Optimal Control Design for the Stabilization of Network Controlled Systems

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***Abstract*— In this paper, the problem of remote output stabilization of networked control systems is investigated. The network is considered as a time-varying delay in the communication channel. An average model of the delay dynamics is supposed to be known and the unpredicted events occurring on the network are introduced as a random input in these dynamics. We propose a constructive control scheme where the deterministic aspect of the network is explicitly taken into account in a predictor-based feedback law. A stochastic descent algorithm is then introduced to set the controller gain according to the non-deterministic part of the delay dynamics. Some simulation results are also presented.**

I. Introduction

The remote output stabilization of systems controlled through a communication network is considered in this work. The network induces a *time-varying delay* in the communication channel and is supposed to be *secured* (the lost packets are re-emitted). We assume that this delay can be modelled as a dynamic system composed of a deterministic part and a colored noise, i.e. a filtered stochastic signal. The deterministic part represents the average network behavior, which can be evaluated using some existing models (see [1], [2] and [3] for examples), some round trip time measurements [4], or be directly set by the user through the transfer protocol and the router's queue management scheme in the specific case of dedicated networks. In order to account for the multiple-users interaction, the difficulty to model large networks (such as internet) and other unplanned events occurring in the communication channel, we introduce a stochastic signal as an exogenous input in the average dynamics, which results in a delay with colored noise. The control setup considered is presented in Figure 1, where the system and the controller interchange measurements and control signals through a lossless communication network.

The control method used in this paper is a predictor-based feedback approach, which sets a finite spectrum on systems with a deterministic time-delay in the input ([5], [6]...). More precisely, we are interested in the results of [7] and [8], where the time-delay model is *explicitly* used in the control design. This control scheme is set with the average network dynamics (no stochastic input) and allows to cope with the deterministic part of the time-delay. Few constraints are set on the choice of the controller gain, which can be used to adjust the closed-loop system bandwidth to the complete network dynamics, including its stochastic nature.

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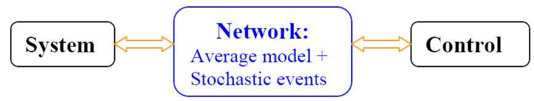

Fig. 1. Overview of the control setup.

After an analysis of the effects of the delay model uncertainties on the closed-loop system, we propose a stochastic approach which sets the controller gain according to the full delay dynamics (deterministic and stochastic). The resulting control gain minimizes a cost function on the output error signal under some robustness constraints imposed by the stochastic behavior of the network. The advantage of this method is then to propose a constructive approach which takes into account the difference between the modelled part of the network and some real measurements of the time-delay while minimizing a cost function.

We consider the remote stabilization of linear systems with a delayed input. More precisely, this class of systems writes as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau(t)), \quad x(0) = x_0 && (1)\\ y(t) &= Cx(t) && (2)\\ \dot{z}(t) &= f(z(t), u_d(t)), \quad z(0) = z_0 && (3)\\ \tau(t) &= h(z(t), u_d(t)) && (4) \end{aligned}$$

where $x \in R^n$ is the state, $u \in R$ the control input, $y \in R^m$ the output, A, B, C are matrices of appropriate dimensions and $u_d(t)$, $f(\cdot)$, $h(\cdot)$ are known class $\mathcal{C}^1$ functions. The pairs (A, B) and (A, C) are assumed to be controllable and observable in the usual sense, respectively, but no assumption is made on the stability of the open-loop system. The internal delay dynamics representing the transmission channel model is described by (3)-(4). The deterministic behavior of the network is modelled by $f(\cdot)$ and $h(\cdot)$, while the stochastic perturbations $\epsilon(t)$ are introduced in the input $u_d(t)$. We suppose that these perturbations satisfy the following conditions

$H1)$ $\epsilon(t)$ is a sequence of independent random variables,
$H2)$ $E|\epsilon(t)|^p$ exists and is bounded in t for each $p > 1$.

