Dynamic Boundary Stabilization of First Order Hyperbolic Systems

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Abstract In this chapter, we address the problem of the dynamic boundary stabilization of linear, quasi-linear and LPV first-order hyperbolic systems. We provide sufficient conditions for the exponential stability for this class of infinite dimensional systems by means of Lyapunov based techniques and matrix inequalities. We develop an applicative example of a temperature boundary control in a Poiseuille flow using some of our main results and we present simulation results that illustrate the efficiency of our approach.

1 Introduction

Variable time-delays are often encountered in many industrial applications such as those where mass transport is present. This phenomenon makes the closed loop control much more challenging motivating the investigation of new strategies to effectively stabilize these systems. Among the potential applications, hydraulic networks [28], multiphase flow [22], road traffic networks [15], gas flow in pipelines [4] or flow regulation in deep pits [32] are of significant importance. Due to the convective nature of the mass transport, it is often modeled by means of balance laws which are described by hyperbolic partial differential equations (PDE) and typically used to express the fundamental dynamics of open conservative systems. The interest in boundary control comes from the fact that measurements in distributed parameter systems are usually not available. It is more common for sensors and actuators to be located at the boundaries.

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Techniques based on Lyapunov functions are commonly used for the stability analysis of infinite dimensional dynamical systems, such as those described by strict hyperbolic partial differential equations. One of the main properties of this class of PDEs is the existence of the so-called Riemann transformation which is a powerful tool for the proof of classical solutions, analysis and control, among other properties [2]. Control results for first-order hyperbolic systems do exist in the literature. For instance, sufficient conditions for controllability and observability of quasilinear hyperbolic systems have been obtained in [20]. The works of Krstic and co-workers (e.g. [18, 17, 23, 30]) on boundary control using backstepping designs, and the works of Coron and co-workers (e.g. [11, 12, 5, 14]) on dissipative boundary conditions, are some of the most representative references for the boundary control of first-order hyperbolic systems. In [21], the boundary control problem is investigated using a frequency domain approach while [1] focuses in the disturbance rejection problem. In [26], a strict time-varying Lyapunov function that allows establishing the asymptotic stability of time-varying hyperbolic systems is formulated for time-invariant boundary conditions and in [25] the boundary control of switched linear hyperbolic systems is considered. Other important references on boundary control are found in [13, 19, 24, 27]. However, results in dynamic boundary stabilization of hyperbolic systems has been less explored and therefore we have proposed some strategies to address this aspects in our previous works [8, 7, 9].

In this chapter, we provide an extension of our previous results and general overview of the dynamic boundary stabilization of hyperbolic systems. We start with a brief description of the class of first-order hyperbolic systems considered in this work. In Section 3, we use Lyapunov-based techniques to establish sufficient conditions for exponential stability of linear, LPV and quasi-linear hyperbolic systems with dynamics associated with their boundary conditions. A polytopic approach is developed for the LPV and quasi-linear hyperbolic systems to guarantee the exponential stability inside a prescribed non-empty convex set. The main results are presented in three theorems using a matrix inequality framework. Finally in Section 4, a reduced physical model of an experimental setup for the regulation of the output temperature in a Poiseuille flow is developed. A temperature boundary control is designed using the main contributions of this work. Simulation results are presented to illustrate the effectiveness of the proposed control strategy.

2 Linear and Quasi-Linear Hyperbolic Systems

Let n be a positive integer and Θ be an open non-empty convex set of $\mathbb{R}^n$. Consider the following class of quasi-linear hyperbolic systems of order n:

$$\partial_t \xi(x,t) + \Lambda(\xi)\partial_x \xi(x,t) = 0 \quad \forall x \in [0,1], t \geq 0 \tag{1}$$

where $\xi : [0,1] \times [0,\infty) \to \mathbb{R}^n$ and Λ is a continuously differentiable diagonal matrix function $\Lambda : \Theta \to \mathbb{R}^{n \times n}$ such that $\Lambda(\xi) = diag(\lambda_1(\xi), \lambda_2(\xi), ..., \lambda_n(\xi))$. Assume the following.

Assumption 1: The following inequalities hold for all $\xi \in \Theta$:

$$0 < \lambda_1(\xi) < \lambda_2(\xi) < ... < \lambda_n(\xi) \tag{2}$$

If $\Lambda(\xi) = \Lambda$, then (1) is a linear hyperbolic system given by:

$$\partial_t \xi(x,t) + \Lambda \partial_x \xi(x,t) = 0 \quad \forall x \in [0,1], t \geq 0 \tag{3}$$

Remark 1. Define the open non-empty convex set of $\mathbb{R}^n$ Ω and consider the general class of quasi-linear hyperbolic systems of order n defined as follows [29]:

$$\partial_t s(x,t) + F(s(x,t)) \partial_x s(x,t) = 0 \tag{4}$$

where $s(x,t) \in \Omega$, and $F : \Omega \to \mathbb{R}^{n \times n}$ is a continuously differentiable function. If (4) is strictly hyperbolic (all eigenvalues of $F(s)$ are different from zero and different between each other), then a bijection $\xi(s) \in \Theta \subset \mathbb{R}^n$ may exist, such that (4) can be transformed into the following system of coupled transport equations (it has been proved that for $n \leq 2$, this bijection always exists [10]):

$$\partial_t \xi_i(x,t) + \lambda_i(\xi(x,t)) \partial_x \xi_i(x,t) = 0, \quad i \in [1,...,n] \tag{5}$$

where $\xi_i(x,t)$ are called the Riemann coordinates of (4), which are constant along the characteristic curves described by:

$$\frac{dx}{dt} = \lambda_i(\xi(x,t)) \tag{6}$$

where $\xi = [\xi_1, \xi_2, ..., \xi_n]^T$. Using the transformation with Riemann coordinates, the system (4) can be expressed in the form of (1).

