Controllability and Invariance of monotone systems for robust ventilation automation in buildings

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Abstract— **The problem considered is the temperature control in a building equipped with UnderFloor Air Distribution (UFAD). Its 0-D model is derived from the energy and mass conservation in each room, and also presents discrete components to describe the disturbances from heat sources and doors opening. Using the monotonicity of this model, we can characterize two concepts of robust control, the *Robust Controllability* and the *Robust Controlled Invariance* introduced in this paper, and determine their limits for control design objectives. The validity of these results is then illustrated in a simulation of a two-room example.**

I. INTRODUCTION

Since the introduction of the concept of intelligent buildings in the 1980s, this topic has been the source of a substantial amount of work [1]. In the particular case of climate regulation in a building, research on modeling, simulation [2] and control [3] of Heating, Ventilating and Air Conditioning (HVAC) systems leads to an improved comfort for the users and a reduction of energy consumption. Compared to traditional ceiling ventilation, the UnderFloor Air Distribution (UFAD) solution that we chose has shown some interesting results on these matters [4].

Various paths have already been explored for the control of HVAC systems in intelligent buildings. When the focus is mainly on control, numerous feedback strategies have been devised, based on simple PID or On/Off control, more robust controllers with the $\mathcal{H}_\infty$ approach [5], or non-linear approaches [6]. For more energy-efficient controllers, we can look for the optimal tradeoff between comfort and energy saving [7], a model-predictive strategy [8], or a fuzzy logic controller [9].

Our goal for future work is to develop a discrete control strategy using a symbolic transformation on the UFAD 0-D model [5]. The purpose of this paper is to establish the theoretical limits for notions of robust control in this framework: the *Controllability*, to reach a state; or the *Invariance*, to stay in an interval. This is achieved by using the monotonicity property [10] verified by our model. The obtained results thus reflect the model properties and do not depend on any specific feedback control strategy.

The paper is organized as follows. In Section II, we describe the 0-D UFAD model, for which we prove the monotonicity property in Section III. In Section IV, we introduce the notions and the main theorems on controllability and invariance. Finally, Section V gives a simulation example to illustrate the previous results.

This work was supported by a PhD scholarship and the research project COHYBA funded by Région Rhône-Alpes.

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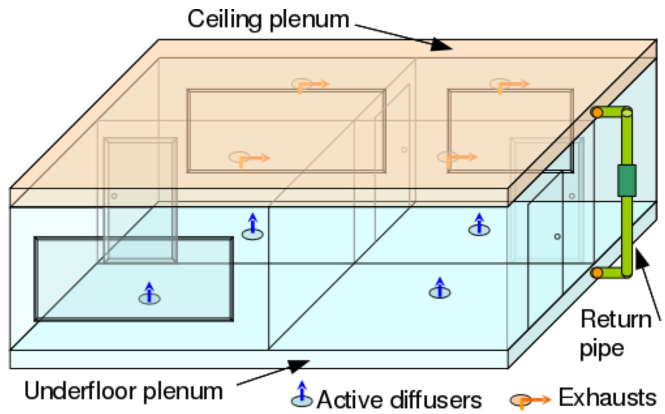

Fig. 1. 4-room building with UnderFloor Air Distribution

II. SYSTEM DESCRIPTION

The system considered is equipped with UnderFloor Air Distribution (UFAD), and is based on the scale model prototype [1] sketched on Fig. 1. It has an underfloor plenum where the air is cooled down and sent into the rooms. The excess of air in each room is pushed into the plenum above the fake ceiling, and sent back to the underfloor plenum through a pipe outside of the building. The control of the individual room temperatures (our control objective) is done through the speed of the underfloor fans, sending cold air from the underfloor plenum to each room. This system is subject to the following disturbances: door opening between the rooms; heat sources in each room that can be on or off.

As in [5], we consider a 0-D model for this ventilation system. Due to the reduced speed and mass of air, we assume that it is incompressible and its kinetic and potential energies can be neglected. We also consider the density and temperature in a room to be uniform. The model is based on energy and mass conservation in each room, expressed in (1) and (2).

$$\frac{dE_i}{dt} = \dot{Q}_i + \sum_k C_p T_k \dot{m}_{k\to i} - \sum_k C_p T_i \dot{m}_{i\to k} \qquad (1)$$

Equation (1) is the first law of thermodynamics applied to the room i. $E_i = \rho_i V_i C_v T_i$ is the room energy, $\dot{Q}_i$ the heat

[1]Built at the physics department of University Joseph Fourier, Grenoble, France.

exchanges, $\dot{m}_{k\to i}$ and $\dot{m}_{i\to k}$ are the mass flow rates entering and leaving room i respectively, with k representing another room or a plenum. The mass flow rates in (1) are positive and associated with the temperature of the room from where the air flow is coming. T_i, V_i and ρ_i are the temperature, volume and air density of room i. C_v and C_p are respectively the constant volume specific heat and constant pressure specific heat.

The heat exchanges considered in this model are of two kinds. The conduction between rooms i and j through a wall of conductivity k, surface A and thickness Δ is given by:

$$\dot{Q}_{cond} = -\frac{kA}{\Delta}(T_i - T_j),$$

where T_j may also represent the temperature of a plenum (T_c for the ceiling; T_u for the underfloor) or of the outside (T_o) [11]. The radiation in room i from a heat source s of emissivity ϵ, temperature T_s and surface A_s writes as follow:

$$\dot{Q}_{rad} = \epsilon\sigma A_s(T_s^4 - T_i^4),$$

where σ is the Stephan-Boltzmann constant [12].

The energy transfer ($\pm C_p T_a \dot{m}_{a\to b}$) induced by a mass flow rate is positive only when the flow is entering the room considered in (1). For a given room i, these energy transfers can be of four types:

- $C_p T_u \dot{m}_{u\to i}$: where the mass flow from the underfloor plenum to room i is forced by the fan, which is our controlled input;
- $-C_p T_i \dot{m}_{i\to c}$: where the mass flow rate corresponds to the air in room i pushed into the ceiling plenum;
- $C_p T_j \dot{m}_{j\to i}$: when the door between rooms i and j is open and $T_i < T_j$;
- $-C_p T_i \dot{m}_{i\to j}$: when the door between rooms i and j is open and $T_i > T_j$.

