

A Nonlinear State Feedback Design for Nonlinear Systems with Input Delay

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Abstract— This paper is devoted to the design of a nonlinear feedback law based on state prediction for nonlinear systems with input time-delay. We successively consider the case of known constant time-delay and the case of time-varying delay in the input. In the case of constant delays and as in the linear case (under the finite-spectrum assignment assumption), a nonlinear distributed-delay control law is obtained. Since the computation of delay-distributed control laws remain a difficult problem as in the linear case, we discuss a control law approximation which is derived by using both a state prediction approximation and the "dynamic inversion" of a fixed point problem. In the case of time-varying delays, we extend the approach proposed in [9] by using a control law similar to the linear case one, together with dynamic inversion of a fixed point problem. Finally two illustrative examples are provided that demonstrate the effectiveness of the approach.

Keywords: Nonlinear control, nonlinear time-delay systems, state feedback, state predictor.

I. INTRODUCTION

A good motivation among others for the study of dynamic systems with input delays may be found in the fact that input delays can strongly impact the performance of control systems. The development of Networked Control Systems (NCS) or distributed control systems is one of the major sources of input delay occurrences [9].

Time-delay systems are known to be infinite-dimensional systems. In the linear case, a time-delay system has in general an infinite number of eigenvalues. Control laws have been proposed to assign a finite number of eigenvalues in closed loop [3]. This approach is called the finite spectrum assignment problem. Solutions to the finite spectrum assignment problem are obtained in term of delay-distributed control laws. However, the implementation of distributed-delay control laws is difficult due to the integral term which cannot be computed explicitly. In [3], it is suggested to approximate the integral by a sum of point-wise delays by using a quadrature rule. However this approach may fail due to the occurrence of unstable poles introduced the discretization procedure [8]. The use of block-pulse functions has also been proposed in [1]. More recently, a safe implementation of delay-distributed control laws has been proposed by using a low-pass filter in the control loop [7].

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In [4] a passivity-based control scheme is proposed for the stabilization of SISO nonlinear systems with input delay. However distributed-delay control laws for nonlinear systems has not yet been extensively studied. Other approaches have been proposed for special cases [5], [6], [10]. The goal of this paper is to propose a control law based on the state prediction in a way very similar to the case of linear systems with both constant and time-varying input delays. In the case of time-varying delays, we extend the approach proposed in [9] to the nonlinear case.

The case of constant time-delay systems is first presented in section II. Then the case of time varying delay systems is discussed in section III. In section IV, we illustrate the effectiveness of the proposed feedback laws with two simulation examples. Finally, some conclusions are given in section V.

II. A STATE FEEDBACK FOR NONLINEAR SYSTEMS WITH CONSTANT INPUT DELAY

Consider the following nonlinear system with input delay:

$$\dot{x}(t) = F(x(t), u(t - \tau)) \quad (1)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, τ is a constant delay supposed to be known and F is a continuously differentiable function. The origin is supposed to be an equilibrium point of the system ($F(0, 0) = 0$) and the system is not necessarily stable. Furthermore, we suppose that the system does not exhibit any "finite escape time" behavior and the non delayed system

$$\dot{x}(t) = F(x(t), u(t)) \quad (2)$$

is stabilizable via state feedback.

A. PROBLEM STATEMENT

We seek for the design of a state feedback law in order to stabilize the system in closed loop. Our goal is to extend the so-called finite spectrum assignment approach already available for linear input-delayed systems.

This approach is based on the following principle: firstly a prediction of the state over one delay interval $x_p(t, t + \tau)$ is computed from the available state $x(t)$ at time t and input controls $u(\theta)$, $\theta \in [t - \tau, t]$. Then the predicted state is used to compute the control law. Consequently, the effect of the delay vanishes and the closed-loop system is no more a time-delay system.

In order to illustrate this approach, consider a linear system with input delay

$$\dot{x}(t) = Ax(t) + Bu(t - \tau) \quad (3)$$

where the pair (A, B) is supposed to be controllable and A is not necessarily Hurwitz.

Clearly, if the state feedback $u(t) = Kx_p(t, t + \tau)$ is applied to the system, the closed-loop system is given by

$$\dot{x}(t) = (A + BK)x(t) \quad (4)$$

which is no more a time-delay system. Since the system is controllable, K can be computed in order to assign a finite stable spectrum to the closed-loop system, using a pole placement technique or any state feedback design (LQR or H_∞ design for example).