We consider the following boundary controls for the quasi-linear hyperbolic system (1):

$$\dot{X}_c = A_c X_c(t) + B_c u_c(t) \tag{7}$$
$$Y_c(t) = C_c X_c(t) + D_c u_c(t)$$

with

$$Y_c(t) = \xi(0,t), \quad u_c = K\xi(1,t) \tag{8}$$

where $X_c \in \mathbb{R}^{n_x}, A_c \in \mathbb{R}^{n_x \times n_x}, B_c \in \mathbb{R}^{n_x \times n}, C_c \in \mathbb{R}^{n \times n_x}, D_c \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{n \times n}, u \in \mathbb{R}^n$ and $n_x \geq 1$. Define the initial condition for (1) and (7) as:

$$\begin{cases} \xi(x,0) = \xi^0(x), & \forall x \in [0,1] \\ X_c(0) = X_c^0 \end{cases} \tag{9}$$

where $\xi^0(x) \in L^2((0,1);\mathbb{R}^n)$ and $X_c^0 \in \mathbb{R}^{n_x}$. It has been proved that there exists a $\delta_0 > 0$ and a $T > 0$ such that for every $\xi^0 \in H^2((0,1),\mathbb{R}^n)$ satisfying $|\xi^0|_{H^2((0,1),\mathbb{R}^n)} < \delta_0$ and the zero-order and one-order compatibility conditions, the Cauchy problem ((1), (7) and (9)) has a unique maximal classical solution satisfying (see e.g. [11] and [16], among other references):

$$|\xi(.,t)|_{H^2} < \delta_0 \quad \forall t \in [0,T) \tag{10}$$

Moreover, for linear hyperbolic systems (3), it holds for $T = +\infty$.

Remark 2. Under Assumptions 1 and static boundary conditions, there is no coupling between the states and thus a controller can be designed for each state separately. However, this is not true for the dynamic boundary conditions (7) since it induces a coupling between the states and motivates further analysis for the control design.

3 Dynamic Boundary Stabilization of Hyperbolic Systems

In this section, we give sufficient conditions for the dynamic boundary stabilization of linear, LPV and quasilinear hyperbolic systems (Sections 3.1, 3.2 and 3.3, respectively). More precisely, we address the problem of finding the boundary control gain K (according to (7)) such that (1) is exponentially stable.

3.1 Stability of Linear Hyperbolic Systems with Dynamic Boundary Conditions

Sufficient conditions for the exponential stability of (3) with boundary conditions (7) and initial condition (9) are obtained with the following theorem:

Theorem 1. *[Stability analysis] Along with Assumption 1, assume that there exist two diagonal positive definite matrices $P_1 \in \mathbb{R}^{n_x \times n_x}$ and $P_2 \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that the following matrix inequality is satisfied,*

$$M = \begin{bmatrix} \begin{array}{c} A_c^T P_1 + P_1 A_c + C_c^T \Lambda P_2 C_c \\ + K D_c^T \Lambda P_2 D_c K + \mu \Lambda P_1 \end{array} & P_1 B_c K + C_c^T \Lambda P_2 D_c K \\ \hline K^T B_c^T P_1 + K^T D_c^T \Lambda P_2 C_c & -e^{-\mu} \Lambda P_2 \end{bmatrix} \preceq 0 \tag{11}$$

Then, there exist two constant scalars $a > 0$ and $b > 0$ such that, for all $\xi^0 \in L^2((0,1); \mathbb{R}^n)$ and $X_c^0 \in \mathbb{R}^{n_x}$, the solution of (3), (7) and (9) satisfies, for all $t \geq 0$

$$||X_c(t)||^2 + ||\xi(t)||_{L^2(0,1)} \leq b e^{-at} \left(||X_c^0||^2 + ||\xi^0||_{L^2(0,1)} \right) \tag{12}$$

Proof: Given the diagonal positive definite matrices P_1 and P_2, consider (as an extension of the Lyapunov function proposed in [12]) the quadratic strict Lyapunov function candidate defined for all continuously differentiable functions $\xi : [0,1] \rightarrow \Theta$ as:

$$V(\xi, X_c) = X_c^T P_1 X_c + \int_0^1 \left(\xi^T P_2 \xi \right) e^{-\mu x} dx \tag{13}$$

where μ is a positive scalar. Note that (13) has some similarities with respect to the Lyapunov function proposed in [27] for boundary control with integral action. Computing the time derivative $\dot{V}$ of V along the classical C^1-solutions of (3) with boundary conditions (7) and initial condition (9), yields to the following:

$$\dot{V} = \dot{X}_c^T P_1 X_c + X_c^T P_1 \dot{X}_c + \int_0^1 \left(\dot{\xi}^T P_2 \xi + \xi^T P_2 \dot{\xi} \right) e^{-\mu x} dx \tag{14}$$

After integration by parts, the following is obtained:

$$\dot{V} = \left(X_c^T \left(A_c^T P_1 + P_1 A_c \right) X_c \right) + \left(\xi(1)^T K^T B_c^T P_1 X_c + X_c^T P_1 B_c K \xi(1) \right)$$
$$- \left[e^{-\mu x} \xi^T \Lambda P_2 \xi \right] \Big|_0^1 - \mu \int_0^1 \left(\xi^T \Lambda P_2 \xi \right) e^{-\mu x} dx \tag{15}$$

where $\xi(1) = \xi(1,t)$. The previous equation can be written using the boundary conditions (7) as follows:

$$\dot{V} = \left(X_c^T \left(A_c^T P_1 + P_1 A_c \right) X_c \right) + \left(\xi(1)^T K^T B_c^T P_1 X_c + X_c^T P_1 B_c K \xi(1) \right)$$
$$- e^{-\mu} \xi(1)^T \Lambda P_2 \xi(1) + X_c^T C_c^T \Lambda P_2 C_c X_c + X_c^T C_c^T \Lambda P_2 D_c K \xi(1)$$
$$+ \xi(1)^T K^T D_c^T \Lambda P_2 C_c X_c + \xi(1)^T K^T D_c^T \Lambda P_2 D_c K \xi(1) - \mu \int_0^1 \left(\xi^T \Lambda P_2 \xi \right) e^{-\mu x} dx$$
$$= -\mu X_c^T \Lambda P_1 X_c - \mu \int_0^1 \left(\xi^T \Lambda P_2 \xi \right) e^{-\mu x} dx + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T M \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \tag{16}$$

where the matrix M is defined as in (11). The matrix inequality $M \preceq 0$ implies that the last term of (16) is always negative or zero. This gives the following inequality:

$$\dot{V} \leq -\mu X_c^T \Lambda P_1 X_c - \mu \int_0^1 \left(\xi^T \Lambda P_2 \xi \right) e^{-\mu x} dx \tag{17}$$

From (2) it can be proved that there always exists a $\rho > 0$ such that $\Lambda - \rho I^{n \times n} \succ 0$ (e.g. ρ could be the smallest eigenvalue of Λ. Moreover, the diagonality of P_1, P_2

and Λ implies that:

$$\dot{V} \leq -\mu\rho V(\xi, X_c) \tag{18}$$

Therefore, the function (13) is a Lyapunov function for the hyperbolic system (3) with boundary conditions (7).

