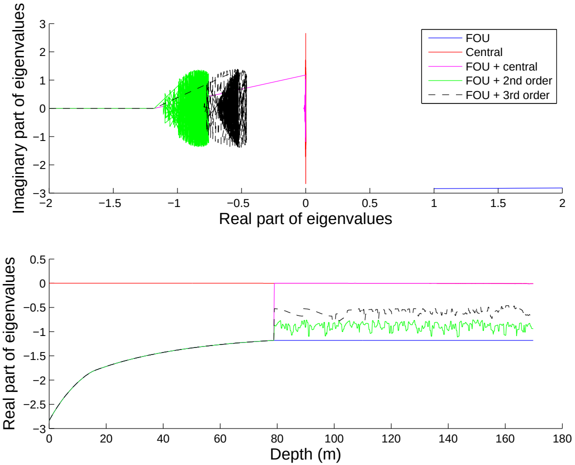
$$\frac{dq}{dt} = Aq + r^{o\to c}$$

- The choice of the discretization scheme directly affects the definition of A and its eigenvalues distribution: need to check stability and precision!

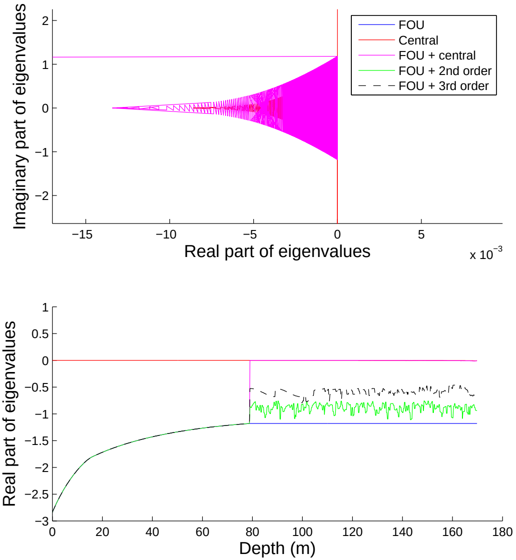
e.g. eig(A) for CH_4 at NEEM with $dt = 1$ month



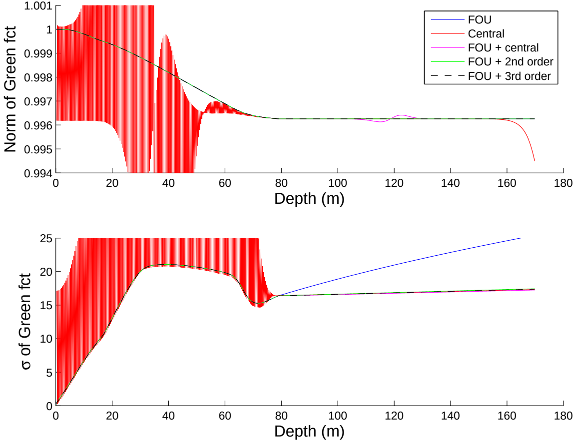
e.g. eig(A) for CH_4 at NEEM with $dt \approx 1$ week



e.g. eig(A) for CH_4 at NEEM with $dt \approx 1$ week, zoom



e.g. Impulse response (Green's function) for CH_4 at NEEM with $dt = 1\,\text{month}$



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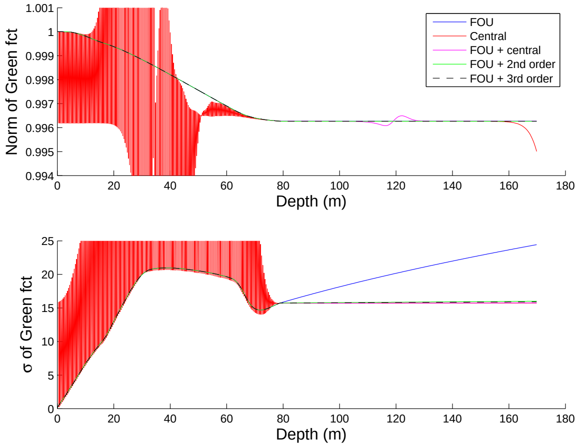


Conclusions on the firn example

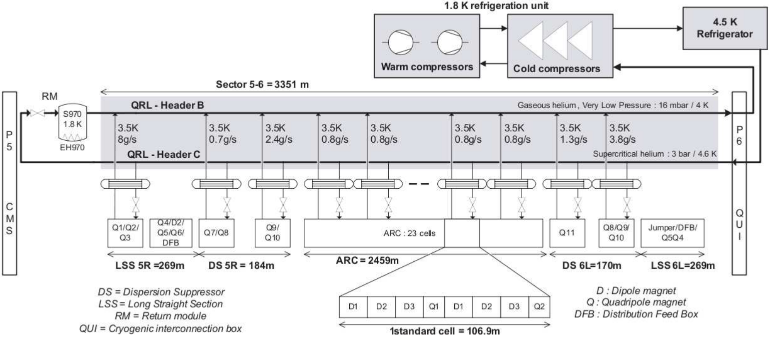
- Solution obtained from classical CFD (computational fluid dynamics) analysis
- Purely advective below the bubbles closure → physically, we want to reflect pure information transport, i.e. delay
- Dynamics represented as an I/O map with the Green function
- The high precision + large sampling time objective is difficult to meet with a direct numerical approach
- Need to extend the result to variable advection speed