$x_p(t, t + \tau)$ is formally given by

$$x_p(t, t + \tau) = e^{A\tau}x(t) + \int_0^\tau e^{A\theta}Bu(t - \theta)d\theta. \quad (5)$$

Then a distributed-delay control law is given by

$$u(t) = K\{e^{A\tau}x(t) + \int_0^\tau e^{A\theta}Bu(t - \theta)d\theta\}. \quad (6)$$

A similar approach can be also applied to nonlinear systems.

If we suppose that a smooth state feedback $u(t) = \Phi(x(t))$ is available for the non delayed system (2) ensuring that the closed-loop system $\dot{x} = F(x, \Phi(x))$ is locally (resp. globally) stable, a prediction of the state at time $t + \tau$ has to be performed in order to cancel the effect of the time-delay and to get a locally (resp. globally) stable finite-dimensional closed-loop system also defined by $\dot{x} = F(x, \Phi(x))$. The main issue remains the computation of the predicted state $x_p(t, t + \tau)$, which is given by

$$x_p(t, t + \tau) = x(t) + \int_t^{t+\tau} F(x_p(t, \theta), u(\theta - \tau))d\theta \quad (7)$$

where $x_p(t_1, t_2)$ is the prediction of x at time $t = t_2$ based on the values of both x and u for $t \leq t_1$. The predicted state $x_p(t, t + \tau)$ may also be defined in term of an operator

$$x_p(t, t + \tau) = \Psi(x(t), \{u(\theta)\}_{\theta \in [t-\tau, t]}) \quad (8)$$

Finally, the control law is given by

$$u(t) = \Phi(\Psi(x(t), \{u(\theta)\}_{\theta \in [t-\tau, t]})) \quad (9)$$

which is very similar to (6).

B. STABILITY RESULT

We are ready to state the following stability theorem:

Theorem 1: Suppose that there exists a smooth state feedback $\Phi(x)$ ensuring that the non delayed system $\dot{x}(t) = F(x(t), u(t))$ is locally (resp. globally) asymptotically stable around the origin: There exist a domain $D \subset R^n$ containing

the origin and a continuously differentiable function $V : D \rightarrow R$ (resp. there exists a continuously differentiable function $V : R^n \rightarrow R$) such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in D - \{0\} \\ (\text{resp. } \forall x \neq 0) \quad (10)$$

$$\dot{V} = \frac{\partial V}{\partial x}F(x, \Phi(x)) < 0, \forall x \in D - \{0\} \\ (\text{resp. } \forall x \neq 0 \text{ and } \|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty). \quad (11)$$

Then the control law given by

$$u(t) = \Phi(\Psi(x(t), \{u(\theta)\}_{\theta \in [t-\tau, t]})) \quad (12)$$

ensures that the delayed system (1) is locally (resp. globally) asymptotically stable around the origin. V is a Lyapunov function of the closed-loop system

$$\dot{x} = F(x, \Phi(\Psi(x(t), \{u(\theta)\}_{\theta \in [t-\tau, t]}))). \quad (13)$$

Proof: Immediate from the previous discussion.

The computation of control law (9) remains a difficult issue since Ψ is a very complicated integral operator (more than in the linear case). We can observe that (9) can be viewed as the on-line computation of a fixed point of the form $u(t) = I(u(t), x(t), \{u(\theta)\}_{\theta \in [t-\tau, t]})$.

Now we focus our attention on the practical implementation of control law (9), based on these observations.

