Input-to-State Stability of a Time-Varying Nonhomogeneous Diffusive Equation Subject to Boundary Disturbances

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***Abstract*— Input-to-state stability (ISS) of a diffusive equation with Dirichlet boundary conditions is shown, in the L^2-norm, with respect to boundary disturbances. In particular, the spatially distributed diffusion coefficients are allowed to be rapidly, yet smoothly, time-varying within a given set. Based on a Lyapunov function for the system with homogeneous boundary conditions, ISS inequalities are derived for the disturbed equation. A heuristic method used to numerically compute weighting functions is discussed. Numerical simulations are presented and discussed to illustrate the implementation of the theoretical results.**

I. INTRODUCTION

Parabolic partial differential equations (PDEs) are used to model a wide array of physical phenomena. Within this class of equations, diffusion or diffusive equations are a very common ocurrence. For most physical systems in which diffusive effects are present, diffusivity coefficients can be approximated as being constant throughout the domain of interest. However, in particular when dealing with nonhomogeneous or anisotropic (direction-dependent) media, the use of distributed coefficients is required. The extension of constant-coefficient results to these cases is not always easy to tackle and can be further complicated when the coefficients are time-varying.

Input-to-state stability (ISS) results in nonlinear finite-dimensional systems have been a long standing research topic and thorough reviews of such results can be found for instance in [16] and [8]. Nevertheless, ISS properties are not restricted to finite-dimensional systems. Some particularly interesting examples that can be cited in an infinite-dimensional setting are: [7], where a frequency-domain approach is used to guarantee ISS properties; [11], where a strict Lyapunov function is constructed for semilinear parabolic PDEs; and [13], where a strict Lyapunov function is used for time-varying hyperbolic PDEs.

The use of Lyapunov functions to study the solutions or properties of infinite-dimensional systems is not new, see for instance [1], but it is still an active research topic. Some interesting results involving Lyapunov functions applied to parabolic equations, other than those already mentioned in the context of ISS results can be found in [3], where a Lyapunov approach is used to prove the existence of a global solution to the heat equation; [10], where Lyapunov functions are used to analyze the regularity and well-posedness of Burgers' equation with a backstepping boundary control; [9], where a Lyapunov function is used to analyze the heat equation with unknown destabilizing parameters and its control extensions in [14] and [15]. Other results not involving parabolic equations are for example [5], where a Lyapunov function is used for the stabilization of a rotating beam, or more recently [6], where the construction of stabilizing boundary controls for a system of conservation laws is tackled using a Lyapunov function, as well as [4], where a Lyapunov function is used for the analysis of nonlinear hyperbolic systems.

There are two main contributions in this paper: the first one is, using the strict Lyapunov function constructed in [2], to set sufficient conditions for ISS, in the L^2-norm, with respect to boundary disturbances in a time-varying nonhomogeneous diffusive equation with rapidly (yet smoothly) time-varying coefficients; the second contribution is providing a heuristic method for numerically computing adequate weighting functions in order to apply the theoretical results. The use of strict Lyapunov functions was chosen since it provides a very natural framework for dealing with robustness issues and eventually considering nonlinearities in the system behaviour.

This article is organized as follows: In Section II, the diffusive equation and boundary disturbances under consideration are presented, as well as the general objective of the paper. In Section III, the main result of the paper, an ISS inequality with respect to boundary disturbances, is presented and a sufficient condition to find a strict Lyapunov function, taken from [2], is recalled. In Section IV, a heuristic method to find a suitable weighting function for exponentially-shaped diffusivity coefficients is presented. In Section V, a weighting function is provided and numerical simulations are presented for the system with and without boundary disturbances.

Notation

Throughout this paper the following notation conventions will be used: Given a function $\xi : (r,t) \mapsto \xi(r,t)$, its partial derivatives with respect to r and t will be denoted ξ_r and ξ_t, respectively; given a function of time $\Xi : t \mapsto \Xi(t)$, the derivative of Ξ with respect to time will be denoted $\dot{\Xi}$; given a function of a spatial variable $g : r \mapsto g(r)$, the first and second derivatives of g with respect to r will be

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denoted $g'(r)$ and $g''(r)$, respectively. For $g \in L^2([0,1])$ the following notation will be used $\|g\|_{L^2} \doteq \left(\int_0^1 g^2(\rho)d\rho\right)^{\frac{1}{2}}$.