⇒ An analytical approach using time-delays (kernel) for the I/O map could solve the problem

CERN example: combined advection and diffusion in travelling wave modeling

Modeling of the very low pressure helium flow in the LHC Cryogenic Distribution Line (QRL) after a quench [Bradu, Gayeta, Niculescu and Witrant, Cryogenics 2010]

LHC sector 5-6 with the main cooling loops for the superconducting magnets:



Method

Assumptions:

- model using Euler equation

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\vec{v}\\ \rho E\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\vec{v}\\ \rho\vec{v}^T\otimes\vec{v}+p\mathbf{I}\\ \rho\vec{v}H-k\nabla T\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \dot{q}\end{bmatrix},$$

- flux according to the x direction only (the main flow direction) : $V=V_x$ and $M=\rho\cdot V_x$
- straight line: neglect QRL curvature (rad. of curv. 4.3 km)
- neglect kinetic component:
  $\rho\cdot|\vec{V}|^2 << P \Rightarrow \rho\cdot\vec{V}^T\otimes\vec{V}+P\cdot I\approx P\cdot I$

Euler equation expressed as a quasilinear 1D hyperbolic PDE:

$$\frac{\partial X(x,t)}{\partial t}+F(X)\cdot\frac{\partial X(x,t)}{\partial x}=Q(x,t)$$

where $X=\begin{bmatrix}\rho & M & E\end{bmatrix}^T$ is the state vector, F is the Jacobian flux matrix and $Q=\begin{bmatrix}0 & 0 & q\end{bmatrix}^T$ is the source vector.

Method (2): space discretization

From the Jacobian (empirical formulation for the helium internal energy)

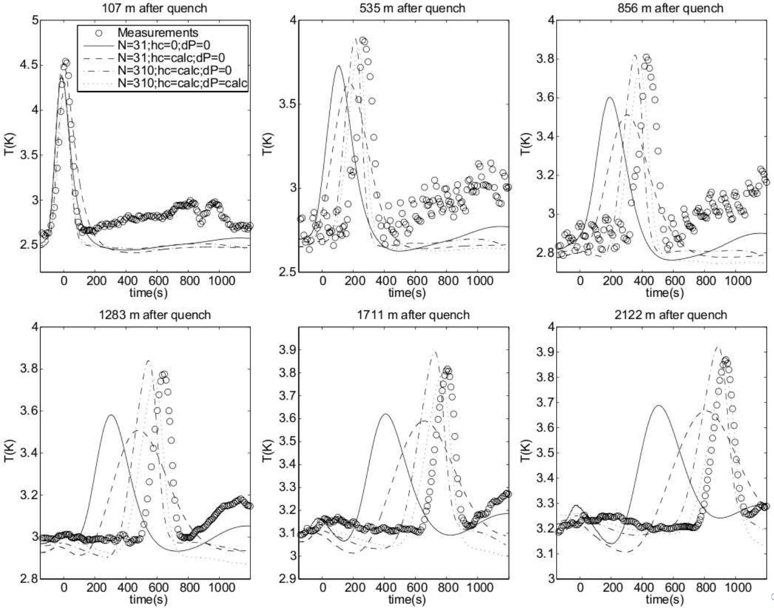
$$F = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)V^2}{2} - u_0\hat{\gamma} & (3-\gamma)V & \hat{\gamma} \\ \hat{\gamma}V^3 - \frac{\gamma V E}{\rho} & \frac{\gamma E}{\rho} - \hat{\gamma}(\frac{3V^2}{2} + u_0) & \gamma V \end{bmatrix}$$

we obtain the state-space matrices from:

$$\dot{X}_i(t) + \frac{A_i(X_i)}{\Delta x} X_i(t) + \frac{B_i(X_i)}{\Delta x} X_{i-1}(t) + \frac{C_i(X_i)}{\Delta x} X_{i+1}(t) = Q_i(t)$$

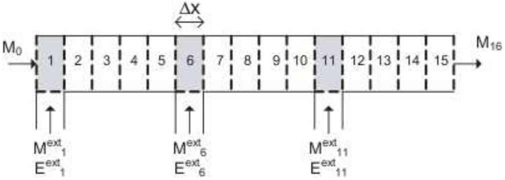
where i denotes the value at x_i and $X_i = \begin{bmatrix} \rho_i & M_i & E_i \end{bmatrix}$
+ add the interconnections with external inputs in Q_i

Temperature transport: Impact of convection heat, hydrostatic pressure and friction pressure drops



Conclusions on the CERN example

- Quasilinear hyperbolic system with 3 states = 3 travelling waves, one in the opposite direction
- The diffusive effect is mostly numerical and should be avoided
- Essential role of mass flow rate & pressure gradient = advection (time-delay) if we consider the temperature transport
- Multiple interconnections = set of interconnected time-delay systems



Single-directional transport

Suppose that we want to control a unidirectional, mostly advective, process through the boundary: can a time-delay approach help?