C. NUMERICAL APPROXIMATION OF THE FEEDBACK LAW

The approximated computation of the predicted state $x_p(t, t + \tau)$ can be performed with any consistent and converging integration method in principle. However we should only consider unconditionally stable methods, which are not sensitive to the integration step in term of stability. Here we consider one step of the backward Euler method (the implicit Euler method) which will give a approximation of the predicted state $\hat{x}_p(t, t + \tau)$ according to

$$\hat{x}_p(t, t + \tau) = x(t) + \tau F(\hat{x}_p(t, t + \tau), u(t)). \quad (14)$$

The problem is now to compute on-line the fixed point $X(t) = (u^T(t), \hat{x}_p^T(t, t + \tau))^T$, solution of

$$\hat{x}_p(t, t + \tau) = x(t) + \tau F(\hat{x}_p(t, t + \tau), u(t)), \quad (15)$$

$$u(t) = \Phi(x(t) + \tau F(\hat{x}_p(t, t + \tau), u(t))) \quad (16)$$

which is of the general form¹

$$X(t) = H(X(t), x(t)). \quad (17)$$

Computation of a fixed point can be traditionally performed by using a Newton-Raphson method, but this technique is time consuming and therefore not appropriate for on-line computation. Here we propose another approach based on "dynamic inversion".

¹We could also consider several steps of any implicit integration schemes. In that case, the fixed point vector will include additional discretization states.

Suppose that we seek for the solution of $G(x, t) = 0$, where G is a nonlinear C^1 -function : $R^n \times [0, +\infty) \rightarrow R^n$ and the Jacobian matrix $\frac{\partial G}{\partial x}$ is supposed to be invertible. The main idea is now to compute the solution of the differential equation

$$\dot{x} + \Lambda G = 0 \quad (18)$$

where Λ is any positive definite matrix ensuring the asymptotic stability of this equation. In the coordinates x , (18) is equivalent to

$$\frac{\partial G}{\partial x} \dot{x} + \frac{\partial G}{\partial t} + \Lambda G(x, t) = 0. \quad (19)$$

Since $\frac{\partial G}{\partial x}$ has full rank, (19) is equivalent to

$$\dot{x} = -\frac{\partial G}{\partial x}^{-1} \left[\frac{\partial G}{\partial t} + \Lambda G(x, t) \right]. \quad (20)$$

The motivation may be found in the fact that if the initial state x_0 is solution of $G(x, t = 0) = 0$, then the trajectory $x(t)$ of (18), is solution of $G(x(t), t) = 0, \forall t > 0$. Since (18) is asymptotically stable, even when the initial state is not a solution of $G(x, t = 0) = 0$, $x(t)$ will reach asymptotically the manifold $G(x, t) = 0$, since the solution of (18) is $G(x, t) = e^{-\Lambda t} G(x(0), 0)$ and $\lim_{t \rightarrow +\infty} G(x(t), t) = 0$ for all $\Lambda > 0$. Λ can be used to control the speed of convergence.

Application of this approach to (17) leads to the state-prediction-based control law given by

$$\dot{X} = \left(I_d - \frac{\partial H}{\partial X} \right)^{-1} \left[\frac{\partial H}{\partial x} F(x(t), u(t - \tau)) - \Lambda (X - H(X, x(t))) \right] \quad (21)$$

$$u(t) = \begin{pmatrix} I_d & 0 \end{pmatrix} X(t) \quad (22)$$

with

$$\frac{\partial H}{\partial X} = \begin{pmatrix} \tau \frac{\partial F}{\partial u} & \tau \frac{\partial F}{\partial x} \\ \tau \frac{\partial \Phi}{\partial x} \frac{\partial F}{\partial u} & \tau \frac{\partial \Phi}{\partial x} \frac{\partial F}{\partial x} \end{pmatrix}$$

and

$$\frac{\partial H}{\partial x} = \begin{pmatrix} I_d \\ \frac{\partial \Phi}{\partial x} \end{pmatrix}.$$

(21)-(22) is nothing else but a dynamic state feedback.

D. DISCUSSION ON THE STABILITY OF THE APPROXIMATE FEEDBACK

An important issue is to establish that state feedback (21)-(22) can stabilize the system in closed-loop.

Firstly we will consider the case when the fixed point problem (15)-(16) is supposed to be solved at each time t without dynamic inversion.

In this case, the main source of approximation comes from the fact that we only get an approximate prediction of $x(t +$

$\tau) = x_p(t, t + \tau)$ with $\hat{x}_p(t, t + \tau)$ here obtained from a simple implicit Euler integration scheme. We will now establish a technical result defining the structure of the error $\hat{x}_p(t, t + \tau) - x_p(t, t + \tau)$.

Proposition 2: The error $\hat{x}_p(t, t + \tau) - x_p(t, t + \tau)$ is of the form

$$\hat{x}_p(t, t + \tau) - x_p(t, t + \tau) = E(\tau, x(t)), \quad (23)$$

with $E(\tau, 0) = 0$.