II. Model and Problem Statement

Consider the following two-dimensional equation with symmetric coefficients (defined in $\Omega \doteq \left\{x=(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2<1\right\}$) expressed in cartesian coordinates:

$$\zeta_t(x,t) = \eta(x,t)\Delta\zeta(x,t), \forall (x,t) \in \Omega\times[0,T) \qquad (1)$$

where Δ is the Laplacian operator. With symmetric disturbed boundary condition:

$$\zeta_\nu(x,t) = \varepsilon(t), \forall (x,t) \in \partial\Omega\times[0,T) \qquad (2)$$

where ζ_ν is the derivative of ζ in the outward normal direction to $\partial\Omega$, and symmetric initial condition belonging to $C^1(\overline{\Omega})$:

$$z(x,0) = z_0(x), \forall x \in \overline{\Omega}. \qquad (3)$$

Under the revolution symmetry condition, system (1)-(3) can be expressed in the following one-dimensional representation, in polar coordinates which will be used as the reference equation throughout this article:

$$z_t = \left[\frac{\eta(r,t)}{r}\left[rz\right]_r\right]_r, \forall (r,t) \in (0,1)\times[0,T) \qquad (4)$$

with disturbed Dirichlet boundary condition:

$$\begin{aligned} z(0,t) &= 0, & \forall t \in [0,T) \\ z(1,t) &= \varepsilon(t), & \forall t \in [0,T) \end{aligned} \qquad (5)$$

where the condition at the center is given by the symmetry and regularity of the solutions, and with initial condition:

$$z(r,0) = z_0(r), \forall r \in [0,1]. \qquad (6)$$

The following assumption is used for the analysis of the well-posedness of the problem:

A1: η is positive and belongs to $C^\infty(\overline{\Omega}\times[0,T])$. ε belongs to $C^\infty([0,T])$.

Based on Theorem 6.2 in [12] (page 228), and using the same procedure as in Section II of [2], we have that:

Proposition 1: *Given A1, for every $z_0 : [0,1] \to \mathbb{R}$ in $L^p([0,1])$, $1<p<\infty$, the evolution equations* (4)-(6) *have a unique solution z in $C^\infty([0,1]\times(0,T))$.*

Remark 1: *The regularity conditions in A1 can be relaxed if the solution z is also allowed to be less regular, but it is beyond the scope of this article. Hereafter, sufficiently regular solutions to* (4)-(6) *are assumed to exist.*

The problem under consideration is then:

Problem 1: *Given a bounded disturbance signal $\varepsilon(t)$ with bounded derivative $\dot{\varepsilon}(t)$, find some bounds to the L^2-norm of the solution z of* (4)-(6).

III. Strict Lyapunov Function and sufficient conditions for Input-to-State Stability

Consider the following candidate Lyapunov function, for $\zeta \in L^2([0,1])$:

$$V(z) \doteq \frac{1}{2}\int_0^1 f(r)z^2 dr \qquad (7)$$

where $f : [0,1] \to \mathbb{R}^+$ is a twice differentiable positive function with bounded second derivative.

Following [16] and other references, V will be said to be a strict Lyapunov function for the undisturbed version of system (4)-(6) if, when setting $\varepsilon(t)=0$ for all $t\in[0,T)$, there exists some positive constant α such that, for every initial condition z_0 as defined in (6):

$$\dot{V} \leq -\alpha V(z(\cdot,t)), \forall t \in [0,T) \qquad (8)$$

where $\dot{V}$ stands for the time derivative of V along the trajectory of the undisturbed system stemming from z_0.

Hereafter, we shall define for any $g \in L^2([0,1])$ a weighted L^2 norm as follows $\|g\|_f \doteq (V(g))^{\frac{1}{2}}$.