We also assume that the solutions of (3)-(4) have the following properties for all $t \geq 0$

$$\begin{aligned} \tau_{max} \geq \tau(t) \geq 0 && (5)\\ 1 - \nu \geq \dot{\tau}(t) && (6) \end{aligned}$$

where $\tau_{max} \geq 0$ is an upper bound of the time-variation of $\tau(t)$ and $\nu > 0$ is an arbitrarily small constant. Note that the first property is ensured from the secured characteristic of the

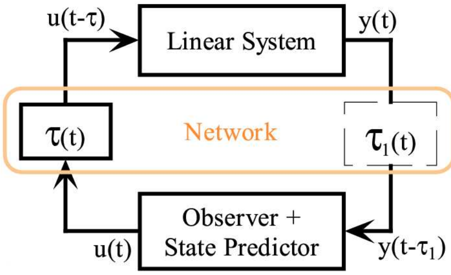

Fig. 2. Delayed output feedback control setups.

network while the second one is a causality issue introduced to deal with the deterministic part of the delay model.

Remark 1.1: If we consider the case of a dedicated multi-user network, some priority level can be attributed to the systems with the fastest dynamics or requiring higher performances. This will induce a reduced time delay and sensitivity of the average delay model to unpredicted events (the bound on the perturbation is reduced).

II. State Predictor Control

We recall in this section the main results of [7] and [8], where the deterministic behavior of the network is compensated using a state predictor with a time-varying horizon. Under the hypothesis that the time-delay model is known, the finite spectrum assignment method allows to remove the delay from the loop by transforming the open-loop time-invariant delayed system into a closed-loop *time-varying* system without delay. The time variation of the resulting linear system comes from the fact that the delay is time-varying. Two control setups are considered: the state feedback case (the full state is available to set the control law) and the output feedback case (only the delayed system output is measurable).

A. State feedback with one delay communication channel

We first consider the control setup where the full state $x(t)$ is available at the sensor side. Defining $\delta(t)$ as

$$\delta(t) \doteq \tau(t + \delta(t))$$

and considering the problem of state feedback stabilization, the finite closed-loop spectrum is set with the control input[1]

$$\begin{aligned} x(t+\delta) &= e^{A\delta}\left[x(t) + e^{At}\int_t^{t+\delta} e^{-A\theta}Bu(\theta - \tau(\theta))d\theta\right] \\ u(t) &= -Kx(t+\delta). \end{aligned} \tag{7}$$

The resulting time-shifted closed-loop system writes as

$$\frac{dx(t+\delta)}{d(t+\delta)} = (A - BK)x(t+\delta) = A_{cl}x(t+\delta)$$

where A_{cl} is the closed loop state matrix. The previous equation is equivalent to

$$\frac{dx(t+\delta)}{dt} = (1+\dot{\delta})A_{cl}x(t+\delta)$$

[1]For simplicity sake, the time dependency of $\tau(t)$ and $\delta(t)$ will be omitted. In the sequel, the notation τ and δ, will be used instead.

which is a *time-varying* system without delay since the state matrix is now $(1+\dot{\delta}(t))A_{cl}$ and time-dependent. The stability of this system is ensured by the following theorem, where a dynamic computation of $\delta(t)$ is also proposed.

Theorem 2.1 ([8]): Consider the system (1) with (A, B) a controllable pair. Assume that the delay dynamics (3)-(4) is such that (5)-(6) hold. Then the feedback control law (7) with

$$\begin{aligned} \dot{\delta}(t) &= -\frac{\lambda}{1 - d\tau(\zeta)/d\zeta}\delta + \frac{d\tau(\zeta)/d\zeta + \lambda\tau(\zeta)}{1 - d\tau(\zeta)/d\zeta} \\ \frac{d\tau}{d\zeta}(\zeta) &= \frac{dh}{d\zeta}(z(\zeta), u_d(\zeta)) \\ \frac{dz}{d\zeta}(\zeta) &= f(z(\zeta), u_d(\zeta)), \quad z(0) = z_0 \end{aligned}$$

where $\zeta(t) \doteq t+\delta(t)$, λ is a positive constant and $\delta(0) = \delta_0$, ensures that the closed-loop system is bounded, and that the state $x(t)$ converges asymptotically to zero.

Remark 2.1: This control setup *explicitely* takes into account the deterministic part of the delay dynamics. This ensures a full use of the available information on the network behaviour and a reduced conservativeness compared to most of the other control methods, where only the bounds of the delay are taken into account.