Proposed strategy:

1. starting from Euler's equation, isolate the variable of interest
2. simplify the model to define an observer/estimation structure
3. use the estimated parameter for a lumped control-oriented model with transport as a delay
4. choose a stabilizing feedback

Volume-averaged model for pressure regulation

- From energy conservation in Euler & phys. hypoteses:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{M}{\rho}\cdot\left(1+\frac{R}{c_v}\right)p\right] + \frac{R}{c_v}\dot{q}$$

- Volume-averaged impact of momentum and density: $\bar{X}(t) \doteq \frac{1}{\mathcal{V}}\oint_{\mathcal{V}} X(v,t)dv$, for $X \doteq \{M, \rho\}$
- Energy losses = pressure losses (friction and exhausts), e.g. $\dot{q}(x,t)R/c_v = s(x,t) + r(t)p(x,t)$
- Leads to the PDE model with boundary (controlled) input:

$$\begin{cases} \tilde{p}_t = c(t)\tilde{p}_x + r(t)\tilde{p} + s(x,t), \\ \tilde{p}(0,\ t) = p_{in}(t),\ \tilde{p}(x,\ 0) = \tilde{p}_0(x) \end{cases}$$

$\Rightarrow$ Given distributed measurements, estimate transport coefficients and set feedback $p_{in}(t)$

Observer-based online parameter estimation [W, Marchand'08]

Theorem (parameter estimation for affine PDE): Consider the class of systems, affine in the parameter

$$\begin{cases} p_t = \mathcal{A}(p, p_x, p_{xx}, u, \vartheta)\vartheta \\ a_1 p_x(0,t) + a_2 p(0,t) = a_3 \\ a_4 p_x(L,t) + a_5 p(L,t) = a_6 \end{cases}$$

with distributed measurements of $p(x,t)$ and for which we want to estimate ϑ. Then

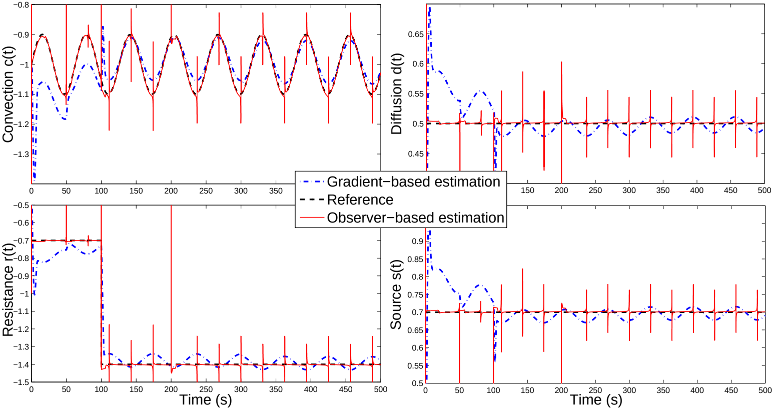
$$\|p(x,t) - \hat{p}(x,t)\|_2^2 = e^{-2(\gamma+\lambda)t}\|p(x,0) - \hat{p}(x,0)\|_2^2$$

if

$$\begin{cases} \hat{p}_t = \mathcal{A}(\hat{p}, \hat{p}_x, \hat{p}_{xx}, u, \hat{\vartheta})\hat{\vartheta} + \gamma(p - \hat{p}) \\ a_1 \hat{p}_x(0,t) + a_2 \hat{p}(0,t) = a_3 \\ a_4 \hat{p}_x(L,t) + a_5 \hat{p}(L,t) = a_6 \\ \dot{\hat{\vartheta}} = \mathcal{A}(\hat{p}, \hat{p}_x, \hat{p}_{xx}, u, \hat{\vartheta})^\dagger[p_t + \lambda(p - \hat{p})] \end{cases}$$

Example: comparison with gradient-descent algorithm

$$p_t = d(t)p_{xx} + c(t)p_x + r(t)p + s(t)p_{ext}(x,t)$$



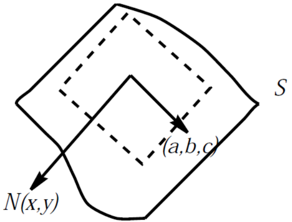
⇒ accurate results & decoupling, singularities when the gradients are zero (in pseudoinverse) - need to add a filter.