A useful technical assumption is introduced:

A2: There exists a weighting function f as defined in (7) such that V is a strict Lyapunov function for system (4)-(6) if $\epsilon(t)=0$ for all $t\in[0,T)$.

The next theorem constitutes the main contribution of this article:

Theorem 2: *Under Assumption A2, the following inequality is satisfied, for all $t_0 \in [0,T)$, by the state of the disturbed system* (4)-(6)*:*

$$\begin{aligned} \|z(\cdot,t)\|_{L^2} \leq\ & ce^{-\frac{\alpha}{2}(t-t_0)}\left[\|z(\cdot,t_0)\|_{L^2} + \frac{1}{\sqrt{3}}|\varepsilon(t_0)|\right] \\ & + c\int_{t_0}^t e^{-\frac{\alpha}{2}(t-\tau)}\|\overline{\varepsilon}(\cdot,\tau)\|_{L^2}d\tau \\ & + \frac{c}{\sqrt{3}}|\varepsilon(t)|, \forall t \in [t_0,T) \end{aligned} \qquad (9)$$

where $\overline{\varepsilon}(r,t) \doteq 2\eta_r(r,t)\varepsilon(t) - r\dot{\varepsilon}(t)$ for all $(r,t)\in[0,1]\times[t_0,T)$, $c \doteq \sqrt{\frac{f_{max}}{f_{min}}}$ and $f_{min} \doteq \min_{r\in[0,1]}\{f(r)\}$, $f_{max} \doteq \max_{r\in[0,1]}\{f(r)\}$.

Proof: Consider an alternative definition of the state variable:

$$\hat{z}(r,t) \doteq z(r,t) - r\varepsilon(t), \forall (r,t) \in [0,1]\times[t_0,T). \qquad (10)$$

The evolution of the new state variable $\hat{z}$ will be given by:

$$\hat{z}_t = \left[\frac{\eta}{r}\left[r\hat{z}\right]_r\right]_r + 2\eta_r\varepsilon - r\dot{\varepsilon}, \forall (r,t) \in (0,1)\times[t_0,T)$$

with Dirichlet boundary conditions:

$$\hat{z}(0,t) = \hat{z}(1,t) = 0, \forall t \in [t_0,T) \qquad (11)$$

and initial condition:

$$\hat{z}(r,t_0) = z(r,t_0) - r\varepsilon(t_0), \forall r \in (0,1). \qquad (12)$$

Consider the function V defined in (7) with the weighting function of A2, and in a manner similar to [10], applied to the reformulated system (11)-(12):

$$V(\hat{z}) \doteq \frac{1}{2}\int_0^1 f(r)\hat{z}^2 dr \tag{13}$$

We compute, for all $t \in [t_0, T)$:

$$\begin{aligned} \dot{V} &= \int_0^1 f(r)\hat{z}\left[\frac{\eta}{r}\left[r\hat{z}\right]_r\right]_r dr \\ &\quad +2\int_0^1 f(r)\hat{z}\eta_r \varepsilon dr \\ &\quad -\int_0^1 f(r)\hat{z}r\dot{\varepsilon}dr. \end{aligned}$$

Using inequality (8) this implies, for all $t \in [t_0, T)$:

$$\begin{aligned} \dot{V} &\leq -\alpha V(\hat{z}) \\ &\quad +2\int_0^1 f(r)\hat{z}\eta_r \varepsilon dr \\ &\quad -\int_0^1 f(r)\hat{z}r\dot{\varepsilon}dr \end{aligned}$$

Using the definition of $\overline{\varepsilon}(r,t)$ in Theorem 2, it can be rewritten, for all $t \in [t_0, T)$, as:

$$\dot{V} \leq -\alpha V(\hat{z}) + \int_0^1 f(r)\hat{z}\overline{\varepsilon}dr$$

where, by the boundedness of $\varepsilon(t)$ and $\dot{\varepsilon}(t)$ in Problem 1, $\overline{\varepsilon}(r,t)$ is uniformly bounded in $[0,1]\times[t_0,T)$.