Integrating the inequality (18) from 0 to t gives the following:

$$V(t) \leq V(0)e^{-\mu\rho t} \tag{19}$$

To obtain the final result (12), we bound the Lyapunov function as follows:

$$\min\{\lambda_{min}(P_1), \lambda_{min}(P_2)\}(||X_c(t)||^2 + ||\xi(t)||_{L^2(0,1)}) \leq$$
$$V(t) \leq \max\{\lambda_{max}(P_1), \lambda_{max}(P_2)\}(||X_c(t)||^2 + ||\xi(t)||_{L^2(0,1)}) \tag{20}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the considered matrices, respectively. Then, using (19) together with (20) gives:

$$||X_c(t)||^2 + ||\xi(t)||_{L^2(0,1)} \leq \frac{\min\{\lambda_{min}(P_1), \lambda_{min}(P_2)\}}{\max\{\lambda_{max}(P_1), \lambda_{max}(P_2)\}} \left(||X_c^0||^2 + ||\xi^0||_{L^2(0,1)}\right) e^{-\mu\rho t} \tag{21}$$

which implies that $a = \mu\rho$ and $b = \min\{\lambda_{min}(P_1), \lambda_{min}(P_2)\} / \max\{\lambda_{max}(P_1), \lambda_{max}(P_2)\}$ in (12). ∎

Note that the matrix inequality (11) considers, through the Lyapunov matrices P_1 and P_2, the dynamic coupling between the system and its boundary conditions. Inequality (11) along with (18) implies that μ is a tuning parameter of the controller design as it explicitly enables to set the convergence speed of the Lyapunov function. Another interesting convergence feature can be deduced from (18): a faster convergence is obtained for larger values of ρ. This implies that hyperbolic systems with high convective velocities converge faster, which is physically consistent.

The following corollary gives a sufficient condition for the design of a stabilizing controller for the particular case where C_c is a diagonal matrix, $D_c = 0$ and $n = n_x$.

Corollary 1. *[Design of a stabilizing controller] Along with Assumptions 1, if C_c is diagonal and $D_c = 0$ and if there exists a diagonal positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that the following linear matrix inequality is satisfied,*

$$\begin{bmatrix} QA_c^T + A_cQ + C_c\Lambda QC_c + \mu\Lambda Q & B_cY \\ Y^TB_c^T & -e^{-\mu}\Lambda Q \end{bmatrix} \preceq 0 \tag{22}$$

where $Y = KQ$, then there exist two constants $\alpha > 0$ and $M > 0$ such that, for all $\xi^0(x) \in L^2((0,1); \mathbb{R}^n)$ and $X_c^0 \in \mathbb{R}^n$, the solution of (3) with boundary conditions (7) and initial condition (9) satisfies (12) for all $t \geq 0$.

Proof: Given a diagonal positive definite matrix P, consider the quadratic strict Lyapunov function candidate defined for all continuously differentiable functions $\xi : [0,1] \rightarrow \Theta$ as:

$$V(\xi, X_c) = X_c^T P X_c + \int_0^1 \left(\xi^T P \xi\right) e^{-\mu x} dx \tag{23}$$

where μ is a positive scalar. Performing the same procedure as the one presented in the proof of Theorem 1 and considering $D_c = 0$, the following equality is obtained

$$\dot{V} = -\mu X_c^T \Lambda P X_c - \mu \int_0^1 \left(\xi^T \Lambda P \xi\right) e^{-\mu x} dx + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T H \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \tag{24}$$

where

$$H = \begin{bmatrix} A_c^T P + P A_c + C_c \Lambda P C_c + \mu \Lambda P & P B_c K \\ K^T B_c^T P & -e^{-\mu} \Lambda P \end{bmatrix} \tag{25}$$

Note that (22) is equivalent to $H \preceq 0$. This is obtained by multiplying both sides of (42) by $diag\left(P^{-1}, P^{-1}\right)$, commuting P^{-1} with C_c (both matrices being diagonal) and performing the variable transformations $Q = P^{-1}$ and $Y = KQ$. Therefore, as $H \preceq 0$ and $\Lambda - \rho I^{n \times n} \succ 0$, the function (23) is a Lyapunov function for the hyperbolic system (3) with boundary conditions (7). ∎

This corollary is interesting because, for systems where C_C is diagonal and $D_c = 0$, it provides a constructive approach to obtain the boundary control gain K using convex optimization algorithms after determining a suitable value of $\mu > 0$ (e.g. chosen to obtain a good performance versus robustness trade-off for the system considered).

Remark 3. The previous results (namely Theorem 1 and Corollary 1) extend to first-order hyperbolic systems with both negative and positive convective speeds ($\lambda_1 < ... < \lambda_m < 0 < \lambda_{m+1} < ... < \lambda_n$) by defining the state description $\xi = \begin{bmatrix} \xi_- \\ \xi_+ \end{bmatrix}$, where $\xi_- \in \mathbb{R}^m$ and $\xi_+ \in \mathbb{R}^{n-m}$, and the variable transformation $\tilde{\xi}(x,t) = \begin{pmatrix} \xi_-(1-x,t) \\ \xi_+(x,t) \end{pmatrix}$.

3.2 Stability of Parameter-Varying Linear Hyperbolic Systems with Dynamic Boundary Conditions

Let Z_φ be a non empty convex set of $\mathbb{R}^l$. Consider the general class of first-order LPV hyperbolic systems of order n defined as follows:

$$\partial_t \xi(x,t) + \Lambda(\varphi) \partial_x \xi(x,t) = 0 \quad \forall x \in [0,1], t \geq 0 \tag{26}$$

where $\xi : [0,1] \times [0,+\infty) \to \Theta$, φ is a varying parameter vector that takes values in the parameter space Z_φ, $\Lambda(\varphi) : Z_\varphi \to \mathbb{R}^{n \times n}$ is a diagonal and invertible matrix function (called the characteristic matrix) such that $\Lambda(\varphi) = diag(\lambda_1(\varphi), \lambda_2(\varphi), ..., \lambda_n(\varphi))$.