Proof: From (15)-(16) and using the implicit function theorem, we can prove that there exists a function ψ such that $\hat{x}_p(t, t + \tau) = \psi(x(t))$. Furthermore, under the assumption that vector field F is analytic, $x_p(t, t + \tau)$ can be expressed in Taylor series

$$x_p(t, t + \tau) = x(t) + \sum_{i=1}^{+\infty} \frac{\tau^i}{i!} \frac{d^i x}{dt^i}(t).$$

We can conclude that $\hat{x}_p(t, t + \tau) - x_p(t, t + \tau)$ depends on both τ and $x(t)$. Since $F(0, 0) = 0$, $E(\tau, 0) = 0$.

We can also show easily that the closed-loop system becomes a nonlinear system with an additive perturbation term depending on the delayed state $x(t - \tau)$. Indeed, the closed-loop system can be expressed as

$$\begin{aligned} \dot{x}(t) &= F(x(t), \Phi(\psi(x(t - \tau)))) \\ &= F(x(t), \Phi(x(t))) \\ &\quad + [F(x(t), \Phi(\psi(x(t - \tau)))) - F(x(t), \Phi(x(t)))] \\ &= F(x(t), \Phi(x(t))) + P(x(t), x(t - \tau)) \end{aligned} \quad (24)$$

We are now ready to state the following stability theorem:

Theorem 3: If the following conditions hold:

- 1) There exists a smooth control law Φ and a Lyapunov function $V(x)$ such that the following assumptions hold [2], $\forall x \in D$, where $D \subset R^n$ is a domain that contains the origin:

- $c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$
- $\frac{\partial V(x)}{\partial x} F(x, \Phi(x)) \leq -c_3 \|x\|^2$
- $\left\| \frac{\partial V(x)}{\partial x} \right\| \leq c_4 \|x\|$

where c_1, c_2, c_3 and c_4 are some positives scalar numbers.

- 2) The perturbation term $P(x(t), x(t - \tau))$ satisfies $\|P(x(t), x(t - \tau))\| \leq \gamma \|x(t - \tau)\|, \forall x(t) \in D$ (this condition may be derived from proposition 2 together with a first-order Taylor development of H).

and if the perturbation is such that $\gamma < \frac{c_3}{c_4}$, then the closed-loop system is locally asymptotically stable.

Proof: We introduce the Lyapunov-Krasovskii function candidate $W(x_t) = V(x) + \frac{c_3}{2} \int_{t-\tau}^t \|x(\theta)\|^2 d\theta$, where $x_t =$

$x(t + \theta), \theta \in [-\tau, 0]$. The time derivative of W is given by

$$\begin{aligned} \dot{W} &= \frac{\partial V(x)}{\partial x} F(x, \Phi(x)) + \frac{\partial V(x)}{\partial x} P(x(t), x(t - \tau)) \\ &+ \frac{c_3}{2} [\|x(t)\|^2 - \|x(t - \tau)\|^2]. \end{aligned} \quad (25)$$

Using the two conditions, we get

$$\begin{aligned} \dot{W} &\leq -c_3 \|x(t)\|^2 + c_4 \gamma \|x(t)\| \|x(t - \tau)\| \\ &+ \frac{c_3}{2} [\|x(t)\|^2 - \|x(t - \tau)\|^2]. \end{aligned} \quad (26)$$

$$\begin{aligned} \dot{W} &\leq - \begin{pmatrix} \|x(t)\| & \|x(t - \tau)\| \end{pmatrix} \\ &\times \begin{pmatrix} \frac{c_3}{2} & -c_4 \frac{\gamma}{2} \\ -c_4 \frac{\gamma}{2} & \frac{c_3}{2} \end{pmatrix} \begin{pmatrix} \|x(t)\| \\ \|x(t - \tau)\| \end{pmatrix}. \end{aligned} \quad (27)$$

Using Sylvester's criterion, we get a sufficient condition for \dot{W} to be negative definite:

$$\gamma < \frac{c_3}{c_4}$$

that concludes the proof.