The last equation implies that, for all $t \in [t_0, T)$:

$$\dot{V} \leq -\alpha V(\hat{z}) + \int_0^1 |f(r)\hat{z}\overline{\varepsilon}|\, dr \tag{14}$$

Using the Cauchy-Schwarz inequality in (14), we have for all $t \in [t_0, T)$:

$$\dot{V} \leq -\alpha V(\hat{z}) + \|\sqrt{f(r)}\hat{z}\|_{L^2}\|\sqrt{f(r)}\overline{\varepsilon}\|_{L^2}$$

which implies:

$$\dot{V} \leq -\alpha V(\hat{z}) + 2\|\hat{z}\|_f\|\overline{\varepsilon}\|_f$$

from which:

$$\frac{d}{dt}\|\hat{z}\|_f \leq -\frac{\alpha}{2}\|\hat{z}\|_f + \|\overline{\varepsilon}\|_f$$

We get for all $t \in [t_0, T)$:

$$\|\hat{z}(\cdot,t)\|_f \leq e^{-\frac{\alpha}{2}(t-t_0)}\|\hat{z}(\cdot,t_0)\|_f + \int_{t_0}^t e^{-\frac{\alpha}{2}(t-\tau)}\|\overline{\varepsilon}(\cdot,\tau)\|_f d\tau$$

Recalling (10), and after some rearrangements, this implies for all $t \in [t_0, T)$:

$$\begin{aligned} \|z(\cdot,t)\|_f &\leq e^{-\frac{\alpha}{2}(t-t_0)}\left[\|z(\cdot,t_0)\|_f + |\varepsilon(t_0)|\|r\|_f\right] \\ &\quad + \int_{t_0}^t e^{-\frac{\alpha}{2}(t-\tau)}\|\overline{\varepsilon}(\cdot,\tau)\|_f d\tau \\ &\quad + |\varepsilon(t)|\|r\|_f \end{aligned} \tag{15}$$

Using the equivalence between the L^2 and $\|\cdot\|_f$ norms, and simply majorating and minorating f by f_{max} and f_{min} respectively, this implies (9) and completes the proof. ■

A simple application of Theorem 2, yields the following corollary:

Corollary 3: *If there is a nonnegative constant t_0 such that for all $t \geq t_0$, ε is zero, the state of the system (4)-(6) converges exponentially fast to zero in the topology of the L^2-norm.*

To give a sufficient condition for Assumption A2 to hold, it is useful to apply Theorem 3.2 from [2] as follows:

Proposition 4: *If there exist f, as defined in (7), and a positive constant α such that, for all $(r,t) \in [0,1]\times[0,T)$:*

$$f''(r)\eta + f'(r)\left[\eta_r - \frac{\eta}{r}\right] + f(r)\left[\frac{\eta_r}{r} - \frac{\eta}{r^2} + \alpha\right] \leq 0 \tag{16}$$

then Assumption A2 holds.

For review purposes, the proof of this result can be found in [2], a preprint version of which is available at http://www.gipsa-lab.fr/∼christophe.prieur/preprint11.pdf.

Remark 2: *Up to this point, no assumption on the shape or behaviour of η has been made other than some regularity requirements. In the next section a particular shape of η, motivated by a physical application, will be treated.*

IV. Finding a Weighting Function

The objective of this section is to propose a heuristic for numerically computing an adequate weighting function so that Assumption A2 holds by verifying the conditions of Proposition 4 for a particular set of diffusivity coefficients. In the rest of this article, the η profile is assumed to be of the form:

$$\eta(r,t) = a(t)e^{\lambda(t)r}, \forall (r,t) \in [0,1]\times[0,T) \tag{17}$$

where $0 < \underline{a} \leq a(t) \leq \overline{a}$, and $0 < \underline{\lambda} \leq \lambda(t) \leq \overline{\lambda}$.

This choice of profiles was physically motivated by the application of magnetic flux profile control in Tokamak plasmas, c.f. section V of [2] for a more detailed discussion.