B. Output feedback with two delay communication channel

We consider the *delayed output* $y(t - \tau_1(t))$ as the feedback signal, where $\tau_1(t)$ is the system to control delay, and assume that τ_1 also satisfies the properties (5)-(6) since both delays are parts of the same network. The assumption that the full state is measurable is released by introducing an observer-based controller, as presented in Figure 2. The observer state $\hat{x}(t-\tau_1)$ is used to evaluate the state-dependent part of the control law. Its dynamics is set by the following Luenberger state-observer for system (1)-(2):

$$\begin{aligned} \dot{\hat{x}}(t-\tau_1) &= A\hat{x}(t-\tau_1) + Bu(t-\tau_1-\tau(t-\tau_1)) \\ &\quad + H\{y(t-\tau_1) - C\hat{x}(t-\tau_1)\} \end{aligned}$$

Defining the resulting observation error

$$\epsilon_o(t-\tau_1) \doteq x(t-\tau_1) - \hat{x}(t-\tau_1)$$

the control law can be expressed as a function of the delayed observation state $\hat{x}(t-\tau_1)$ as

$$u(t) = -K\left[e^{A(\delta+\tau_1)}\hat{x}(t-\tau_1) + I(t)\right]$$

where

$$I(t) \doteq e^{A(t+\delta)}\int_{t-\tau_1}^{t+\delta} e^{-A\theta}Bu(\theta-\tau(\theta))d\theta$$

is the integral part of the control law, and $E = A - HC$ is a matrix with assignable eigenvalues (from the observability property of the system).

The complete closed-loop dynamics including the observer is

$$\dot{x}(t+\delta) = A_{cl}x(t+\delta) + BKe^{A(\delta+\tau_1)}\epsilon_o(t-\tau_1) \tag{8}$$

$$\dot{\epsilon}_o(t-\tau_1) = E\epsilon_o(t-\tau_1) \tag{9}$$

The stability of this closed-loop system (detailed in [7]) is ensured if both delays satisfy (5)-(6), $\dot{\tau}_1 > -1\,\forall t$ and the matrices A_{cl} and E are Hurwitz.

C. Trajectory tracking

The previous results are easily extended to the output tracking case. Let $y_r(t)$ be the desired trajectory to be followed and assume that $\dot{y}_r(t)$, is bounded and known. Consider the system (1)-(2) and the particular case of system with a relative degree one[2], i.e. $CB \neq 0$. Assume also that the resulting zero-dynamics of dimension $n-1$ is rendered stable by a first state feedback, following the notation in (7). Then the following control law,

$$u(\zeta) = -Kx(\zeta) + \frac{1}{CB}\left[-CA_{cl}x(\zeta) - k\tilde{y}(\zeta) + \dot{y}_r(\zeta)\right]$$

with $\tilde{y}(t) = y(t) - y_r(t)$ and $k > 0$, leads to the closed-loop output error equation

$$\dot{\tilde{y}}(\zeta) = -k\tilde{y}(\zeta)$$

The stability of this equation follows from Theorem 2.1. The resulting zero dynamics ($\tilde{y} \to 0$)

$$\dot{x}(\zeta) = \left[A - \frac{BC}{CB}(A + kI)\right]x(\zeta) + \frac{B}{CB}(\dot{y}_r(\zeta) - ky_r(\zeta))$$

is stable due to the boundedness of $\dot{y}_r(\zeta)$, and the assumption on k. Finally, by following the procedure detailed in the previous subsection, the control law can be written as

$$u(t) = -\bar{K}x(t + \delta(t)) + \bar{y}_r(t + \delta(t))$$

with

$$\begin{aligned} \bar{K} &= K + C\frac{(A_{cl} + kI)}{CB} = C\frac{(A + kI)}{CB} \\ \bar{y}_r(\zeta) &= \frac{1}{C^T B}\left[ky_r(\zeta) + \dot{y}_r(\zeta)\right] \end{aligned}$$

III. Perturbed closed-loop equations

Denote by $\tau(t)$ (respectively $\tau_1(t)$) the actual time-delay, and by $\hat{\tau}(t)$ (respectively $\hat{\tau}_1(t)$) the estimated one (obtained from the delay model prediction), used to compute the predictor's horizon $\delta(t)$ and the control law. Robustness with respect to inaccuracies on the delay prediction can be asset as follows.