Now we have to establish closed-loop stability of the dynamic control(21)-(22). For this goal, we can invoke Tikhonov's singular perturbation theorem [2]. Indeed, for $\Lambda = \lambda I_d$, with $\lambda > 0$ large enough, the system in closed-loop with (21)-(22) is a singularly perturbed system:

$$\dot{x}(t) = F(x(t), u(t - \tau)) \quad (28)$$

$$\begin{aligned} \epsilon \dot{X} &= (I_d - \frac{\partial H}{\partial X})^{-1} [\epsilon \frac{\partial H}{\partial x} F(x(t), u(t - \tau)) \\ &- (X - H(X, x(t)))] \end{aligned} \quad (29)$$

$$u(t) = \begin{pmatrix} I_d & 0 \end{pmatrix} X(t) \quad (30)$$

with $\epsilon = \frac{1}{\lambda}$.

III. A STATE FEEDBACK FOR NONLINEAR SYSTEMS WITH TIME-VARYING INPUT DELAY

A. PROBLEM STATEMENT

In this section, we consider the following problem

$$\dot{x}(t) = F(x(t), u(t - \tau(t))) \quad (31)$$

$$\dot{\tau} = G(t, \tau) \quad (32)$$

$$0 \leq \tau(t) \leq \tau_{max} \quad (33)$$

$$\sup_{t \in \mathbb{R}^+} \dot{\tau}(t) < 1. \quad (34)$$

Again we suppose that the system does not exhibit any "finite escape time" behavior.

In fact the idea of state prediction is still applicable in this case. However the prediction horizon cannot be τ anymore but a time-varying prediction horizon $\delta(t)$ such that $\delta(t) = \tau(t + \delta(t))$.

This condition is used to ensure that a control input $u(t)$ can be computed since in this case $u(t - \tau(t + \delta(t)) + \delta(t)) = u(t)$ [9].

We will show that the stability of the closed-loop system expressed in the time coordinate $t + \delta(t)$

$$\dot{x}(t + \delta(t)) = F(x(t + \delta(t)), \Phi(x(t + \delta(t)))) \quad (35)$$

can be guaranteed under some conditions, if there exists a smooth feedback $\Phi(x)$ ensuring the closed-loop stability of the non delayed system $\dot{x}(t) = F(x(t), u(t))$.

The main issue also remains the computation of the predicted state $x_p(t, t + \delta(t))$, which is given by

$$x_p(t, t + \delta(t)) = x(t) + \int_t^{t + \delta(t)} F(x_p(t, \theta), u(\theta - \tau(\theta))) d\theta. \quad (36)$$

The predicted state $x_p(t, t + \delta(t))$ may also be defined in term of an operator

$$x_p(t, t + \delta(t)) = \Psi(x(t), \{u(\theta)\}_{\theta \in [t - \tau, t]}) \quad (37)$$

Finally, the control law is given by

$$u(t) = \Phi(\Psi(x(t), \{u(\theta)\}_{\theta \in [t - \tau, t]})) \quad (38)$$

which is very similar to (9).

B. STABILITY RESULT

In order to prove closed-loop stability, we need the following technical result:

Theorem 4: Consider the following nonlinear system

$$x'(\zeta) = f(x(\zeta)) \quad (39)$$

where $x'(\zeta) = \frac{d}{d\zeta} x(\zeta)$, with $\zeta = t + \delta(t)$ for $t \geq 0$ and $\delta(0) = \delta_0$.

If the three following conditions hold:

- 1) There exists a Lyapunov function $V(x)$ such that the following assumptions hold [2], $\forall x \in D$, where $D \subset \mathbb{R}^n$ is a domain that contains the origin:

$$\bullet \quad c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

$$\bullet \quad \frac{\partial V(x)}{\partial x} f(x) \leq -c_3 \|x\|^2$$

where c_1, c_2 and c_3 are some positives scalar numbers.