Proposition 5: *With η defined as in (17), a sufficient condition to apply Proposition 4 is the existence of a twice differentiable positive function $f : [0,1] \to \mathbb{R}^+$ with bounded second derivative such that the following inequality is verified:*

$$f''(r) + f'(r)\left[\lambda - \frac{1}{r}\right] + f(r)\left[\frac{\lambda}{r} - \frac{1}{r^2} + \epsilon\right] \leq 0 \tag{18}$$

for every $(r,\lambda) \in [0,1]\times[\underline{\lambda},\overline{\lambda}]$ and some positive constant ϵ.

Proof: This proposition results from multiplying (18) by $a(t)e^{\lambda r}$ and setting $\alpha \doteq \epsilon \inf_{(r,t,\lambda)\in[0,1]\times[0,T)\times[\underline{\lambda},\overline{\lambda}]}\left\{a(t)e^{\lambda r}\right\} \geq \epsilon\underline{a} > 0$ ■

A twice differentiable positive function $f : [0,1] \to \mathbb{R}^+$ satifies (18) if there exists $w(r,\lambda) \leq 0$ for all $\lambda \in [\underline{\lambda},\overline{\lambda}]$ such that, for all $(r,\lambda) \in [0,1]\times[\underline{\lambda},\overline{\lambda}]$ the following equation is verified:

$$\begin{bmatrix} f \\ f' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{1}{r^2} - \frac{\lambda}{r} - \epsilon & \frac{1}{r} - \lambda \end{bmatrix}\begin{bmatrix} f \\ f' \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(r,\lambda) \tag{19}$$

In order to avoid testing the condition for all $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, the following result is needed:

Proposition 6: *Given a twice differentiable positive function $f : [0,1] \rightarrow \mathbb{R}^+$, the following two conditions are equivalent:*

i: *There exists $w(r,\lambda) \le 0$ such that* (19) *is verified for all $(r,\lambda) \in [0,1] \times [\underline{\lambda}, \overline{\lambda}]$.*

ii: *There exists $w_2(r) \le 0$ such that the following equation is verified for all $r \in [0,1]$:*

$$\begin{bmatrix} f \\ f' \end{bmatrix}' = A(f,r) \begin{bmatrix} f \\ f' \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_2(r) \qquad (20)$$

where:

$$A(f,r) = \begin{cases} \begin{bmatrix} 0 & 1 \\ \frac{1}{r^2} - \frac{\underline{\lambda}}{r} - \epsilon & \frac{1}{r} - \underline{\lambda} \end{bmatrix} & \text{if } sw(f,r) \le 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{1}{r^2} - \frac{\overline{\lambda}}{r} - \epsilon & \frac{1}{r} - \overline{\lambda} \end{bmatrix} & \text{if } sw(f,r) > 0 \end{cases}$$

where $sw(f,r) \doteq \frac{f(r)}{r} + f'(r)$.

Proof: The proof stems from the fact that the left-hand side of (18), which is equivalent to (19), is linear in λ and therefore attains its maximum at either $\overline{\lambda}$ or $\underline{\lambda}$. It is easy to verify that the switching condition in matrix $A(f,r)$ corresponds to the sign of the partial derivative of the left-hand side of (18) with respect to λ. Therefore, $w_2(r) = max_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \{w(r,\lambda)\}$ for all $r \in [0,1]$. ■

Remark 3: *The easiest way to find a function f that satisfies condition* (20) *would be to fix some initial conditions for f and f', set $w_2(r) = 0$ for all $r \in [0,1]$, and solve the resulting equation. Nevertheless, this yields solutions with a singularity at the origin, as can be seen in Figure 1 for $\underline{\lambda} = \overline{\lambda} = 4$.*

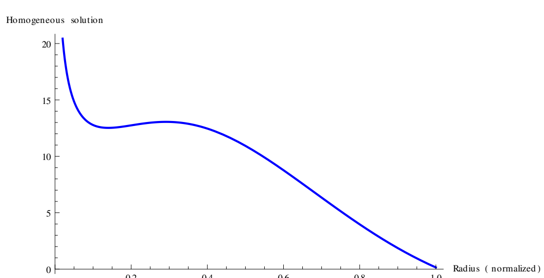

Fig. 1. Function f obtained by numerically solving the homogeneous equation (19) for a single value of $\lambda = 4$.