Method of characteristics, some background [J. Levandosky course[1]]

Solve linear, quasilinear and nonlinear first-order PDEs.
E.g. for the first order linear equation:

$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$
$$\Leftrightarrow \underbrace{[a(x,y), b(x,y), c(x,y)]}_{\mathcal{T}(x,y)} \cdot \underbrace{[u_x(x,y), u_y(x,y), -1]}_{\mathcal{N}(x,y)} = 0$$

- the surface $S \doteq \{x, y, u(x,y)\}$ has a normal vector $\mathcal{N}$ and is defined by the tangent plane $\mathcal{T}(x,y)$

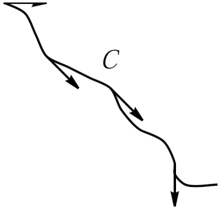


[1] http://www.stanford.edu/class/math220a/handouts/firstorder.pdf

Method of characteristics (2)

- construct a curve $\mathcal{C}$ (= integral or characteristic curve) parameterized by s s.t. it is tangent to $\mathcal{T}(x(s), y(s))$ at each point (x, y, z):

$$\frac{dx}{ds} = a(x(s), y(s)), \quad \frac{dy}{ds} = b(x(s), y(s)), \quad \frac{dz}{ds} = c(x(s), y(s))$$



$\Rightarrow$ Form the surface $S = \{x, y, z\}$ (integral surface) by solving the set of ODE

Characteristics for a time-delay model [W, Niculescu'10]

Consider the advective-resistive flow:

$$\zeta_t(x,t) + \bar{\mathcal{A}}_1(t)\zeta_x(x,t) = -\bar{\mathcal{S}}_{i,1}(t)\zeta(x,t)$$

with $\zeta(0,t) = u(t)$, $\zeta(x,0) = \psi(x)$. Applying the method of characteristics with the new independent variable θ as

$$\zeta(\theta) \doteq \zeta(x(\theta), t(\theta))$$

It follows that (solution including time axis)

$$\zeta(L,t) \doteq u(t-\theta_f)exp\left(-\int_0^{\theta_f} \bar{\mathcal{S}}_{i,1}(\eta)\, d\eta\right) \text{, with } L = \int_{t-\theta_f}^{t} \bar{\mathcal{A}}_1(\eta)\, d\eta$$

The line-average state $\bar{\zeta}(t) \doteq \int_0^L \zeta(\eta,t)\, d\eta$ is provided by the Delay Differential Equation

$$\frac{d}{dt}\bar{\zeta} = \bar{\mathcal{A}}_1(t)\left[u(t) - u(t-\theta_f)exp\left(-\int_0^{\theta_f} \bar{\mathcal{S}}_{i,1}(\eta)\, d\eta\right)\right] - \bar{\mathcal{S}}_{i,1}(t)\bar{\zeta}$$

Tracking feedback controller design

Control problem: design a feedback such that the average distributed pressure: $\bar{\zeta}(t) = \frac{1}{L}\int_0^L \zeta(x,t)dx$ tracks the reference $\bar{\zeta}_r(t)$.

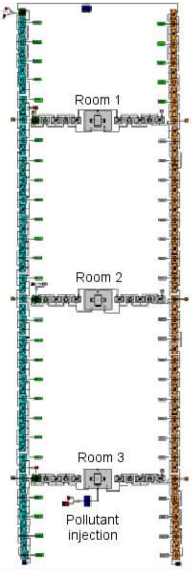
Achieved if (solving a robustified Cauchy system):

$$\dot{\bar{\zeta}}(t) - \dot{\bar{\zeta}}_r(t) + \lambda(\bar{\zeta}(t) - \bar{\zeta}_r(t)) = 0$$

$\rightarrow$ ensures exponential convergence $|\bar{\zeta}(t) - \bar{\zeta}_r| = |\bar{\zeta}(0) - \bar{\zeta}_r|e^{-\lambda t}$
Using the previous DDE and solving for $u(t)$, it follows that (e.g. with $\dot{\bar{\zeta}}_r = 0$):

$$\begin{aligned} \frac{d}{dt}\bar{\zeta} &= L\bar{\mathcal{A}}_1(t)\left[u(t) - u(t-\theta_f)e^{-\int_0^{\theta_f}\bar{\mathcal{S}}_{i,1}(\eta)d\eta}\right] - \bar{\mathcal{S}}_{i,1}(t)\bar{\zeta} \\ u(t) &= -\frac{L}{\bar{\mathcal{A}}_1(t)}\left[-\bar{\mathcal{S}}_{i,1}(t)\bar{\zeta}(t) + \lambda(\bar{\zeta}(t) - \bar{\zeta}_r)\right] + \zeta(L,t) \end{aligned}$$

Mining ventilation example: reference model



Simulator properties:

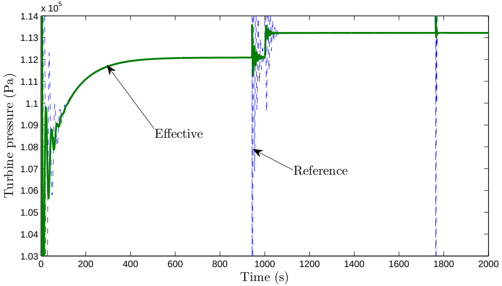
- ventilation shafts ≈ 28 control volumes (CV), 3 extraction levels
- regulation of the turbine and fans
- flows, pressures and temperatures measured in each CV
- Computation 34× faster than real-time

