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Dynamic Boundary Stabilization of Linear and Quasi-Linear Hyperbolic Systems

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Abstract— **Systems governed by hyperbolic partial differential equations with dynamics associated with their boundary conditions are considered in this paper. These infinite dimensional systems can be described by linear or quasi-linear hyperbolic equations. By means of Lyapunov based techniques, some sufficient conditions are derived for the exponential stability of such systems. A polytopic approach is developed for quasi-linear hyperbolic systems in order to guarantee stability in a region of attraction around an equilibrium point, given specific bounds on the parameters. The main results are illustrated on the model of an isentropic inviscid flow.**

I. Introduction

Techniques based on Lyapunov function are commonly used for the stability analysis of dynamical systems, such as those described by partial differential equations (PDE). Many distributed physical systems are described by strict hyperbolic PDEs. For example, the conservation laws describing process evolution in open conservative systems are described by hyperbolic PDEs. One of the main properties of hyperbolic systems is the existence of the so-called Riemann transformation, which is a powerful tool for the proof of classical solutions, analysis and control, among other properties [1]. Among the potential applications, hydraulic networks [11], road traffic networks [13], gas flow in pipelines [3] or flow regulation in deep pits [19] are of significant importance.

The stability problem of boundary control in hyperbolic systems has been considered extensively in the literature, as reported in [10] [9] [15], among other references. Most results consider that the boundary control can react fast enough when compared to the travel time of waves. More precisely, no time response limitation is taken into account at the boundary conditions. For many applications (e.g. [12] [4]), the wave travel can be considered much slower than the actuator time response. A static relationship can then be established between the control input and the boundary condition. Nevertheless, there are applications where the dynamics associated with the boundary control cannot be neglected (e.g. when using a resistor to control the temperature of an airflow). Discrete approximations of this kind of systems have been used to address this problem in [6].

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The stability problem of linear and quasi-linear hyperbolic systems in presence of dynamic behavior at the boundary conditions is addressed in this work. The asymptotic stability of this class of hyperbolic systems is demonstrated using Riemann coordinates along with a Lyapunov function formulation. The sufficient conditions for the stability are obtained in terms of the boundary conditions dynamics. These conditions are presented in a linear matrix inequality (LMI) framework.

The theoretical results are applied on an homogeneous system of conservation laws with the aim of designing a stabilizing boundary control for an isentropic and inviscid flow in a pipe with constant cross section. More precisely, the problem of the regulation of the air pressure, density and speed inside a pipe is addressed. The physical model is a strict quasi-linear hyperbolic system since the presence of friction or thermal sources is not considered. This model is based on the Euler equations, which are commonly used in compressible flow dynamics to describe the flow transport in ducts.

The paper is organized as follows. First, Section II describes the linear and quasi-linear hyperbolic systems considered. In Section III, the main stability results for linear hyperbolic systems with dynamic behavior at the boundary conditions are established. An extension for quasi-linear hyperbolic systems is developed in Section IV using a polytopic approach. Finally, the main result is applied to the boundary regulation of pressure, density and speed in a pipe with isentropic and inviscid air flow (Section V).

II. Problem Formulation

Let n be a positive integer and Ω be an open non-empty convex set of $\mathbb{R}^n$. Consider the general class of quasi-linear hyperbolic systems of order n defined as follows [17]:

$$\partial_t s(x,t) + F(s(x,t))\partial_x s(x,t) = 0 \tag{1}$$

where ∂_t and ∂_x denote the partial derivative with respect to t and x respectively, $s(x,t) \in \Omega$, and $F : \Omega \to \mathbb{R}^{n\times n}$ is a continuously differentiable function called the characteristic matrix of (1). If (1) is strictly hyperbolic, then a bijection $\xi(s) \in \Theta \subset \mathbb{R}^n$ exists, at least locally, such that the system can be transformed into a system of coupled transport equations [8]:

$$\partial_t \xi_i(x,t) + \lambda_i(\xi(x,t))\partial_x \xi_i(x,t) = 0 \qquad (2)$$
$$i \in [1, ..., n]$$

where $\xi_i(x,t)$ are called the Riemann coordinates of (1), which are constant along the characteristic curves described by:

$$\frac{dx}{dt} = \lambda_i(\xi(x,t)) \qquad (3)$$

where $\xi = [\xi_1, \xi_2, ..., \xi_n]^T$.