A. State feedback

Using the delay estimate $\hat{\tau}(t)$, the control law is set with

$$\begin{aligned} u(t) &= -Kp(t) \\ p(t) &\doteq e^{A\delta}x(t) + e^{A(t+\delta)}\int_t^{t+\delta} e^{-A\theta}Bu(\theta - \hat{\tau}(\theta))d\theta \quad (10) \\ \delta(t) &= \hat{\tau}(t + \delta(t)) \end{aligned}$$

The predicted state dynamics is obtained by deriving (10) with respect to time. This gives:

$$\dot{p}(t) = A_n(t)p(t) + A_d(t)[p(t - \hat{\tau}) - p(t - \tau)] \quad (11)$$

[2]the results can be easily extended to other system with different relative degree under the appropriate assumptions

with $A_n(t) \doteq (1 + \dot{\delta}(t))(A - BK)$ and $A_d(t) \doteq e^{A\delta}BK$. Note that (11) is a time-varying delayed system, written in the classical form, with two delayed states. A proper choice of the feedback gain K that renders the matrix $A - BK$ Hurwitz is sufficient to stabilize the system if $\hat{\tau} = \tau$. Otherwise, we have to find the proper feedback gain K that takes into account the effect of $A_d(t)$ in order to render the system robust to uncertainties on the time-delay.

B. Output feedback

The observer-based control law is designed using the time-delay estimates $\hat{\tau}_1(t)$ and $\hat{\tau}(t)$, and the observer state $\hat{x}(t - \hat{\tau}_1)$. The observer state is obtained from the dynamics

$$\begin{aligned} \dot{\hat{x}}(\hat{\zeta}_1) &= A\hat{x}(\hat{\zeta}_1) + Bu(\hat{\zeta}_1 - \hat{\tau}(\hat{\zeta}_1)) \\ &\quad + H\{y(t - \tau_1) - C\hat{x}(\hat{\zeta}_1)\} \end{aligned}$$

where $\hat{\zeta}_1(t) \doteq t - \hat{\tau}_1(t)$. The corresponding observation error

$$\hat{\epsilon}_o(t) \doteq x(\hat{\zeta}_1) - \hat{x}(\hat{\zeta}_1)$$

has the dynamics

$$\begin{aligned} \dot{\hat{\epsilon}}_o &= (1 - \dot{\hat{\tau}}_1)[\hat{A}_{cl}\hat{\epsilon}_o + HC\{x(t - \tau_1) - x(\hat{\zeta}_1)\} \\ &\quad + B\{u(\hat{\zeta}_1 - \tau(\hat{\zeta}_1)) - u(\hat{\zeta}_1 - \hat{\tau}(\hat{\zeta}_1)\}] \quad (12) \end{aligned}$$

The control law is now set as

$$u(t) = -Kp_o(t) + Ke^{A(\delta + \hat{\tau}_1)}\hat{\epsilon}_o(t) \quad (13)$$

where

$$p_o(t) \doteq e^{A(\delta+\hat{\tau}_1)}x(\hat{\zeta}_1) + e^{A(t+\delta)}\int_{t-\hat{\tau}_1}^{t+\delta} e^{-A\theta}Bu(\theta - \hat{\tau}(\theta))d\theta$$

and the dynamics of the predicted state is derived as

$$\begin{aligned} \dot{p}_o(t) &= (1 + \dot{\delta})[(A - BK)p_o(t) + BKe^{A(\delta+\hat{\tau}_1)}\hat{\epsilon}_o(t)] \\ &\quad + \dot{\hat{\zeta}}_1 e^{A(\delta+\hat{\tau}_1)}B[u(\hat{\zeta}_1 - \tau(\hat{\zeta}_1)) - u(\hat{\zeta}_1 - \hat{\tau}(\hat{\zeta}_1))] \quad (14) \end{aligned}$$

The closed-loop dynamics is then obtained from (12) and (14) with $u(t)$ given by (13). Note that if $\hat{\tau} = \tau$ and $\hat{\tau}_1 = \tau_1$, this dynamics is equivalent to (8)-(9).

IV. Stochastic Descent Approach

The aim of this section is to propose a numerical approach to the robustness problem formulated above. Time-delay uncertainty is modelled as a random perturbation ϵ which is introduced into the average delay model $\hat{\tau}(t)$ to obtain the actual time-delay $\tau(t)$. Using some previous results established in [9] for the stochastic approximation problem, an algorithm based on the stochastic gradient is proposed. This algorithm computes an optimal (in the mean square sense) parameter vector κ. The stability of the perturbed system depends on the convergence of the proposed algorithm, which is ensured by the so-called *ordinary differential equation (ODE) method*, first introduced in [10].