Assumption 2: Assume that the following inequalities hold for all $\varphi \in Z_\varphi$:

$$0 < \lambda_1(\varphi) < ... < \lambda_n(\varphi) \tag{27}$$

Consider the following dynamic boundary conditions for (26):

$$\dot{X}_c = A_c(\varphi)X_c + B_c(\varphi)u \tag{28}$$
$$Y_c = C_c X_c + D_c u$$

with

$$Y_c = \xi(0,t), \quad u = K\xi(1,t) \tag{29}$$

where $X_c \in \mathbb{R}^{n_x}$, $A_c : Z_\varphi \to \mathbb{R}^{n_x \times n_x}$, $B_c : Z_\varphi \to \mathbb{R}^{n_x \times n}$, $C_c \in \mathbb{R}^{n \times n_x}$, $D_c \in \mathbb{R}^{n \times n}$, $K \in \mathbb{R}^{n \times n}$, $u \in \mathbb{R}^n$ and $n_x \geq 1$. Define the initial condition for (26) as done in (9). Let the polytope Z_φ be defined as follows:

$$Z_\varphi := \{ [\varphi_1, ..., \varphi_l]^T \in \mathbb{R}^l \mid \varphi_i \in [\overline{\varphi}_i, \underline{\varphi}_i], \forall i = 1, ..., l \} \tag{30}$$

for given $\overline{\varphi}_i$, $\underline{\varphi}_i$ and $l \in \mathbb{N}^+$. We thus consider that all the admissible values of the vector φ are constrained in a hyperrectangle in the parameter space Z_φ. Consider the polytopic linear representation of the parameter varying characteristic matrix for all $\varphi \in Z_\varphi$ [3]:

$$\Lambda(\varphi) = \sum_{i=1}^{N_\varphi} \alpha_i(\varphi)\Lambda(w_i) \tag{31}$$

where $w_i \in Z_\varphi$ are the $N_\varphi = 2^l$ vertices of the polytope formed by all extremities ($\overline{\varphi}_i$ and $\underline{\varphi}_i$) of each varying parameter $\varphi \in Z_\varphi$, $\sum_{i=1}^{2^l} \alpha_i(\varphi)\Lambda(w_i) : Z_\varphi \to \mathbb{R}^{n \times n}$ and $\alpha_i(\varphi)$ is a scheduling function $\alpha_i : Z_\varphi \to [0,1]$ previously defined in (**??**) - (**??**). The polytopic representation (31) can also be considered for the matrices $A_c(\varphi)$ and $B_c(\varphi)$ of the boundary conditions (28) using the same scheduling function (**??**). Define the polytopic linear representation (PLR) of the parameter varying hyperbolic system (26) with boundary conditions (28) as follows:

$$\partial_t \xi(x,t) + \sum_{i=1}^{N_\varphi} \alpha_i(\varphi)\Lambda(w_i)\partial_x \xi(x,t) = 0 \tag{32}$$
$$\forall \varphi \in Z_\varphi, \quad \forall x \in [0,1], \quad t \geq 0$$

with boundary conditions

$$\dot{X}_c = \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) A_c(w_i) X_c + \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) B_c(w_i) u$$

$$Y_c = C_c X_c + D_c u$$

(33)

Based on the PLR (32) - (33), the following theorem states a sufficient condition to ensure the exponential stability for system (26) with boundary conditions (28) and initial condition (9) for all $\varphi \in Z_\varphi$.

Theorem 2. *[Stability analysis] Along with Assumptions 2, assume that there exists two diagonal positive definite matrices $P_1 \in \mathbb{R}^{n_x \times n_x}$ and $P_2 \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that the following matrix inequality is satisfied, for all $i = 1, ..., N_\varphi$,*

$$M_i = \begin{bmatrix} \begin{matrix} A_c(w_i)^T P_1 + P_1 A_c(w_i) + C_c^T \Lambda(w_i) P_2 C_c \\ + K D_c^T \Lambda(w_i) P_2 D_c K + \mu \Lambda(w_i) P_1 \end{matrix} & P_1 B_c(w_i) K + C_c^T \Lambda(w_i) P_2 D_c K \\ K^T B_c(w_i)^T P_1 + K^T D_c^T \Lambda(w_i) P_2 C_c & -e^{-\mu} \Lambda(w_i) P_2 \end{bmatrix} \preceq 0$$

(34)

Then there exist two constant scalars $a > 0$ and $b > 0$ such that, for all $\xi^0 \in L^2((0,1); \mathbb{R}^n)$ and $X_c^0 \in \mathbb{R}^{n_x}$, the solution of (26), (28) and (9) satisfies (12), for all $t \geq 0$.

Proof: Consider once again the Lyapunov function candidate (13). Computing the time derivative $\dot{V}$ of V along the classical C^1-solutions of (26) with boundary conditions (28) and initial conditions (9), gives the following:

$$\dot{V} = \dot{X}_c^T P_1 X_c + X_c^T P_1 \dot{X}_c + \int_0^1 \left(\dot{\xi}^T P_2 \xi + \xi^T P_2 \dot{\xi} \right) e^{-\mu x} dx$$

(35)

After integration by parts and taking into account the PLR (26) and (28), the following is obtained:

$$\dot{V} = \sum_{i=1}^{2^l} \alpha_i(\varphi) \Big[\left(X_c^T \left(A_c(w_i)^T P_1 + P_1 A_c(w_i) \right) X_c \right) + \left(\xi(1)^T K^T B_c(w_i)^T P_1 X_c \right.$$

$$+ X_c^T P_1 B_c(w_i) K \xi(1) \big) - \left[e^{-\mu x} \xi^T \Lambda(w_i) P_2 \xi \right] \big|_0^1 - \mu \int_0^1 \left(\xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \Big]$$

(36)