Case study:

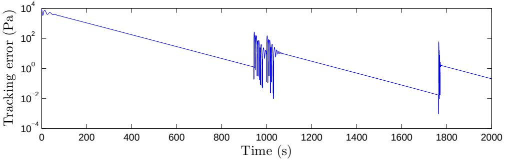
- 1^{st} level fan not used (natural airflow), 2^{nd} operated at 1000 s (150 rpm) and 3^{rd} runs continuously (200 rpm)
- CO pollution injected in 3^{rd} level
- measurement of flow speed, pressure, temperature and pollution at the surface and extraction levels

Feedback control results for mine ventilation

Reference and effective turbine output pressure:



Feedback tracking error:



$\Rightarrow$ Sensible to initial conditions and some numerical integration errors but exponential convergence verified!

Information transport

Physical models

Telegrapher's equation for homogeneous transmission line:

$$\begin{bmatrix} V_t \\ I_t \end{bmatrix} + \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} V_z \\ I_z \end{bmatrix} = 0$$

- The wave propagation is 2 ways and characterized by the line impedence $Z = \sqrt{L/C}$, propagation velocity $v = 1/\sqrt{LC}$ and time-delay $\tau = \sqrt{LC}$ (supposing unit length)
- The scattering variables (orthogonal) $S_+ = f(t + z/v)$ and $S_- = g(t - z/v)$, for some continuous functionals f and g, describe the incident and reflected waves, from:

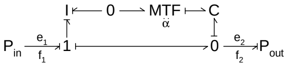
$$\frac{\partial S_+}{\partial t} - v\frac{\partial S_+}{\partial z} = 0, \quad \frac{\partial S_-}{\partial t} + v\frac{\partial S_-}{\partial z} = 0$$

Physical models (2)

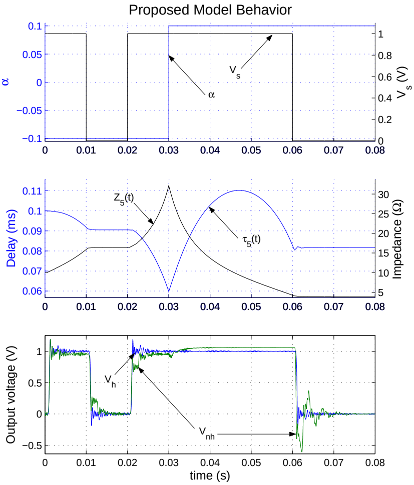
Consider the heterogeneous Telegrapher's equation [Witrant, van der Schaft, Stramigioli, Ph.D.'05].

$$\begin{bmatrix} V_t \\ I_t \end{bmatrix} + \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} V_z \\ I_z \end{bmatrix} = \alpha(t) \frac{VI}{2} \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}$$

where local L and C variations are captured with $\alpha(t)$ in the elementary cell [Ph.D.'05]:



Induce wave reflections and time-varying delays.



Communication models

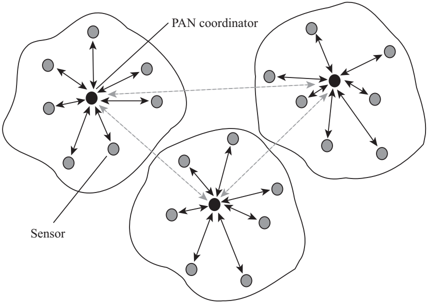
I.e. Fluid-flow model for the network [Misra et al. 2000, Hollot and Chait 2001]: TCP with proportional active queue management (AQM) set the window size W and queue length q variations as

$$\begin{aligned}
\frac{dW_i(t)}{dt} &= \frac{1}{R_i(t)} - \frac{W_i(t)}{2}\frac{W_i(t-R_i(t))}{R_i(t-R_i(t))}p_i(t), \\
\frac{dq(t)}{dt} &= -C_r + \sum_{i=1}^{N}\frac{W_i(t)}{R_i(t)}, \quad q(t_0) = q_0,
\end{aligned}$$

where $R_i(t) \doteq \dfrac{q(t)}{C_r} + T_{pi}$ is the round trip time, C_r the link capacity, $p_i(t) = K_p q(t - R_i(t))$ the packet discard function and T_{pi} the constant propagation delay. The average time-delay is $\tau_i = \frac{1}{2}R_i(t)$

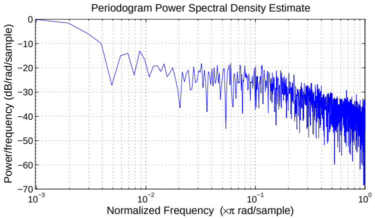
Wireless Sensor Networks

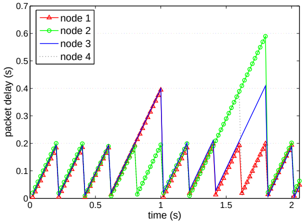
[Park, di Marco, Soldati, Fischione, Johansson'09...]