A. Output error minimization using a stochastic gradient

A classical control problem is to find the appropriate controller that ensures a reference trajectory tracking. This is done in this section by choosing a cost function $E_\epsilon J$ which reflects the variance of the system output from a given reference trajectory on a given time interval $[0, T]$, where ϵ is a random process defined on the same interval. The time interval has to be chosen large enough to ensure that there is no drift on the system. Consider the system with the measured output $y_m(\kappa, t, \epsilon)$ where κ is the design parameter vector. We want to find κ such that the system output is as close as possible to the desired trajectory $y_{ref}(t)$ in the sense of the minimum of the output error variance. The variance of output error $y_{ref}(t) - y_m(\kappa, t, \epsilon)$ writes as

$$E_\epsilon J(\kappa, \epsilon)$$

where E_ϵ is the expectation with respect to random process ϵ, and

$$J(\kappa, \epsilon) = \frac{1}{T}\int_0^T ||y_{ref}(t) - y_m(\kappa, t, \epsilon)||^2 dt$$

The output error is then minimized for κ^* satisfying

$$\kappa^* = \arg\min_\kappa E_\epsilon J(\kappa, \epsilon)$$

This optimization problem can be solved with a stochastic descent algorithm (see [11] for example), using the sensitivity of $y_m(t)$ with respect to κ

$$S(\kappa, t, \epsilon) \doteq \frac{\partial y_m}{\partial \kappa}$$

The stochastic gradient writes as

$$\nabla J(\kappa, \epsilon) = -2\int_0^T (y_{ref} - y_m)^T S dt \qquad (15)$$

and the optimal parameter κ^* is obtained by moving along the steepest slope $-\nabla J(\kappa, \epsilon)$ with a step α, which as to be small enough to ensure that

$$\dot{\kappa} = -\alpha \nabla J(\kappa, \epsilon) \qquad (16)$$

converges to κ^*. This step is chosen according to the *damped Newton's method* [12] and writes as

$$\alpha \doteq (\Psi J(\kappa, \epsilon) + \upsilon I)^{-1}$$

where υ is a positive constant introduced to ensure strict positiveness and $\Psi J(\kappa, \epsilon)$ is the pseudo-Hessian, derived using the Gauss-Newton approximation as

$$\Psi J(\kappa, \epsilon) = 2\int_0^T S(\kappa, t, \epsilon) S(\kappa, t, \epsilon)^T dt$$

Remark 4.1: The convergence of the previous algorithm, commonly used in least square problems, is ensured when $\epsilon = 0$ from the fact that $\Psi J(\kappa, \epsilon) \geq 0$ and the use of the positive constant υ to compensate the singularity point $\Psi J(\cdot) = 0$. The convergence speed of the algorithm is inversely proportional to the design parameter υ but choosing this parameter too small creates some oscillations in the solution $\kappa(t)$ of (16).

This problem can be related to an identification problem, where the unknown parameters are the design parameters. It is well posed if the design parameters satisfy some identifiability conditions, such as those proposed in [13] for the case of time-delay systems.

B. Application to perturbed dynamical systems

The previous method is now applied to dynamical systems where the dynamics is perturbed by a stochastic process ϵ (i.e. a process with uniform probability distribution). The main idea is to minimize $J(\kappa_l, \epsilon_l(t))$ for various error scenarios of $\epsilon_l(t)$, where $l = 1 \ldots N$ is the scenario index, defined on the interval $[0, T]$ (ideally an infinity to get the minimization of the variance $E_\epsilon J(.)$), to determine the robustness of the closed-loop system with respect to the stochastic process ϵ.
The system considered for each scenario writes as

$$\begin{cases} \dot{x}_m &= f_m(x_m, \kappa_l, \epsilon_l(t)) \\ y_m &= g_m(x_m, \kappa_l) \end{cases}$$

where $f_m(.)$ describes the closed-loop dynamics of the model state $x_m(t)$ and $g_m(.)$ sets the output; both functions are continuous. Furthermore, we suppose that the model dynamics is such that the system has a bounded state for a bounded input. In order to minimize J, we use the stochastic gradient (15) where the output sensitivity function S_l is given by

$$S_l = \frac{\partial g_m}{\partial x_m}\frac{\partial x_m}{\partial \kappa_l} + \frac{\partial g_m}{\partial \kappa_l}$$

where $\dfrac{\partial x_m}{\partial \kappa_l}$ is the state sensitivity, which is obtained by solving on $[0, T]$ the ODE

$$\begin{cases} \dot{x}_m &= f_m(x_m, \kappa_l, \epsilon_l(t)) \\ \dfrac{d}{dt}\left[\dfrac{\partial x_m}{\partial \kappa}\right] &= \dfrac{\partial f_m}{\partial x_m}\dfrac{\partial x_m}{\partial \kappa_l} + \dfrac{\partial f_m}{\partial \kappa_l} \end{cases} \qquad (17)$$