- 2) $\infty > \delta_M \geq \delta(t) \geq 0$ (this assumption holds if $0 \leq \tau(t) \leq \tau_{max}$)

- 3) $\infty > \dot{\delta}(t) \geq -1$ (this assumption holds if $\sup_{t \in \mathbb{R}^+} \dot{\tau}(t) < 1$)

then,

$$\lim_{t \rightarrow \infty} \|x(t + \delta(t))\| = 0,$$

$$\forall t \geq \delta_0.$$

Proof: Firstly we note that

$$\frac{d}{dt}x(\zeta) = \frac{d}{d\zeta}x(\zeta)(1 + \dot{\delta}(t))f(x(\zeta)).$$

Then the time derivative of the Lyapunov function:

$$\dot{V} = (1 + \dot{\delta}(t)) \frac{\partial V}{\partial x} f(x(\zeta)).$$

From the second assumption of condition 1), one gets

$$\dot{V} \leq -(1 + \dot{\delta}(t))c_3\|x(\zeta)\|^2.$$

Substituting assumption 1 of 1) in the last inequality and integrating from 0 to t , one gets

$$V(t) \leq V(0) \exp(-\phi(t)),$$

where:

$$\phi(t) = \frac{c_3}{c_1} \int_0^t (1 + \dot{\delta}(\theta)) d\theta.$$

Again using assumption 1 of 1), the following inequality holds

$$\|x(\zeta)\|^2 \leq \frac{c_2}{c_1} \|x(\delta_0)\|^2 \exp(-\phi(t)).$$

One can establish that $\phi(t) \rightarrow +\infty$, when $t \rightarrow +\infty$, since $\phi(t) = \frac{c_3}{c_2}(t + \delta(t) - \delta_0)$, where δ is positive and bounded from condition 2). We can now conclude that,

$$\lim_{t \rightarrow \infty} \|x(t + \delta(t))\| = 0$$

This result may be applied to demonstrate the stability of the closed-loop system:

Corollary 5: Under the conditions of theorem 4 and if the system does not diverge in finite time, the closed-loop system (31) with (38) has a bounded trajectory which converges towards 0. If the conditions of (4) hold globally, then the convergence property holds globally.

Proof: Immediate. If the system does not diverge in finite time, $x(\delta_0)$ is bounded for every bounded δ_0 , then the result holds.

C. NUMERICAL APPROXIMATION OF THE FEEDBACK LAW

An extension of (17) is possible by including the fixed point problem $\delta(t) = \tau(t + \delta(t))$

$$\delta(t) = \tau(t + \delta(t)), \quad (40)$$

$$\hat{x}_p(t, t + \delta(t)) = x(t) + \delta(t)F(\hat{x}_p(t, t + \delta(t)), u(t)), \quad (41)$$

$$u(t) = \Phi(x(t) + \delta(t)F(\hat{x}_p(t, t + \delta(t)), u(t))). \quad (42)$$

Then a dynamic feedback similar to (21)-(22) can be derived. Stability of this feedback can be performed in a manner similar to the case of constant input delays. We can show that the approximation of the predicted state at $t + \delta(t)$,

$\hat{x}_p(t, t + \delta(t))$ induces a perturbation depending on the delayed state. The closed-loop system can be expressed as

$$\begin{aligned} \dot{x}(t) &= F(x(t), \Phi(\psi(x(t - \tau)))) \\ &= F(x(t), \Phi(x(t))) \\ &\quad + [F(x(t), \Phi(\psi(x(t - \tau)))) - F(x(t), \Phi(x(t)))] \\ &= F(x(t), \Phi(x(t))) + P(x(t), x(t - \tau)) \end{aligned} \quad (43)$$

Theorem 6: If the following conditions hold:

1) There exists a smooth control law Φ and a Lyapunov function $V(x)$ such that the following assumptions hold [2], $\forall x \in D$, where $D \subset R^n$ is a domain that contains the origin:

- $c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$
- $\frac{\partial V(x)}{\partial x} F(x, \Phi(x)) \leq -c_3\|x\|^2$
- $\|\frac{\partial V(x)}{\partial x}\| \leq c_4\|x\|$

where c_1, c_2, c_3 and c_4 are some positives scalar numbers.

2) The perturbation term $P(x(t), x(t - \tau))$ satisfies $\|P(x(t), x(t - \tau))\| \leq \gamma\|x(t - \tau)\|, \forall x(t) \in D$

and if the perturbation is such that $\gamma < \sqrt{1 - \frac{c_3}{c_4}}$, then the closed-loop system is locally asymptotically stable.