If we note $\dot{m}_{d_{ij}}$ the flow going through the door section $A_{d_{ij}}$ (always from the high to the low temperature room: $\dot{m}_{d_{ij}} = \dot{m}_{j\to i}$ when $T_i < T_j$; $\dot{m}_{d_{ij}} = \dot{m}_{i\to j}$ when $T_i > T_j$), its expression is derived from Bernoulli's equation as $\dot{m}_{d_{ij}} = \rho A_{d_{ij}}\sqrt{\frac{2R}{\mathcal{M}}|T_i - T_j|}$, with $\mathcal{M}$ the molar mass of air, and $R = C_p - C_v$. The mass conservation in room i is expressed by the following equation:

$$\dot{m}_{u\to i} - \dot{m}_{i\to c} + \sum_{j\in\mathcal{N}_i} sign(T_j - T_i)\dot{m}_{d_{ij}} = 0, \quad (2)$$

where $\mathcal{N}_i$ is the set of rooms adjacent to room i. Equation (2) allows us to replace the unknown flow going into the ceiling by its expression depending on the other known flows.

We describe the state of the discrete disturbances using two boolean inputs: $\delta_{d_{ij}} = 1$ when the door between rooms i and j is open; and $\delta_{s_i} = 1$ when the heat source in room i is active. To simplify the notations, we introduce the continuously differentiable function h:

$$h(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^{3/2} & \text{if } x > 0. \end{cases}$$

The temperature dynamics is thus finally obtained as:

$$\begin{aligned} \rho V_i C_v \frac{dT_i}{dt} = & -\alpha_{u_i}(T_i - T_u) - \alpha_{c_i}(T_i - T_c) \\ & - \sum_{j\in\mathcal{N}_i} \alpha_{i,j}(T_i - T_j) - \alpha_{o_i}(T_i - T_o) \\ & + C_p\dot{m}_{u\to i}(T_u - T_i) + \delta_{s_i}\epsilon\sigma A_{s_i}(T_{s_i}^4 - T_i^4) \\ & + \sum_{j\in\mathcal{N}_i} \delta_{d_{ij}} C_p \rho A_{d_{ij}} \sqrt{\frac{2R}{\mathcal{M}}} * h(T_j - T_i), \quad (3) \end{aligned}$$

where $\alpha_x = k_x A_x/\Delta_x$ is a conduction factor with x representing the connection between a room and either another room, a plenum or the outside.

Equation (3) describes a dynamical system of state $T = [T_1, \ldots, T_n]$. The inputs for this system are of three kinds. The controlled input $u \in \mathbb{R}^n$ (where we note $u_i = \dot{m}_{u\to i} \ge 0$ to simplify the notations) corresponds to the mass flow rates sent by the underfloor fans into each room. The vector of exogenous inputs is $w \in \mathbb{R}^p$ ($p = n+3$) and gathers T_u, T_c, T_o and the surface temperature of the sources T_{s_i}. These temperatures are considered as known exogenous inputs, controlled by external loops or measured. We assume that the underfloor temperature is controlled so that at all time, its value is set to $T_u \le min(T_i)$, otherwise we would not be able to cool down some of the rooms. To consider s_i as a heat source, we also assume that its surface temperature is always $T_{s_i} \ge T_i$. The last input vector $\delta \in \mathbb{R}^q$ contains all the boolean variables representing the disturbances: $\delta_{d_{ij}}$ for the doors, and δ_{s_i} for the heat sources.

III. MONOTONICITY

We consider the dynamical system of state $x \in \mathbb{R}^n$ and input $v \in \mathbb{R}^m$ defined by the differential equations $\dot{x} = f(x, v)$. In the case of our UFAD problem, since the booleans δ can also be considered as taking their values in $\mathbb{R}^q$, we have $x = T$ and $v = [u, w, \delta]$. Let $\Phi(t, x_0, \mathbf{v})$ be the state trajectory for the initial condition x_0 and input function $\mathbf{v} : \mathbb{R}^+ \to \mathbb{R}^m$. An ordering $\succeq_x$ for the state is defined by a positive cone $K_x \subset \mathbb{R}^n$ such that $x \succeq_x x' \Leftrightarrow x - x' \in K_x$. We can take a similar ordering for the input functions: $\mathbf{v} \succeq_v \mathbf{v}' \Leftrightarrow \forall t \ge 0,\ \mathbf{v}(t) \succeq_v \mathbf{v}'(t)$. The system is monotone, as in [10], if the following holds $\forall t \ge 0$:

$$x \succeq_x x',\ \mathbf{v} \succeq_v \mathbf{v}' \ \Rightarrow\ \Phi(t, x, \mathbf{v}) \succeq_x \Phi(t, x', \mathbf{v}').$$

Let the ordering be defined by an orthant of the state space:

$$x \succeq_x x' \ \Leftrightarrow\ \forall i \in \{1, \ldots, n\},\ (-1)^{\varepsilon_i}(x_i - x_i') \ge 0,$$

with $\varepsilon \in \{0,1\}^n$. Similarly, we take $\gamma \in \{0,1\}^m$ for the input space. This leads to a characterization of the monotonicity using the differential equations of the system, and without needing an explicit expression of its trajectories.

Proposition 1: [10] The system defined by $\dot{x} = f(x, v)$ is monotone if and only if, $\forall i \in \{1, \ldots, n\},\ \forall j \ne i,\ \forall k \in \{1, \ldots, m\}$,

$$\forall x \in \mathbb{R}^n,\ \forall v \in \mathbb{R}^m, \begin{cases} (-1)^{\varepsilon_i + \varepsilon_j} \dfrac{\partial f_i}{\partial x_j}(x, v) \ge 0 \\ (-1)^{\varepsilon_i + \gamma_k} \dfrac{\partial f_i}{\partial v_k}(x, v) \ge 0. \end{cases}$$

For our system (3), we consider the following four orderings:

$$T \succeq_T T' \;\Leftrightarrow\; \forall i,\; T_i \geq T'_i \tag{4}$$
$$\mathbf{u} \succeq_u \mathbf{u}' \;\Leftrightarrow\; \forall t \geq 0,\; \forall k,\; \mathbf{u_k}(t) \leq \mathbf{u'_k}(t) \tag{5}$$
$$\mathbf{w} \succeq_w \mathbf{w}' \;\Leftrightarrow\; \forall t \geq 0,\; \forall k,\; \mathbf{w_k}(t) \geq \mathbf{w'_k}(t) \tag{6}$$
$$\boldsymbol{\delta} \succeq_\delta \boldsymbol{\delta}' \;\Leftrightarrow\; \forall t \geq 0,\; \forall k,\; \boldsymbol{\delta}_\mathbf{k}(t) \geq \boldsymbol{\delta}'_\mathbf{k}(t). \tag{7}$$