Since setting $w_2(r) = 0$ does not suffice to find adequate weighting functions, a more structured approach was developed to tackle this problem. In order to compute a weight verifying condition ii of Proposition 6, initial conditions will be set at $r = 1$ and the equation will be solved backwards up to $r = 0$ using Algorithm 1.

Algorithm 1:

1: *Set numerical values for the initial conditions at $r = 1$, $f(1)$ and $f'(1)$, and for ϵ.*

2: *Evaluate $\frac{f(r)}{r} + f'(r)$ and fix the value of the dynamic matrix $A(f,r)$ accordingly, using* (20).

3: *Find a numerical solution going backwards until hitting a zero-crossing of $\frac{f(r)}{r} + f'(r)$, setting $w_2(r) = 0$, and verifying that $f(r)$ remains positive. Otherwise, change the initial conditions or the value of ϵ.*

4: *Use the values of $f(r)$ and $f'(r)$ at the zero-crossings of $\frac{f(r)}{r} + f'(r)$ as initial values for the next step in solving the equation, switching the dynamic matrix but keeping $w_2(r) = 0$, always verifying that $f(r)$ remains positive and bounded.*

5: *Repeat 3-4 until either reaching $r = 0$ or finding a point such that both elements in the lower row of the A matrix are positive, as well as f and f', with $f(r) - rf'(r) > 0$. If no such point exists before $r = 0$, change the initial conditions or the value of ϵ and start over.*

6: *If $r = 0$ has not been reached yet, complete the solution by setting $w_2(r)$ to have $f''(r) = 0$ for the remaining interval, in order to avoid singularities in the solution near zero.*

Remark 4: *Although this heuristic does not guarantee finding an adequate weighting function, it does provide a methodic framework to search for one and, in practice, yields good results, as shown in the next section.*

The conclusion of this section is that Algorithm 1 gives a practical way for numerically testing condition ii in Proposition 6, which in turn, by Proposition 5 implies the conditions of Proposition 4 are verified for η defined as in (17). Proposition 4 being a sufficient condition for A2, Theorem 2 follows and Problem 1 is solved.

V. Numerical Application

A. Weighting Function

We shall set to find an adequate weighting function for $a(t) \in [0.0093, 0.0121]$, $\lambda \in [4, 7.3]$ (see (17)). This implies a 55% increase in the allowable range for $\eta(1,t)$ with respect to the objective set in [2].

The initial conditions at $r = 1$ were chosen as $f_1 = 0.15$, $f'_1 = -15$, and a suitable weighting function was found for a maximum value of $\epsilon = 5.3$. Given the initial conditions, the solution was obtained first using the dynamic matrix with $\underline{\lambda}$ and then switching dynamics at $r \approx 0.51988$. For $r \in [0, 0.015]$, f'' was set to 0 using $w(r)$. The resulting weighting function, numerically computed using Mathematica, can be seen in Figure 2. The piecewise continuous and bounded second derivative of the weighting function is also shown in Figure 3. The maximum value of f is ~ 12.4037 and the minimum is 0.15, which means the constant c used for the norm equivalence and in (9) has a value of ~ 9.09347. Other functions with a much lower value of c can be found, but usually there is a compromise between this constant and the guaranteed value for ϵ.

In order to test that this function verifies the conditions of Proposition 5, the value of the left-hand side of the inequality was plotted for values of $(r,\lambda) \in [0,1] \times [4, 7.3]$. The result can be seen in Figure 4. It is interesting to note that for each value of r, the critical value of λ in the inequality is the one used to compute the weighting function. Only at values of r

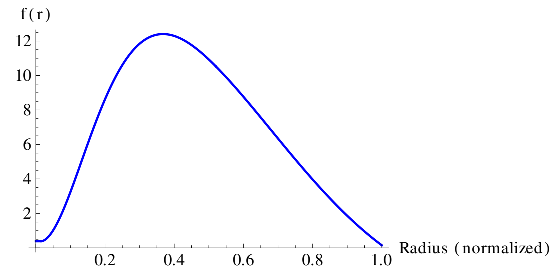

Fig. 2. Function f obtained using the heuristic.