The previous equation can be written using the boundary conditions (28) as follows:

$$\dot{V} = \sum_{i=1}^{2^l} \alpha_i(\varphi) \Big[\left(X_c^T \left(A_c(w_i)^T P_1 + P_1 A_c(w_i) \right) X_c \right) + \left(\xi(1)^T K^T B_c(w_i)^T P_1 X_c \right.$$

$$+ X_c^T P_1 B_c(w_i) K \xi(1) \Big) - e^{-\mu} \xi(1)^T \Lambda(w_i) P_2 \xi(1) + X_c^T C_c^T \Lambda(w_i) P_2 C_c X_c$$

$$+ X_c^T C_c^T \Lambda(w_i) P_2 D_c K \xi(1) + \xi(1)^T K^T D_c^T \Lambda(w_i) P_2 C_c X_c$$

$$+ \xi(1)^T K^T D_c^T \Lambda(w_i) P_2 D_c K \xi(1) - \mu \int_0^1 \left(\xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \Big] \qquad (37)$$

$$= \sum_{i=1}^{2^l} \alpha_i(\varphi) \Big[-\mu X_c^T \Lambda(w_i) P_1 X_c - \mu \int_0^1 \left(\xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx$$

$$+ \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T M_i \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \Big]$$

where the matrix M_i is defined as in (34). The definition $\alpha_i \geq 0$ and the matrix inequality $M_i \preceq 0$ from (34) imply that the last term of (37) is always negative or zero. This gives the following inequality:

$$\dot{V} \leq \sum_{i=1}^{2^l} \alpha_i(\varphi) \left[-\mu X_c^T \Lambda(w_i) P_1 X_c - \mu \int_0^1 \left(\xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \right] \qquad (38)$$

From (27), it can be proved that there always exists a $\rho > 0$ such that $\Lambda(\varphi) - \rho I^{n \times n} \succ 0$ (e.g. ρ could be the smallest eigenvalue of $\Lambda(\varphi)$ over Z_φ). Moreover, the diagonality of P_1, P_2 and Λ implies that:

$$\dot{V} \leq -\mu \rho V(\xi, X_c) \qquad (39)$$

Therefore, the function (13) is a Lyapunov function for the hyperbolic system (26) with boundary conditions (28). ∎

Note that Theorem 2 allows verifying whether the boundary control gain K stabilizes the hyperbolic system (26) with boundary conditions (28) for all the varying parameters that belong to the convex set Z_φ.

As done for linear hyperbolic system, the following corollary gives sufficient conditions for the design of a stabilizing controller for the particular case where C_c is a diagonal matrix, $D_c = 0$ and $n = n_x$.

Corollary 2. *[Design of a stabilizing controller] Along with Assumptions 2, if C_c is diagonal and $D_c = 0$ and if there exists a diagonal positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that the following linear matrix inequality is satisfied, for all $i \in 1, ..., N_\varphi$,*

$$\begin{bmatrix} QA_c(w_i)^T + A_c(w_i)Q + C_c\Lambda(w_i)QC_c + \mu\Lambda(w_i)Q & B_c(w_i)Y \\ Y^T B_c(w_i)^T & -e^{-\mu}\Lambda(w_i)Q \end{bmatrix} \preceq 0 \qquad (40)$$

where $Y = KQ$, then there exist two constants $\alpha > 0$ and $M > 0$ such that, for all $\xi^0(x) \in L^2((0,1); \mathbb{R}^n)$ and $X_c^0 \in \mathbb{R}^n$, the solution of (26) *with boundary conditions* (28) *and initial condition* (9) *satisfies* (12) *for all $t \geq 0$.*

Proof: Consider once again the Lyapunov function candidate (23). Performing the same procedure as the one presented in the proof of Theorem 2 and considering $D_c = 0$, the following equality is obtained

$$\dot{V} = \sum_{i=1}^{2^l} \alpha_i(\varphi) \left[-\mu X_c^T \Lambda(w_i) P X_c - \mu \int_0^1 \left(\xi^T \Lambda(w_i) P \xi \right) e^{-\mu x} dx + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T H_i \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \right] \tag{41}$$

where

$$H_i = \begin{bmatrix} A_c(w_i)^T P + P A_c(w_i) + C_c \Lambda(w_i) P C_c + \mu \Lambda(w_i) P & P B_c(w_i) K \\ K^T B_c(w_i)^T P & -e^{-\mu} \Lambda(w_i) P \end{bmatrix} \tag{42}$$

Note that (22) is equivalent to $H_i \preceq 0$. This is obtained by multiplying both sides of (42) by $diag\left(P^{-1}, P^{-1}\right)$, commuting P^{-1} with C_c (both matrices being diagonal) and performing the variable transformations $Q = P^{-1}$ and $Y = KQ$. Therefore, as $H_i \preceq 0$ and $\Lambda(\varphi) - \rho I^{n \times n} \succ 0$, the function (23) is a Lyapunov function for the hyperbolic system (26) with boundary conditions (28). ∎

Remark 4. The previous results (namely Theorem 2 and Corollary 2) extend to first-order hyperbolic systems with both negative and positive convective speeds $(\lambda_1(\varphi) < ... < \lambda_m(\varphi) < 0 < \lambda_{m+1}(\varphi) < ... < \lambda_n(\varphi))$ by defining the state description $\xi = \begin{bmatrix} \xi_- \\ \xi_+ \end{bmatrix}$, where $\xi_- \in \mathbb{R}^m$ and $\xi_+ \in \mathbb{R}^{n-m}$, and the variable transformation $\tilde{\xi}(x,t) = \begin{pmatrix} \xi_-(1-x,t) \\ \xi_+(x,t) \end{pmatrix}$.

3.3 Stability of Quasi-Linear Hyperbolic Systems with Dynamic Boundary Conditions

Coron and co-workers have found some sufficient conditions to exponentially stabilize one-dimensional quasi-linear hyperbolic systems with static boundary conditions on a bounded interval by means of the following Lyapunov function (for more details refer to [11]).

$$V(\xi) = V_1(\xi) + V_2(\xi, \xi_x) + V_3(\xi, \xi_x, \xi_{xx}) \tag{43}$$

where

$$V_1(\xi) = \int_0^1 \left(\xi^T Q(\xi)\xi\right) e^{-\mu x} dx, \quad V_2(\xi, \xi_x) = \int_0^1 \left(\xi_x^T R(\xi)\xi_x\right) e^{-\mu x} dx,$$

$$V_3(\xi, \xi_x, \xi_{xx}) = \int_0^1 \left(\xi_{xx}^T S(\xi)\xi_{xx}\right) e^{-\mu x} dx \tag{44}$$

and $Q(\xi)$, $R(\xi)$ and $S(\xi)$ are symmetric positive definite matrices. In this section, we study the stability of system (1) with dynamic boundary conditions (7) in a different way. We represent the non-linear characteristic matrix $\Lambda(\xi)$ in a uncertain linear form described by a linear PDE that contains state variations, which are only known to belong to a prescribed bounded domain around the equilibrium. This allows using the results given in Section 3.2 for the study of the local stability of quasi-linear hyperbolic systems.