A. Quasi-Linear Hyperbolic Systems with Dynamic Boundary Conditions

Consider the following quasi-linear hyperbolic equation in Riemann coordinates:

$$\partial_t \xi(x,t) + \Lambda(\xi)\partial_x \xi(x,t) = 0 \quad \forall\, x \in [0,1], t \geq 0 \qquad (4)$$

where Λ is a diagonal matrix function $\Lambda : \Theta \rightarrow \mathbb{R}^{n\times n}$ such that $\Lambda(\xi) = diag(\lambda_1(\xi), \lambda_2(\xi), ..., \lambda_n(\xi))$ with

$$\lambda_1(\xi) < ... < \lambda_m(\xi) < 0 < \lambda_{m+1}(\xi) < ... < \lambda_n(\xi)$$
$$\forall\, \xi \in \Theta \qquad (5)$$

If $\Lambda(\xi) = \Lambda$, then (4) is a linear hyperbolic equation given by:

$$\partial_t \xi(x,t) + \Lambda\partial_x \xi(x,t) = 0 \quad \forall\, x \in [0,1], t \geq 0 \qquad (6)$$

The state description can be partitioned as: $\xi = \begin{bmatrix} \xi_- \\ \xi_+ \end{bmatrix}$ where $\xi_- \in \mathbb{R}^m$ and $\xi_+ \in \mathbb{R}^{n-m}$. Define:

$$\Lambda^+ = diag(|\lambda_1(\xi)|, |\lambda_2(\xi)|, ..., |\lambda_n(\xi)|) \qquad (7)$$

The problem of stability of linear hyperbolic systems has been considered by [10], [16] and [4], among others, using the static boundary conditions:

$$\begin{pmatrix} \xi_-(1,t) \\ \xi_+(0,t) \end{pmatrix} = G \begin{pmatrix} \xi_-(0,t) \\ \xi_+(1,t) \end{pmatrix} \qquad (8)$$

where the map $G : \mathbb{R}^{n\times n} \rightarrow \mathbb{R}^{n\times n}$ vanishes at 0. Linear hyperbolic systems with dynamics associated with their boundary conditions are less explored in the literature, although there are approaches using finite-dimensional approximations such as in [6], where this kind of systems has been successfully stabilized. Instead of (8), consider the following dynamics for the boundary conditions:

$$\dot{X}_c = AX_c + Bu \qquad (9)$$
$$Y_c = X_c$$

with

$$X_c = \begin{pmatrix} \xi_-(1,t) \\ \xi_+(0,t) \end{pmatrix}, \quad u = KY_\xi, \quad Y_\xi = \begin{pmatrix} \xi_-(0,t) \\ \xi_+(1,t) \end{pmatrix} \qquad (10)$$

where $K \in \mathbb{R}^{n\times n}$ and $u \in \mathbb{R}^n$. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ are given matrices. The Cauchy problem (4) - (9) has a unique classical solution [10] if there exists a $\delta_0 > 0$ and a continuously differentiable function $\xi^0 : [0,1] \rightarrow \Theta$ such that the zero-order and one-order compatibility conditions are satisfied and $|\xi^0|_{H^2((0,1,R^n))} < \delta_0$. Thus, the initial condition can be defined for (4) - (9) as:

$$\xi(x,0) = \xi^0(x), \quad X_c(0) = X_c^0, \quad \forall\, x \in [0,1] \qquad (11)$$

III. Stability of Linear Hyperbolic Systems with Dynamic Boundary Conditions

The aim of this section is to use Lyapunov functions to state some sufficient conditions for the exponential stability of (6), (9) and (11). The main results obtained for linear hyperbolic systems are presented in the following theorem:

Theorem 1. *Consider the system* (6), (9) *and* (11). *Assume that there exists a diagonal positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that the following LMI is satisfied*

$$\begin{bmatrix} QA^T + AQ + \Lambda^+ Q & BY \\ Y^T B^T & -\Lambda^+ Q \end{bmatrix} \prec 0 \qquad (12)$$

where $K = YQ^{-1}$. Then there exist two constants $\alpha > 0$ and $M > 0$ such that, for all continuously differentiable functions $\xi^0 : [0,1] \rightarrow \Theta$ satisfying the zero-order and one-order compatibility conditions, the solution of (6), (9) *and* (11) *satisfies, for all $t \geq 0$,*

$$||X_c(t)||^2 + ||\xi(x,t)||_{L^2(0,1)} \leq$$
$$Me^{-\alpha t}\left(||X_c^0||^2 + ||\xi^0(x)||_{L^2(0,1)}\right) \qquad (13)$$