The stochastic descent algorithm is now used with the damped Newton's method to find the optimal parameter vector κ^*, which is the solution of

$$\begin{aligned} \kappa_{l+1} &= \kappa_l - \alpha_l \nabla J(\epsilon_l) \\ \alpha_l &= (\Psi J(\kappa_l, \epsilon_l) + \upsilon I)^{-1} \end{aligned} \qquad (18)$$

for l sufficiently large. Note that l sets the corresponding error scenario ϵ_l and α_l is the algorithm step established with the Newton's method.

Remark 4.2: The initial value of the parameter κ_1 has to be chosen so that the system (17) has a continuous solution on the time interval considered. The proposed algorithm then ensures that this parameter converges to its optimal value for the set of stochastic perturbations considered.
The bound on the perturbations appears *implicitly* in the algorithm since its convergence implies that the proposed control setup can stabilize the system for the considered set of perturbations. This is a numerical stability result.

V. Feedback gain design

We now apply the previous method to explore the robustness of (1)-(2) controlled by (7) with respect to some uncertainties in the delay model (3)-(4). This problem is formulated as in Section III-A: the closed-loop state considered contains the dynamics of the variables $\{p,\ \delta,\ \hat{z},\ \hat{\tau},\ z,\ \tau\}$ and the design parameter vector is the feedback gain $K(l)$. In order to simplify this system, we consider the specific case where $\tau(t) = z(t)$ and $\hat{\tau}(t) = \hat{z}(t)$ (the general case is easily obtained by introducing $h(.)$ and its derivative). The system model is then described by $x_m = [p\ \delta\ \hat{\tau}\ \tau]^T$ and its dynamics writes as

$$\frac{d}{dt}\begin{bmatrix} p \\ \delta \\ \hat{\tau} \\ \tau \end{bmatrix} = \begin{bmatrix} A_n p + A_d[p(t-\tau) - p(t-\hat{\tau})] \\ \dfrac{-\lambda\delta + f(\hat{\tau}(\zeta), u_d(\zeta)) + \lambda\hat{\tau}(\zeta)}{1 - f(\hat{\tau}(\zeta), u_d(\zeta))} \\ f(\hat{\tau}, u_d) \\ f(\tau, u_d, \epsilon_l) \end{bmatrix} \tag{19}$$

where λ is a positive real and $\hat{\tau}'(\zeta) = f(\hat{\tau}(\zeta), u_d(\zeta))$ is directly introduced in the derivative of δ.

In order to study the robustness of the predicted state with respect to ϵ and according to (2), the system output is $y_m(t) = Cp(t)$. Noticing that p is the only state of x_m depending on K, the sensitivity function is then given by

$$S_l = C\frac{\partial p}{\partial K}$$

which is computed by solving the set of ODEs

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial p}{\partial K}\right] &= \frac{\partial \dot{p}}{\partial x_m}\frac{\partial x_m}{\partial K} + \frac{\partial \dot{p}}{\partial K} \\ &= \frac{\partial \dot{p}}{\partial p}\frac{\partial p}{\partial K} + \frac{\partial \dot{p}}{\partial K} \\ &= (1+\dot{\delta})(A - BK)\frac{\partial p}{\partial K} - (1+\dot{\delta})Bp^T \\ &\quad + e^{A\delta}B[p(t-\tau) - p(t-\hat{\tau})]^T \end{aligned} \tag{20}$$

The solution of the previous equation can now be used to compute the stochastic gradient ∇J from (15) and to find the controller gain in (18) that minimizes the output error.