Proof: We introduce the Lyapunov-Krasovskii function candidate $W(x_t) = V(x) + \frac{c_3}{2} \int_{t-\tau}^t \|x(\theta)\|^2 d\theta$, where $x_t = x(t + \theta), \theta \in [-\tau, 0]$. The time derivative of W is given by

$$\begin{aligned} \dot{W} &= \frac{\partial V(x)}{\partial x} F(x, \Phi(x)) + \frac{\partial V(x)}{\partial x} P(x(t), x(t - \tau)) \\ &\quad + \frac{c_3}{2} [\|x(t)\|^2 - (1 - \dot{\tau}(t))\|x(t - \tau)\|^2]. \end{aligned} \quad (44)$$

Using the two conditions and $\sup_{t \in R^+} \dot{\tau}(t) = \bar{\tau} < 1$, we get

$$\begin{aligned} \dot{W} &\leq -c_3\|x(t)\|^2 + c_4\gamma\|x(t)\|\|x(t - \tau)\| \\ &\quad + \frac{c_3}{2} [\|x(t)\|^2 - (1 - \dot{\tau}(t))\|x(t - \tau)\|^2]. \end{aligned} \quad (45)$$

$$\begin{aligned} \dot{W} &\leq - \left(\|x(t)\| \quad \|x(t - \tau)\| \right) \\ &\quad \times \begin{pmatrix} \frac{c_3}{2} & -c_4\frac{\gamma}{2} \\ -c_4\frac{\gamma}{2} & (1 - \bar{\tau})\frac{c_3}{2} \end{pmatrix} \begin{pmatrix} \|x(t)\| \\ \|x(t - \tau)\| \end{pmatrix}. \end{aligned} \quad (46)$$

Using Sylvester's criterion, we get a sufficient condition for \dot{W} to be negative definite:

$$\gamma < \sqrt{1 - \bar{\tau}} \frac{c_3}{c_4}$$

that concludes the proof.

Again Tikhonov's theorem can be invoked to prove stability of the dynamic inversion of the dynamic controller based on the so-called dynamic inversion.

IV. SOME ILLUSTRATIVE EXAMPLES

In order to demonstrate the effectiveness of the here-proposed approach, we consider the control of a simple pendulum. We suppose that the control input is delayed.

The dynamics of the studied simple pendulum is given by:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) - \sin(x_1(t)) + u(t) \\ y &= x_1\end{aligned}\quad (47)$$

The nominal control (without input delay) is obtained from input-output linearization:

$$\begin{aligned}u &= \Phi(x_1, x_2) \\ &= x_2(t) + \sin(x_1(t)) - k_1 x_2(t) - k_2(x_1(t) - x_1^d)\end{aligned}\quad (48)$$

This control law renders the system equivalent to

$$\ddot{y} + k_1 \dot{y} + k_2(y - x_1^d) = 0 \quad (49)$$

Obviously, the closed-loop system is asymptotically stable for any positive k_1 and k_2 . (48) will be used in what follows.

A. STABILIZATION OF A SIMPLE PENDULUM WITH CONSTANT INPUT DELAY

We consider the following control problem:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) - \sin(x_1(t)) + u(t - \tau) \\ y &= x_1\end{aligned}\quad (50)$$

The dynamic feedback (21)-(22) based on both (48) and (15)-(16) is simulated with $\tau = 1$, $\Lambda = 10$, $k_1 = 2$, $k_2 = 1$ and $x_1^d = \frac{\pi}{3}$. The figure 1 illustrates the effectiveness of the control.

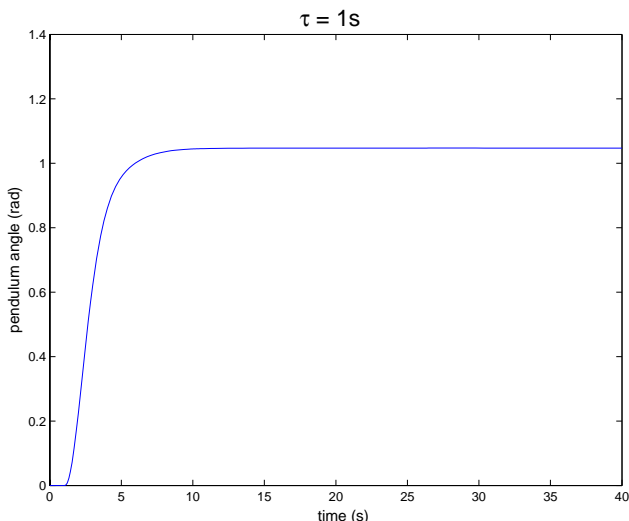


Fig. 1. Pendulum angle response

B. STABILIZATION OF THE SIMPLE PENDULUM WITH TIME-VARYING INPUT DELAY

We consider the following control problem:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) - \sin(x_1(t)) + u(t - \tau(t)) \\ y &= x_1 \\ \tau(t) &= 1 + 0.5 \cos(t)\end{aligned}\quad (51)$$