Proof. Considering the system (6), it is possible to replace $\xi(x,t)$ by $\begin{pmatrix} \xi_-(1-x,t) \\ \xi_+(x,t) \end{pmatrix}$ and obtain a PDE whose corresponding diagonal characteristic matrix function is Λ^+. Therefore, it can be may assumed without loss of generality that $m = 0$ and $\Lambda^+ = \Lambda$ and that the boundary conditions (9) have the following form:

$$\dot{X}_c = AX_c + Bu \qquad (14)$$
$$X_c = \xi(0,t), \quad u = KY_\xi, \quad Y_\xi = \xi(1,t)$$

Given a diagonal positive definite matrix P, consider, as an extension of the Lyapunov function proposed in [9], the quadratic Lyapunov function candidate defined for all continuously differentiable functions $\xi : [0,1] \rightarrow \Theta$ as:

$$V(\xi, X_c) = X_c^T P X_c + \int_0^1 \left(\xi^T P \xi\right) e^{-\mu x} dx \qquad (15)$$

where μ is a positive scalar that is precised below. Computing the time derivative of V along the solutions of (6), (9) and (11) yields to:

$$\begin{aligned}\dot{V} =& X_c^T\left(A^TP+PA\right)X_c+Y_\xi^TK^TB^TPX_c\\&+X_c^TPBKY_\xi-\left[e^{-\mu x}\xi^T\Lambda P\xi\right]\Big|_0^1\\&-\mu\int_0^1\left(\xi^T\Lambda P\xi\right)e^{-\mu x}dx\end{aligned}\qquad(16)$$

(16) can be written in terms of the boundary conditions dynamics (14) as follows:

$$\begin{aligned}\dot{V} =& -\mu X_c^T\Lambda P X_c-\mu\int_0^1\left(\xi^T\Lambda P\xi\right)e^{-\mu x}dx\\&+\begin{bmatrix}X_c\\Y_\xi\end{bmatrix}^T\left[\begin{array}{c|c}A^TP+PA & PBK\\+\Lambda P+\mu\Lambda P & \\\hline K^TB^TP & -e^{-\mu}\Lambda P\end{array}\right]\\&\times\begin{bmatrix}X_c\\Y_\xi\end{bmatrix}\end{aligned}\qquad(17)$$

Note that (12) is equivalent to consider that

$$\begin{bmatrix}A^TP+PA+\Lambda P & PBK\\K^TB^TP & -\Lambda P\end{bmatrix}\prec 0\qquad(18)$$

which is obtained by multiplying both sides of (18) by $diag\left(P^{-1},P^{-1}\right)$ and performing the change of variable $Q=P^{-1}$ and $Y=KQ$. Thus, for a small enough and positive μ, the third term of (17) is always negative. From (5) it can be proved that there always exists an $\epsilon>0$ such that $\Lambda>\epsilon I^{n\times n}$ (e.g ϵ could be the smallest eigenvalue of Λ). Moreover, the diagonality of P and Λ imply that:

$$\dot{V}\leq-\mu\epsilon V(\varepsilon)\qquad(19)$$

Therefore for a sufficiently small $\mu>0$, the function (15) is a Lyapunov function for the hyperbolic system (6), (9), and (11). ■

IV. Stability of Quasi-Linear Hyperbolic Systems with Dynamic Boundary Conditions

A proof of the Lyapunov stability of (4) under the static boundary conditions (8) has been investigated in details in [10], assuming that $\rho_1(G^{'}(0))<1$ (where $\rho_1(G^{'}(0))=Inf\{||\Delta G^{'}(0)\Delta^{-1}||;\Delta\in D_{n,+}\}$, where $D_{n,+}$ denotes the set of $n\times n$ real diagonal positive definite matrices), and using as a Lyapunov function candidate:

$$V(\xi)=V_1(\xi)+V_2(\xi,\xi_x)+V_3(\xi,\xi_x,\xi_{xx})\qquad(20)$$

with

$$\begin{aligned}V_1(\xi)&=\int_0^1\left(\xi^TQ(\xi)\xi\right)e^{-\mu x}dx\\V_2(\xi,\xi_x)&=\int_0^1\left(\xi_x^TR(\xi)\xi_x\right)e^{-\mu x}dx\\V_3(\xi,\xi_x,\xi_{xx})&=\int_0^1\left(\xi_{xx}^TS(\xi)\xi_{xx}\right)e^{-\mu x}dx\end{aligned}\qquad(21)$$

where $Q(\xi)$, $R(\xi)$ and $S(\xi)$ are symmetric positive definite matrices. The stability of system (4) with dynamics associated with the boundary conditions is studied in a different way by introducing a polytopic approach in the characteristic matrix $\Lambda(\xi)$.

Define a non empty convex set $\Xi\subset\Theta$ and a map $T:\Xi\rightarrow Z_\varphi$. Consider the following polytopic linear representation of the nonlinear characteristic matrix:

$$\Lambda(\xi)=\sum_{i=1}^{2^l}\alpha_i(\varphi)\Lambda(w_i)\qquad(22)$$

$\forall\,\xi\in\Xi$ and therefore $\forall\,\varphi\in Z_\varphi$, where φ is a varying parameter vector that takes values in the parameter space Z_φ (a convex set) such that [2]:

$$Z_\varphi:=\{[\varphi_1,...,\varphi_l]^T\in\mathbb{R}^l,\varphi_i\in[\overline{\varphi}_i,\underline{\varphi}_i]\,\forall\,i=1...l\}\qquad(23)$$

where l is the number of varying parameters, $\alpha_i(\varphi)$ is a scheduling function $\alpha_i:Z_\varphi\rightarrow[0,1]$, w_i are the $2^l=N_\varphi$ vertices of the polytope defined by all extremities of each varying parameter $\varphi\in Z_\varphi$ and $\sum_{i=1}^{2^l}\alpha_i(\varphi)\Lambda(w_i):Z_\varphi\rightarrow\mathbb{R}^{n\times n}$. In general, all the admissible values of the vector φ are constrained in an hyperrectangle in the parameter space Z_φ. The scheduling functions $\alpha_i(\varphi)$ are defined as (see [2]):

$$\alpha_i(\varphi)=\frac{\prod_{k=1}^l|\varphi_k-C(w_i)_k|}{\prod_{k=1}^l|\overline{\varphi}_k-\underline{\varphi}_k||}\qquad(24)$$

where:

$$C(w_i)_k=\left\{\varphi_k\left|\begin{array}{c}\varphi_k=\overline{\varphi}_k\text{ if }(w_i)_k=\underline{\varphi}_k\\\varphi_k=\underline{\varphi}_k\text{ otherwise}\}\end{array}\right.\right\}\qquad(25)$$

which has the following properties:

$$\alpha_i(\varphi)\geq0,\quad\sum_{i=1}^{2^l}\alpha_i(\varphi)=1\qquad(26)$$

Consider (4) as an equivalent parameter varying hyperbolic system defined by:

$$\begin{aligned}&\partial_t\xi(x,t)+\sum_{i=1}^{2^l}\alpha_i(\varphi)\Lambda(w_i)\partial_x\xi(x,t)=0\\&\forall\,\varphi\in Z_\varphi,\quad\forall\,x\in[0,1],\quad t\geq0\end{aligned}\qquad(27)$$

It is clear that φ depends on ξ. However, as long as ξ remains in the set Ξ, the varying parameters φ_i can be considered as independent varying parameters (LPV framework [5]) that change the characteristic matrix. This results in a conservative tool for stability analysis because a whole polytope is stabilized instead of only the vertices [14]. Using (27), the following theorem states some sufficient conditions to ensure exponential stability for system (4), (9) and (11) in a defined region Z_φ.