- IEEE 802.15.4, Markov chain model, network & control codesign
- Communication constraints = time-delay + packet loss

Delays characterization [W, Park and Johansson, Springer'10]

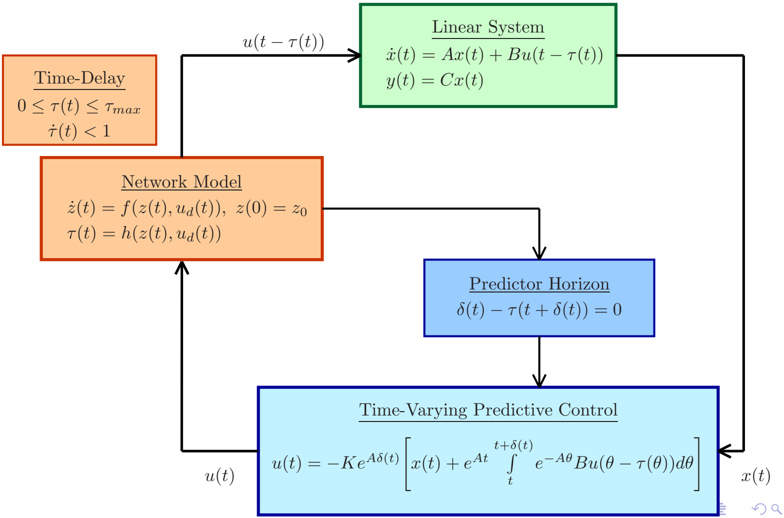




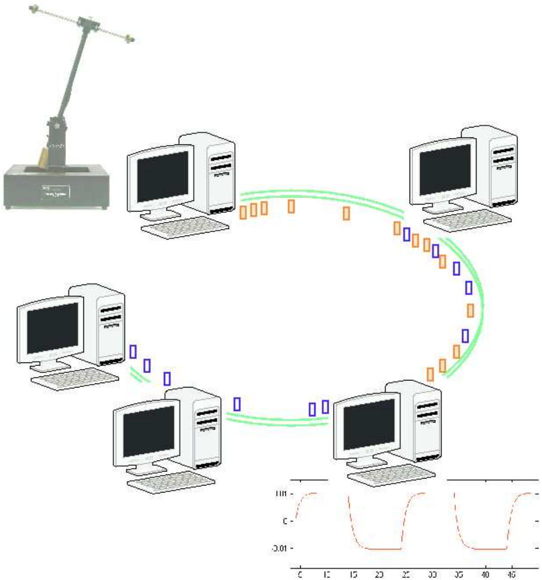
- Three-frequencies jitter & KUMSUM Kalman estimation
- Synchronous/async. cases
- Packet losses as time-delays

Feedback design

I.e. finite-spectrum assignment [Michiels, Ph.D.'02] with online adaptation of the horizon of a MPC feedback scheme with robust gain design [Witrant et al., Ph.D.'05, TAC'07]

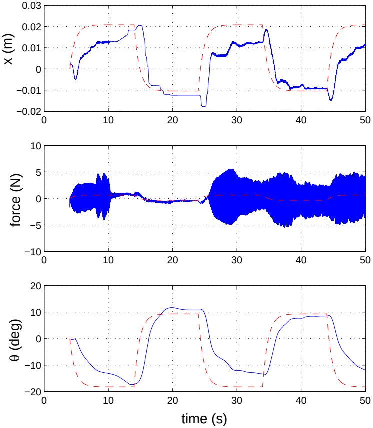


Example: control of an inverted pendulum over a network

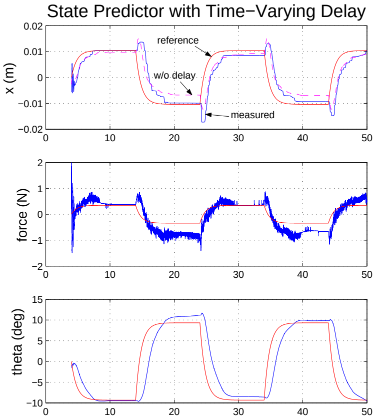


Experimental results for the inverted pendulum

Control over a network with 2 users (LQR gain design):



(a) Predictor with fixed horizon.

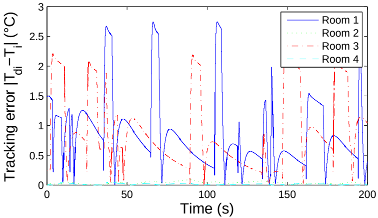


(b) Predictor with time-varying horizon.