To represent the variation of the characteristic matrix with respect to the state, let us define the convex set Z_ξ as follows:

$$Z_\xi := \{[\xi_1, ..., \xi_n]^T \in \mathbb{R}^n \mid \xi_i \in [\overline{\xi}_i, \underline{\xi}_i], \forall i = 1, ..., n\} \tag{45}$$

where $\overline{\xi}_i$ and $\underline{\xi}_i$ are some maximal and minimal allowable values for ξ_i, respectively. In other words, the matrix $\Lambda(\xi)$ belongs to the matrix variation domain [6]:

$$D_\xi = \left\{ \Lambda : \Lambda = \sum_{i=1}^{2^n} \beta_i \Lambda(v_i), \beta_i \geq 0, \sum_{i=1}^{2^n} \beta_i = 1 \right\} \tag{46}$$

where $v_i \in Z_\xi$ are the $N_\xi = 2^n$ vertices of the polytope formed by all the state extremities $\overline{\xi}_i$ and $\underline{\xi}_i$. From this formulation we propose the following theorem, which gives sufficient conditions for the exponential stability of (1) with boundary conditions (7) and initial conditions (9) over the characteristic matrix variation domain D_ξ.

Theorem 3. *[Stability analysis] Along with Assumptions 1, assume that there exists two diagonal positive definite matrices $P_1 \in \mathbb{R}^{n_x \times n_x}$ and $P_2 \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that the following matrix inequality is satisfied, for all $i = 1, ..., N_\xi$,*

$$\begin{bmatrix} \begin{array}{c} A_c^T P_1 + P_1 A_c + C_c^T \Lambda(v_i) P_2 C_c \\ + K D_c^T \Lambda(v_i) P_2 D_c K + \mu \Lambda(v_i) P_1 \end{array} & P_1 B_c K + C_c^T \Lambda(v_i) P_2 D_c K \\ \hline K^T B_c^T P_1 + K^T D_c^T \Lambda(v_i) P_2 C_c & -e^{-\mu} \Lambda(v_i) P_2 \end{bmatrix} \preceq 0 \tag{47}$$

Then there exist two constant scalars $a > 0$ and $b > 0$ such that, for all $\xi^0 \in Z_\xi$ and $X_c^0 \in \mathbb{R}^{n_x}$, the solution of (1), (7) and (9) satisfies (12), for all $t \geq 0$.

Proof: The proof of this theorem is very similar to the one developed for Theorem 2. The main difference is that in this case, the stability is proved over the convex set of the state variation Z_ξ instead of the parameter space Z_φ. ∎

Similarly to the Corollaries 1 and 2, the following corollary can be formulated for quasilinear hyperbolic systems:

Corollary 3. *[Design of a stabilizing controller] Along with Assumption 1, if C_c is diagonal and $D_c = 0$ and if there exists a diagonal positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that the following linear matrix inequality is satisfied, for all $i \in 1, ..., N_\xi$,*

$$\begin{bmatrix} QA_c^T + A_cQ + C_c\Lambda(v_i)QC_c + \mu\Lambda(v_i)Q & B_cY \\ Y^TB_c^T & -e^{-\mu}\Lambda(v_i)Q \end{bmatrix} \preceq 0 \qquad (48)$$

where $Y = KQ$, then there exist two constants $\alpha > 0$ and $M > 0$ such that, for all $\xi^0(x) \in Z_\xi$ and $X_c^0 \in \mathbb{R}^n$, the solution of (1) with boundary conditions (7) and initial condition (9) satisfies (12), for all $t \geq 0$.

This corollary completes our results on dynamic boundary stabilization of hyperbolic systems.

4 Boundary Temperature Control Example

To further investigate the phenomenon of fluid transport in a Poiseuille flow with dynamics at the boundary conditions, an experimental setup has been designed to test and validate advanced control strategies. Figure 1 shows the schematic of the proposed device.
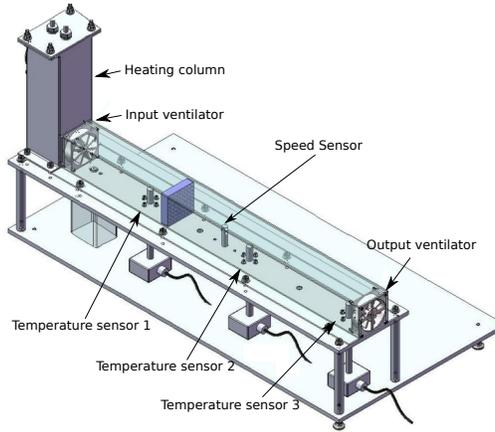
Fig. 1: Schematic of the experimental setup

This device mainly consists of a heating column encasing a resistor, a tube, two ventilators, a gas speed meter and distributed temperature sensors. The control problem is to regulate the outlet temperature of the tube by driving the power dissipated on the heating resistor at different air flow speeds (exogenous inputs produced by fans) through the tube. Only the outlet temperature and the flow speed will be considered as measurements for the closed loop boundary control strategy.

The modeling of the experimental setup presented in Figure 1 is done by considering two subsystems: the heating column and the tube. A zero-dimensional model (0-D) (control volume approach with heat exchanges coming from the heating resistor) represents the heating column. For the tube, a one dimensional (1-D) transport model is used to describe the gas density variations in the tube.