Example 5.1: Consider the second order system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t - \tau(t)) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

with the average and the actual time-delays dynamics given by

$$\begin{aligned} \dot{\hat{\tau}}(t) &= -2\hat{\tau} + \frac{1}{2}v(t,0), \quad \hat{\tau}(0) = \frac{1}{4} \\ \dot{\tau}(t) &= -2\tau + \frac{1}{2}v(t,\epsilon_l(t)), \quad \tau(0) = \frac{1}{8} \end{aligned}$$

with $v(t,\epsilon_l(t)) \doteq 1 + sin(\pi t) + \epsilon_l(t)$ and

$$\epsilon_l(t) \doteq [0.4 - 0.8rand(l)]rand(t)$$

where $rand(i)$ is a random number between 0 and 1 computed at the instant i, and l is the scenario considered. The reference output is chosen as $y_{ref} = 1$ and the initial controller gain is $K(1) = [1\ \ 1]$. For each error scenario l, the dynamic equations (19) are evaluated along with the sensibility function (20) to compute the stochastic gradient (15). The corresponding gain is then obtained from (18) with $\upsilon = 1$ and its convergence is shown in Figure 3. This figure shows the effectiveness of the proposed algorithm, which converges in about 10 iterations and remains almost constant afterward (small variations due to the stochastic process).

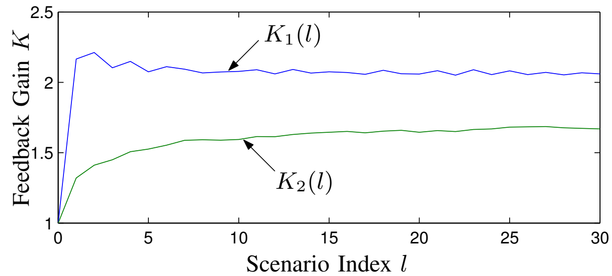

Fig. 3. Convergence of the controller gain.

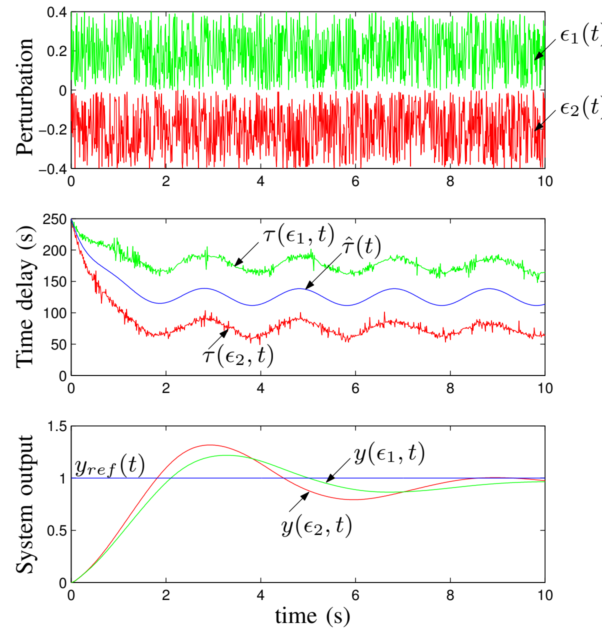

Fig. 4. System response to a stochastic error.

Setting the controller gain to the final value obtained from the proposed algorithm $K(30) = [2.0602\quad 1.6693]$, the time-delay and the corresponding system output are computed for the maximum and minimum error average value and presented in Figure 4. More precisely, we consider the error scenarios $\epsilon_1(t) = 0.4\,rand(t)$ and $\epsilon_2(t) = -0.4\,rand(t)$. Note that the system output converges instantaneously since the system considered here is the predicted state (the delay effect doesn't appear in the initial output response). This simulation shows the effectiveness of the proposed controller to stabilize a system when a random error is introduced in the delay dynamics.

VI. Observer Gain Design

The stochastic descent method proposed in Section IV is applied here to the observer-based control scheme to determine the optimal observer gain that ensures the robustness of the closed-loop system with respect to some uncertainties in the time-delay model. In order to simplify the proposed results, we assume that both channels are parts of the same network and induce the same time-delay in the transmitted signal. This implies the following simplification: $\tau_1(t) = \tau(t)$ and $\hat{\tau}(t) = \hat{\tau}_1(t)$. The closed-loop system is described by the state $x_{mo} = [x \ p_o \ \hat{\epsilon}_o \ \delta \ \hat{\tau} \ \tau]^T$ and its dynamics (derived in the previous sections) writes as