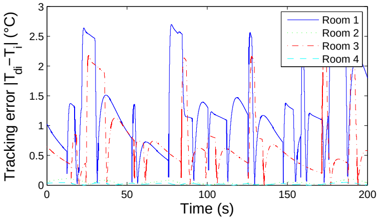
Example 2: room temperature control over multi-hop WSN in intelligent buildings [W, Mocanu and Sename, TdS'09]

$$\frac{dx}{dt} = (A_1 + A_2(u))x + (B_1 + B_2(u))u + B_w w + P\mathcal{P}(x - Ux) + s + \mathcal{H}(Y, x)$$

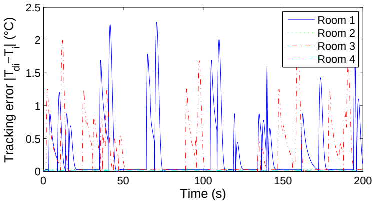
Simulation results [W, Di Marco, Park and Briat, IMA'10]



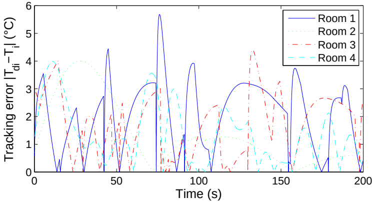
(a) Synchronous mode (Mixed-sensitivity H_∞)



(b) Asynchronous mode (Mixed-sensitivity H_∞)



(c) BRL with weights, post check for TD stability (Skelton et al., 1997)



(d) Delay constraint during the design (Seuret, 2009), scale $\times$ 2

Conclusions on 1-direction and information transport

- Direct equivalence between time-delays and advective transport for single-directional transport and homogenous (possibly time-varying) coefficients
- Equivalence obtained for two waves in homogeneous transmission lines from the scattering variables
- Time-delay with router feedback in communication networks
- Possible online addaptation of the controller's size according to the delay variations with the predictor architecture
- Strong impact of signal distorsion in WSN may call for robustness rather than time-delay compensation

Travelling waves modeling

The conservative form of Euler equations:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \vec{M} \\ E \end{bmatrix} + \vec{\nabla}\cdot\begin{bmatrix} \rho\cdot\vec{V} \\ \rho\cdot\vec{V}^T\otimes\vec{V}+P\cdot I \\ \rho\cdot\vec{V}\cdot\left(u+\frac{P}{\rho}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q \end{bmatrix}$$

writes in 1-D for a straight line topology and neglecting the kinetic effects (V^2) as:

$$\frac{\partial\zeta}{\partial t} + \mathcal{A}_1(\zeta,\mathbf{x},t)\nabla\zeta = u$$

where $\zeta = [\ \rho \quad M \quad E\ ]^T$, $u = [\ 0 \quad 0 \quad q\ ]^T$ and $\mathcal{A}_1$ is the Jacobian flux matrix [Hirsh'90] (ideal gas hyp.):

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)V^2}{2} & (3-\gamma)V & \hat{\gamma} \\ \hat{\gamma}V^3 - \frac{\gamma VE}{\rho} & \frac{\gamma E}{\rho} - \frac{3\hat{\gamma}V^2}{2} & \gamma V \end{bmatrix}$$

Decoupled model

- The eigenvalues of the Jacobian define the traveling waves, going into two directions: $\lambda_1(\zeta) = V - c$, $\lambda_2(\zeta) = V$ and $\lambda_3(\zeta) = V + c$
- Using a change of coordinates $\bar{\zeta}$ given by the Riemann invariants, we obtain a quasi-linear hyperbolic formulation with (isentropic case):

$$\mathcal{A}_1 = \begin{bmatrix} \lambda_1(\bar{\zeta}) & 0 & 0 \\ 0 & \lambda_2(\bar{\zeta}) & 0 \\ 0 & 0 & \lambda_3(\bar{\zeta}) \end{bmatrix}$$

$\Rightarrow$ For linear systems with appropriate boundary conditions, can be analyzed as a set of time-delay systems

Dynamic Boundary Stabilization of Quasi-Linear Hyperbolic Systems [Castillo, W, Prieur and Dugard, CDC'12]

Consider the system

$$\partial_t \xi(x,t) + \Lambda(\xi)\partial_x \xi(x,t) = 0 \quad \forall\, x \in [0,1], t \geq 0$$

where $\xi \in \Theta$, Λ is a diagonal matrix function $\Lambda : \Theta \to \mathbb{R}^{n\times n}$ such that $\Lambda(\xi) = diag(\lambda_1(\xi), \lambda_2(\xi), ..., \lambda_n(\xi))$ with

$$\underbrace{\lambda_1(\xi) < ... < \lambda_m(\xi)}_{\xi_-} < 0 < \underbrace{\lambda_{m+1}(\xi) < ... < \lambda_n(\xi)}_{\xi_+}, \quad \forall\, \xi \in \Theta$$

with BC

$$\begin{aligned} \dot{X}_c &= A\, X_c + B\, K\, Y_\xi \\ Y_c &= X_c \end{aligned} \; ; \quad \underbrace{\begin{pmatrix} \xi_-(1,t) \\ \xi_+(0,t) \end{pmatrix}}_{Y_c} = G \underbrace{\begin{pmatrix} \xi_-(0,t) \\ \xi_+(1,t) \end{pmatrix}}_{Y_\xi}$$

and IC $\xi(x,0) = \xi^0(x), \quad X_c(0) = X_c^0, \quad \forall\, x \in [0,1].$

Consider the Lyapunov function candidate ($P > 0$):