Theorem 1: With the orderings (4) to (7), the dynamical system defined by (3) is monotone:

$$\forall\, T_0 \succeq_T T'_0,\; \mathbf{u} \succeq_u \mathbf{u}',\; \mathbf{w} \succeq_w \mathbf{w}',\; \boldsymbol{\delta} \succeq_\delta \boldsymbol{\delta}',\; \forall t \geq 0,$$
$$\Phi(t, T_0, \mathbf{u}, \mathbf{w}, \boldsymbol{\delta}) \succeq_T \Phi(t, T'_0, \mathbf{u}', \mathbf{w}', \boldsymbol{\delta}').$$

Proof: Using Proposition 1, we show that the system is monotone for the chosen orderings. With respect to the state variables, the partial derivatives give, $\forall i \neq j$:

$$\rho V_i C_v \frac{\partial f_i}{\partial T_j} = \alpha_{i,j} + \delta_{d_{ij}} C_p \rho A_{d_{ij}} \sqrt{\frac{2R}{\mathcal{M}}} \frac{\partial h(T_j - T_i)}{\partial T_j} \geq 0,$$

so the only condition on the state ordering is that the chosen orthant is either $(\mathbb{R}^+)^n$ or $(\mathbb{R}^-)^n$. To keep it simple and consistent with the physics of our system, we take the natural ordering induced by the positive orthant (4).

With respect to the controlled input, we obtain for our ventilation problem:

$$\forall i,\; \rho V_i C_v \frac{\partial f_i}{\partial u_i} = C_p(T_u - T_i) \leq 0, \qquad \forall k \neq i,\; \frac{\partial f_i}{\partial u_k} = 0.$$

With the ordering already chosen for the state (4), Proposition 1 implies that we choose the negative orthant $(\mathbb{R}^-)^n$ for the controlled input ordering, as in (5). We can note that for a heating problem ($T_u \geq max(T_i)$), we should choose the natural ordering: $\forall i,\; \mathbf{u_i}(t) \geq \mathbf{u'_i}(t)$.

For the exogenous inputs, we have:

$$\forall k,\; \forall i,\; \frac{\partial f_i}{\partial w_k} \geq 0.$$

So we need to choose the same orthant for T and $\mathbf{w}$. With the state ordering (4) known, we also need to take the natural ordering (6) for the exogenous inputs.

Since we are only interested in the extremal values 0 and 1, and it is not required in [10] for the inputs to be continuous, we compute the partial derivatives with respect to the boolean inputs as if they were defined for some continuous variables $\delta \in \mathbb{R}^q$. The partial derivatives obtained for both door and source booleans are always positive. Therefore, as for $\mathbf{w}$, Proposition 1 implies that we choose the ordering defined by the positive orthant for (7). ■

IV. CONTROLLABILITY AND INVARIANCE

We want to study the possibility of controlling the system so that it has a given behavior. Depending on the desired behavior, the notion of controllability can have various forms. In this section, we see two of them: controlling the system on a given point; or keeping the state in an interval.

A. Robust Invariance

All the inputs are considered bounded. Either because of physical constraints ($\delta,\ u,\ T_{s_i}$), because they are controlled (T_u), or due to observations ($T_c,\ T_o$). For a given variable $a \in \mathbb{R}^b$, we define the b-dimension interval $[\underline{a}, \overline{a}]$ according to the natural ordering (induced by the positive orthant $(\mathbb{R}^+)^b$). For the corresponding function $\mathbf{a} : \mathbb{R}^+ \to \mathbb{R}^b$, we also use the notation $\mathbf{a} \in [\underline{a}, \overline{a}]$ instead of $\forall t \geq 0,\ \mathbf{a}(t) \in [\underline{a}, \overline{a}]$.

Definition 1 (Robust Invariance): The system is said to be *Robust Invariant* in an interval $[\underline{T_r}, \overline{T_r}]$ if,

$$\forall T_0 \in [\underline{T_r}, \overline{T_r}],\; \forall \mathbf{w} \in [\underline{w}, \overline{w}],\; \forall \boldsymbol{\delta} \in [\underline{\delta}, \overline{\delta}],\; \forall \mathbf{u} \in [\underline{u}, \overline{u}],$$
$$\forall t \geq 0,\; \Phi(t, T_0, \mathbf{u}, \mathbf{w}, \boldsymbol{\delta}) \in [\underline{T_r}, \overline{T_r}].$$

For all bounded external conditions (w and δ) and controlled inputs (u), the state cannot leave this interval. So this interval contains all the equilibria of the system. However, it does not mean that all points in the interval are reachable.

Proposition 2: If $[\underline{T_r}, \overline{T_r}]$ is defined by $f(\underline{T_r}, \overline{u}, \underline{w}, \underline{\delta}) = 0$ and $f(\overline{T_r}, 0, \overline{w}, \overline{\delta}) = 0$, it is the smallest interval in which the system is *Robust Invariant*.

Proof: The monotonicity of our system, in Theorem 1, implies that

$$\forall T_0 \in [\underline{T_r}, \overline{T_r}],\; \forall \mathbf{w} \in [\underline{w}, \overline{w}],\; \forall \boldsymbol{\delta} \in [\underline{\delta}, \overline{\delta}],\; \forall \mathbf{u} \in [0, \overline{u}],\; \forall t,$$
$$\Phi(t, \overline{T_r}, 0, \overline{w}, \overline{\delta}) \succeq_T \Phi(t, T_0, \mathbf{u}, \mathbf{w}, \boldsymbol{\delta}) \succeq_T \Phi(t, \underline{T_r}, \overline{u}, \underline{w}, \underline{\delta}).$$

Since $f(\underline{T_r}, \overline{u}, \underline{w}, \underline{\delta}) = 0$ and $f(\overline{T_r}, 0, \overline{w}, \overline{\delta}) = 0$, we have $\Phi(t, \underline{T_r}, \overline{u}, \underline{w}, \underline{\delta}) = \underline{T_r}$ and $\Phi(t, \overline{T_r}, 0, \overline{w}, \overline{\delta}) = \overline{T_r}$. So $\forall t \geq 0,\ \Phi(t, T_0, \mathbf{u}, \mathbf{w}, \boldsymbol{\delta}) \in [\underline{T_r}, \overline{T_r}]$, and $[\underline{T_r}, \overline{T_r}]$ is indeed a *Robust Invariant* interval. The boundaries $\underline{T_r}$ and $\overline{T_r}$ are equilibria of the system, so we cannot find a smaller *Robust Invariant* interval. ■

In what follows, we use the notation $[\underline{T}, \overline{T}]$, for the interval in which we want to control the system.