$$\frac{d}{dt}\begin{bmatrix} x \\ p_o \\ \\ \hat{\epsilon}_o \\ \\ \delta \\ \\ \hat{\tau} \\ \tau \end{bmatrix} = \begin{bmatrix} Ax(t) + Bu(t-\tau(t)) \\ (1+\dot{\delta})[A_{cl}p_o(t) + BKe^{A(\delta+\hat{\tau})}\hat{\epsilon}_o(t)] \\ +\dot{\hat{\zeta}}e^{A(\delta+\hat{\tau})}B[u(\hat{\zeta}-\tau(\hat{\zeta})) - u(\hat{\zeta}-\hat{\tau}(\hat{\zeta}))] \\ (1-\dot{\hat{\tau}})[\hat{A}_{cl}\hat{\epsilon}_o + HC\{x(t-\tau) - x(\hat{\zeta})\} \\ +B\{u(\hat{\zeta}-\tau(\hat{\zeta})) - u(\hat{\zeta}-\hat{\tau}(\hat{\zeta})\}] \\ \dfrac{-\lambda\delta + f(\hat{\tau}(\zeta), u_d(\zeta)) + \lambda\hat{\tau}(\zeta)}{1 - f(\hat{\tau}(\zeta), u_d(\zeta))} \\ f(\hat{\tau}, u_d) \\ f(\tau, u_d, \epsilon) \end{bmatrix} \tag{21}$$

with $\hat{\zeta} = t - \hat{\tau}$ and $u(t) = -Kp_o(t) + Ke^{A(\delta+\hat{\tau})}\hat{\epsilon}_o(t)$. In order to get the sensitivity function S, we first have to compute the state sensitivity with respect to the observer gain H. Note that the first three states of (21) are coupled while the remaining ones do not depend on $(x, p_o, \hat{\epsilon}, H)$; it follows that

$$\frac{\partial x_m}{\partial H_l} = \left[\frac{\partial x}{\partial H_l} \ \frac{\partial p_o}{\partial H_l} \ \frac{\partial \hat{\epsilon}_o}{\partial H_l} \ 0 \ 0 \ 0\right]^T$$

Introducing the reduced state $x_m^*(t) \doteq [x \ p_o \ \hat{\epsilon}_o]^T$, we have the dynamics

$$\frac{d}{dt}\left[\frac{\partial x_m^*}{\partial H_l}\right] = \frac{\partial f_m^*}{\partial x_m^*}\frac{\partial x_m^*}{\partial H_l} + \frac{\partial f_m^*}{\partial H_l}$$

with

$$\begin{aligned} \frac{\partial f_m^*}{\partial x_m^*} &= \begin{bmatrix} A & -BK & BKe^{A(\delta+\hat{\tau})} \\ 0 & A_n & e^{A\hat{\tau}}A_d + \dot{\hat{\zeta}}e^{A(\delta+\hat{\tau})}BK\Phi(t) \\ 0 & 0 & (1-\dot{\hat{\tau}})[\hat{A}_{cl} + BK\Phi(t)] \end{bmatrix} \\ \frac{\partial f_m^*}{\partial H_l} &= \begin{bmatrix} 0 & 0 & (1-\dot{\hat{\tau}})C[\{x(t-\tau) - x(\hat{\zeta})\} - \hat{\epsilon}_o] \end{bmatrix}^T \end{aligned}$$

where

$$\Phi(t) \doteq e^{A[\delta(\hat{\zeta}-\tau(\hat{\zeta}))+\hat{\tau}(\hat{\zeta}-\tau(\hat{\zeta}))]} - e^{A[\delta(\hat{\zeta}-\hat{\tau}(\hat{\zeta}))+\hat{\tau}(\hat{\zeta}-\hat{\tau}(\hat{\zeta}))]}$$

The system output is $y_m = Cp_o(t)$ and the resulting sensitivity function writes as

$$S_l = C\frac{\partial p_o}{\partial H_l}$$

which is computed from (21) with

$$\frac{\partial p_o}{\partial H_l} = [0 \quad 1 \quad 0]\frac{\partial x_m^*}{\partial H_l}$$

For a given controller gain, the system gradient and pseudo hessian are finally computed from the previous equation to set the algorithm (18). The resulting observer gain minimizes the output error.

VII. Conclusion

We formulated the problem of remote stabilization through communication networks as a time-delay problem, where the delays are time-varying and modelled as a dynamical system with stochastic inputs. A control law based on a state predictor with a time-varying horizon is used to compensate the deterministic behavior of the network. The effect of the stochastic perturbations on the closed loop system is then modelled to design a stochastic descent algorithm based on the Newton's method. This algorithm allows us the find the optimal gain which minimizes the difference between the system output and a reference output. The state and output feedback cases are both considered and some simulation results illustrate the state feedback case.

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