Theorem 2. *Consider the system* (4), (9) *and* (11). *Assume that there exists a diagonal positive definite matrix* $Q \in \mathbb{R}^{n\times n}$ *such that the following LMI is satisfied* $\forall i \in [1, ..., N_\varphi]$

$$\begin{bmatrix} QA^T + AQ + \Lambda^+(w_i)Q & BY \\ Y^T B^T & -\Lambda^+(w_i)Q \end{bmatrix} \prec 0 \qquad (28)$$

where $K = YQ^{-1}$. *Then there exist two constants* $\alpha > 0$ *and* $M > 0$ *such that, for all continuously differentiable functions* $\xi^0 : [0,1] \to \Xi$ *satisfying the zero-order and one-order compatibility conditions, the solution of* (4), (9) *and* (11) *satisfies* (13) *for all* $t \geq 0$

Proof. Consider once again the Lyapunov function candidate (15). Computing the time derivative of V along the solutions of (4), (9) and (11) yields the following:

$$\begin{aligned} \dot{V} &= X_c^T \left(A^T P + PA\right) X_c + Y_\xi^T K^T B^T P X_c \\ &+ X_c^T PBKY_\xi - \sum_{i=1}^{2^l} \alpha_i(\varphi) \left[e^{-\mu x} \xi^T \Lambda(w_i) P \xi\right] \Big|_0^1 \\ &- \sum_{i=1}^{2^l} \alpha_i(\varphi) \mu \int_0^1 \left(\xi^T \Lambda(w_i) P \xi\right) e^{-\mu x} dx \end{aligned} \qquad (29)$$

Using the same procedure performed in the proof of Theorem 1, assuming once again that $\mu > 0$ is small enough and the fact that by definition, $\sum_{i=1}^{2^l} \alpha_i(\varphi) = 1$ and $\alpha_i \geq 0$ gives:

$$\dot{V} \leq -\mu\epsilon V(\varepsilon) \qquad (30)$$

where ϵ in this case could be the smallest eigenvalue of $\Lambda(w_i)\,\forall i \in [1, ..., N_\varphi]$). This proves that (15) is a Lyapunov function for the system (4), (9) and (11). ■

The polytopic approach guarantees the stability and robustness of the quasi-linear hyperbolic system in a determined region Z_φ, which cannot be achieved with the approach presented in Theorem 1 as it would only guarantee the stability of (4), (9) and (11) in a small enough neighborhood around the equilibrium.

V. Illustrating Example: Isentropic Inviscid Flow in a Pipe with Constant Cross Section

In this section, the air flow inside a pipe with a constant cross section is modeled by the Euler equations. The stabilization problem is solved using a boundary control computed using Riemann coordinates as presented in Sections III and IV. Two boundary controllers are designed: one to stabilize the system in an arbitrarily small neighborhood around a steady-state equilibrium (Theorem 1) and a second one to stabilize the system in a region in the parameter space around the system's equilibrium (Theorem 2).

Consider the Euler equations expressed in terms of the primitive variables [18]: density (ρ), speed (u) and pressure (p),

$$\partial_t W + A(W)\partial_x W + C(W) = 0 \qquad (31)$$

where

$$W = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}; \quad A(W) = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & a^2\rho & u \end{bmatrix}$$

$a = \sqrt{\frac{\gamma p}{\rho}}$ is the speed of sound, γ is the specific heat ratio and $C(W)$ is a function that describes the friction losses and heat exchanges. The isentropic case is analyzed, then $C(W) = 0$. The eigenvalues of the characteristic matrix $A(W)$, called the characteristic velocities, are:

$$\lambda_1(W) = u + a,\ \lambda_2(W) = u,\ \lambda_3(W) = u - a \qquad (32)$$

and their respective Riemann invariants (see [18]):

$$a + \frac{\gamma - 1}{2}u, \quad \sqrt{\frac{p}{\rho^\gamma}}, \quad a - \frac{\gamma - 1}{2}u \qquad (33)$$

Assume that the velocities (32) verify:

$$\lambda_3(W) < 0 < \lambda_2(W) < \lambda_1(W) \qquad (34)$$

which characterizes (31) as a strict hyperbolic system and ensures the existence of a transformation to the Riemann coordinates. Consider the following change of coordinates:

$$\begin{aligned} \xi_1 &= \sqrt{\frac{\gamma p}{\rho}} + \frac{\gamma - 1}{2}u - \sqrt{\frac{\gamma \tilde{p}}{\tilde{\rho}}} - \frac{\gamma - 1}{2}\tilde{u} \\ \xi_2 &= \sqrt{\frac{p}{\rho^\gamma}} - \sqrt{\frac{\tilde{p}}{\tilde{\rho}^\gamma}} \\ \xi_3 &= \sqrt{\frac{\gamma p}{\rho}} - \frac{\gamma - 1}{2}u - \sqrt{\frac{\gamma \tilde{p}}{\tilde{\rho}}} + \frac{\gamma - 1}{2}\tilde{u} \end{aligned} \qquad (35)$$