B. Robust Controllability

We define the *Robust Controllability* with the states which are reachable by the system for all the external conditions ($\mathbf{w}$ and $\boldsymbol{\delta}$).

Definition 2 (Robust Controllability): The system is said to be *Robust Controllable* in a set $\mathcal{S}$ if,

$$\forall T \in \mathcal{S},\; \forall T_0 \in \mathcal{S},\; \forall \mathbf{w} \in [\underline{w}, \overline{w}],\; \forall \boldsymbol{\delta} \in [\underline{\delta}, \overline{\delta}],$$
$$\exists \mathbf{u} \in [0, \overline{u}]\;, \exists t \geq 0 \mid \Phi(t, T_0, \mathbf{u}, \mathbf{w}, \boldsymbol{\delta}) = T.$$

Using the monotonicity, we can obtain a new characterization of the *Robust Controllability*.

Theorem 2: The system is *Robust Controllable* in a set $\mathcal{S}$, if

$$\forall T \in \mathcal{S},\; \forall i \in \{1, \ldots, n\},\; \begin{cases} f_i(T, \overline{u_i}, \overline{w}, \overline{\delta}) < 0 \\ f_i(T, 0, \underline{w}, \underline{\delta}) > 0 \end{cases}$$

Proof: Firstly, we prove that the following equation is implied by the one involved in Theorem 2.

$$\forall i \in \{1,\dots,n\},\ \forall w \in [\underline{w},\overline{w}],\ \forall \delta \in [\underline{\delta},\overline{\delta}],$$
$$\exists u_i^a \geq u_i^b \in [0,\overline{u_i}] \mid \begin{cases} f_i(T,u_i^a,w,\delta) < 0 \\ f_i(T,u_i^b,w,\delta) > 0 \end{cases} \tag{8}$$

The existence of u_i^a and u_i^b is a mere reformulation since we can take $u_i^a = \overline{u_i}$ and $u_i^b = 0$. Then we obtain (8) by using the partial derivatives with respect to the exogenous or boolean inputs which are positive (see the proof of Theorem 1): given T and u_i, $\forall w \in [\underline{w},\overline{w}]$, $\forall \delta \in [\underline{\delta},\overline{\delta}]$,

$$f_i(T,u_i,\overline{w},\overline{\delta}) \geq f_i(T,u_i,w,\delta) \geq f_i(T,u_i,\underline{w},\underline{\delta}).$$

Equation (8) implies that for any $T_0 \in \mathcal{S}$ in the neighborhood of $T \in \mathcal{S}$, and for all bounded external conditions w and δ, we can force the vector field f to point in the direction of T, with $f \neq 0$. This implies the condition for the *Robust Controllability* given in Definition 2. ■

Even though we do not prove what follows because it has no utility for this paper, we can note that we actually have an equivalence between Theorem 2 and (8) which leads to the following remark.

Remark 1: If Definition 2' is Definition 2 restricted to the open control set ($\exists u \in]0,\overline{u}[\mid \Phi(t,T_0,\mathbf{u},\mathbf{w},\boldsymbol{\delta}) = T$), and Theorem 2' is Theorem 2 with non-strict inequalities ($f_i(T,\overline{u_i},\overline{w},\overline{\delta}) \leq 0$; $f_i(T,0,\underline{w},\underline{\delta}) \geq 0$),

Definition 2' ⇒ Theorem 2 ⇒ Definition 2 ⇒ Theorem 2'.

If the system is *Robust Controllable* at a state T, this result means that for all bounded external conditions w and δ, we can warm up and cool down the temperature of each room, or more precisely we can at least prevent the temperature from increasing and decreasing (since we are referring to the last implication of Remark 1).

C. Robust Controlled Invariance

This notion is less restrictive than the *Robust Controllability*, because here we only want to keep the state in a given interval $[\underline{T},\overline{T}]$.

Definition 3 (Robust Controlled Invariance): The system is said to be *Robust Controlled Invariant* in $[\underline{T},\overline{T}]$ if,

$$\forall T_0 \in [\underline{T},\overline{T}],\ \forall \mathbf{w} \in [\underline{w},\overline{w}],\ \forall \boldsymbol{\delta} \in [\underline{\delta},\overline{\delta}],$$
$$\exists \mathbf{u} \in [0,\overline{u}] \mid \forall t \geq 0,\ \Phi(t,T_0,\mathbf{u},\mathbf{w},\boldsymbol{\delta}) \in [\underline{T},\overline{T}].$$

In a similar way than for Theorem 2, we can obtain new conditions for the *Robust Controlled Invariance*.

Theorem 3: The system is *Robust Controlled Invariant* in $[\underline{T},\overline{T}]$ if and only if

$$\forall i \in \{1,\dots,n\},\ \begin{cases} f_i(\overline{T},\overline{u_i},\overline{w},\overline{\delta}) \leq 0 \\ f_i(\underline{T},0,\underline{w},\underline{\delta}) \geq 0. \end{cases}$$

Proof: We want our system to be invariant in a given set. Since the state T varies continuously and there is no delay in our model, it is enough to ensure that the vector field f points toward the interior when the state is on the boundary of the set. Our control set has the simple form of an interval $[\underline{T},\overline{T}]$, so keeping f toward the interior can be split into its n components: if $T_i \in \{\underline{T_i},\overline{T_i}\}$ we want to ensure that f_i points toward the interior of the interval $[\underline{T_i},\overline{T_i}]$. This can be expressed in terms of scalar conditions: for each room i, we need $f_i \geq 0$ if $T_i = \underline{T_i}$; and $f_i \leq 0$ if $T_i = \overline{T_i}$. According to (3), u_i is the only controlled input with an influence on f_i. With all these considerations, Definition 3 can be expressed as:

$$\forall i \in \{1,\dots,n\},\ \forall j \neq i,\ \forall T_j \in [\underline{T_j},\overline{T_j}],$$
$$\forall w \in [\underline{w},\overline{w}],\ \forall \delta \in [\underline{\delta},\overline{\delta}],\ \exists u_i^a, u_i^b \in [0,\overline{u_i}] \mid$$
$$\begin{cases} T_i = \overline{T_i} \Rightarrow f_i(\overline{T_i},(T_j)_{j\neq i},u_i^a,w,\delta) \leq 0 \\ T_i = \underline{T_i} \Rightarrow f_i(\underline{T_i},(T_j)_{j\neq i},u_i^b,w,\delta) \geq 0. \end{cases} \tag{9}$$