4.1 Heating Column Model

Figure 2 presents the schematic of the 0-D control volume approach considered for the heating column.
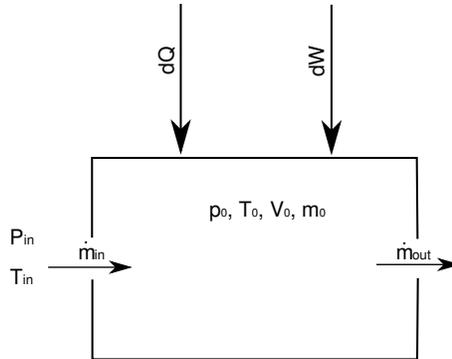
Fig. 2: Schematic of the control volume approach

Consider the internal energy of a perfect gas:

$$U_0 = C_v m_0 T_0 \tag{49}$$

where U_0 is the gas internal energy, T_0 the gas temperature, m_0 the mass inside the column and C_v the specific heat of the gas for constant volume. The time derivative of (49) is:

$$\dot{U}_0 = C_v m_0 \dot{T}_0 + C_v T_0 \dot{m}_0 \tag{50}$$

Using the first law of thermodynamics, the dynamics of the internal energy of the gas inside the column can also be given by:

$$\dot{U}_0 = \sum h_i \dot{m}_i + dQ + dW \tag{51}$$

where h_i is the specific enthalpy getting in and out of the volume with a mass flow rate $\dot{m}_i$, dQ quantifies the heat exchanges and dW is the work done by the gas. In the case of the heating column, there are two flows interacting with the volume, the input mass flow rate $\dot{m}_{in}$ and the output mass flow rate $\dot{m}_{out}$. As the gas does not perform any work, then $dW = 0$. In order to write (51) in terms of temperature, the specific enthalpy of a gas, defined by $h = C_p T$, where C_p is the specific heat constant at constant pressure, is used. Therefore, (51) can be expressed as:

$$\dot{U}_0 = C_p T_{in} \dot{m}_{in} - C_p T_0 \dot{m}_{out} + dQ \tag{52}$$

where T_{in} is the heating column input temperature. To simplify the model, consider the following two hypotheses:

Hypotheses 1:

- The pressure dynamics is much faster than the temperature dynamics, which allows considering a quasi-static behavior of the mass and pressure;
- $p_0 \approx p_{in}$, where p_{in} is the input pressure;

Hypotheses 1 allow writing (50) and (52), respectively as:

$$\dot{U}_0 = C_v m_0 \dot{T}_0 \tag{53}$$

$$\dot{U}_0 = C_p \dot{m}_{in}(T_{in} - T_0) + dQ \tag{54}$$

To simplify, overall in the 1-D model (see next sub-section), the temperature dynamics can be expressed in terms of the gas density by introducing the following change of variable (perfect gases law):

$$\rho_0 = \frac{p_{in}}{RT_0} \tag{55}$$

Taking the time derivative of (55) yields

$$\dot{T}_0 = -\frac{R}{p_{in}} T_0^2 \dot{\rho}_0 \tag{56}$$

Equalizing (53) and (54) and using (55) - (56) along with the perfect gases law to replace the mass inside the control volume m_0 in terms of the pressure and the specific gas constant R, the following is obtained:

$$\dot{\rho}_0 = -\frac{R\gamma T_{in}\dot{m}_{in}}{p_{in}V_0}\rho_0 - \frac{R}{p_{in}V_0 C_v}\rho_0 dQ + \frac{\gamma\dot{m}_{in}}{V_0} \tag{57}$$

where $\rho_0 = m_0/V_0$ is the density inside the heating column, V_0 is the column volume and $\gamma = \frac{C_p}{C_v}$.

4.2 Tube Model

To model the dynamics of the fluid inside the tube, the one-dimensional Euler equations are considered for a perfect gas and a constant tube cross section. These equations can be written in terms of the primitive variables (density ρ, particle speed u and pressure p) as follows (see [31]):

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial x} + \mathbf{C}(\mathbf{V}) = 0 \tag{58}$$

$$\mathbf{V} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} ; \mathbf{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & a^2\rho & u \end{bmatrix} ; \mathbf{C} = \begin{bmatrix} 0 \\ G \\ (\gamma-1)\rho(q+uG) \end{bmatrix}$$

where $a = \sqrt{\frac{\gamma p}{\rho}}$ is the speed of sound, G is a term associated with the friction losses and q is a term associated with the wall heat exchanges. In order to simplify (58), the following hypothesis are considered:

Hypotheses 2:

- the propagation speed of the entropy wave (average energy and mass) inside the tube is much slower than the sound speed $u << a$;
- the pressure inside the tube is considered constant (equal to the atmospheric pressure) because the pressure differential introduced by the fans is very small;
- the heat exchanges and the friction in the tube are neglected: $q = 0$ and $G = 0$.

Hypotheses 2, imply $\frac{\partial u(x,t)}{\partial x} \approx 0$ and $\frac{\partial p(x,t)}{\partial x} = 0$. This reduces system (58) to the following convection equation:

$$\frac{\partial \rho(x,t)}{\partial t} + u(t)\frac{\partial \rho(x,t)}{\partial x} = 0 \tag{59}$$

where $u(t)$ is the time-varying convection parameter of (59). The gas speed $u(t)$ in the tube is measured. Using Hypotheses 2 allows expressing the input mass flow rate as:

$$\dot{m}_{in} = u(t)\rho(0,t)A_t \tag{60}$$

where A_t is the tube cross section area. With (60), the boundary conditions of (59) can be expressed as:

$$\begin{aligned}\dot{\rho}_0 &= -\frac{R\gamma T_{in}u(t)\rho(0,t)A_t}{p_{in}V_0}\rho_0 - \frac{R}{p_{in}V_0 C_v}\rho_0 dQ \\ &+ \frac{\gamma u(t)\rho(0,t)A_t}{V_0}; \quad \rho(0,t) = \rho_0\end{aligned} \tag{61}$$

4.3 Output Temperature Boundary Control

The output temperature boundary control is designed for (59) with boundary conditions (61). Define the density error as:

$$\xi = \rho - \rho_{ref} \tag{62}$$

where ρ_{ref} is the desired output density. It is easy to show that system (59) with boundary conditions (61) can be expressed in an LPV form as follows:

$$\frac{\partial \xi(x,t)}{\partial t} + \Lambda(\varphi)\frac{\partial \xi(x,t)}{\partial x} = 0 \tag{63}$$

with boundary conditions:

$$\dot{\xi}_0 = A_c(\varphi)\xi_0 + B_c(\varphi)dQ + E_c(\varphi) \tag{64}$$

and with the varying parameters defined as follows:

$$\varphi_1 = u(t), \quad \varphi_2 = \rho_0, \quad \varphi_3 = \rho_0 u(t) \tag{65}$$