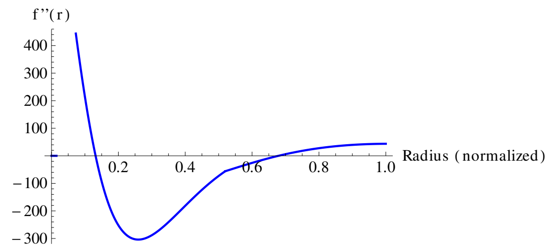

Fig. 3. Piecewise continuous second derivative of function f obtained using the heuristic.

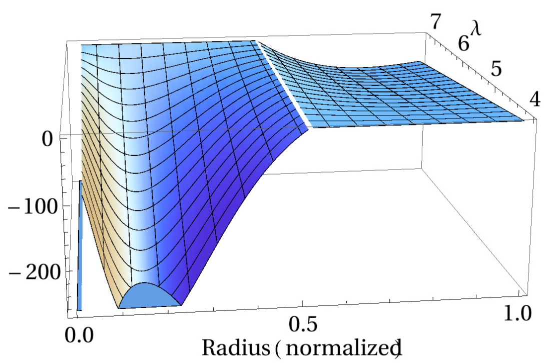

Fig. 4. Numerical test of the conditions of Proposition 5.

close to zero is the slack variable w different from zero at all values of λ, avoiding the singularity in $f(r)$, as desired.

B. Simulations

Using the weighting function found in the previous subsection, the evolution of the diffusion equation was simulated using Matlab. The numerical scheme used is an explicit finite-difference method with space and time steps such that the CFL condition, $\max_{(r,t)\in[0,1]\times[0,T)}\{\eta\}\frac{\Delta t}{(\Delta x)^2} \leq 0.5$, is verified.

First, choosing the minimum values for the diffusion coefficients and without disturbances, the evolution of the distributed state can be seen in Figure 5. With this simulation, the evolution of the Lyapunov function and the equivalent rate of convergence are shown in figures 6 and 7, respectively. For this initial condition, the guaranteed rate of convergence is ~ 23 times smaller than the obtained one. Considering the fact that the condition imposed in Proposition 4 was verified at every point and that the diffusivity at

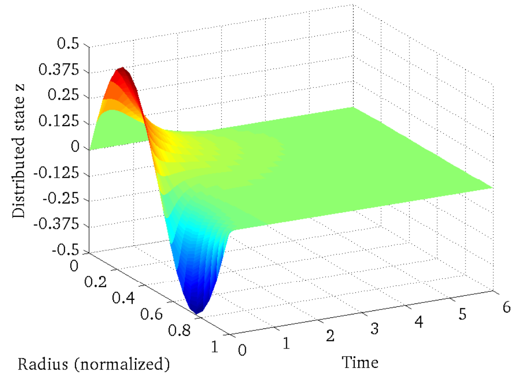

Fig. 5. Evolution of the state with no disturbances and minimum diffusivity.

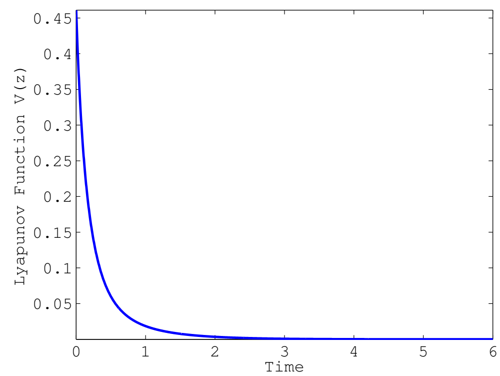

Fig. 6. Evolution of the Lyapunov function with no disturbances and minimum diffusivity.