Similarly to the proof of Theorem 2 implying (8), we can prove that the conditions in Theorem 3 imply (9) by taking the worst cases for w, δ and T_j (since $\partial f_i/\partial T_j \geq 0$). For the converse, the existence of u_i^a and u_i^b (such that (9) is verified) is also true for particular combinations of $((T_j)_{j\neq i},w,\delta)$: $((\overline{T_j})_{j\neq i},\overline{w},\overline{\delta})$ and $((\underline{T_j})_{j\neq i},\underline{w},\underline{\delta})$. With the the partial derivatives $\partial f_i/\partial u_i \leq 0$ computed in the proof of Theorem 1, given the other variables T, w and δ, we have $\forall u_i \in [0,\overline{u_i}]$,

$$f_i(T,0,w,\delta) \geq f_i(T,u_i,w,\delta) \geq f_i(T,\overline{u_i},w,\delta),$$

which implies the conditions from Theorem 3. ■

Theorem 3 thus states that if the extremal values of the controller allow us to keep the system in $[\underline{T},\overline{T}]$ in the worst possible cases, then the invariance in the interval is verified for any other condition.

D. Controllable spaces

We consider the $2n$ conditions defined in Theorem 2. If we take them separately and replace the inequalities by equalities, each condition defines a manifold of dimension $n-1$ splitting the state space $\mathbb{R}^n$ in two halves. The condition taken from Theorem 2 is satisfied only on one side of the manifold, and this manifold sets the controllability limit for the corresponding action (cooling down T_i if $u_i = \overline{u_i}$; warming up T_i if $u_i = 0$). We define the controllable spaces as the half spaces induced by the $(n-1)$-manifolds.

Definition 4 (Controllable spaces): A controllable space $\mathcal{C}_i(u_i \in \{0,\overline{u_i}\}) \subset \mathbb{R}^n$ is the half space where the system is controllable with the input u_i:

$$T \in \mathcal{C}_i(\overline{u_i}) \quad \Leftrightarrow \quad f_i(T,\overline{u_i},\overline{w},\overline{\delta}) \leq 0,$$

$$T \in \mathcal{C}_i(0) \quad \Leftrightarrow \quad f_i(T,0,\underline{w},\underline{\delta}) \geq 0.$$

An immediate consequence of this definition is a new result on the *Robust Controllability*.

Proposition 3: The system is *Robust Controllable* in a set $\mathcal{S}$ if

$$\mathcal{S} \subset Interior\left(\left(\bigcap_i \mathcal{C}_i(\overline{u_i})\right) \cap \left(\bigcap_i \mathcal{C}_i(0)\right)\right).$$

For the *Robust Controlled Invariance*, we can replace Theorem 3 by the corresponding conditions on the controllable spaces.

Proposition 4: The system is *Robust Controlled Invariant* in $[\underline{T}, \overline{T}]$ if and only if

$$\begin{cases} \underline{T} \in \bigcap_i \mathcal{C}_i(0) \\ \overline{T} \in \bigcap_i \mathcal{C}_i(\overline{u_i}) \end{cases}$$

So Proposition 4 indicates where to choose the extremal values of our control interval $[\underline{T}, \overline{T}]$ for the system to be *Robust Controlled Invariant*.

If we also consider an interval for the *Robust Controllability*, then according to Propositions 3 and 4, we have the following result.

Proposition 5: If the system is *Robust Controllable* in $[\underline{T}, \overline{T}]$, then

$$\begin{cases} \text{it is also } \textit{Robust Controlled Invariant} \text{ in } [\underline{T}, \overline{T}]; \\ \text{it is } \textit{Robust Controllable} \text{ in any } [\underline{T'}, \overline{T'}] \subset [\underline{T}, \overline{T}]. \end{cases}$$

V. RESULTS

A. Model and controller

We created a two-room model of this system using MATLAB® and Simulink®. While simplified, this model still covers all the important features of the system (conduction with the exogenous inputs and the other room; heat sources; door between the rooms) and it is easier to display the results in the state space. The building considered has an area of $12 \times 4\ m^2$ and the rooms are $2.5\ m$ high. Room 1 is a square of side $4\ m$, and room 2 covers the remaining surface, which is twice as big.

To avoid modeling the variations of the exogenous inputs, we consider them as constants in this simple model. We choose $T_u = 5\,°\text{C}$, $T_o = 35\,°\text{C}$ and the heat sources in each room are $1\ m^2$ surfaces with $T_{s_i} = 200\,°\text{C}$. For the ceiling temperature, we take a coarse approximation as the mean of the outside temperature and the desired temperature for each room. The maximal mass flow rate sent by the fans into each room is $\overline{u_i} = 1\ kg/s$. Since we have three boolean inputs (one source in each room and the door), we run the simulations in order to meet all 8 possible combinations as in Fig. 2, and switch from one to the next every 200 seconds to leave enough time for the system to stabilize.

All the definitions and results in the previous section are independent of the chosen control strategy. Our goal here is to establish the limits of what can be achieved for the robust control of the system. This is why, in order to use it as a point of comparison for more advanced control methods (to be developed), we choose the simplest robust controller to implement, even though it is probably far from being the most energy-efficient controller.