$$V(\xi, X_c) = X_c^T P X_c + \int_0^1 \left(\xi^T P \xi\right) e^{-\mu x} dx$$

Theorem

Assume that there exists a diagonal $Q \in \mathbb{R}^{n\times n} > 0$ and a matrix $Y \in \mathbb{R}^{n\times n}$ such that, $\forall i \in [1, ..., N_\varphi]$

$$\left[\begin{array}{cc} QA^T + AQ + \Lambda(w_i)Q & BY \\ Y^T B^T & -\Lambda(w_i)Q \end{array}\right] \prec 0$$

where $\Lambda(w_i)$ is a polytopic representation of $\Lambda(\xi)$
Let $K = YQ^{-1}$, then there exist two constants $\alpha > 0$ and $M > 0$ such that, for all continuously differentiable functions $\xi^0 : [0, 1] \rightarrow \Xi$ satisfying the zero-order and one-order compatibility conditions, the solution of satisfies, for all $t \geq 0$,

$$\|X_c(t)\|^2 + \|\xi(x,t)\|_{L^2(0,1)} \leq Me^{-\alpha t}\left(\|X_c^0\|^2 + \|\xi^0(x)\|_{L^2(0,1)}\right)$$

Example on flow dynamics: Video

A change of reference from $\tilde{V} = [1.16, 20, 100000]^T$ to $\tilde{V} = [1.2, 30, 105000]^T$ is introduced.

Complex models with spectral decomposition for Magnetohydrodynamics (MHD) stabilization

I.e. Resistive-wall mode physics in RFP: from MHD instability to perturbed ODE

- Linear stability investigated by periodic spectral decomposition

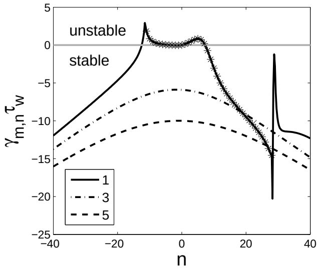
$$\mathbf{b}(r,t) = \sum_{mn} \mathbf{b}_{mn}(r) e^{j(t\omega + m\theta + n\phi)}$$

Fourier eigenmodes $\mathbf{b}_{mn}(r)$ with growth-rate $\gamma_{mn} = j\omega_{mn}$,

- Ideal MHD modes:

$$\tau_{mn}\dot{b}^r_{mn} - \tau_{mn}\gamma_{mn}b^r_{mn} = b^{r,ext}_{mn}$$

b^r_{mn}: radial component of perturbed field, $b^{r,ext}_{mn}$: external active coil, τ_{mn}: penetration time.



Growth-rates $\tau_w\gamma_{mn}$. *: Integer-n non-resonant positions (RWMs) for $m = 1$.

Stability analysis and delay effects [Olofsson, W, Briat, Niculescu and Brunsell, CDC'08, IOP PPCF'10]

Closed-loop dynamics with multiple delays and time-scales:

- Infinite spectrum of the Delay Differential Equation

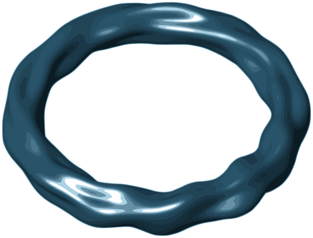
$$\det \Delta(s) = \det\left| sI - \mathcal{A}_0 - \sum_{i=1}^{n} \mathcal{A}_i e^{-s\tau_i} \right| = 0$$

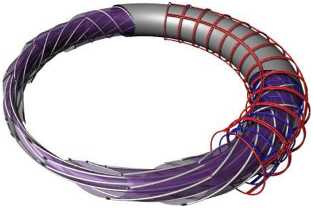
- Mode-control and perfect decoupling: SISO dynamics (fixed gains)

$$G_{mn}(s) = \frac{1}{\tau_{mn}s - \tau_{mn}\gamma_{mn}} \frac{1}{\tau_c s + 1} \frac{1}{\tau_a s + 1} e^{-s\tau_h}$$

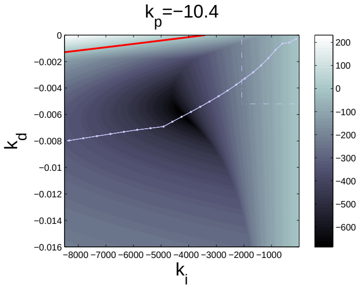
$\rightarrow$ fictitious but useful for disturbance rejection and resonant-field amplification analysis

Experimental results





- Direct Eigenvalue Optimization
- Two different parameterizations, implicitly assigning the closed-loop performance and control-input norm
- Robust: convergence within 10 – 30 iterations



⇒ *44 % reduction of average field energy at the expense of higher input power* (+28 %).

Conclusions

- Physical delay is often a major issue in transport phenomena; Time-delay systems can often be usefull
- Its proper inclusion in the feedback architecture compensates advection and losses can be dealt with an integral action
- Information transport strongly affected by information losses and needs robustness
- Capturing the traveling wave requires finer modeling

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