Only φ_1 is measured. φ_2 and φ_3 are considered as uncertain parameters as no temperature measurement is taken inside the heating column. Strictly speaking, system (64) is quasi-LPV because one of the parameters is a state. However, as ρ_0 can be easily bounded from the knowledge of the operating conditions of the experimental setup, then the system can be considered as an LPV one. Define the control input as:

$$dQ = F(\varphi, \rho_{ref}) + K\xi(L,t) \tag{66}$$

where $F(\varphi, \rho_{ref})$ is defined as:

$$
\begin{aligned}
F(\varphi, \rho_{ref}) &= -B_c(\varphi)^{-1} E_c(\varphi) \\
&= C_v \gamma u(t) A_t \left(\frac{p_{in}}{R} - T_{in}\rho_{ref} \right)
\end{aligned}
\tag{67}
$$

This yields to the system (59) with boundary conditions (61), which corresponds to the system considered in Theorem 1. Note that dQ is independent of the uncertain parameters φ_2 and φ_3, which is crucial for the boundary control implementation. Define the convex subset Z_φ in order to operate the experimental setup between the temperatures of 290 K and 323 K and a flow speed between 0.63 m/s and 3.82 m/s:

$$
\begin{aligned}
Z_\varphi := \{ & [\varphi_1, \varphi_2, \varphi_3]^T \in \mathbb{R}^3, \varphi_1 \in [0.63, 3.82], \\
& \varphi_2 \in [0.968, 1.08], \varphi_3 \in [0.61, 4.12] \}
\end{aligned}
\tag{68}
$$

In order to design the boundary control for system (59) with boundary conditions (61), consider the control architecture presented in Figure 3.
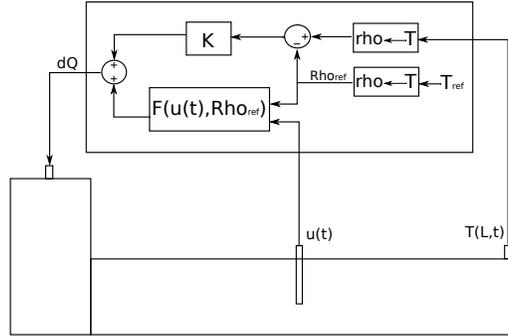
Fig. 3: Flow Tube Control Architecture

Consider the following system parameters: $p_{in} = 1 \times 10^5$ Pa, $T_{in} = 300$ K, $V_0 = 4 \times 10^{-3}$ m^3, $A_t = 6.4 \times 10^{-3}$ m^2, and $L = 1.5$ m where L is the tube's length. Applying Theorem 1 leads to the following control gain and Lyapunov function parameter P:

$$K = -654, \quad P = 1 \tag{69}$$

which ensure that the equilibrium $\xi = 0$ is exponentially stable $\forall \varphi \in Z_\varphi$. In order to illustrate the effectiveness of the proposed boundary control strategy, some simulation results of system (59) with boundary conditions (61) and the boundary control (69) are presented for different flow speeds. The results obtained are presented in Figures 4 and 5. A change of temperature reference from $300K$ to $320K$ (which can be transformed into a density reference using the perfect gases law) is introduced at $1s$.
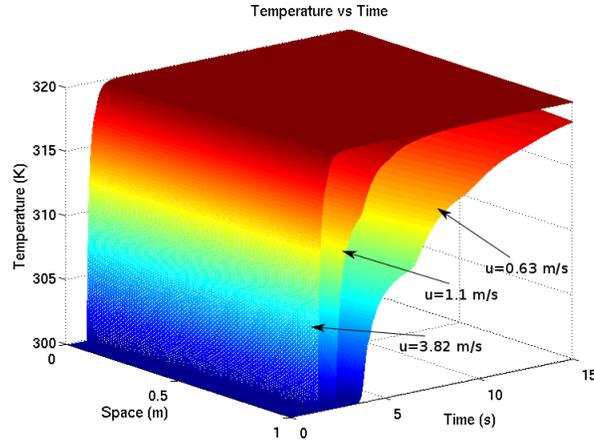
Fig. 4: Output temperature boundary control results for 3 different flow speed
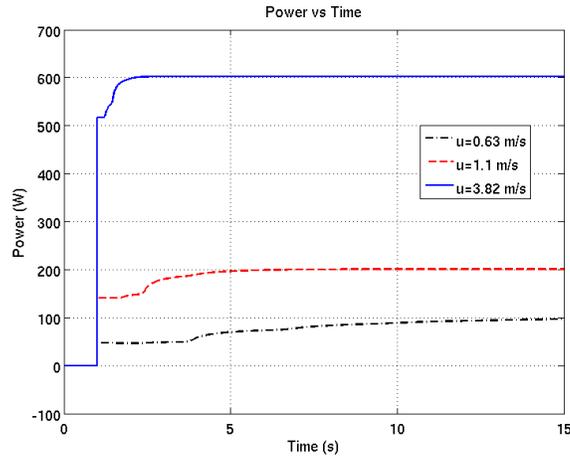
Fig. 5: Control Input for 3 different flow speed

As depicted in Figure 4, the system effectively follows the change of reference for the different flow speeds. The faster the flow speed, the faster the convergence, as the fluid transport time is smaller. Figure 5 shows the respective control inputs obtained for the simulation results. It appears that the power dissipated by the heating resistor has to be greater as the flow speed increases. This is due to the fact that in this case, the gas residence time inside the heating column is smaller and the amount of energy absorbed by the gas is less important.

5 Chapter Summary

This chapter focused on the stabilization of linear, LPV and quasi-linear first-order hyperbolic systems with dynamic boundary conditions and their application to temperature boundary control in a Poiseuille flow. We found sufficient conditions for boundary control design for linear, LPV and quasi-linear strict hyperbolic systems with n rightward convecting PDEs with dynamic boundary conditions. An extension of the strict Lyapunov function proposed in [12] was used to demonstrate the exponential stability of this class of infinite dimensional systems. In Theorem 1, we presented the sufficient conditions for the boundary control design for linear hyperbolic systems with dynamic boundary control, for all $\xi^0 : [0,1] \to \Theta$. In Theorem 2, a polytopic formulation was considered to state sufficient conditions for boundary control design for LPV hyperbolic systems with LPV dynamics at the boundary conditions over a convex set Z_φ. Then, in Theorem 3, by representing the non-linear characteristic matrix $\Lambda(\xi)$ in an uncertain linear form, sufficient conditions for exponential stability were found for all $\xi^0 \in Z_\xi$. Finally, an application of the main results of this chapter are illustrated by developing a temperature boundary control for an experimental setup specially designed to study the mass transport. Simulation results have shown the effectiveness of the proposed dynamic boundary control techniques.

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