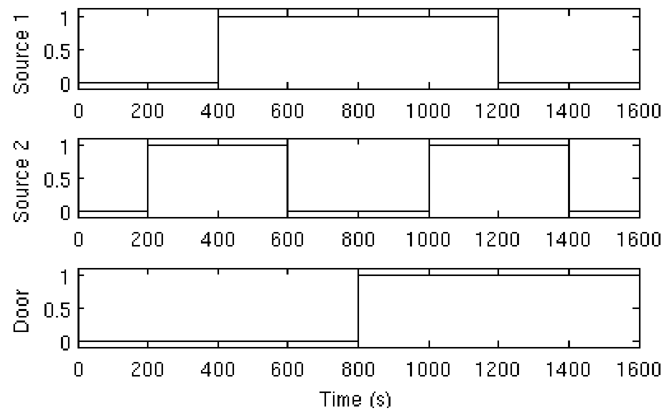

Fig. 2. Switching of the three disturbances.

Definition 5 (Decentralized Bang-Bang Controller):

$$\forall i \in \{1, \ldots, n\}, \quad \begin{cases} T_i \geq \overline{T_i} \Rightarrow u_i = \overline{u_i} \\ T_i \leq \underline{T_i} \Rightarrow u_i = 0 \end{cases}$$

This is a *Bang-Bang* control strategy because we only use the extremal values of the controlled inputs, and it is decentralized since only the temperature T_i has an influence on the choice of the corresponding input flow u_i. In Definition 5, the switches occur when the temperature of a room goes over (resp. below) an upper threshold $\overline{T_i}$ (resp. lower threshold $\underline{T_i}$). If we consider the *Robust Controllability* at a point, we have $\overline{T_i} = \underline{T_i}$ and such a control strategy implies a infinite number of switches in finite time in order to stay on the target state. Therefore, in our application on temperature control, it seems wiser and more realistic to use the *Robust Controlled Invariance* in a control interval $[\underline{T}, \overline{T}]$. In our Simulink model, this control interval is defined as a desired state T_d and an allowed variation ΔT_d around this state: $\underline{T_i} = T_{d,i} * (1 - \Delta T_d)$ and $\overline{T_i} = T_{d,i} * (1 + \Delta T_d)$.

B. Controllable spaces

The main results on the model previously described are displayed on Fig. 3, representing the state space T_1-T_2 in Celsius degrees. The dashed rectangle is the smallest *Robust Invariant* interval $[\underline{T_r}, \overline{T_r}]$ as in Proposition 2.

As defined in IV-D, we consider the $(n-1)$-manifolds representing the controllability limits. Since we are in a 2-dimensional example, these 1-manifolds are four curves. On Fig. 3, the manifolds are the four solid curves associated with a text box containing their name (using the same notation as the controllable spaces in Definition 4, $\mathcal{M}_i(u_i \in \{0, \overline{u_i}\})$). According to the ordering of the boolean variables chosen for the monotonicity (7), the coldest situation is when there is no disturbance. Therefore, the equations for the heating manifolds are linear (see the model equation (3)) and $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ are straight lines. On the other hand, $\mathcal{M}_1(\overline{u_1})$ and $\mathcal{M}_2(\overline{u_2})$ are non-linear, and there is a discontinuity in their slope when the curves cross $T_1 = T_2$ since the mass

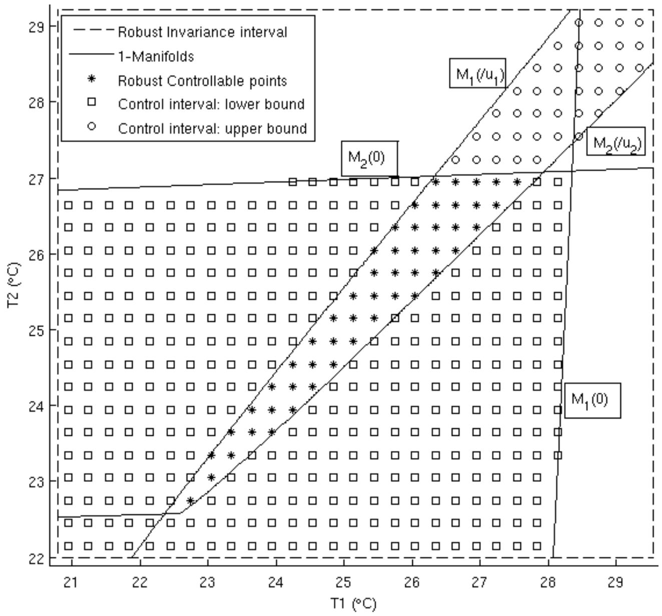

Fig. 3. 1-Manifolds and controllable spaces.

flow rate going through the door only appears in the equation of the coldest room.

On Fig. 3, the controllable spaces from Definition 4 are as following:

- $\mathcal{C}_1(0)$: we can warm up T_1 on the left of $\mathcal{M}_1(0)$;
- $\mathcal{C}_2(0)$: we can warm up T_2 below $\mathcal{M}_2(0)$;
- $\mathcal{C}_1(\overline{u_1})$: we can cool down T_1 on the right of $\mathcal{M}_1(\overline{u_1})$;
- $\mathcal{C}_2(\overline{u_2})$: we can cool down T_2 above $\mathcal{M}_2(\overline{u_2})$.

Therefore, the region filled with squares ($\square$) is the intersection $\mathcal{C}_1(0) \cap \mathcal{C}_2(0)$ between both heating controllable spaces, and the region filled with circles ($\circ$) is the intersection $\mathcal{C}_1(\overline{u_1}) \cap \mathcal{C}_2(\overline{u_2})$ between the cooling controllable spaces. Finally, the region filled with stars (*) is the intersection of both previous areas: $(\bigcap_i \mathcal{C}_i(\overline{u_i})) \cap (\bigcap_i \mathcal{C}_i(0))$. According to Proposition 3, the system is *Robust Controllable* at any state of the region with stars. This result has been confirmed by running simulations on our Simulink model, using small control interval (with an allowed variation $\Delta T_d \approx 0.1\%$) centered on the target state. Proposition 4 also indicates that the system is *Robust Controlled Invariant* if the control interval $[\underline{T}, \overline{T}]$ is chosen such that its lower bound $\underline{T}$ is in the region with squares, and its upper bound $\overline{T}$ is in the region with circles.