where $\tilde{W} = [\tilde{\rho}, \tilde{u}, \tilde{p}]^T$ is an arbitrary steady-state. With these new coordinates (ξ_1, ξ_2, ξ_3), the dynamics (31) can be rewritten in the quasi-linear hyperbolic form (4). Since the change of coordinates (35) is a mapping, ρ, u and p can be expressed in terms of the Riemann invariants as:

$$u = \frac{\xi_1 - \xi_3 + (\gamma-1)\tilde{u}}{\gamma - 1}, \quad p = \rho^\gamma \left(\xi_2 + \sqrt{\frac{\tilde{p}}{\tilde{\rho}^\gamma}}\right)^2$$
$$\rho = \left(\frac{\xi_1 - \frac{\gamma-1}{2}(u-\tilde{u}) + \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}}{\sqrt{\gamma}\left(\xi_2 + \sqrt{\frac{\tilde{p}}{\tilde{\rho}^\gamma}}\right)}\right)^{\frac{2}{\gamma-1}} \tag{36}$$

Note that the equilibrium $[\tilde{\rho}, \tilde{u}, \tilde{p}]^T$ expressed in terms of Riemann coordinates is $[0, 0, 0]^T$.

A. Boundary Control for the Quasi-Linear Model

A boundary control for equation (31) is designed with proved stability in a region described by a polytope around an equilibrium point. Define the dynamic boundary conditions (9) with:

$$A = \begin{bmatrix} -300 & 0 & 13 \\ 0 & -40 & 0 \\ 4 & 0 & -300 \end{bmatrix}$$
$$B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{37}$$

The characteristic matrix defined in terms of physical quantities can be expressed as:

$$\Lambda(W) = \begin{bmatrix} a+u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & a-u \end{bmatrix} \tag{38}$$

Define the varying parameter $\varphi = [a, u]^T$, which is enough to describe the propagation speed of the Riemann invariants. Define the limits on each parameter as $[\overline{a}, \underline{a}]$ and $[\overline{u}, \underline{u}]$ to describe the region Z_φ where the stability of system (31) is ensured by using Theorem 2. Consider the equilibrium $\tilde{W} = [1.16, 20, 100000]^T$: $\tilde{W}$ imposes at the equilibrium point $\tilde{u} = 20$ and $\tilde{a} = 347$. The region Z_φ is then defined by setting the following limits of each parameter:

$$\overline{a} = 355,\ \underline{a} = 340,\ \overline{u} = 40,\ \underline{u} = 5 \tag{39}$$

Applying Theorem 2, the following controller is obtained:

$$K_1 = \begin{bmatrix} -14.35 & 0 & -1.54 \\ 0 & -0.07 & 0 \\ -2.12 & 0 & -14.5 \end{bmatrix} \tag{40}$$

with the respective diagonal positive definite matrix associated with the Lyapunov function (15):

$$P^{-1} = Q = \begin{bmatrix} 0.026 & 0 & 0 \\ 0 & 0.19 & 0 \\ 0 & 0 & 0.026 \end{bmatrix} \tag{41}$$

To illustrate this result, numerical simulations of (31) with (9) and (40) are performed with an initial condition $W_0 = W(x, 0) \in \Xi$. The MacCormack numerical method combined with a time varying diminishing (TVD) scheme has been considered for the PDE simulation, along with a modified method of characteristics for the boundary conditions resolution. For more detail on the simulator and its validation see [7]. Define $W_0 = [1.168, 21, 101000]^T \in \Xi$.

Figures 1a, 1b and 1c present the results obtained in the numerical simulations using the feedback gain (40). The states ρ, u and p reach effectively the desire equilibrium in finite time, while satisfying the condition $\varphi \in Z_\varphi \, \forall t > 0$ (Figures 1b and 2a).