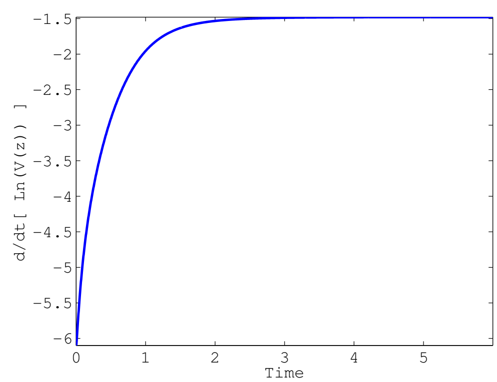

Fig. 7. Exponential convergence rate for $V(z)$ with no disturbances and minimum diffusivity.

the right boundary is ~ 55 times the one at the center, this level of conservatism is not unexpected.

Next, introducing a boundary disturbance equal to $\varepsilon(t) = 0.1 + 0.05\sin(4.56\pi t)$ and letting the diffusivity coefficient vary with $a(t) = 0.0107 - 0.0014\cos(4\pi t)$ and $\lambda(t) = 5.65 + 1.65\sin(2\pi t)$ the resulting evolution can be seen in Figure 8 and in Figure 9. The behaviour of the Lyapunov function is shown in Figure 10. The Lyapunov function

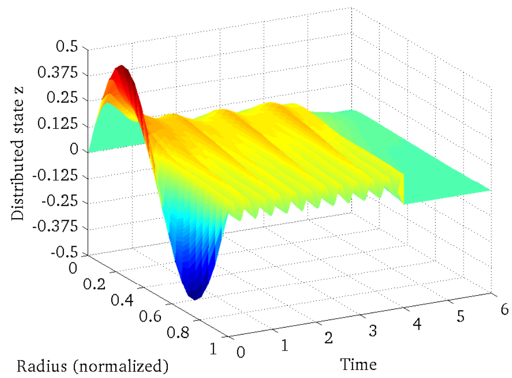

Fig. 8. Evolution of the state with disturbances and time-varying diffusivity.

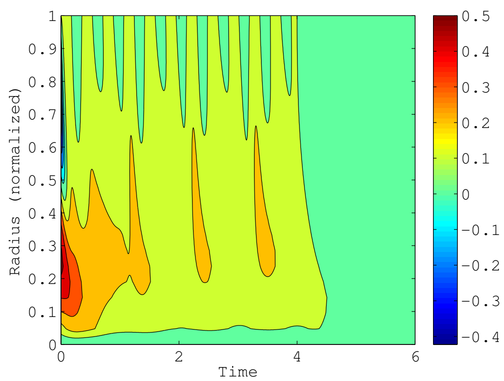

Fig. 9. Contour plot of the evolution of the state with disturbances and time-varying diffusivity.

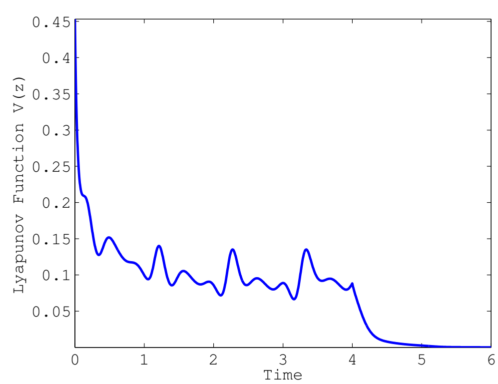

Fig. 10. Evolution of the Lyapunov function with disturbances and time-varying diffusivity.

remains bounded while the disturbance is present and, when the disturbance is cut at $t = 4$ the exponential convergence to zero is verified.

VI. Conclusions and Perspectives

In this article, ISS-like inequalities with respect to boundary disturbances are derived for a diffusive equation with singular coefficients stemming from a change between cartesian and polar coordinates. The ISS condition is obtained by means of a strict Lyapunov function for the undisturbed system. Another contribution of this article is a detailed account of the method used to numerically find suitable weighting functions in order to implement the obtained results for certain diffusivity profiles. Simulation results were obtained by discretizing the system using a finite-difference method.

Further works will tackle the problem of reducing the conservatism of this approach in order to better estimate the convergence rates, thus refining the ISS inequalities for the system. Also, extensions to other forms of diffusivity profiles is desirable.

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