C. Simulation examples

Even though Proposition 4 is written as two conditions, each is the intersection between two controllable spaces, so there are actually four constraints: $\underline{T} \in \mathcal{C}_1(0)$; $\underline{T} \in \mathcal{C}_2(0)$; $\overline{T} \in \mathcal{C}_1(\overline{u_1})$; $\overline{T} \in \mathcal{C}_2(\overline{u_2})$. Each of these conditions corresponds to being able to control one of the temperature in one particular direction (heating or cooling down). Here, we consider an example where only three of these conditions are verified. We choose $T_d = (25; 24)$ and the allowed variation $\Delta T_d = 5\%$. We obtain the following values for the control interval boundaries:

$$\underline{T} = \begin{pmatrix} 23.75 \\ 22.80 \end{pmatrix} \qquad \overline{T} = \begin{pmatrix} 26.25 \\ 25.20 \end{pmatrix},$$

and we can check on Fig. 3 that three of the four conditions are indeed verified, but $\overline{T} \notin \mathcal{C}_2(\overline{u_2})$.

Fig. 4 gives the results of the simulation of the system when we try to keep it in the control interval. The initial conditions for the room temperatures are taken at the outside temperature. As shown in Fig. 2, this simulation covers all eight possible combinations of the disturbances. The door is closed during the first half of the simulation, and it opens at 800 seconds. The coldest situation of the disturbances (with regard to the monotonicity) is between 0 and 200 seconds, and the hottest case between 1000 and 1200 seconds.

If we look at the evolution of T_1 on the top-left graph of Fig. 4, we can see that for all disturbances, we can always control the system to keep $T_1 \in [\underline{T_1}, \overline{T_1}]$. Also, we are able to keep T_2 above its lower bound $\underline{T_2}$, even in the coldest case. These remarks are consistent with the fact that the following three conditions are satisfied: $\underline{T} \in \mathcal{C}_1(0)$; $\overline{T} \in \mathcal{C}_1(\overline{u_1})$; $\underline{T} \in \mathcal{C}_2(0)$. On the other hand, we notice that when the disturbances are bringing too much heat (here, when the door is open), we cannot keep T_2 below $\overline{T_2}$, even with the maximal ventilation $\overline{u_2}$. This behavior is explained by the fact that the last condition is not verified: $\overline{T} \notin \mathcal{C}_2(\overline{u_2})$. Therefore, all the results of this simulation are consistent with the theoretical conditions to obtain a *Robust Controlled Invariant* system.

In the previous example, the system was not *Robust Controlled Invariant* in $[\underline{T}, \overline{T}]$ since one of the conditions ($\overline{T} \in \mathcal{C}_2(\overline{u_2})$) was not verified. This means that in some parts of the control interval, we were not able to maintain the state in the interval. However, with a simple operation we can modify $[\underline{T}, \overline{T}]$ to obtain a new interval (which is a subset of the initial control interval) in which the system is *Robust Controlled Invariant*. In our case, we want a new upper bound satisfying $\overline{T'} \in \mathcal{C}_2(\overline{u_2}) \cap [\underline{T}, \overline{T}]$. The easiest way to obtain that is to reduce $\overline{T_1}$ to $\overline{T_1'}$ such that $(\overline{T_1'}, \overline{T_2}) \in \mathcal{C}_1(\overline{u_1}) \cap \mathcal{C}_2(\overline{u_2})$. We can see on Fig. 3 that the largest of such sub-interval is for $\overline{T_1'} \approx 25.8$.

We can consider another control strategy which is more realistic than the *Decentralized Bang-Bang* controller. For each room, the controlled input u_i is set as proportional to T_i, with saturations when u_i reaches its boundaries $\overline{u_i}$ and 0. Therefore, the controlled input follows the same behavior as the one described in Definition 5, with the additional condition:

$$T_i \in [\underline{T_i}, \overline{T_i}] \quad \Rightarrow \quad u_i(T_i) = \overline{u_i} * \frac{T_i - \underline{T_i}}{\overline{T_i} - \underline{T_i}}.$$

Fig. 5 shows the simulation for this control strategy, with the same conditions as those used for Fig. 4. We can see that the variations of the temperatures and controls are much smoother, which improves both the comfort and the lifespan

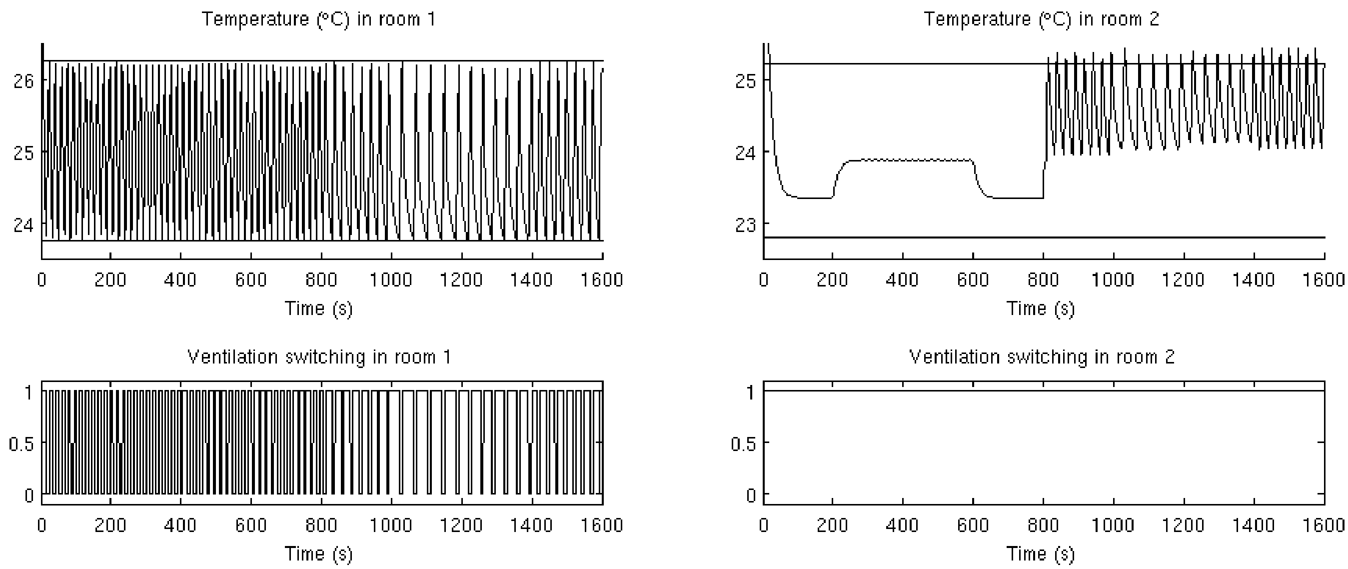

Fig. 4. Simulation of the system with a *Decentralized Bang-Bang* controller.