To illustrate the differences between Theorem 1 and 2, a controller K_2 is derived using Theorem 1 at $\tilde{W}$. K_2 thus guarantees the system stability in a neighborhood that is close enough to the equilibrium $\tilde{W}$, considering Λ to be constant. The following K_2 is obtained:

$$K_2 = \begin{bmatrix} -30.86 & 0 & -1.95 \\ 0 & -0.6 & 0 \\ -2.10 & 0 & -32.4 \end{bmatrix} \tag{42}$$

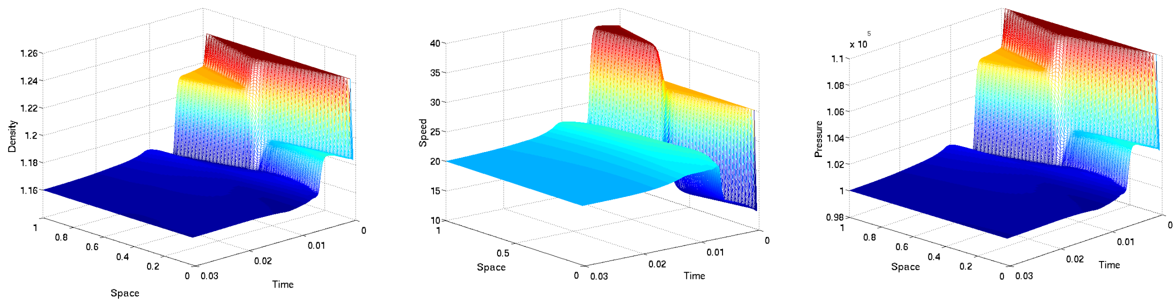
with the corresponding Lyapunov matrix (15):

$$P^{-1} = Q = \begin{bmatrix} 0.013 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \tag{43}$$

Both controllers (K_1 and K_2) are simulated with the same conditions starting from the initial condition $W_0 = [1.168, 21, 101000]^T$. The results are presented in Figures 2b and 2c, which are the projections in time of the speed profiles. Note that the polytopic controller presents a better response (in terms on time response and smoothness) together with the guarantee of stability.

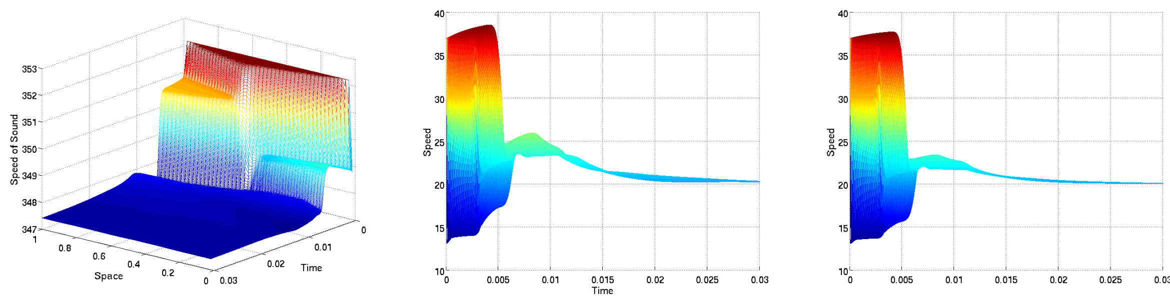
VI. Conclusion

In this paper, some sufficient conditions were derived for the exponential stability of a linear and a quasi-linear hyperbolic PDE system with dynamics associated with the boundary conditions. The stability analysis has been done using a Lyapunov function which allows expressing the stability conditions in an LMI framework. A polytopic approach was implemented to guarantee the stability of quasi-linear hyperbolic system inside a defined polytope. A simulation example has shown the effectiveness of the contributions presented in this work and the advantages in terms of convergence and robustness of the polytopic approach. This work has many applications in different systems governed by hyperbolic PDE. A natural extension can be the sufficient conditions for stability of quasi-linear hyperbolic systems with non-linear dynamical behavior at the boundaries. Many questions are



(a) Density profile with Theorem 2 (b) Speed profile with Theorem 2 (c) Pressure profile with Theorem 2

Fig. 1: Time evolution of the density, speed and pressure profile using the controller for quasi-linear hyperbolic systems



(a) Speed of sound profile with Theorem 2 (b) Projection of the speed profile with Theorem 1 (c) Projection of the speed profile with Theorem 2

Fig. 2: Comparison of the speed profiles using both approaches

still open. In particular, the derivation of boundary observers for hyperbolic systems with linear and non-linear dynamic boundary control seems to be a challenging issue.

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