The dynamic feedback (21)-(22) based on both (48) and (40)-(42) is simulated with $\Lambda = 10$, $k_1 = 2$, $k_2 = 1$ and $x_1^d = \frac{\pi}{3}$. The figure 2 shows that the system is stabilized.

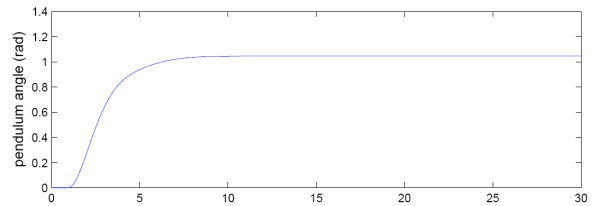


Fig. 2. Pendulum angle response

V. CONCLUSIONS

A novel approach based on both state-prediction and the so-called "dynamic inversion" has been proposed for the control of nonlinear systems with constant or time-varying input delay. The effectiveness of this approach has been demonstrated on two illustrative examples.

Future works will be devoted to observer-based control design using this approach and to state-dependent input delayed systems of the form:

$$\dot{x}(t) = F(x(t), u(t - \tau(x(t)))) \quad (52)$$

Again in this case it seems to be possible to introduce a dynamic state feedback based on the solution of a fixed point problem.

REFERENCES

- [1] A. Fattouh, O. Sename and J.M. Dion, "Pulse Controller Design For Linear Systems With Delayed State and Control", in *proceedings of IFAC Symposium on System Structure and Control*, Praha, 2001.
- [2] H.K. Khalil, "Nonlinear systems", *Prentice-Hall, Inc*, 1996, pp. 149.
- [3] A.Z. Manitius and A.W. Olbrot, "Finite Spectrum Assignment Problem for systems with delays", *IEEE Trans. Automat. Contr.*, Vol. 24, No 4, april 1979.
- [4] L. Maza-Casas, M. Velasco-Villa and J.A. Alvarez-Gallegos, "Compensation of Input Time Delay For a Class of Nonlinear Systems", in *Proceedings of 39 th IEEE Conference on Decision and Control*, Sydney, dec. 2000.
- [5] F. Mazenc, S. Mondie and S.-I. Niculescu, "Global asymptotic stabilization for chains of integrators with a delay in the input", *IEEE Trans. Automat. Contr.*, Vol. 48, pp. 57-63, 2003.
- [6] F. Mazenc, S. Mondie and R. Francisco, "Global asymptotic stabilization of feedforward systems with delay in the input", in *Proceedings of the 42th IEEE Conference on Decision and Control*, Maui, Hawaii USA, 2003, pp. 4020-4025.

- [7] S. Mondie and W. Michiels, "Finite Spectrum Assignment of Unstable Time-Delay Systems With a Safe Implementation", *IEEE Trans. Automat. Contr.*, Vol. 48, No 12, dec. 2003.
- [8] V. Van Assche, M. Dambrine and J.F. Lafay, "Some Problems Arising in the Implementation of Distributed-Delay Control Law", in *Proceedings of 38th IEEE Conference on Decision and Control*, Phoenix, AZ, dec 1999.
- [9] E. Witrant, C. Canudas de Wit, D. Georges and M. Alamir, "Remote stabilization via time-varying communication delays: Application to TCP networks", in *Proceedings of the IEEE Conference on Control Applications*, Taipei, Taiwan, 2004.
- [10] X. Zhang, H. Gao, and C. Zhang, "Global Asymptotic Stabilization of Feedforward nonlinear Systems With a Delay in the Input", *Int. Journal of Systems Science*, Vol. 37 , No 3, February 2006, pp. 141 - 148.