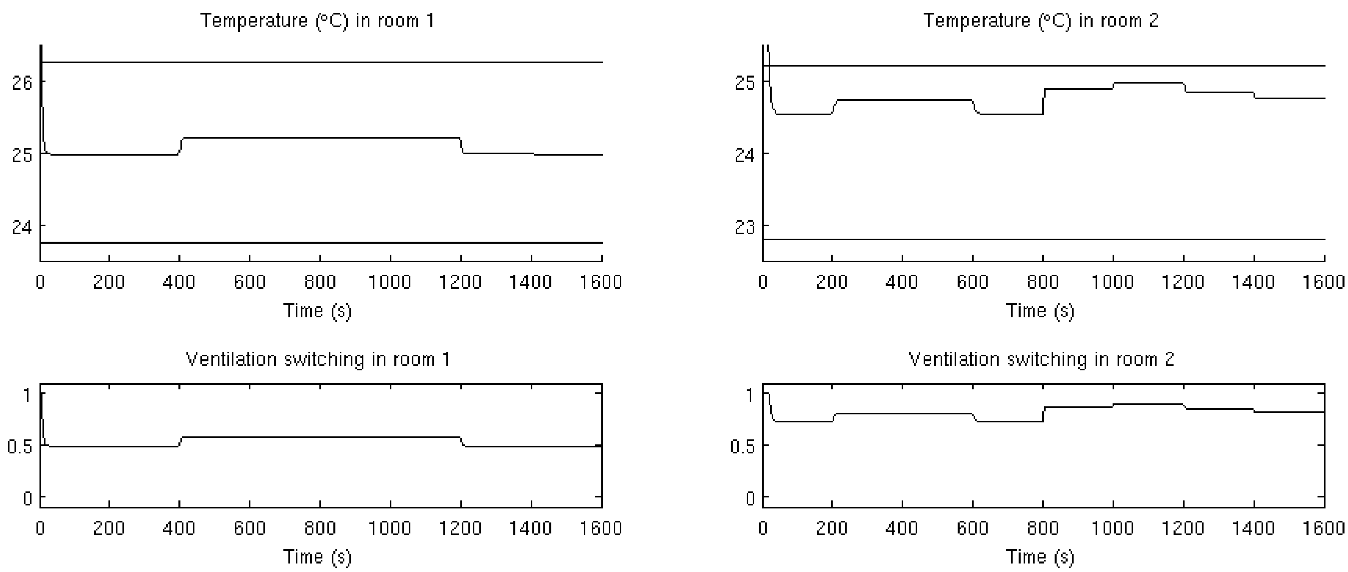

Fig. 5. Simulation of the system with a linear controller with saturations.

of the fans. Also, we can notice that even in the hottest case for the disturbances (between 1000 and 1200 seconds), T_2 stays in its interval despite the missing condition for the *Robust Controlled Invariance*. This comes from the fact that these conditions consider the hottest case not only on the disturbances, but also on the temperature of the adjacent rooms. With this controller, the temperatures are not oscillating and in this case T_1 is far enough from its hottest value $\overline{T_1}$ to allow T_2 to stay in the interval. Therefore, without making the system *Robust Controlled Invariant*, this new control strategy has more chances keeping the temperatures in their control intervals, even when some of the invariance conditions are not verified.

VI. Conclusion

In this paper, we introduced a monotone model for the temperature evolution in a building equipped with Under-Floor Air Distribution. Using the monotonicity property, we then proposed characterizations for *Robust Controllability* and *Robust Controlled Invariance*, and we established the limits for the robust control of such system. Finally, we confirmed these results on a two-room model using a *Decentralized Bang-Bang* controller.

Such simple robust controller was chosen in order to be used as a point of comparison for more advanced and energy-efficient methods, such as using a symbolic model associated with a discrete controller. We will also have the possibility to run experimental validations of the current and future methods on the already existing UFAD-building prototype.

References

[1] J.K.W. Wong, H. Li and S.W. Wang, Intelligent building research: a review, *Automation in Construction*, Vol. 4, No. 1, 2004, pp 143-159.

[2] M. Trcka and J.L.M. Hensen, Overview of HVAC system simulation, *Automation in Construction*, Vol. 19, No. 2, 2010, pp 93-99.

[3] H. Mirinejad, S.H. Sadati, M. Ghasemian and H. Torab, Control Techniques in Heating, Ventilating and Air Conditioning (HVAC) Systems, *Journal of Computer Science*, Vol. 4, No. 9, 2008, pp 777-783.

[4] F.S. Bauman and A. Daly, *Underfloor Air Distribution (UFAD) Design Guide*, ASHRAE, Atlanta, USA, 2003.

[5] E. Witrant, P. Di Marco, P. Park and C. Briat, Limitations and performances of robust control over WSN: UFAD control in intelligent buildings, *IMA Journal of Mathematical Control and Information*, Vol. 27, No. 4, 2010, pp 527-543.

[6] B. Arguello-Serrano and M. Velez-Reyes, Nonlinear control of a heating, ventilating, and air conditioning system with thermal load estimation, *IEEE Transactions on Control Systems Technology*, Vol. 7, No. 1, 1999, pp 56-63.

[7] J.M. House and T.F. Smith, Optimal control of building and HVAC systems, *American Control Conference*, Vol. 6, 1995, pp 4326-4330.

[8] F. Oldewurtel, A. Parisio, C.N. Jones, M. Morari, D. Gyalistras, M. Gwerder, V. Stauch, B. Lehmann and K. Wirth, Energy efficient building climate control using Stochastic Model Predictive Control and weather predictions, *American Control Conference*, 2010, pp 5100-5105.

[9] Z. Yu and A. Dexter, Simulation based predictive control of low-energy building systems using two-stage optimization, *IBPSA Building Simulation Conference*, 2009, pp 1562-1568.

[10] D. Angeli and E.D. Sontag, Monotone Control Systems, *IEEE Transactions on Automatic Control*, Vol. 48, No. 10, 2003, pp 1684-1698.

[11] F.C. McQuiston, J.D. Parker and J.D. Spitler, *Heating, Ventilating and Air Conditioning Analysis and Design*, Wiley, Ed. 6, 2005.

[12] G.J. Levermore, *Building Energy Management Systems: Applications to low-energy HVAC and natural ventilation control*, E&FN Spon, Ed. 2, 2000.